



SAPIENZA
UNIVERSITÀ DI ROMA

Integer Linear Programming

Exercises 4 — MIP heuristics

Worked exercises on chapters 10–10, with complete solutions. Each exercise cites the catalogue problem it uses.

Fabio Furini

fabiofurini.github.io

Department of Computer, Control and Management Engineering — Sapienza University of Rome

Online version and downloadable material:
fabiofurini.github.io/integer-linear-programming/
github.com/fabiofurini/integer-linear-programming

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Part V

MIP heuristics



MIP heuristics

10.1 Rounding and diving

Exercise 10.1 — Rounding and diving ($P5$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ & 6x_1 + 21x_2 \leq 77 \\ & 8x_1 + 3x_2 \leq 36 \\ & x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

Questions

1. Compute the optimal solution $\tilde{\mathbf{x}}$ of the LP relaxation and its value $z(\text{LP})$.
2. Apply rounding: round $\tilde{\mathbf{x}}$ to the nearest integer point, check whether it is feasible, and then round it down.
3. Apply fractional diving: at each step solve the LP relaxation with the variables fixed so far, fix the least fractional variable to its nearest integer (ties: smallest index), and repeat until the LP solution is integer.
4. Compute the optimality gap of the solution of the dive with respect to the upper bound $\lfloor z(\text{LP}) \rfloor$, and prove that the solution is optimal by examining the possible values of x_2 .

Solution

1. The two constraints are tight at the optimal vertex: from $8x_1 + 3x_2 = 36$ we get $x_2 = 12 - \frac{8}{3}x_1$, and $6x_1 + 21x_2 = 77$ becomes $6x_1 + 252 - 56x_1 = 77$, i.e., $x_1 = \frac{7}{2}$ and

$$x_2 = \frac{8}{3}:$$

$$\tilde{\mathbf{x}} = \left(\frac{7}{2}, \frac{8}{3}\right), \quad z(\text{LP}) = 7 + 8 = 15.$$

2. $\left(\left\lceil \frac{7}{2} \right\rceil, \left\lceil \frac{8}{3} \right\rceil\right) = (4, 3)$ violates the second constraint: $8 \cdot 4 + 3 \cdot 3 = 41 > 36$. **Rounding fails.** Rounding down gives $(3, 2)$: $60 \leq 77$ and $30 \leq 36$, feasible, of value 12.
3. **Step 1.** The distances from the nearest integer are $\frac{1}{2}$ for x_1 and $\frac{1}{3}$ for x_2 : fix $x_2 := 3$.
Step 2. With $x_2 = 3$ the constraints become $6x_1 \leq 14$ and $8x_1 \leq 27$, so $\tilde{\mathbf{x}} = \left(\frac{7}{3}, 3\right)$, of value $\frac{14}{3} + 9 = \frac{41}{3}$. Fix $x_1 := \left\lceil \frac{7}{3} \right\rceil = 3$.
Step 3. The LP solution $(2, 3)$, of value 13, is integer: the dive returns $(2, 3)$.
4. The upper bound is $\lfloor 15 \rfloor = 15$, so the gap is $\frac{15-13}{13} \approx 15.4\%$. To prove optimality, let x_2 vary: $x_2 \geq 4$ is infeasible ($21 \cdot 4 = 84 > 77$); $x_2 = 3$ gives $x_1 \leq 2$, value at most 13; $x_2 = 2$ gives $x_1 \leq 3$, value at most 12; $x_2 = 1$ gives $x_1 \leq 4$, value at most 11; $x_2 = 0$ gives $x_1 \leq 4$, value at most 8. Hence $z(\text{ILP}) = 13$ and $(2, 3)$ is **optimal**.

Exercise 10.2 — Rounding and diving (P3)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{array}{rclcl} \max & 2x_1 & + & 4x_2 & \\ & -x_1 & + & x_2 & \leq 2 \\ & 8x_1 & + & 2x_2 & \leq 19 \\ & x_1, & & x_2 & \in \mathbb{Z}_{\geq 0} \end{array}$$

Questions

1. Compute the optimal solution $\tilde{\mathbf{x}}$ of the LP relaxation and its value $z(\text{LP})$.
2. Round $\tilde{\mathbf{x}}$ to the nearest integer point: is it feasible? Round it down instead, and give the value.
3. Apply fractional diving (ties: smallest index). Does it return a solution?
4. Repeat the dive taking the opposite decision at the first step, and continue with the rule of fractional diving. Then prove that the solution is optimal by examining the possible values of x_1 .

