

SAPIENZA
UNIVERSITÀ DI ROMA

Integer Linear Programming

Exercises 2 — Exact methods

Worked exercises on chapters 6–8, with complete solutions. Each exercise cites the catalogue problem it uses.

Fabio Furini

fabiofurini.github.io

Department of Computer, Control and Management Engineering — Sapienza University of Rome

Online version and downloadable material:
fabiofurini.github.io/integer-linear-programming/
github.com/fabiofurini/integer-linear-programming

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Part III

Exact methods

Chapter 6

The cutting plane algorithm with Gomory fractional cuts

Exercise 6.1 — Cutting-plane algorithm ($P1$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 1$ linear constraint:

$$\begin{array}{rrcrcl} \max & x_1 & + & 8\,x_2 & & \\ & 3\,x_1 & + & 6\,x_2 & \leq & 16 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

and the following optimal simplex dictionary of its LP relaxation in standard form:

$$\begin{cases} x_2 = \dfrac{8}{3} - \dfrac{1}{2}\,x_1 - \dfrac{1}{6}\,x_3 \\ z = \dfrac{64}{3} - 3\,x_1 - \dfrac{4}{3}\,x_3 \end{cases}$$

Questions

1. Execute the cutting-plane algorithm based on Gomory fractional cuts using:

   (a) one Gomory fractional cut at each iteration;

   (b) the largest right-hand side as the cut selection strategy.

   Solve the LP problems after adding the Gomory fractional cut using the dual simplex algorithm. Graphically show the optimal solution of the LP problem at each iteration.

2. Graphically show the effect of each Gomory fractional cut on the LP relaxation of the ILP problem in canonical form.

Given a row $i \in \mathcal{B}$ of a simplex dictionary associated with a feasible basic solution such that $\bar{b}_i$ is fractional, the linear inequality

$$\sum_{j \in \mathcal{N}} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor)\, x_j \geq \bar{b}_i - \lfloor \bar{b}_i \rfloor$$

is called the **Gomory fractional cut** generated by row i.

Solution

Execution of the Cutting-Plane Algorithm to solve the ILP problem.

- **Initialization**

  Let us consider the LP_1 problem in standard form:

$$\begin{array}{lrcrcrcr} \max & x_1 & + & 8\,x_2 & & & & \\ & 3\,x_1 & + & 6\,x_2 & + & x_3 & = & 16 \\ & x_1, & & x_2, & & x_3 & \geq & 0 \end{array}$$

  and its optimal simplex dictionary:

$$\begin{cases} x_2 = \dfrac{8}{3} - \dfrac{1}{2}\, x_1 - \dfrac{1}{6}\, x_3 \\ z = \dfrac{64}{3} - 3\, x_1 - \dfrac{4}{3}\, x_3 \end{cases}$$

  The optimal solution $\tilde{\boldsymbol{x}}(\, LP_1\,)$ is **fractional** and the LP_1 problem is **feasible**.

  We have the upper bound on the optimal value of the ILP problem:

$$z(\, ILP\,) \leq \underbrace{\left\lfloor \frac{64}{3} \right\rfloor}_{=21}$$

  For the LP_1 problem in canonical form, we have:

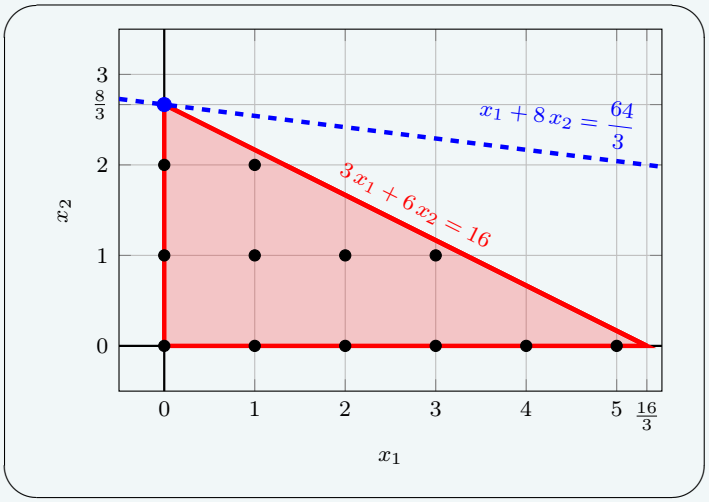
$$\begin{array}{lrcrcrcr} \max & x_1 & + & 8\,x_2 & & \\ & 3\,x_1 & + & 6\,x_2 & \leq & 16 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

An optimal solution $\tilde{\boldsymbol{x}}(\, LP_1\,)$ is:

$$\tilde{x}_1 = 0, \qquad \tilde{x}_2 = \frac{8}{3}$$

The optimal value is:

$$z(\; LP_1\;) = \frac{64}{3}$$



- **Iteration 1**

a) **Cut generation phase**

We have the following simplex dictionary associated with a fractional basic feasible solution of the LP_1 problem in standard form:

$$\begin{cases} x_2 = \dfrac{8}{3} - \dfrac{1}{2}\,x_1 - \dfrac{1}{6}\,x_3 \\ z = \dfrac{64}{3} - 3\,x_1 - \dfrac{4}{3}\,x_3 \end{cases}$$

where

$$\bar{\boldsymbol{A}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} \end{pmatrix} \qquad \bar{\boldsymbol{b}} = \begin{pmatrix} \frac{8}{3} \end{pmatrix}$$

– The Gomory fractional cut generated by the first row is:

$$\underbrace{\left(\frac{1}{2} - \left\lfloor \frac{1}{2} \right\rfloor\right)}_{\frac{1}{2}-0=\frac{1}{2}} x_1 + \underbrace{\left(\frac{1}{6} - \left\lfloor \frac{1}{6} \right\rfloor\right)}_{\frac{1}{6}-0=\frac{1}{6}} x_3 \geq \underbrace{\left(\frac{8}{3} - \left\lfloor \frac{8}{3} \right\rfloor\right)}_{\frac{8}{3}-2=\frac{2}{3}}$$

$$\frac{1}{2}\,x_1 + \frac{1}{6}\,x_3 \geq \frac{2}{3} \quad \Rightarrow \quad 3\,x_1 + x_3 \geq 4$$

By subtracting a surplus variable, we obtain the Gomory fractional cut in standard form:

$$\frac{1}{2}\,x_1 + \frac{1}{6}\,x_3 - x_4 = \frac{2}{3} \quad \Rightarrow \quad 3\,x_1 + x_3 - 6\,x_4 = 4$$

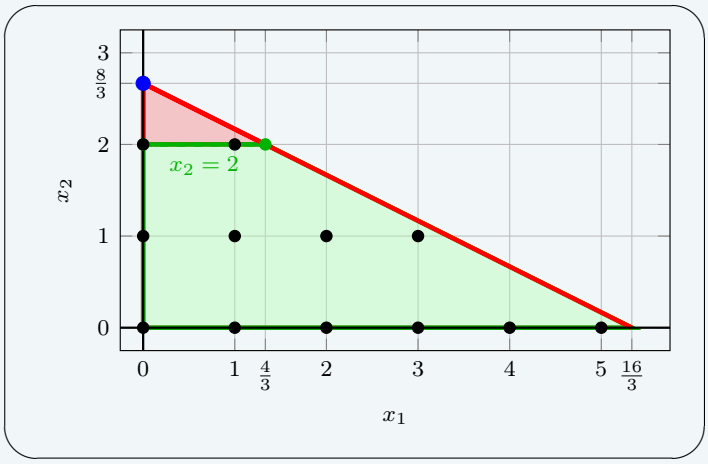
From the standard form of the LP_1 problem, we have:

$$x_3 = 16 - 3\,x_1 - 6\,x_2.$$

Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$3\,x_1 + (16 - 3\,x_1 - 6\,x_2) \geq 4 \quad \Rightarrow \quad x_2 \leq 2$$

Effect of the Gomory fractional cut generated by the first row on the LP_1 problem in canonical form:



– The Gomory fractional cut generated by the objective-function row is:

$$\underbrace{(3 - \lfloor 3 \rfloor)}_{3-3=0}\,x_1 + \underbrace{\left(\frac{4}{3} - \left\lfloor \frac{4}{3} \right\rfloor\right)}_{\frac{4}{3}-1=\frac{1}{3}}\,x_3 \geq \underbrace{\left(\frac{64}{3} - \left\lfloor \frac{64}{3} \right\rfloor\right)}_{\frac{64}{3}-21=\frac{1}{3}}$$

$$\frac{1}{3}\,x_3 \geq \frac{1}{3} \quad \Rightarrow \quad x_3 \geq 1$$

By subtracting a surplus variable, we obtain the Gomory fractional cut in standard form:

$$\frac{1}{3}\,x_3 - x_4 = \frac{1}{3} \quad \Rightarrow \quad x_3 - 3\,x_4 = 1$$

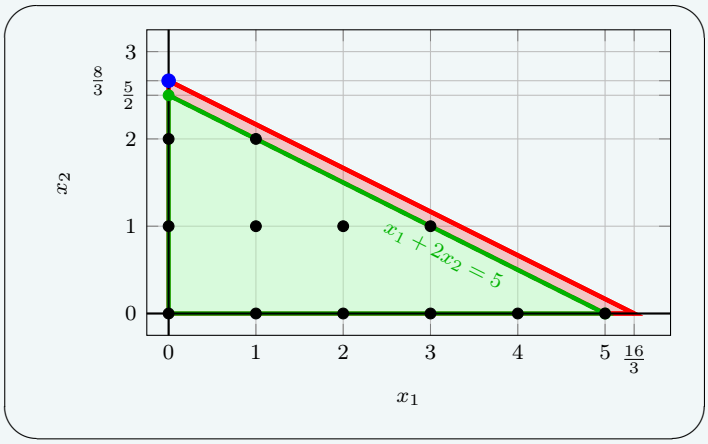
From the standard form of the LP_1 problem, we have:

$$x_3 = 16 - 3\,x_1 - 6\,x_2,$$

Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$16 - 3\,x_1 - 6\,x_2 \geq 1 \quad \Rightarrow \quad x_1 + 2\,x_2 \leq 5$$

Effect of the Gomory fractional cut generated by the objective-function row on the LP_1 problem in canonical form:



We select the Gomory fractional cut with the largest right-hand side and we create the LP_2 problem by adding it to the LP_1 problem.

b) **LP re-optimization phase**

Let us consider the LP_2 problem in standard form:

$$\begin{array}{lrcrcrcrcl}
\max & x_1 & + & 8\,x_2 & & & & & & \\
& 3\,x_1 & + & 6\,x_2 & + & x_3 & & & = & 16 \\
& & & x_2 & & & + & x_4 & = & 2 \\
& x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}$$

For the added Gomory fractional cut we have:

$$3\,x_1 + x_3 - 6\,x_4 = 4 \quad \Rightarrow \quad x_4 = -\frac{2}{3} + \frac{1}{2}\,x_1 + \frac{1}{6}\,x_3$$

We then add this equation to the previous optimal dictionary and we obtain the following dictionary :

$$\begin{cases}
x_2 = \frac{8}{3} - \frac{1}{2}\,x_1 - \frac{1}{6}\,x_3 \\
x_4 = -\frac{2}{3} + \frac{1}{2}\,x_1 + \frac{1}{6}\,x_3 \\
z = \frac{64}{3} - 3\,x_1 - \frac{4}{3}\,x_3
\end{cases}$$

where $\mathcal{B} = \{2, 4\}$ and $\mathcal{N} = \{1, 3\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_2 = \frac{8}{3}, \quad \bar{x}_4 = -\frac{2}{3}, \qquad \bar{x}_1 = 0, \quad \bar{x}_3 = 0 \quad \text{of value} \quad \frac{64}{3}$$

- the basic dual feasible solution is:

$$\bar{\pi}_2 = 0, \quad \bar{\pi}_4 = 0, \qquad \bar{\pi}_1 = 3, \quad \bar{\pi}_3 = \frac{4}{3} \quad \text{of value} \quad \frac{64}{3}$$

If we increase the value of π_4, we can get a better dual solution. We have:

$$\begin{cases} \pi_1 &= 3 - \frac{1}{2}\pi_4 \\ \pi_3 &= \frac{4}{3} - \frac{1}{6}\pi_4 \end{cases} \qquad \begin{cases} \underbrace{3 - \frac{1}{2}\pi_4 \geq 0}_{\pi_1 \geq 0} \\ \underbrace{\frac{4}{3} - \frac{1}{6}\pi_4 \geq 0}_{\pi_3 \geq 0} \end{cases} \qquad \begin{cases} \pi_4 &\leq 6 \\ \pi_4 &\leq 8 \end{cases}$$

By increasing the value of π_4 to $\min\{6, 8\} = 6$, the value of π_1 decreases to 0. Then we have that:

> π_4 becomes a dual basic variable and accordingly x_4 becomes a primal non basic variable
>
> π_1 becomes a dual non-basic variable and accordingly x_1 becomes a primal basic variable

We have:

$$x_4 = -\frac{2}{3} + \frac{1}{2}x_1 + \frac{1}{6}x_3 \quad \Longrightarrow \quad x_1 = \frac{4}{3} - \frac{1}{3}x_3 + 2x_4$$

Substituting into the other equations of the primal dictionary, we get:

$$x_2 = \frac{8}{3} - \frac{1}{2}\left(\frac{4}{3} - \frac{1}{3}x_3 + 2x_4\right) - \frac{1}{6}x_3 = 2 - x_4$$

$$z = \frac{64}{3} - 3\left(\frac{4}{3} - \frac{1}{3}x_3 + 2x_4\right) - \frac{4}{3}x_3 = \frac{52}{3} - \frac{1}{3}x_3 - 6x_4$$

Accordingly, we now have the following optimal primal dictionary:

$$\begin{cases} x_1 &= \frac{4}{3} - \frac{1}{3}x_3 + 2x_4 \\ x_2 &= 2 - x_4 \\ z &= \frac{52}{3} - \frac{1}{3}x_3 - 6x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$ and

– the basic primal optimal solution is:

$$\tilde{x}_1 = \frac{4}{3}, \quad \tilde{x}_2 = 2, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \qquad \text{of value } \frac{52}{3}$$

– the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = \frac{1}{3}, \quad \tilde{\pi}_4 = 6 \qquad \text{of value } \frac{52}{3}$$

The optimal solution $\tilde{\boldsymbol{x}}(\, LP_2\,)$ is **fractional** and the LP_2 problem is **feasible**.

We have the upper bound on the optimal value of the ILP problem:

$$z(\, ILP\,) \;\leq\; \underbrace{\left\lfloor \frac{52}{3} \right\rfloor}_{=17}$$

For the LP_2 problem in canonical form, we have:

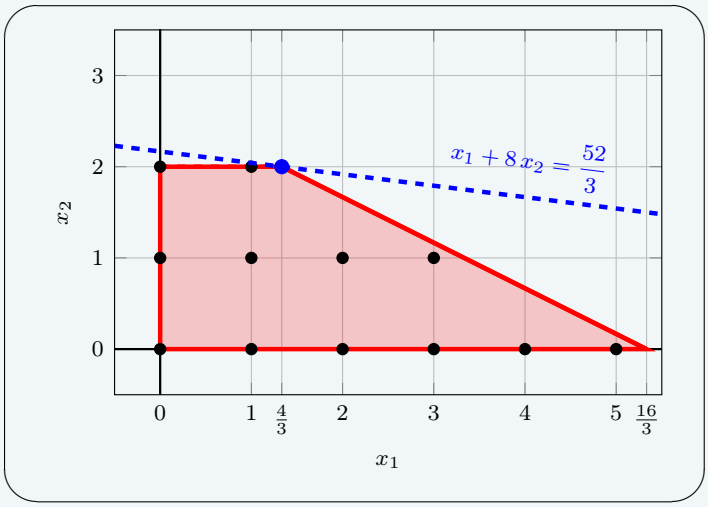
$$\begin{array}{lrcrcl} \max & x_1 & + & 8\,x_2 & & \\ & 3\,x_1 & + & 6\,x_2 & \leq & 16 \\ & & & x_2 & \leq & 2 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

An optimal solution $\tilde{\boldsymbol{x}}(\, LP_2\,)$ is:

$$\tilde{x}_1 = \frac{4}{3}, \qquad \tilde{x}_2 = 2$$

The optimal value is:

$$z(\, LP_2\,) = \frac{52}{3}$$



- **Iteration 2**

a) **Cut generation phase**

We have the following simplex dictionary associated with a fractional basic feasible solution of the LP_2 problem in standard form:

$$\begin{cases} x_1 = \frac{4}{3} - \frac{1}{3}\, x_3 + 2\, x_4 \\ x_2 = 2 - x_4 \\ z = \frac{52}{3} - \frac{1}{3}\, x_3 - 6\, x_4 \end{cases}$$

where

$$\bar{\boldsymbol{A}} = \begin{pmatrix} \frac{1}{3} & -2 \\ 0 & 1 \end{pmatrix} \qquad \bar{\boldsymbol{b}} = \begin{pmatrix} \frac{4}{3} \\ 2 \end{pmatrix}$$

– The Gomory fractional cut generated by the first row is:

$$\underbrace{\left(\frac{1}{3} - \left\lfloor \frac{1}{3} \right\rfloor\right)}_{\frac{1}{3}-0=\frac{1}{3}} x_3 + \underbrace{(-2 - \lfloor -2 \rfloor)}_{-2-(-2)=0} x_4 \geq \underbrace{\left(\frac{4}{3} - \left\lfloor \frac{4}{3} \right\rfloor\right)}_{\frac{4}{3}-1=\frac{1}{3}}$$

$$\frac{1}{3}\, x_3 \geq \frac{1}{3} \quad \Rightarrow \quad x_3 \geq 1$$

By subtracting a surplus variable, we obtain the Gomory fractional cut in standard form:

$$\frac{1}{3}\, x_3 - x_5 = \frac{1}{3} \quad \Rightarrow \quad x_3 - 3\, x_5 = 1$$

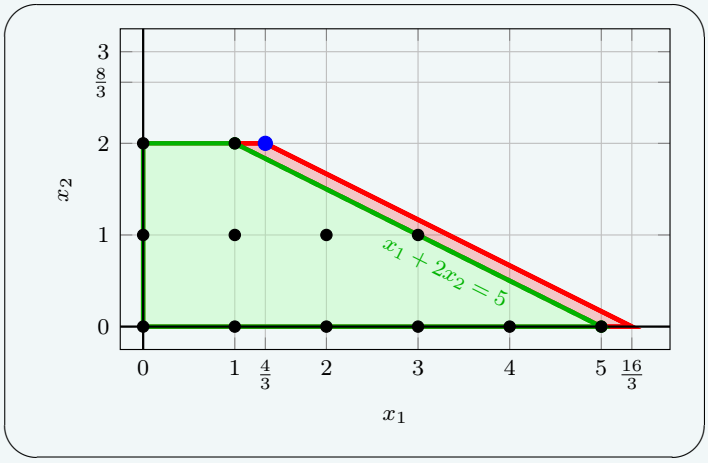
From the standard form of the LP_2 problem, we have:

$$x_3 = 16 - 3\, x_1 - 6\, x_2,$$

Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$(16 - 3\, x_1 - 6\, x_2) \geq 1 \quad \Rightarrow \quad x_1 + 2\, x_2 \leq 5.$$

Effect of the Gomory fractional cut generated by the first row on the LP_2 problem in canonical form:



– The Gomory fractional cut generated by the objective-function row is:

$$\underbrace{\left(\frac{1}{3} - \left\lfloor \frac{1}{3} \right\rfloor\right)}_{\frac{1}{3}-0=\frac{1}{3}} x_3 + \underbrace{(6 - \lfloor 6 \rfloor)}_{6-6=0} x_4 \geq \underbrace{\left(\frac{52}{3} - \left\lfloor \frac{52}{3} \right\rfloor\right)}_{\frac{52}{3}-17=\frac{1}{3}}$$

$$\frac{1}{3}\, x_3 \geq \frac{1}{3} \quad \Rightarrow \quad x_3 \geq 1$$

This Gomory fractional cut is the same as the one generated from the first row.

We select the Gomory fractional cut with the largest right-hand side and we create the LP_3 problem by adding it to the LP_2 problem.

b) **LP re-optimization phase**

Let us consider the LP_3 problem in standard form:

$$\begin{array}{lrcrcrcrcrcl}
\max & x_1 & + & 8\,x_2 & & & & & & & & \\
& 3\,x_1 & + & 6\,x_2 & + & x_3 & & & & & = & 16 \\
& & & x_2 & & & + & x_4 & & & = & 2 \\
& x_1 & + & 2\,x_2 & & & & & + & x_5 & = & 5 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
\end{array}$$

For the added Gomory fractional cut we have:

$$x_3 - 3\,x_5 = 1 \quad \Rightarrow \quad x_5 = -\frac{1}{3} + \frac{1}{3}\, x_3.$$

We then add this equation to the previous optimal dictionary and we obtain the following dictionary:

$$\begin{cases} x_1 = \frac{4}{3} - \frac{1}{3}\, x_3 + 2\, x_4 \\ x_2 = 2 - x_4 \\ x_5 = -\frac{1}{3} + \frac{1}{3}\, x_3 \\ z = \frac{52}{3} - \frac{1}{3}\, x_3 - 6\, x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{4}{3}, \quad \bar{x}_2 = 2, \quad \bar{x}_5 = -\frac{1}{3}, \qquad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \qquad \text{of value } \frac{52}{3}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_5 = 0, \qquad \bar{\pi}_3 = \frac{1}{3}, \quad \bar{\pi}_4 = 6 \qquad \text{of value } \frac{52}{3}$$

If we increase the value of π_5, we can get a better dual solution. We have:

$$\pi_3 = \frac{1}{3} - \frac{1}{3}\pi_5 \qquad \Longrightarrow \qquad \underbrace{\frac{1}{3} - \frac{1}{3}\pi_5 \geq 0}_{\pi_3 \geq 0} \qquad \Longrightarrow \qquad \pi_5 \leq 1$$

By increasing the value of π_5 to 1, the value of π_3 decreases to 0.
Then we have that:

π_5 becomes a dual basic variable and accordingly x_5 becomes a primal non-basic variable

π_3 becomes a dual non-basic variable and accordingly x_3 becomes a primal basic variable

We have:

$$x_5 = -\frac{1}{3} + \frac{1}{3}\, x_3 \quad \Longrightarrow \quad x_3 = 1 + 3\, x_5.$$

Substituting in the other equations of the primal dictionary, we get:

$$x_1 = \frac{4}{3} - \frac{1}{3}(1 + 3x_5) + 2x_4 = 1 + 2x_4 - x_5,$$

$$x_2 = 2 - x_4,$$

$$z = \frac{52}{3} - \frac{1}{3}(1 + 3x_5) - 6x_4 = 17 - 6x_4 - x_5.$$

Accordingly, we now have the following optimal primal dictionary:

$$\begin{cases} x_1 = 1 + 2\,x_4 - x_5 \\ x_2 = 2 - x_4 \\ x_3 = 1 + 3\,x_5 \\ z = 17 - 6\,x_4 - x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$.

- the basic primal optimal solution is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 2, \quad \tilde{x}_3 = 1, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \qquad \text{of value } 17$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 6, \quad \tilde{\pi}_5 = 1 \qquad \text{of value } 17$$

The optimal solution $\tilde{\boldsymbol{x}}(\,LP_3\,)$ is **integer**. Accordingly, an optimal solution $\tilde{\boldsymbol{x}}(\,ILP\,)$ is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 2$$

The optimal value $z(\,ILP\,)$ is 17.