Solution

1. Both constraints are tight at the optimal vertex: $x_2 = 2 + x_1$ and $8x_1 + 2x_2 = 19$ give

$$10x_1 = 15:$$

$$\tilde{x} = \left(\frac{3}{2}, \frac{7}{2}\right), \quad z(\text{LP}) = 3 + 14 = 17.$$

2. $\left(\left\lceil \frac{3}{2} \right\rceil, \left\lceil \frac{7}{2} \right\rceil\right) = (2, 4)$ satisfies the first constraint ($2 \leq 2$) but violates the second: $16 + 8 = 24 > 19$. **Rounding fails.** Rounding down gives $(1, 3)$: $2 \leq 2$ and $14 \leq 19$, feasible, of value 14.

3. **Step 1.** Both variables are at distance $\frac{1}{2}$: fix $x_1 := \left\lceil \frac{3}{2} \right\rceil = 2$.

Step 2. With $x_1 = 2$ the constraints become $x_2 \leq 4$ and $2x_2 \leq 3$, so $\tilde{x} = \left(2, \frac{3}{2}\right)$, of value 10. Fix $x_2 := \left\lceil \frac{3}{2} \right\rceil = 2$.

Step 3. With $x_1 = 2$ and $x_2 = 2$ the second constraint is $16 + 4 = 20 > 19$: the LP is infeasible, **the dive fails.**

4. **Step 1.** Fix $x_1 := 1$ instead of 2. The constraints become $x_2 \leq 3$ and $2x_2 \leq 11$, so $\tilde{x} = (1, 3)$, of value 14, which is integer: the dive returns $(1, 3)$. The first decision of the first dive, $\left\lceil \frac{3}{2} \right\rceil = 2$, was the wrong one.

Optimality: the second constraint gives $x_1 \leq 2$. For $x_1 = 0$ it is $x_2 \leq 2$, value at most 8; for $x_1 = 1$ it is $x_2 \leq 3$, value at most 14; for $x_1 = 2$ it is $x_2 \leq 1$, value at most 8. Hence $z(\text{ILP}) = 14$ and $(1, 3)$ is **optimal**.

10.2 Feasibility pump

Exercise 10.3 — Feasibility pump 1

Consider the integer program

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{subject to} \quad & 6x_1 - 2x_2 \leq 16 \\ & 6x_1 - 7x_2 \geq 8 \\ & x_1 \in \{0, 1, \dots, 5\} \\ & x_2 \in \{0, 1, \dots, 5\} \end{aligned}$$

(Recall that the L_1 distance from a point to a segment that is neither horizontal, vertical, nor of slope ± 1 is minimized at an endpoint or where the segment crosses the horizontal or the vertical line through the point.)

Questions

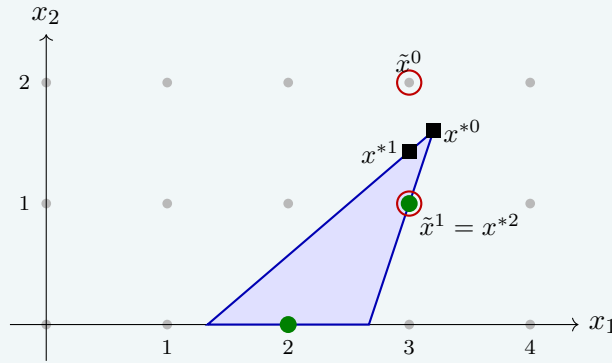
1. Draw the region $\mathcal{L}$ of the LP relaxation and its integer points, and compute the optimal solution x^{*0} of the LP relaxation.
2. Round x^{*0} to the nearest integer point $\tilde{x}^0$, and check whether it is feasible.

3. Apply the feasibility pump: at each iteration compute the projection x^{*k} , i.e., the point of $\mathcal{L}$ that minimizes the L_1 distance $\Delta(x, \tilde{x}^{k-1})$, round it, and, if the rounding gives back the same integer point, move the component with the largest $|x_j^{*k} - \tilde{x}_j^{k-1}|$ by one unit toward x_j^{*k} . Stop when the distance is 0.
4. Is the solution found by the pump optimal? Compare its value with the LP bound.

Solution

1. The constraint $6x_1 - 7x_2 \geq 8$ with $x_2 \geq 0$ requires $x_1 \geq \frac{4}{3}$, and $6x_1 - 2x_2 \leq 16$ with $x_2 \geq 0$ gives $x_1 \leq \frac{8}{3}$ on the axis. $\mathcal{L}$ is the triangle with vertices $(\frac{4}{3}, 0)$, $(\frac{8}{3}, 0)$ and the intersection of the two lines, $(\frac{16}{5}, \frac{8}{5})$. Its integer points are $(2, 0)$ and $(3, 1)$. The objective $x_1 + x_2$ is maximized at the vertex

$$x^{*0} = \left(\frac{16}{5}, \frac{8}{5}\right), \quad z_{LP}(I) = \frac{24}{5} = 4.8.$$



2. $\tilde{x}^0 = ([3.2], [1.6]) = (3, 2)$: $6 \cdot 3 - 2 \cdot 2 = 14 \leq 16$, but $6 \cdot 3 - 7 \cdot 2 = 4 < 8$: infeasible.
3. **Iteration 1.** Projection of $(3, 2)$. The vertical line $x_1 = 3$ crosses $\mathcal{L}$ for $18 - 2x_2 \leq 16$ and $18 - 7x_2 \geq 8$, i.e., between $(3, 1)$ and $(3, \frac{10}{7})$: the closest is $(3, \frac{10}{7})$, at distance $2 - \frac{10}{7} = \frac{4}{7}$. The horizontal line $x_2 = 2$ does not cross $\mathcal{L}$ ($x_1 \leq \frac{10}{3}$ and $x_1 \geq \frac{11}{3}$). The vertex $(\frac{16}{5}, \frac{8}{5})$ is at distance $\frac{1}{5} + \frac{2}{5} = \frac{3}{5} > \frac{4}{7}$, the other vertices are farther. Hence $x^{*1} = (3, \frac{10}{7})$, $\Delta = \frac{4}{7}$, and $\tilde{x}^1 = [x^{*1}] = (3, 1)$, a new point.

Iteration 2. $(3, 1) \in \mathcal{L}$ ($16 \leq 16$, $11 \geq 8$): $\Delta = 0$, the pump stops with $(3, 1)$, of value 4, after two projections.

4. The other integer point $(2, 0)$ has value 2, so $(3, 1)$ is **optimal**. The LP bound gives $z(I) \leq \lfloor 4.8 \rfloor = 4$ as well, which certifies the optimality without enumerating the integer points.

Exercise 10.4 — Feasibility pump 2

Consider the integer program

$$\max \quad 4x_1 + 4x_2$$

$$\begin{aligned}
\text{subject to } & -3x_1 + 5x_2 \leq 11 \\
& 4x_1 - 3x_2 \leq -1 \\
& x_1 \in \{0, 1, \dots, 5\} \\
& x_2 \in \{0, 1, \dots, 5\}
\end{aligned}$$

(Recall that the L_1 distance from a point to a segment that is neither horizontal, vertical, nor of slope ± 1 is minimized at an endpoint or where the segment crosses the horizontal or the vertical line through the point.)

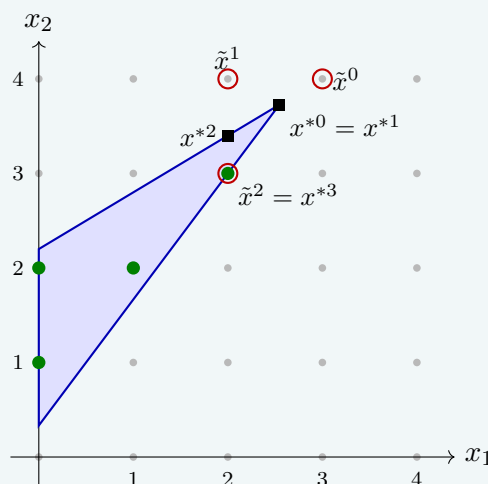
Questions

1. Draw the region $\mathcal{L}$ of the LP relaxation and its integer points, and compute the optimal solution x^{*0} of the LP relaxation.
2. Round x^{*0} to the nearest integer point $\tilde{x}^0$, and check whether it is feasible.
3. Apply the feasibility pump: at each iteration compute the projection x^{*k} , i.e., the point of $\mathcal{L}$ that minimizes the L_1 distance $\Delta(x, \tilde{x}^{k-1})$, round it, and, if the rounding gives back the same integer point, move the component with the largest $|x_j^{*k} - \tilde{x}_j^{k-1}|$ by one unit toward x_j^{*k} . Stop when the distance is 0.
4. How many times does the pump need a flip? Is the solution found optimal?