For the LP_3 problem in canonical form, we have:

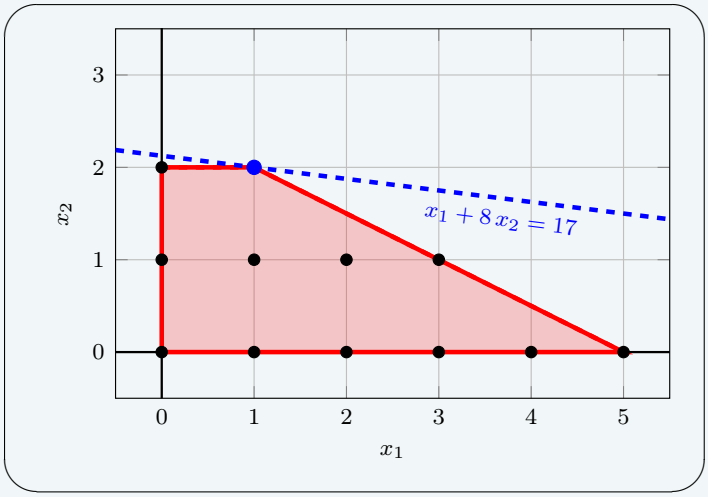
$$\begin{array}{lrcrcl} \max & x_1 & + & 8\,x_2 & & \\ & 3\,x_1 & + & 6\,x_2 & \leq & 16 \\ & & & x_2 & \leq & 2 \\ & x_1 & + & 2\,x_2 & \leq & 5 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

An optimal solution $\tilde{\boldsymbol{x}}(\,LP_3\,)$ is:

$$\tilde{x}_1 = 1, \qquad \tilde{x}_2 = 2$$

The optimal value is:

$$z(\,LP_3\,) = 17$$



Exercise 6.2 — Cutting-plane algorithm ($P2$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{array}{lrcrcrr} \max & & x_1 & + & 3 & x_2 & \\ & -4 & x_1 & + & 10 & x_2 & \leq & 35 \\ & 3 & x_1 & + & & x_2 & \leq & 12 \\ & & x_1, & & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

and an optimal dictionary of its LP relaxation in standard form:

$$\begin{cases} x_1 = \dfrac{5}{2} + \dfrac{1}{34}\, x_3 - \dfrac{5}{17}\, x_4 \\ x_2 = \dfrac{9}{2} - \dfrac{3}{34}\, x_3 - \dfrac{2}{17}\, x_4 \\ z = 16 - \dfrac{4}{17}\, x_3 - \dfrac{11}{17}\, x_4 \end{cases}$$

Questions

1. Execute the cutting-plane algorithm based on Gomory fractional cuts using:
   (a) one Gomory fractional cut at each iteration;
   (b) the largest right-hand side as the cut selection strategy.

   Solve the LP problems after adding the Gomory fractional cut using the dual simplex algorithm. Graphically show the optimal solution of the LP problem at each iteration.

2. Graphically show the effect of each Gomory fractional cut on the LP relaxation of the ILP problem in canonical form.

Given a row $i \in \mathcal{B}$ of a simplex dictionary associated with a feasible basic solution such that $\bar{b}_i$ is fractional, the linear inequality

$$\sum_{j \in \mathcal{N}} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor)\, x_j \geq \bar{b}_i - \lfloor \bar{b}_i \rfloor$$

is called the **Gomory fractional cut** generated by row i.

Solution

Execution of the Cutting-Plane Algorithm to solve the ILP problem.

- **Initialization**

  Let us consider the LP_1 problem in standard form:

$$\begin{array}{lrcrcrcrcl} \max & x_1 & + & 3 & x_2 & & & & & \\ & -4 & x_1 & + & 10 & x_2 & + & x_3 & = & 35 \\ & 3 & x_1 & + & & x_2 & & & + & x_4 & = & 12 \\ & & x_1, & & & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

  and its optimal simplex dictionary:

$$\begin{cases} x_1 = \dfrac{5}{2} + \dfrac{1}{34}\, x_3 - \dfrac{5}{17}\, x_4 \\ x_2 = \dfrac{9}{2} - \dfrac{3}{34}\, x_3 - \dfrac{2}{17}\, x_4 \\ z = 16 - \dfrac{4}{17}\, x_3 - \dfrac{11}{17}\, x_4 \end{cases}$$

  The optimal solution $\tilde{\boldsymbol{x}}(\, LP_1\,)$ is **fractional** and the LP_1 problem is **feasible**.

  We have the upper bound on the optimal value of the ILP problem:

$$z(\, ILP\,) \leq 16$$

  For the LP_1 problem in canonical form, we have:

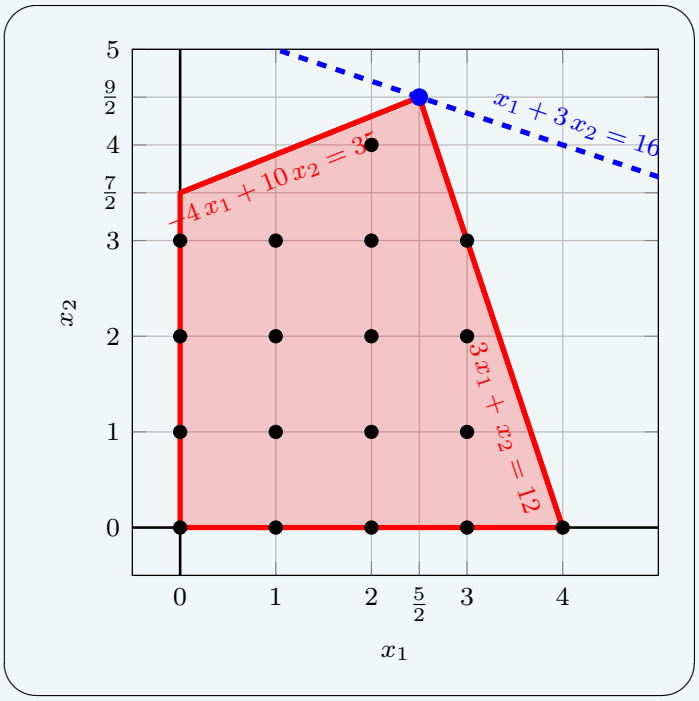
$$\begin{array}{lrcrcrl} \max & x_1 & + & 3 & x_2 & & \\ & -4\,x_1 & + & 10 & x_2 & \leq & 35 \\ & 3\,x_1 & + & & x_2 & \leq & 12 \\ & x_1, & & & x_2 & \geq & 0 \end{array}$$

An optimal solution $\tilde{\boldsymbol{x}}(\,LP_1\,)$ is:

$$\tilde{x}_1 = \frac{5}{2}, \qquad \tilde{x}_2 = \frac{9}{2}$$

The optimal value is:

$$z(\,LP_1\,) = 16$$



- **Iteration 1**

a) **Cut generation phase**

We have the following simplex dictionary associated with a fractional basic feasible solution of the LP_1 problem in standard form:

$$\begin{cases} x_1 = \frac{5}{2} + \frac{1}{34}\,x_3 - \frac{5}{17}\,x_4 \\ x_2 = \frac{9}{2} - \frac{3}{34}\,x_3 - \frac{2}{17}\,x_4 \\ z = 16 - \frac{4}{17}\,x_3 - \frac{11}{17}\,x_4 \end{cases}$$

where

$$\bar{\boldsymbol{A}} = \begin{pmatrix} -\frac{1}{34} & \frac{5}{17} \\ \frac{3}{34} & \frac{2}{17} \end{pmatrix} \qquad \bar{\boldsymbol{b}} = \begin{pmatrix} \frac{5}{2} \\ \frac{9}{2} \end{pmatrix}$$

– The Gomory fractional cut generated by the first row is:

$$\underbrace{\left(-\frac{1}{34} - \left\lfloor -\frac{1}{34} \right\rfloor\right)}_{-\frac{1}{34}-(-1)=\frac{33}{34}} x_3 + \underbrace{\left(\frac{5}{17} - \left\lfloor \frac{5}{17} \right\rfloor\right)}_{\frac{5}{17}-0=\frac{5}{17}} x_4 \geq \underbrace{\left(\frac{5}{2} - \left\lfloor \frac{5}{2} \right\rfloor\right)}_{\frac{5}{2}-2=\frac{1}{2}}$$

$$\frac{33}{34}\, x_3 + \frac{5}{17}\, x_4 \geq \frac{1}{2} \quad \Rightarrow \quad 33\, x_3 + 10\, x_4 \geq 17$$

By subtracting a surplus variable, we obtain the Gomory fractional cut in standard form:

$$\frac{33}{34}\, x_3 + \frac{5}{17}\, x_4 - x_5 = \frac{1}{2} \quad \Rightarrow \quad 33\, x_3 + 10\, x_4 - 34\, x_5 = 17$$

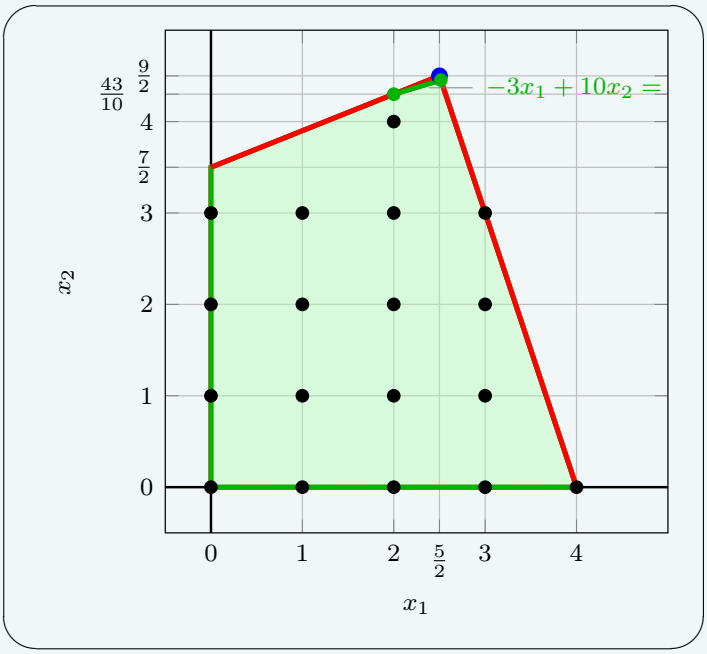
From the standard form of the LP_1 problem, we have:

$$x_3 = 35 + 4\, x_1 - 10\, x_2, \qquad x_4 = 12 - 3\, x_1 - x_2.$$

Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$33\,(35 + 4\, x_1 - 10\, x_2) + 10\,(12 - 3\, x_1 - x_2) \geq 17 \quad \Rightarrow \quad -3\, x_1 + 10\, x_2 \leq 37$$

Effect of the Gomory fractional cut generated by the first row on the LP_1 problem in canonical form:



– The Gomory fractional cut generated by the second row is:

$$\underbrace{\left(\frac{3}{34} - \left\lfloor \frac{3}{34} \right\rfloor\right)}_{\frac{3}{34} - 0 = \frac{3}{34}} x_3 + \underbrace{\left(\frac{2}{17} - \left\lfloor \frac{2}{17} \right\rfloor\right)}_{\frac{2}{17} - 0 = \frac{2}{17}} x_4 \geq \underbrace{\left(\frac{9}{2} - \left\lfloor \frac{9}{2} \right\rfloor\right)}_{\frac{9}{2} - 4 = \frac{1}{2}}$$

$$\frac{3}{34}\, x_3 + \frac{2}{17}\, x_4 \geq \frac{1}{2} \quad \Rightarrow \quad 3\, x_3 + 4\, x_4 \geq 17$$

By subtracting a surplus variable, we obtain the Gomory fractional cut in standard form:

$$\frac{3}{34}\, x_3 + \frac{2}{17}\, x_4 - x_5 = \frac{1}{2} \quad \Rightarrow \quad 3\, x_3 + 4\, x_4 - 34\, x_5 = 17$$

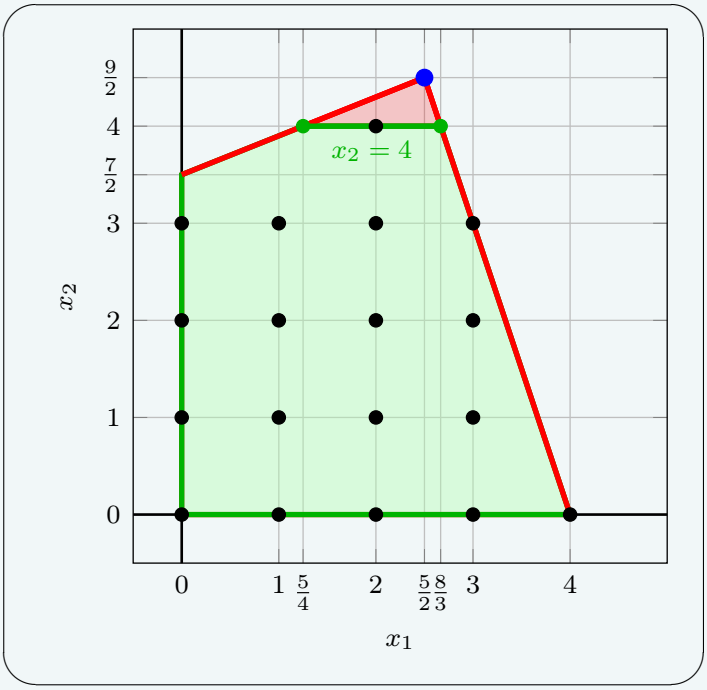
From the standard form of the LP_1 problem, we have:

$$x_3 = 35 + 4\, x_1 - 10\, x_2, \qquad x_4 = 12 - 3\, x_1 - x_2.$$

Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$3\,(35 + 4\, x_1 - 10\, x_2) + 4\,(12 - 3\, x_1 - x_2) \geq 17 \quad \Rightarrow \quad x_2 \leq 4$$