Solution

1. With $x_1 \geq 0$, $\mathcal{L}$ is the triangle with vertices $(0, \frac{1}{3})$ and $(0, \frac{11}{5})$, on the axis $x_1 = 0$, and the intersection of the two lines, $(\frac{28}{11}, \frac{41}{11})$. Its integer points are $(0, 1)$, $(0, 2)$, $(1, 2)$, $(2, 3)$. The objective is maximized at

$$x^{*0} = \left(\frac{28}{11}, \frac{41}{11}\right), \quad z_{LP}(I) = \frac{276}{11} \approx 25.09.$$



2. $\tilde{x}^0 = ([2.55], [3.73]) = (3, 4)$: $-9 + 20 = 11 \leq 11$, but $12 - 12 = 0 > -1$: infeasible.

3. **Iteration 1.** Projection of $(3, 4)$. The vertical line $x_1 = 3$ would need $x_2 \geq \frac{13}{3}$ and $x_2 \leq 4$: no crossing. The horizontal line $x_2 = 4$ would need $x_1 \geq 3$ and $x_1 \leq \frac{11}{4}$: no crossing. The closest point is the vertex $x^*1 = (\frac{28}{11}, \frac{41}{11})$, at distance $\frac{5}{11} + \frac{3}{11} = \frac{8}{11}$. Its rounding is $(3, 4)$ again: a **short cycle**. The largest difference is on x_1 ($\frac{5}{11} > \frac{3}{11}$), which moves toward $\frac{28}{11}$: $\tilde{x}^1 = (2, 4)$.

Iteration 2. Projection of $(2, 4)$. The vertical line $x_1 = 2$ crosses $\mathcal{L}$ for $x_2 \geq 3$ and $x_2 \leq \frac{17}{5}$: the closest point is $(2, \frac{17}{5})$, at distance $\frac{3}{5}$. The horizontal line $x_2 = 4$ does not cross $\mathcal{L}$, and the vertex $(\frac{28}{11}, \frac{41}{11})$ is at distance $\frac{6}{11} + \frac{3}{11} = \frac{9}{11} > \frac{3}{5}$. Hence $x^*2 = (2, \frac{17}{5})$ and $\tilde{x}^2 = (2, 3)$.

Iteration 3. $(2, 3) \in \mathcal{L}$ ($-6 + 15 = 9 \leq 11$, $8 - 9 = -1 \leq -1$): $\Delta = 0$, the pump stops with $(2, 3)$, of value 20.

4. The pump needs **one flip**: without it, it would cycle forever between $(3, 4)$ and the vertex x^*0 . The values of the integer points are 4, 8, 12, 20: $(2, 3)$ is **optimal**.

Exercise 10.5 — Feasibility pump 3

Consider the integer program

$$\begin{aligned} \max \quad & 4x_1 + 2x_2 \\ \text{subject to} \quad & 5x_1 + 4x_2 \leq 9 \\ & -7x_1 + 3x_2 \leq 5 \\ & x_1 \in \{0, 1, \dots, 5\} \\ & x_2 \in \{0, 1, \dots, 5\} \end{aligned}$$

(Recall that the L_1 distance from a point to a segment that is neither horizontal, vertical, nor of slope ± 1 is minimized at an endpoint or where the segment crosses the horizontal or the vertical line through the point.)