Effect of the Gomory fractional cut generated by the second row on the LP_1 problem in canonical form:



Both Gomory fractional cuts have the same right-hand side, equal to $\frac{1}{2}$. We select the Gomory fractional cut generated by the second row and we create the LP_2 problem by adding it to the LP_1 problem.

b) **LP re-optimization phase**

Let us consider the LP_2 problem in standard form:

$$\begin{array}{lrcrcrcrcrcr} \max & & x_1 & + & 3\ x_2 & & & & & & & \\ & -4 & x_1 & + & 10\ x_2 & + & x_3 & & & & = & 35 \\ & 3 & x_1 & + & x_2 & & & + & x_4 & & = & 12 \\ & & & & x_2 & & & & & + & x_5 & = & 4 \\ & & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0 \end{array}$$

For the added Gomory fractional cut we have:

$$3\ x_3 + 4\ x_4 - 34\ x_5 = 17 \qquad \Rightarrow \quad x_5 = -\frac{1}{2} + \frac{3}{34}\ x_3 + \frac{2}{17}\ x_4$$

We then add this equation to the previous optimal dictionary and we obtain the following dictionary :

$$\begin{cases} x_1 = \frac{5}{2} + \frac{1}{34}\ x_3 - \frac{5}{17}\ x_4 \\ x_2 = \frac{9}{2} - \frac{3}{34}\ x_3 - \frac{2}{17}\ x_4 \\ x_5 = -\frac{1}{2} + \frac{3}{34}\ x_3 + \frac{2}{17}\ x_4 \\ z = 16 - \frac{4}{17}\ x_3 - \frac{11}{17}\ x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{5}{2}, \quad \bar{x}_2 = \frac{9}{2}, \quad \bar{x}_5 = -\frac{1}{2}, \qquad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad 16$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_5 = 0, \qquad \bar{\pi}_3 = \frac{4}{17}, \quad \bar{\pi}_4 = \frac{11}{17} \quad \text{of value} \quad 16$$

If we increase the value of π_5, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 = \frac{4}{17} - \frac{3}{34}\pi_5 \\ \pi_4 = \frac{11}{17} - \frac{2}{17}\pi_5 \end{cases} \qquad \begin{cases} \underbrace{\frac{4}{17} - \frac{3}{34}\pi_5 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{11}{17} - \frac{2}{17}\pi_5 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_5 \leq \frac{8}{3} \\ \pi_5 \leq \frac{11}{2} \end{cases}$$

By increasing the value of π_5 to $\min\left\{\frac{8}{3}, \frac{11}{2}\right\} = \frac{8}{3}$, the value of π_3 decreases to 0. Then we have that:

π_5 becomes a dual basic variable and accordingly x_5 becomes a primal non basic variable

π_3 becomes a dual non-basic variable and accordingly x_3 becomes a primal basic variable

We have:

$$x_5 = -\frac{1}{2} + \frac{3}{34}x_3 + \frac{2}{17}x_4 \quad \text{then} \quad x_3 = \frac{17}{3} - \frac{4}{3}x_4 + \frac{34}{3}x_5.$$

Substituting in the other equations of the primal dictionary, we get:

$$x_1 = \frac{5}{2} + \frac{1}{34}\left(\frac{17}{3} - \frac{4}{3}x_4 + \frac{34}{3}x_5\right) - \frac{5}{17}x_4 = \frac{8}{3} - \frac{1}{3}x_4 + \frac{1}{3}x_5$$

$$x_2 = \frac{9}{2} - \frac{3}{34}\left(\frac{17}{3} - \frac{4}{3}x_4 + \frac{34}{3}x_5\right) - \frac{2}{17}x_4 = 4 - x_5$$

$$z = 16 - \frac{4}{17}\left(\frac{17}{3} - \frac{4}{3}x_4 + \frac{34}{3}x_5\right) - \frac{11}{17}x_4 = \frac{44}{3} - \frac{1}{3}x_4 - \frac{8}{3}x_5$$

Accordingly, we now have the following optimal primal dictionary:

$$\begin{cases} x_1 = \frac{8}{3} - \frac{1}{3}x_4 + \frac{1}{3}x_5 \\ x_2 = 4 - x_5 \\ x_3 = \frac{17}{3} - \frac{4}{3}x_4 + \frac{34}{3}x_5 \\ z = \frac{44}{3} - \frac{1}{3}x_4 - \frac{8}{3}x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = \frac{8}{3}, \quad \tilde{x}_2 = 4, \quad \tilde{x}_3 = \frac{17}{3}, \qquad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad \frac{44}{3}$$

– the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \qquad \tilde{\pi}_4 = \frac{1}{3}, \quad \tilde{\pi}_5 = \frac{8}{3} \quad \text{of value} \quad \frac{44}{3}$$

The optimal solution $\tilde{\boldsymbol{x}}(\, LP_2\,)$ is **fractional** and the LP_2 problem is **feasible**.

We have the upper bound on the optimal value of the ILP problem:

$$z(\, ILP\,) \;\leq\; \underbrace{\left\lfloor \frac{44}{3} \right\rfloor}_{=14}$$

For the LP_2 problem in canonical form, we have:

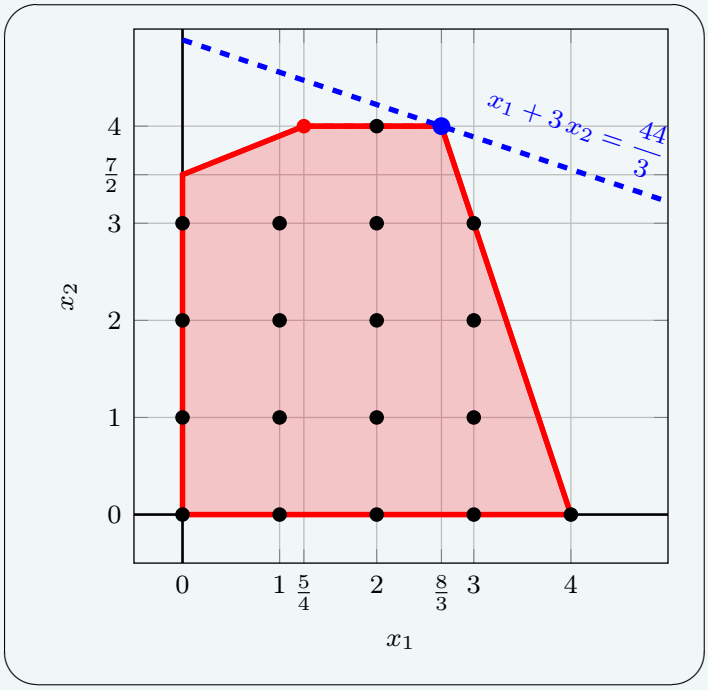
$$\begin{array}{rrcrcrl} \max & x_1 & + & 3 & x_2 & & \\ & -4\ x_1 & + & 10 & x_2 & \leq & 35 \\ & 3\ x_1 & + & & x_2 & \leq & 12 \\ & & & & x_2 & \leq & 4 \\ & x_1, & & & x_2 & \geq & 0 \end{array}$$

An optimal solution $\tilde{\boldsymbol{x}}(\, LP_2\,)$ is:

$$\tilde{x}_1 = \frac{8}{3}, \qquad \tilde{x}_2 = 4$$

The optimal value is:

$$z(\, LP_2\,) = \frac{44}{3}$$



- **Iteration 2**

a) **Cut generation phase**

We have the following simplex dictionary associated with a fractional basic feasible solution of the LP_2 problem in standard form:

$$\begin{cases} x_1 = \frac{8}{3} - \frac{1}{3}\,x_4 + \frac{1}{3}\,x_5 \\ x_2 = 4 - x_5 \\ x_3 = \frac{17}{3} - \frac{4}{3}\,x_4 + \frac{34}{3}\,x_5 \\ z = \frac{44}{3} - \frac{1}{3}\,x_4 - \frac{8}{3}\,x_5 \end{cases}$$

where

$$\bar{\boldsymbol{A}} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 \\ \frac{4}{3} & -\frac{34}{3} \end{pmatrix} \qquad \bar{\boldsymbol{b}} = \begin{pmatrix} \frac{8}{3} \\ 4 \\ \frac{17}{3} \end{pmatrix}$$

– The Gomory fractional cut generated by the first row is:

$$\underbrace{\left(\frac{1}{3} - \left\lfloor \frac{1}{3} \right\rfloor\right)}_{\frac{1}{3}-0=\frac{1}{3}} x_4 + \underbrace{\left(-\frac{1}{3} - \left\lfloor -\frac{1}{3} \right\rfloor\right)}_{-\frac{1}{3}+1=\frac{2}{3}} x_5 \geq \underbrace{\left(\frac{8}{3} - \left\lfloor \frac{8}{3} \right\rfloor\right)}_{\frac{8}{3}-2=\frac{2}{3}}$$

$$\frac{1}{3}\,x_4 + \frac{2}{3}\,x_5 \geq \frac{2}{3} \quad \Rightarrow \quad x_4 + 2\,x_5 \geq 2$$

By subtracting a surplus variable, we obtain the Gomory fractional cut in standard form:

$$\frac{1}{3}\,x_4 + \frac{2}{3}\,x_5 - x_6 = \frac{2}{3} \quad \Rightarrow \quad x_4 + 2\,x_5 - 3\,x_6 = 2$$

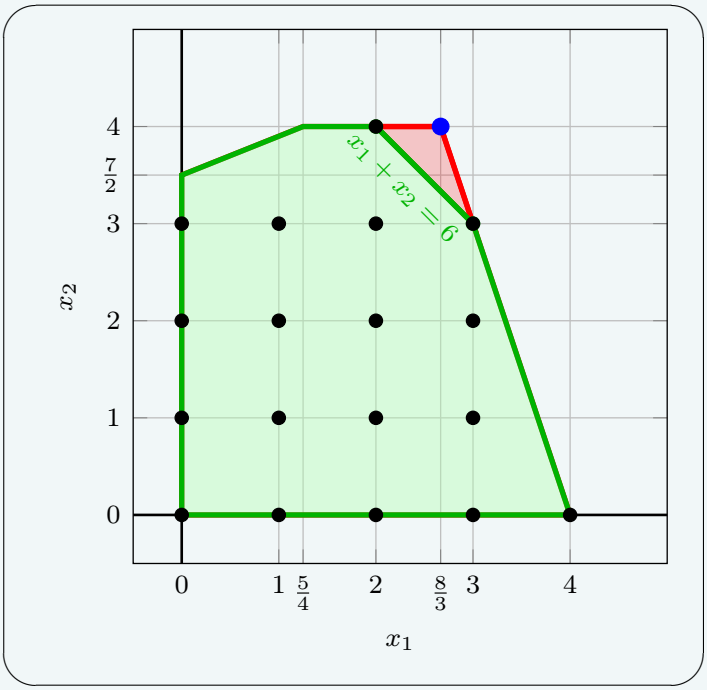
From the standard form of the LP_2 problem, we have:

$$x_4 = 12 - 3\,x_1 - x_2, \qquad x_5 = 4 - x_2.$$

Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$(12 - 3\,x_1 - x_2) + 2\,(4 - x_2) \geq 2 \quad \Rightarrow \quad x_1 + x_2 \leq 6.$$

Effect of the Gomory fractional cut generated by the first row on the LP_2 problem in canonical form:



– The Gomory fractional cut generated by the third row is:

$$\underbrace{\left(\frac{4}{3} - \left\lfloor \frac{4}{3} \right\rfloor\right)}_{\frac{4}{3}-1=\frac{1}{3}} x_4 + \underbrace{\left(-\frac{34}{3} - \left\lfloor -\frac{34}{3} \right\rfloor\right)}_{-\frac{34}{3}+12=\frac{2}{3}} x_5 \geq \underbrace{\left(\frac{17}{3} - \left\lfloor \frac{17}{3} \right\rfloor\right)}_{\frac{17}{3}-5=\frac{2}{3}}$$

$$\frac{1}{3}\, x_4 + \frac{2}{3}\, x_5 \geq \frac{2}{3} \quad \Rightarrow \quad x_4 + 2\, x_5 \geq 2$$

This is the same Gomory fractional cut obtained with the first generating row.

We select the Gomory fractional cut with the largest right-hand side and we create the LP_3 problem by adding it to the LP_2 problem.

b) **LP re-optimization phase**

Let us consider the LP_3 problem in standard form:

$$
\begin{array}{lrcrcrcrcrcrcr}
\max & x_1 & + & 3 & x_2 & & & & & & & & \\
 & -4 & x_1 & + & 10 & x_2 & + & x_3 & & & & = & 35 \\
 & 3 & x_1 & + & & x_2 & & + & x_4 & & & = & 12 \\
 & & & & & x_2 & & & & + & x_5 & = & 4 \\
 & & x_1 & + & & x_2 & & & & & + & x_6 = & 6 \\
 & & x_1, & & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 \geq & 0
\end{array}
$$

For the added Gomory fractional cut we have:

$$x_4 + 2\,x_5 - 3\,x_6 = 2 \qquad \Rightarrow \qquad x_6 = -\frac{2}{3} + \frac{1}{3}x_4 + \frac{2}{3}\,x_5$$

We then add this equation to the previous optimal dictionary and we obtain the following dictionary :

$$
\begin{cases}
x_1 = \dfrac{8}{3} - \dfrac{1}{3}\,x_4 + \dfrac{1}{3}\,x_5 \\
x_2 = 4 - x_5 \\
x_3 = \dfrac{17}{3} - \dfrac{4}{3}\,x_4 + \dfrac{34}{3}\,x_5 \\
x_6 = -\dfrac{2}{3} + \dfrac{1}{3}\,x_4 + \dfrac{2}{3}\,x_5 \\
z = \dfrac{44}{3} - \dfrac{1}{3}\,x_4 - \dfrac{8}{3}\,x_5
\end{cases}
$$

where $\mathcal{B} = \{1, 2, 3, 6\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{8}{3}, \quad \bar{x}_2 = 4, \quad \bar{x}_3 = \frac{17}{3}, \quad \bar{x}_6 = -\frac{2}{3}, \qquad \bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{44}{3}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_6 = 0, \qquad \bar{\pi}_4 = \frac{1}{3}, \quad \bar{\pi}_5 = \frac{8}{3} \quad \text{of value} \quad \frac{44}{3}$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$
\begin{cases}
\pi_4 = \dfrac{1}{3} - \dfrac{1}{3}\pi_6 \\
\pi_5 = \dfrac{8}{3} - \dfrac{2}{3}\,\pi_6
\end{cases}
\qquad
\begin{cases}
\underbrace{\dfrac{1}{3} - \dfrac{1}{3}\pi_6 \geq 0}_{\pi_4 \geq 0} \\
\underbrace{\dfrac{8}{3} - \dfrac{2}{3}\,\pi_6 \geq 0}_{\pi_5 \geq 0}
\end{cases}
\qquad
\begin{cases}
\pi_6 \leq 1 \\
\pi_6 \leq 4
\end{cases}
$$

By increasing the value of π_6 to $\min\{1, 4\} = 1$, the value of π_4 decreases to 0. Then we have that:

π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non basic variable

π_4 becomes a dual non-basic variable and accordingly x_4 becomes a primal basic variable

We have:

$$x_6 = -\frac{2}{3} + \frac{1}{3}x_4 + \frac{2}{3}\,x_5 \quad \text{then} \quad x_4 = 2 - 2x_5 + 3x_6.$$

Substituting in the other equations of the primal dictionary, we get:

$$x_1 = \frac{8}{3} - \frac{1}{3}(2 - 2x_5 + 3x_6) + \frac{1}{3}x_5 = 2 + x_5 - x_6$$

$$x_3 = \frac{17}{3} - \frac{4}{3}(2 - 2x_5 + 3x_6) + \frac{34}{3}x_5 = 3 + 14x_5 - 4x_6$$

$$z = \frac{44}{3} - \frac{1}{3}(2 - 2x_5 + 3x_6) - \frac{8}{3}x_5 = 14 - 2x_5 - x_6$$

Accordingly, we now have the following optimal primal dictionary:

$$\begin{cases} x_1 = 2 + x_5 - x_6 \\ x_2 = 4 - x_5 \\ x_3 = 3 + 14x_5 - 4x_6 \\ x_4 = 2 - 2x_5 + 3x_6 \\ z = 14 - 2x_5 - x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4\}$ and $\mathcal{N} = \{5, 6\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 4, \quad \tilde{x}_3 = 3, \quad \tilde{x}_4 = 2, \qquad \tilde{x}_5 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad 14$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 0, \qquad \tilde{\pi}_5 = 2 \quad \tilde{\pi}_6 = 1 \quad \text{of value} \quad 14$$

The optimal solution $\tilde{\boldsymbol{x}}(\,LP_3\,)$ is **integer**. Accordingly, an optimal solution $\tilde{\boldsymbol{x}}(\,ILP\,)$ is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 4$$

The optimal value $z(\,ILP\,)$ is 14.

For the LP_3 problem in canonical form, we have:

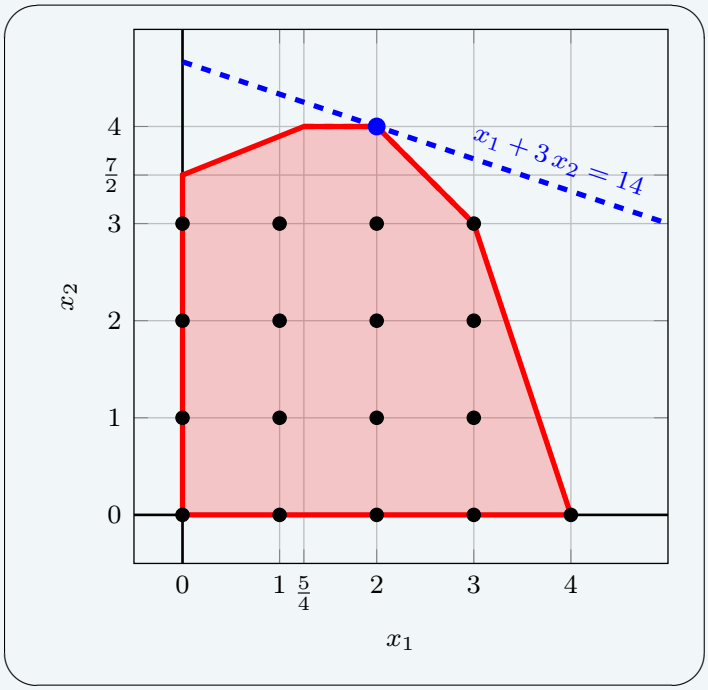
$$
\begin{array}{lrcrcrl}
\max & x_1 & + & 3 & x_2 & & \\
 & -4\ x_1 & + & 10 & x_2 & \leq & 35 \\
 & 3\ x_1 & + & & x_2 & \leq & 12 \\
 & & & & x_2 & \leq & 4 \\
 & x_1 & + & & x_2 & \leq & 6 \\
 & x_1, & & & x_2 & \geq & 0
\end{array}
$$

An optimal solution $\tilde{\boldsymbol{x}}(\, LP_3\,)$ is:

$$\tilde{x}_1 = 2, \qquad \tilde{x}_2 = 4$$

The optimal value is:

$$z(\, LP_3\,) = 14$$



Exercise 6.3 — Cutting-plane algorithm ($P3$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{array}{rrcrcrcl} \max & 2\,x_1 & + & 4\,x_2 & & & \\ & -\,x_1 & + & x_2 & \leq & 2 \\ & 8\,x_1 & + & 2\,x_2 & \leq & 19 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

and an optimal dictionary of its LP relaxation in standard form:

$$\begin{cases} x_1 = \frac{3}{2} + \frac{1}{5}\,x_3 - \frac{1}{10}\,x_4 \\ x_2 = \frac{7}{2} - \frac{4}{5}\,x_3 - \frac{1}{10}\,x_4 \\ z = 17 - \frac{14}{5}\,x_3 - \frac{3}{5}\,x_4 \end{cases}$$

Questions

1. Execute the cutting-plane algorithm based on Gomory fractional cuts using:
   (a) one Gomory fractional cut at each iteration;
   (b) the largest right-hand side as the cut selection strategy.

   Solve the LP problems after adding the Gomory fractional cut using the dual simplex algorithm. Graphically show the optimal solution of the LP problem at each iteration.

2. Graphically show the effect of each Gomory fractional cut on the LP relaxation of the ILP problem in canonical form.

Given a row $i \in \mathcal{B}$ of a simplex dictionary associated with a feasible basic solution such that $\bar{b}_i$ is fractional, the linear inequality

$$\sum_{j \in \mathcal{N}} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor)\, x_j \geq \bar{b}_i - \lfloor \bar{b}_i \rfloor$$

is called the **Gomory fractional cut** generated by row i.

Solution

Execution of the Cutting-Plane Algorithm to solve the ILP problem.

- **Initialization**

Let us consider the LP_1 problem in standard form:

$$\begin{array}{lrcrcrcrcl} \max & 2 & x_1 & + & 4 & x_2 & & & & \\ & - & x_1 & + & & x_2 & + & x_3 & & = & 2 \\ & 8 & x_1 & + & 2 & x_2 & & & + \; x_4 & = & 19 \\ & & x_1, & & & x_2, & & x_3, & & x_4 \geq & 0 \end{array}$$

and its optimal simplex dictionary:

$$\begin{cases} x_1 = \frac{3}{2} + \frac{1}{5} x_3 - \frac{1}{10} x_4 \\ x_2 = \frac{7}{2} - \frac{4}{5} x_3 - \frac{1}{10} x_4 \\ z = 17 - \frac{14}{5} x_3 - \frac{3}{5} x_4 \end{cases}$$

The optimal solution $\tilde{\boldsymbol{x}}(\, LP_1\,)$ is **fractional** and the LP_1 problem is **feasible**.