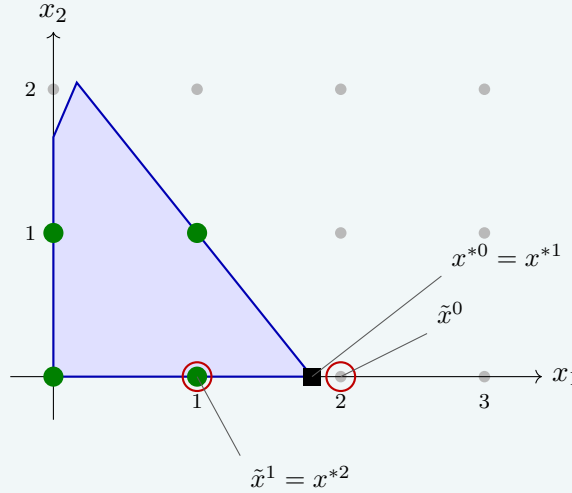
Questions

1. Draw the region $\mathcal{L}$ of the LP relaxation and its integer points, and compute the optimal solution x^*0 of the LP relaxation.
2. Round x^*0 to the nearest integer point $\tilde{x}^0$, and check whether it is feasible.
3. Apply the feasibility pump: at each iteration compute the projection x^{*k} , i.e., the point of $\mathcal{L}$ that minimizes the L_1 distance $\Delta(x, \tilde{x}^{k-1})$, round it, and, if the rounding gives back the same integer point, move the component with the largest $|x_j^{*k} - \tilde{x}_j^{k-1}|$ by one unit toward x_j^{*k} . Stop when the distance is 0.
4. Find the optimal solution by inspecting the integer points, and compute the gap of the solution of the pump with respect to the LP bound. Why does the pump miss the optimum?

Solution

1. $\mathcal{L}$ is the quadrilateral with vertices $(0,0)$, $(\frac{9}{5},0)$, $(\frac{7}{43}, \frac{88}{43})$ (intersection of the two lines) and $(0, \frac{5}{3})$. Its integer points are $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$. The objective values of the vertices are 0 , $\frac{36}{5}$, $\frac{204}{43} \approx 4.74$, $\frac{10}{3}$, so

$$x^{*0} = \left(\frac{9}{5}, 0\right), \quad z_{LP}(I) = \frac{36}{5} = 7.2.$$



2. $\tilde{x}^0 = ([1.8], 0) = (2, 0)$: $10 > 9$, infeasible.
3. **Iteration 1.** With $\tilde{x}^0 = (2, 0)$, $\Delta(x, \tilde{x}^0) = |x_1 - 2| + x_2$. The vertical line $x_1 = 2$ does not cross $\mathcal{L}$; the horizontal line $x_2 = 0$ crosses it on the segment from $(0,0)$ to $(\frac{9}{5}, 0)$, whose closest point is $(\frac{9}{5}, 0)$, at distance $\frac{1}{5}$; the other vertices are farther. So $x^{*1} = (\frac{9}{5}, 0) = x^{*0}$, whose rounding is $(2, 0)$ again: a short cycle. The flip moves x_1 toward $\frac{9}{5}$: $\tilde{x}^1 = (1, 0)$.

Iteration 2. $(1, 0) \in \mathcal{L}$: $\Delta = 0$, the pump stops with $(1, 0)$, of value 4.

4. The values of the integer points are 0, 4, 2, 6: the optimum is $(1, 1)$, with $z(I) = 6$ ($5 + 4 = 9 \leq 9$, $-7 + 3 \leq 5$). With respect to the LP bound $\lfloor 7.2 \rfloor = 7$ the gap of the pump's solution is $\frac{7-4}{7} \approx 42.9\%$ (its true relative error is $\frac{6-4}{6} \approx 33.3\%$). The pump looks only at the distance from $\mathcal{L}$, never at the objective after the first LP: it stops at the first feasible point it meets. Improving it is the job of methods such as local branching or proximity search.

10.3 RINS and local branching

Exercise 10.6 — RINS, local branching and proximity search (P23)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 3$ constraints:

$$\begin{aligned} \max \quad & x_1 + 4x_2 \\ & 3x_1 + 8x_2 \leq 24 \\ & -x_1 \leq -1 \\ & \quad -x_2 \leq -1 \\ & x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

The incumbent solution is $\bar{x} = (1, 2)$.

Questions

1. Compute the value of the incumbent, the optimal solution $\tilde{x}$ of the LP relaxation and the upper bound $\lfloor z(\text{LP}) \rfloor$.
2. Apply RINS with $\theta = 1$: list the variables that are fixed, write the sub-MIP with the cutoff constraint and solve it.
3. Apply local branching with $k = 1$, $k_{\max} = 3$, $k_{\text{step}} = 1$ and $\theta = 1$. Write the local branching constraints explicitly.
4. Apply proximity search with $\theta = 1$. Compare the three methods, and say which of them proves the optimality of the solution found.