We have the upper bound on the optimal value of the ILP problem:

$$z(\, ILP\,) \;\leq\; 17$$

For the LP_1 problem in canonical form, we have:

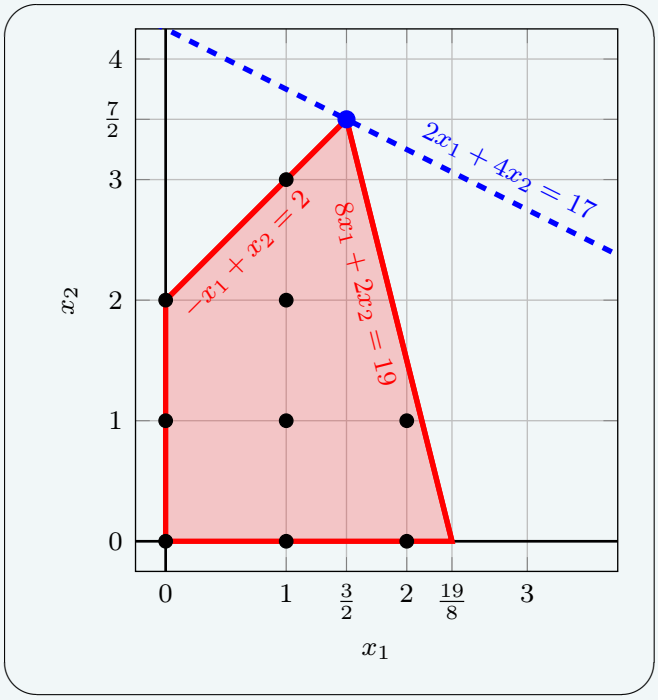
$$\begin{array}{lrcrcrcl} \max & 2 \; x_1 & + & 4 \; x_2 & & & \\ & - \; x_1 & + & x_2 & \leq & 2 \\ & 8 \; x_1 & + & 2 \; x_2 & \leq & 19 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

An optimal solution $\tilde{\boldsymbol{x}}(\, LP_1\,)$ is:

$$\tilde{x}_1 = \frac{3}{2}, \qquad \tilde{x}_2 = \frac{7}{2}$$

The optimal value is:

$$z(\, LP_1\,) = 17$$



- **Iteration 1**

a) **Cut generation phase**

We have the following simplex dictionary associated with a fractional basic feasible solution of the LP_1 problem in standard form:

$$\begin{cases} x_1 = \frac{3}{2} + \frac{1}{5} x_3 - \frac{1}{10} x_4 \\ x_2 = \frac{7}{2} - \frac{4}{5} x_3 - \frac{1}{10} x_4 \\ z = 17 - \frac{14}{5} x_3 - \frac{3}{5} x_4 \end{cases}$$

where

$$\bar{\boldsymbol{A}} = \begin{pmatrix} -\frac{1}{5} & \frac{1}{10} \\ \frac{4}{5} & \frac{1}{10} \end{pmatrix} \qquad \bar{\boldsymbol{b}} = \begin{pmatrix} \frac{3}{2} \\ \frac{7}{2} \end{pmatrix}$$

– The Gomory fractional cut generated by the first row is:

$$\underbrace{\left(-\frac{1}{5} - \left\lfloor -\frac{1}{5} \right\rfloor\right)}_{-\frac{1}{5}-(-1)=\frac{4}{5}} x_3 + \underbrace{\left(\frac{1}{10} - \left\lfloor \frac{1}{10} \right\rfloor\right)}_{\frac{1}{10}-0=\frac{1}{10}} x_4 \geq \underbrace{\left(\frac{3}{2} - \left\lfloor \frac{3}{2} \right\rfloor\right)}_{\frac{3}{2}-1=\frac{1}{2}}$$

$$\frac{4}{5}\,x_3 + \frac{1}{10}\,x_4 \geq \frac{1}{2} \quad \Rightarrow \quad 8\,x_3 + x_4 \geq 5$$

By subtracting a surplus variable, we obtain the Gomory fractional cut in standard form:

$$\frac{4}{5}\,x_3 + \frac{1}{10}\,x_4 - x_5 = \frac{1}{2} \quad \Rightarrow \quad 8\,x_3 + x_4 - 10\,x_5 = 5$$

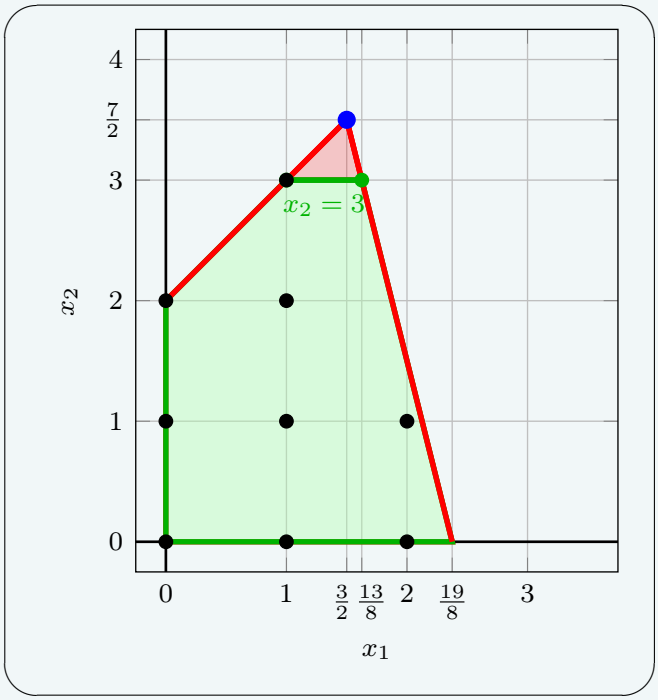
From the standard form of the LP_1 problem, we have:

$$x_3 = 2 + x_1 - x_2, \qquad x_4 = 19 - 8\,x_1 - 2\,x_2.$$

Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$8\,(2 + x_1 - x_2) + (19 - 8\,x_1 - 2\,x_2) \geq 5 \quad \Rightarrow \quad x_2 \leq 3.$$

Effect of the Gomory fractional cut generated by the first row on the LP_1 problem in canonical form:



– The Gomory fractional cut generated by the second row is:

$$\underbrace{\left(\frac{4}{5} - \left\lfloor\frac{4}{5}\right\rfloor\right)}_{\frac{4}{5}-0=\frac{4}{5}}\,x_3 + \underbrace{\left(\frac{1}{10} - \left\lfloor\frac{1}{10}\right\rfloor\right)}_{\frac{1}{10}-0=\frac{1}{10}}\,x_4 \geq \underbrace{\left(\frac{7}{2} - \left\lfloor\frac{7}{2}\right\rfloor\right)}_{\frac{7}{2}-3=\frac{1}{2}}$$

$$\frac{4}{5}\,x_3 + \frac{1}{10}\,x_4 \geq \frac{1}{2} \quad \Rightarrow \quad 8\,x_3 + x_4 \geq 5$$

This is the same Gomory fractional cut obtained with the first generating row.

We select the Gomory fractional cut with the largest right-hand side and we create the LP_2 problem by adding it to the LP_1 problem.

b) **Dual simplex reoptimization phase**

Let us consider the LP_2 problem in standard form:

$$\begin{array}{lrcrcrcrcrcr}
\max & 2 & x_1 & + & 4 & x_2 & & & & & & \\
 & - & x_1 & + & & x_2 & + & x_3 & & & = & 2 \\
 & 8 & x_1 & + & 2 & x_2 & & & + & x_4 & = & 19 \\
 & & & & & x_2 & & & & + \; x_5 & = & 3 \\
 & & x_1, & & & x_2, & & x_3, & & x_4, & \geq & 0
\end{array}$$

For the added Gomory fractional cut we have:

$$8\,x_3 + x_4 - 10\,x_5 = 5 \quad \Rightarrow \quad x_5 = -\frac{1}{2} + \frac{4}{5}\,x_3 + \frac{1}{10}\,x_4$$

We then add this equation to the previous optimal dictionary and we obtain the following dictionary :

$$\begin{cases}
x_1 = \frac{3}{2} + \frac{1}{5}\,x_3 - \frac{1}{10}\,x_4 \\
x_2 = \frac{7}{2} - \frac{4}{5}\,x_3 - \frac{1}{10}\,x_4 \\
x_5 = -\frac{1}{2} + \frac{4}{5}\,x_3 + \frac{1}{10}\,x_4 \\
z = 17 - \frac{14}{5}\,x_3 - \frac{3}{5}\,x_4
\end{cases}$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{3}{2}, \quad \bar{x}_2 = \frac{7}{2}, \quad \bar{x}_5 = -\frac{1}{2}, \qquad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad 17$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_5 = 0, \qquad \bar{\pi}_3 = \frac{14}{5}, \quad \bar{\pi}_4 = \frac{3}{5} \quad \text{of value} \quad 17$$

If we increase the value of π_5, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 &= \frac{14}{5} - \frac{4}{5}\pi_5 \\ \pi_4 &= \frac{3}{5} - \frac{1}{10}\pi_5 \end{cases} \qquad \begin{cases} \underbrace{\frac{14}{5} - \frac{4}{5}\pi_5 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{3}{5} - \frac{1}{10}\pi_5 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_5 &\leq \frac{7}{2} \\ \pi_5 &\leq 6 \end{cases}$$

By increasing the value of π_5 to $\min\left\{\frac{7}{2}, 6\right\} = \frac{7}{2}$, the value of π_3 decreases to 0. Then we have that:

- π_5 becomes a dual basic variable and accordingly x_5 becomes a primal non basic variable
- π_3 becomes a dual non-basic variable and accordingly x_3 becomes a primal basic variable

We have:

$$x_5 = -\frac{1}{2} + \frac{4}{5}x_3 + \frac{1}{10}x_4 \quad \Longrightarrow \quad x_3 = \frac{5}{8} - \frac{1}{8}x_4 + \frac{5}{4}x_5.$$

Substituting into the other equations of the primal dictionary, we get:

$$x_1 = \frac{3}{2} + \frac{1}{5}\left(\frac{5}{8} - \frac{1}{8}x_4 + \frac{5}{4}x_5\right) - \frac{1}{10}x_4 = \frac{13}{8} - \frac{1}{8}x_4 + \frac{1}{4}x_5$$

$$x_2 = \frac{7}{2} - \frac{4}{5}\left(\frac{5}{8} - \frac{1}{8}x_4 + \frac{5}{4}x_5\right) - \frac{1}{10}x_4 = 3 - x_5$$

$$z = 17 - \frac{14}{5}\left(\frac{5}{8} - \frac{1}{8}x_4 + \frac{5}{4}x_5\right) - \frac{3}{5}x_4 = \frac{61}{4} - \frac{1}{4}x_4 - \frac{7}{2}x_5$$

Accordingly, we now have the following optimal primal dictionary:

$$\begin{cases} x_1 &= \frac{13}{8} - \frac{1}{8}x_4 + \frac{1}{4}x_5 \\ x_2 &= 3 - x_5 \\ x_3 &= \frac{5}{8} - \frac{1}{8}x_4 + \frac{5}{4}x_5 \\ z &= \frac{61}{4} - \frac{1}{4}x_4 - \frac{7}{2}x_5 \end{cases}$$

The optimal solution $\tilde{\boldsymbol{x}}(\,LP_2\,)$ is **fractional** and the LP_2 problem is **feasible**.

We have the upper bound on the optimal value of the ILP problem:

$$z(\,ILP\,) \;\leq\; \underbrace{\left\lfloor \frac{61}{4} \right\rfloor}_{=15}$$

For the LP_2 problem in canonical form, we have:

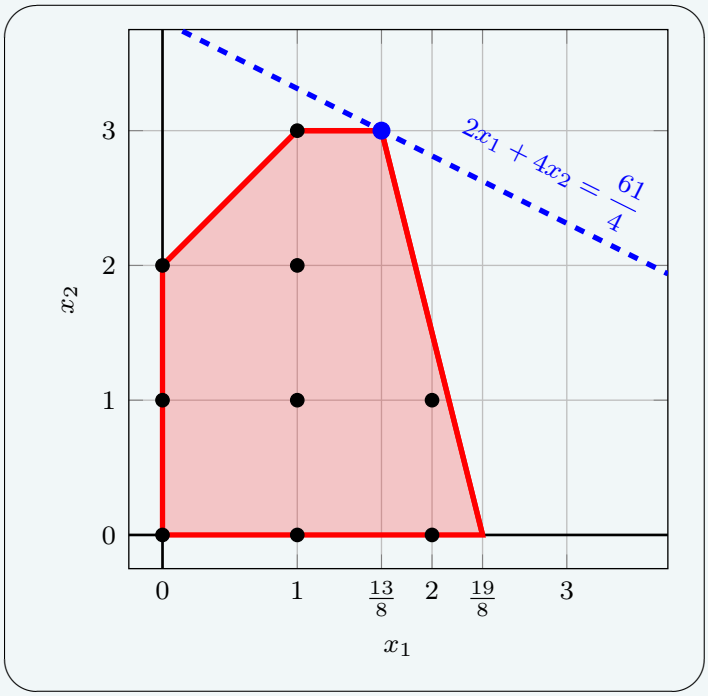
$$\begin{array}{lrcrcrl} \max & 2\,x_1 & + & 4\,x_2 & & & \\ & -\,x_1 & + & x_2 & \leq & 2 & \\ & 8\,x_1 & + & 2\,x_2 & \leq & 19 & \\ & & & x_2 & \leq & 3 & \\ & x_1, & & x_2 & \geq & 0 & \end{array}$$