Solution

1. The incumbent is feasible ($3 + 16 = 19 \leq 24$), of value $1 + 8 = 9$. The LP optimum is $x_1 = 1$ (its lower bound) and $x_2 = \frac{21}{8}$:

$$\tilde{x} = \left(1, \frac{21}{8}\right), \quad z(\text{LP}) = 1 + \frac{21}{2} = \frac{23}{2}, \quad \lfloor z(\text{LP}) \rfloor = 11.$$

2. $\tilde{x}_1 = 1 = \bar{x}_1$, while $\tilde{x}_2 = \frac{21}{8} \neq 2 = \bar{x}_2$: RINS fixes $x_1 := 1$. The sub-MIP is

$$\max \quad 1 + 4x_2 \quad \text{s.t.} \quad 3 + 8x_2 \leq 24, \quad 1 + 4x_2 \geq 10, \quad x_2 \in \mathbb{Z}_{\geq 1}.$$

The first constraint gives $x_2 \leq 2$, the cutoff $x_2 \geq 3$: the sub-MIP is infeasible and RINS returns the incumbent, of value 9.

3. With $\bar{x}^1 = (1, 2)$ the constraint $\Delta(x, \bar{x}^1) \leq 1$ reads $|x_1 - 1| + |x_2 - 2| \leq 1$, with the auxiliary variables $d_1 \geq x_1 - 1$, $d_1 \geq 1 - x_1$, $d_2 \geq x_2 - 2$, $d_2 \geq 2 - x_2$, $d_1 + d_2 \leq 1$; the cutoff is $x_1 + 4x_2 \geq 10$.

$k = 1$. The neighbours at distance 1 are $(2, 2)$, $(0, 2)$, $(1, 3)$, $(1, 1)$. Only $(2, 2)$ is feasible and has value 10: $\bar{x}^2 = (2, 2)$. Add $\Delta(x, \bar{x}^1) \geq 2$.

$k = 1, 2, 3$, **around** $\bar{x}^2$. The cutoff is $x_1 + 4x_2 \geq 11$. For $x_2 = 2$ the first constraint gives $x_1 \leq \frac{8}{3}$, value at most 10; for $x_2 \geq 3$ it is violated; for $x_2 = 1$ the cutoff needs $x_1 \geq 7$, which violates it. No solution: the method stops with $(2, 2)$, of value 10.

4. **Iteration 1.** Cutoff $x_1 + 4x_2 \geq 10$: the closest solution is $(2, 2)$, at distance 1, of value 10.

Iteration 2. Cutoff $x_1 + 4x_2 \geq 11$: infeasible, as shown in question 3. By Proposition ??, 10 is **optimal**.

Comparison: RINS fails, because the variable x_1 on which the LP and the incumbent agree is the one that must change; local branching finds the optimum with $k = 1$ but, stopping at $k_{\max}$, does not prove it; proximity search finds it and proves it. The bound $\lfloor z(\text{LP}) \rfloor = 11$ alone does not prove it.

Exercise 10.7 — RINS, local branching and proximity search (P4)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 5x_1 + 3x_2 \\ & 2x_1 + x_2 \leq 14 \\ & x_1 + 8x_2 \leq 32 \\ & x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

The incumbent solution is $\bar{x} = (5, 3)$.

Questions

1. Compute the value of the incumbent, the optimal solution $\tilde{x}$ of the LP relaxation and the upper bound $\lfloor z(\text{LP}) \rfloor$.
2. Apply RINS with $\theta = 1$: which variables are fixed, and what does the sub-MIP return?
3. Apply local branching with $k = 1$ and then, after adding the reverse constraint, with $k = 2$.
4. Apply proximity search with $\theta = 1$. Which methods prove the optimality of the solution found?

Solution

1. The incumbent is feasible ($13 \leq 14$, $29 \leq 32$), of value 34. Both constraints are tight

at the LP optimum: $x_2 = 14 - 2x_1$ and $x_1 + 8x_2 = 32$ give $15x_1 = 80$:

$$\tilde{x} = \left(\frac{16}{3}, \frac{10}{3}\right), \quad z(\text{LP}) = \frac{80}{3} + 10 = \frac{110}{3}, \quad \lfloor z(\text{LP}) \rfloor = 36.$$

2. $\frac{16}{3} \neq 5$ and $\frac{10}{3} \neq 3$: no variable is fixed, the sub-MIP is the whole problem with the cutoff $5x_1 + 3x_2 \geq 35$. Its solution is $(6, 2)$ ($12 + 2 = 14$, $6 + 16 = 22$), of value 36. Since $36 = \lfloor z(\text{LP}) \rfloor$, the solution is **optimal**.