An optimal solution $\tilde{\boldsymbol{x}}(\,LP_2\,)$ is:

$$\tilde{x}_1 = \frac{13}{8}, \qquad \tilde{x}_2 = 3$$

The optimal value is:

$$z(\,LP_2\,) = \frac{61}{4}$$



- **Iteration 2**

a) **Cut generation phase**

We have the following simplex dictionary associated with a fractional basic feasible solution of the LP_2 problem in standard form:

$$\begin{cases} x_1 = \frac{13}{8} - \frac{1}{8}\, x_4 + \frac{1}{4}\, x_5 \\ x_2 = 3 - x_5 \\ x_3 = \frac{5}{8} - \frac{1}{8}\, x_4 + \frac{5}{4}\, x_5 \\ z = \frac{61}{4} - \frac{1}{4}\, x_4 - \frac{7}{2}\, x_5 \end{cases}$$

where

$$\bar{\boldsymbol{A}} = \begin{pmatrix} \frac{1}{8} & -\frac{1}{4} \\ 0 & 1 \\ \frac{1}{8} & -\frac{5}{4} \end{pmatrix} \qquad \bar{\boldsymbol{b}} = \begin{pmatrix} \frac{13}{8} \\ 3 \\ \frac{5}{8} \end{pmatrix}$$

– The Gomory fractional cut generated by the first row is:

$$\underbrace{\left(\frac{1}{8} - \left\lfloor \frac{1}{8} \right\rfloor\right)}_{\frac{1}{8}-0=\frac{1}{8}} x_4 + \underbrace{\left(-\frac{1}{4} - \left\lfloor -\frac{1}{4} \right\rfloor\right)}_{-\frac{1}{4}-(-1)=\frac{3}{4}} x_5 \geq \underbrace{\left(\frac{13}{8} - \left\lfloor \frac{13}{8} \right\rfloor\right)}_{\frac{13}{8}-1=\frac{5}{8}}$$

$$\frac{1}{8}\, x_4 + \frac{3}{4}\, x_5 \geq \frac{5}{8} \quad \Rightarrow \quad x_4 + 6\, x_5 \geq 5$$

By subtracting a surplus variable, we obtain the Gomory fractional cut in standard form:

$$\frac{1}{8}\, x_4 + \frac{3}{4}\, x_5 - x_6 = \frac{5}{8} \quad \Rightarrow \quad x_4 + 6\, x_5 - 8\, x_6 = 5$$

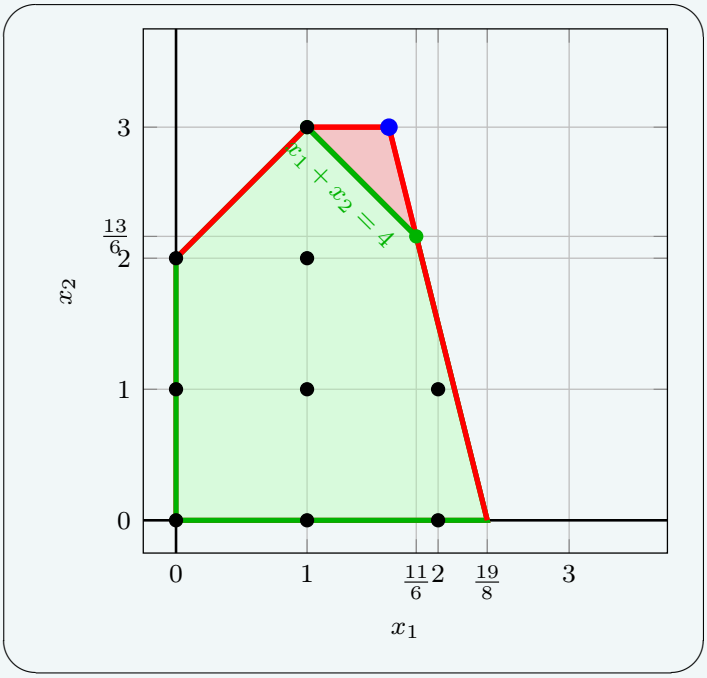
From the standard form of the LP_2 problem, we have:

$$x_4 = 19 - 8\, x_1 - 2\, x_2, \qquad x_5 = 3 - x_2.$$

Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$(19 - 8\, x_1 - 2\, x_2) + 6\, (3 - x_2) \geq 5 \quad \Rightarrow \quad x_1 + x_2 \leq 4.$$

Effect of the Gomory fractional cut generated by the first row on the LP_2 problem in canonical form:



– The Gomory fractional cut generated by the third row is:

$$\underbrace{\left(\frac{1}{8} - \left\lfloor \frac{1}{8} \right\rfloor\right)}_{\frac{1}{8}-0=\frac{1}{8}} x_4 + \underbrace{\left(-\frac{5}{4} - \left\lfloor -\frac{5}{4} \right\rfloor\right)}_{-\frac{5}{4}+2=\frac{3}{4}} x_5 \geq \underbrace{\left(\frac{5}{8} - \left\lfloor \frac{5}{8} \right\rfloor\right)}_{\frac{5}{8}-0=\frac{5}{8}}$$

$$\frac{1}{8}\, x_4 + \frac{3}{4}\, x_5 \geq \frac{5}{8} \quad \Rightarrow \quad x_4 + 6\, x_5 \geq 5$$

This is the same Gomory fractional cut obtained with the first generating row.

We select the Gomory fractional cut with the largest right-hand side and we create the LP_3 problem by adding it to the LP_2 problem.

b) **Dual simplex reoptimization phase**

Let us consider the LP_3 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcr}
\max & 2 & x_1 & + & 4 & x_2 & & & & & & & & \\
 & - & x_1 & + & & x_2 & + & x_3 & & & & & = & 2 \\
 & 8 & x_1 & + & 2 & x_2 & & & + & x_4 & & & = & 19 \\
 & & & & & x_2 & & & & & + & x_5 & = & 3 \\
 & & x_1 & + & & x_2 & & & & & & + \; x_6 & = & 4 \\
 & & x_1, & & & x_2, & & x_3, & & x_4, & & x_5, \quad x_6 & \geq & 0
\end{array}$$

For the added Gomory fractional cut we have:

$$x_4 + 6\,x_5 - 8\,x_6 = 5 \quad \Rightarrow \quad x_6 = -\frac{5}{8} + \frac{1}{8}\,x_4 + \frac{3}{4}\,x_5$$

We then add this equation to the previous optimal dictionary and we obtain the following dictionary :

$$\begin{cases} x_1 = \frac{13}{8} - \frac{1}{8}\,x_4 + \frac{1}{4}\,x_5 \\ x_2 = 3 - x_5 \\ x_3 = \frac{5}{8} - \frac{1}{8}\,x_4 + \frac{5}{4}\,x_5 \\ x_6 = -\frac{5}{8} + \frac{1}{8}\,x_4 + \frac{3}{4}\,x_5 \\ z = \frac{61}{4} - \frac{1}{4}\,x_4 - \frac{7}{2}\,x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 6\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{13}{8}, \quad \bar{x}_2 = 3, \quad \bar{x}_3 = \frac{5}{8}, \quad \bar{x}_6 = -\frac{5}{8}, \qquad \bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{61}{4}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_6 = 0, \qquad \bar{\pi}_4 = \frac{1}{4}, \quad \bar{\pi}_5 = \frac{7}{2} \quad \text{of value} \quad \frac{61}{4}$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_4 = \frac{1}{4} - \frac{1}{8}\,\pi_6 \\ \pi_5 = \frac{7}{2} - \frac{3}{4}\,\pi_6 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{4} - \frac{1}{8}\,\pi_6 \geq 0}_{\pi_4 \geq 0} \\ \underbrace{\frac{7}{2} - \frac{3}{4}\,\pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 \leq 2 \\ \pi_6 \leq \frac{14}{3} \end{cases}$$

By increasing the value of π_6 to $\min\left\{2, \frac{14}{3}\right\} = 2$, the value of π_4 decreases to 0. Then we have that:

π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non basic variable

π_4 becomes a dual non-basic variable and accordingly x_4 becomes a primal basic variable

We have:

$$x_6 = -\frac{5}{8} + \frac{1}{8}\, x_4 + \frac{3}{4}\, x_5 \quad \Longrightarrow \quad x_4 = 5 - 6x_5 + 8x_6.$$

Substituting into the other equations of the primal dictionary, we get:

$$x_1 = \frac{13}{8} - \frac{1}{8}\,(5 - 6x_5 + 8x_6) + \frac{1}{4}x_5 = 1 + x_5 - x_6$$

$$x_3 = \frac{5}{8} - \frac{1}{8}\,(5 - 6x_5 + 8x_6) + \frac{5}{4}x_5 = 2\, x_5 - x_6$$

$$z = \frac{61}{4} - \frac{1}{4}\,(5 - 6x_5 + 8x_6) - \frac{7}{2}x_5 = 14 - 2x_5 - 2x_6$$

Accordingly, we now have the following optimal primal dictionary:

$$\begin{cases} x_1 = 1 + x_5 - x_6 \\ x_2 = 3 - x_5 \\ x_3 = 0 + 2\, x_5 - x_6 \\ x_4 = 5 - 6\, x_5 + 8\, x_6 \\ z = 14 - 2\, x_5 - 2\, x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4\}$ and $\mathcal{N} = \{5, 6\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 3, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 5, \qquad \tilde{x}_5 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad 14$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 0, \qquad \tilde{\pi}_5 = 2, \quad \tilde{\pi}_6 = 2 \quad \text{of value} \quad 14$$

The optimal solution $\tilde{\boldsymbol{x}}(\, LP_3\,)$ is **integer** and the LP_3 problem is **feasible**.

We have the upper bound on the optimal value of the ILP problem:

$$z(\, ILP\,) \;\leq\; 14$$

For the LP_3 problem in canonical form, we have:

$$\begin{array}{rrcrcl} \max & 2\, x_1 & + & 4\, x_2 & & \\ & -\, x_1 & + & x_2 & \leq & 2 \\ & 8\, x_1 & + & 2\, x_2 & \leq & 19 \\ & & & x_2 & \leq & 3 \\ & x_1 & + & x_2 & \leq & 4 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

An optimal solution $\tilde{\boldsymbol{x}}(\, LP_3 \,)$ is:

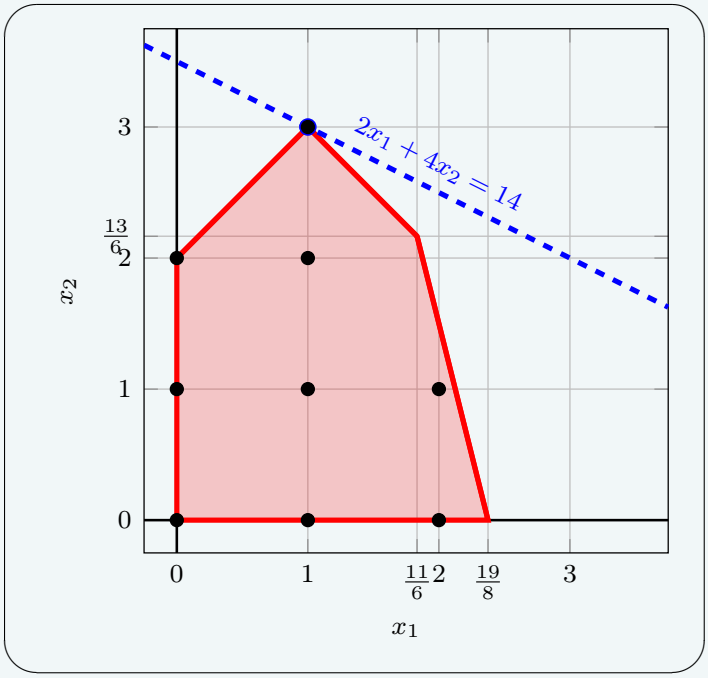
$$\tilde{x}_1 = 1, \qquad \tilde{x}_2 = 3$$

The optimal value is:

$$z(\, LP_3 \,) = 14$$

The optimal solution $\tilde{\boldsymbol{x}}(\, LP_3 \,) = (1, 3)$ is **integer**. Therefore, it is also an optimal solution of the ILP problem.

$$z(\, ILP \,) = 14.$$



Exercise 6.4 — Cutting-plane algorithm ($P4$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{array}{rrcrcrl} \max & 5 & x_1 & + & 3 & x_2 & \\ & 2 & x_1 & + & & x_2 & \leq & 14 \\ & & x_1 & + & 8 & x_2 & \leq & 32 \\ & & x_1, & & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

and the following optimal simplex dictionary of its LP relaxation in standard form:

$$\begin{cases} x_1 = \frac{16}{3} - \frac{8}{15} x_3 + \frac{1}{15} x_4 \\ x_2 = \frac{10}{3} + \frac{1}{15} x_3 - \frac{2}{15} x_4 \\ z = \frac{110}{3} - \frac{37}{15} x_3 - \frac{1}{15} x_4 \end{cases}$$

Questions

1. Execute the cutting-plane algorithm based on Gomory fractional cuts using:
   (a) one Gomory fractional cut at each iteration;
   (b) the largest right-hand side as the cut selection strategy.