3. $\bar{x}^1 = (5, 3)$. $k = 1$. The neighbours at distance 1 are $(4, 3)$, $(6, 3)$, $(5, 2)$, $(5, 4)$: $(6, 3)$ violates the first constraint ($15 > 14$), $(5, 4)$ the second ($37 > 32$), and $(4, 3)$, $(5, 2)$ have values 29 and $31 < 35$. No solution; add $\Delta(x, \bar{x}^1) \geq 2$ and set $k = 2$.

$k = 2$. The solution $(6, 2)$, at distance 2, has value $36 \geq 35$: $\bar{x}^2 = (6, 2)$. No solution of value at least 37 exists, since $37 > \lfloor z(\text{LP}) \rfloor$: the method continues to $k_{\max}$ and stops with 36.

4. **Iteration 1.** Cutoff $5x_1 + 3x_2 \geq 35$: no solution at distance 1 (question 3), and $(6, 2)$ is at distance 2: it is returned, of value 36.

Iteration 2. Cutoff ≥ 37 : infeasible. By Proposition ??, 36 is **optimal**.

RINS and proximity search find the optimum in one step; the bound $\lfloor z(\text{LP}) \rfloor = 36$ also proves it, here without any further work. Local branching finds it with $k = 2$ and does not need to prove it: the neighbourhood of radius 1 is not enough.

Exercise 10.8 — Proximity search ($P3$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 2x_1 + 4x_2 \\ & -x_1 + x_2 \leq 2 \\ & 8x_1 + 2x_2 \leq 19 \\ & x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

The incumbent solution is $\bar{x} = (0, 0)$. Among the solutions of the proximity sub-MIP at the same distance, choose the one of larger value.

Questions

1. Compute the value of the incumbent and the upper bound $\lfloor z(\text{LP}) \rfloor$ (the LP optimum is $(\frac{3}{2}, \frac{7}{2})$).
2. Apply the first iteration of proximity search with $\theta = 1$.
3. Apply the following iterations until the sub-MIP is infeasible.
4. Conclude: is the last solution optimal? How many sub-MIPs were solved?

Solution

1. The incumbent has value 0. The LP value is $z(\text{LP}) = 3 + 14 = 17$, so $\lfloor z(\text{LP}) \rfloor = 17$.
2. **Iteration 1.** Cutoff $2x_1 + 4x_2 \geq 1$. Two feasible points are at distance 1 from $(0, 0)$: $(1, 0)$, of value 2, and $(0, 1)$, of value 4. Choose $(0, 1)$.
3. **Iteration 2.** Cutoff $2x_1 + 4x_2 \geq 5$, around $(0, 1)$. At distance 1: $(0, 2)$, of value 8, and $(1, 1)$, of value 6. Choose $(0, 2)$.
Iteration 3. Cutoff ≥ 9 , around $(0, 2)$. At distance 1: $(0, 3)$ violates $-x_1 + x_2 \leq 2$; $(1, 2)$ has value 10 and is feasible ($1 \leq 2$, $12 \leq 19$). Return $(1, 2)$.
Iteration 4. Cutoff ≥ 11 , around $(1, 2)$. At distance 1: $(2, 2)$ violates the second constraint ($20 > 19$); $(1, 3)$ has value 14 and is feasible ($2 \leq 2$, $14 \leq 19$). Return $(1, 3)$.
Iteration 5. Cutoff ≥ 15 , around $(1, 3)$. The second constraint gives $x_1 \leq 2$, and for $x_1 = 0, 1, 2$ the values are at most 8, 14, 8: infeasible.
4. The last solution $(1, 3)$, of value 14, is **optimal** by Proposition ??: 5 sub-MIPs were solved, four of them with a solution. The bound 17 alone leaves a gap of $\frac{17-14}{14} \approx 21.4\%$. Each step moves by one unit because the cutoff is small: a larger θ would ask for larger moves and fewer iterations.