   Solve the LP problems after adding the Gomory fractional cut using the dual simplex algorithm. Graphically show the optimal solution of the LP problem at each iteration.

2. Graphically show the effect of each Gomory fractional cut on the LP relaxation of the ILP problem in canonical form.

Given a row $i \in \mathcal{B}$ of a simplex dictionary associated with a feasible basic solution such that $\bar{b}_i$ is fractional, the linear inequality

$$\sum_{j \in \mathcal{N}} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_j \geq \bar{b}_i - \lfloor \bar{b}_i \rfloor$$

is called the **Gomory fractional cut** generated by row i.

Solution

Execution of the Cutting-Plane Algorithm to solve the ILP problem.

- **Initialization**

  Let us consider the LP_1 problem in standard form:

$$\begin{array}{lrcrcrcrcl} \max & 5 & x_1 & + & 3 & x_2 & & & & \\ & 2 & x_1 & + & & x_2 & + & x_3 & & = 14 \\ & & x_1 & + & 8 & x_2 & & & + x_4 & = 32 \\ & & x_1, & & & x_2, & & x_3, & x_4 & \geq 0 \end{array}$$

  and its optimal simplex dictionary:

$$\begin{cases} x_1 = \frac{16}{3} - \frac{8}{15} x_3 + \frac{1}{15} x_4 \\ x_2 = \frac{10}{3} + \frac{1}{15} x_3 - \frac{2}{15} x_4 \\ z = \frac{110}{3} - \frac{37}{15} x_3 - \frac{1}{15} x_4 \end{cases}$$

The optimal solution $\tilde{\boldsymbol{x}}(\, LP_1 \,)$ is **fractional** and the LP_1 problem is **feasible**.

We have the upper bound on the optimal value of the ILP problem:

$$z(\, ILP \,) \leq \underbrace{\left\lfloor \frac{110}{3} \right\rfloor}_{=36}$$

For the LP_1 problem in canonical form, we have:

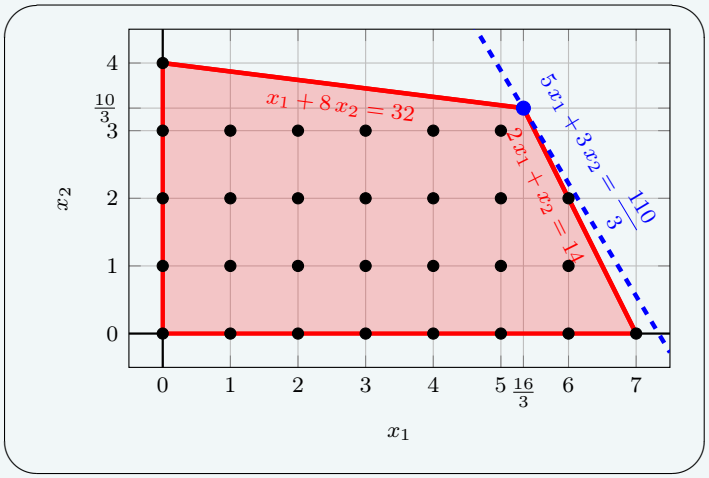
$$\begin{array}{rrcrcrl} \max & 5 & x_1 & + & 3 & x_2 & \\ & 2 & x_1 & + & & x_2 & \leq 14 \\ & & x_1 & + & 8 & x_2 & \leq 32 \\ & & x_1, & & & x_2 & \geq 0 \end{array}$$

An optimal solution $\tilde{\boldsymbol{x}}(\, LP_1 \,)$ is:

$$\tilde{x}_1 = \frac{16}{3}, \qquad \tilde{x}_2 = \frac{10}{3}$$

The optimal value is:

$$z(\, LP_1 \,) = \frac{110}{3}$$



- **Iteration 1**

a) **Cut generation phase**

We have the following simplex dictionary associated with a fractional basic feasible solution of the LP_1 problem in standard form:

$$\begin{cases} x_1 = \frac{16}{3} - \frac{8}{15}\, x_3 + \frac{1}{15}\, x_4 \\ x_2 = \frac{10}{3} + \frac{1}{15}\, x_3 - \frac{2}{15}\, x_4 \\ z = \frac{110}{3} - \frac{37}{15}\, x_3 - \frac{1}{15}\, x_4 \end{cases}$$

where

$$\bar{\boldsymbol{A}} = \begin{pmatrix} \frac{8}{15} & -\frac{1}{15} \\ -\frac{1}{15} & \frac{2}{15} \end{pmatrix} \qquad \bar{\boldsymbol{b}} = \begin{pmatrix} \frac{16}{3} \\ \frac{10}{3} \end{pmatrix}$$

– The Gomory fractional cut generated by the first row is:

$$\underbrace{\left(\frac{8}{15} - \left\lfloor \frac{8}{15} \right\rfloor\right)}_{\frac{8}{15} - 0 = \frac{8}{15}} x_3 + \underbrace{\left(-\frac{1}{15} - \left\lfloor -\frac{1}{15} \right\rfloor\right)}_{-\frac{1}{15} - (-1) = \frac{14}{15}} x_4 \geq \underbrace{\left(\frac{16}{3} - \left\lfloor \frac{16}{3} \right\rfloor\right)}_{\frac{16}{3} - 5 = \frac{1}{3}}$$

$$\frac{8}{15}\, x_3 + \frac{14}{15}\, x_4 \geq \frac{1}{3} \quad \Rightarrow \quad 8\, x_3 + 14\, x_4 \geq 5$$

From the standard form of the LP_1 problem, we have:

$$x_3 = 14 - 2\, x_1 - x_2, \qquad x_4 = 32 - x_1 - 8\, x_2.$$

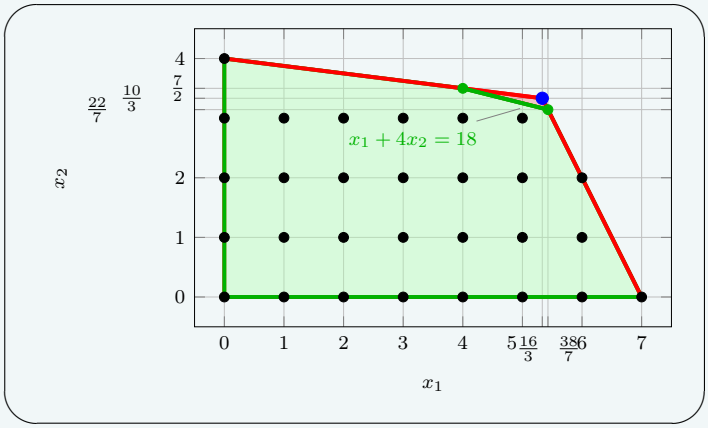
Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$8\,(14 - 2\, x_1 - x_2) + 14\,(32 - x_1 - 8\, x_2) \geq 5 \quad \Rightarrow \quad 2\, x_1 + 8\, x_2 \leq 37$$

Dividing by the greatest common divisor 2 of the coefficients and rounding down the right-hand side, we obtain the cut in primitive form:

$$x_1 + 4\, x_2 \leq \left\lfloor \frac{37}{2} \right\rfloor = 18$$

Effect of the Gomory fractional cut generated by the first row on the LP_1 problem in canonical form:



– The Gomory fractional cut generated by the second row is:

$$\underbrace{\left(-\frac{1}{15} - \left\lfloor -\frac{1}{15} \right\rfloor\right)}_{-\frac{1}{15}-(-1)=\frac{14}{15}} x_3 + \underbrace{\left(\frac{2}{15} - \left\lfloor \frac{2}{15} \right\rfloor\right)}_{\frac{2}{15}-0=\frac{2}{15}} x_4 \geq \underbrace{\left(\frac{10}{3} - \left\lfloor \frac{10}{3} \right\rfloor\right)}_{\frac{10}{3}-3=\frac{1}{3}}$$

$$\frac{14}{15}\, x_3 + \frac{2}{15}\, x_4 \geq \frac{1}{3} \quad \Rightarrow \quad 14\, x_3 + 2\, x_4 \geq 5$$

From the standard form of the LP_1 problem, we have:

$$x_3 = 14 - 2\, x_1 - x_2, \qquad x_4 = 32 - x_1 - 8\, x_2.$$

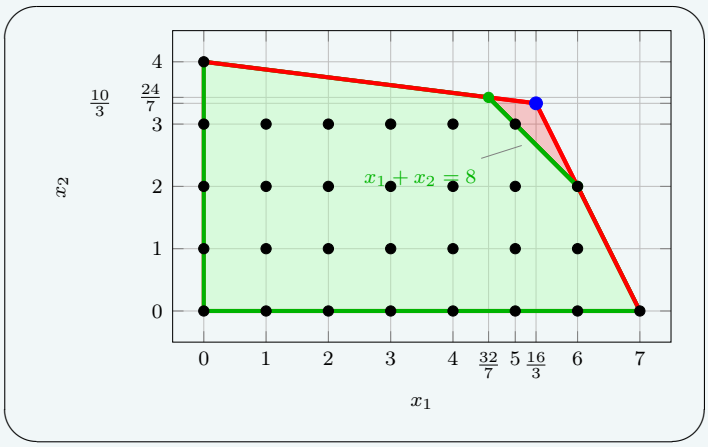
Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$14\,(14 - 2\, x_1 - x_2) + 2\,(32 - x_1 - 8\, x_2) \geq 5 \quad \Rightarrow \quad 2\, x_1 + 2\, x_2 \leq 17$$

Dividing by the greatest common divisor 2 of the coefficients and rounding down the right-hand side, we obtain the cut in primitive form:

$$x_1 + x_2 \leq \left\lfloor \frac{17}{2} \right\rfloor = 8$$

Effect of the Gomory fractional cut generated by the second row on the LP_1 problem in canonical form:



Both Gomory fractional cuts have the same right-hand side, equal to $\frac{1}{3}$. We select the Gomory fractional cut generated by the second row and we create the LP_2 problem by adding it to the LP_1 problem.

b) **Dual simplex reoptimization phase**

Let us consider the LP_2 problem in standard form:

$$\begin{array}{lrcrcrcrcrcl} \max & 5\,x_1 & + & 3\,x_2 & & & & & & & & \\ & 2\,x_1 & + & x_2 & + & x_3 & & & & & = & 14 \\ & x_1 & + & 8\,x_2 & & & + & x_4 & & & = & 32 \\ & x_1 & + & x_2 & & & & & + & x_5 & = & 8 \\ & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0 \end{array}$$

The slack variable of the added primitive cut is $x_5 = 8 - x_1 - x_2$. Substituting the rows of x_1 and x_2 of the previous optimal dictionary, we have:

$$x_5 = 8 - \left(\frac{16}{3} - \frac{8}{15}\,x_3 + \frac{1}{15}\,x_4\right) - \left(\frac{10}{3} + \frac{1}{15}\,x_3 - \frac{2}{15}\,x_4\right) = -\frac{2}{3} + \frac{7}{15}\,x_3 + \frac{1}{15}\,x_4$$

We then add this equation to the previous optimal dictionary and we obtain the following dictionary:

$$\begin{cases} x_1 = \frac{16}{3} - \frac{8}{15}\,x_3 + \frac{1}{15}\,x_4 \\ x_2 = \frac{10}{3} + \frac{1}{15}\,x_3 - \frac{2}{15}\,x_4 \\ x_5 = -\frac{2}{3} + \frac{7}{15}\,x_3 + \frac{1}{15}\,x_4 \\ z = \frac{110}{3} - \frac{37}{15}\,x_3 - \frac{1}{15}\,x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{16}{3}, \quad \bar{x}_2 = \frac{10}{3}, \quad \bar{x}_5 = -\frac{2}{3}, \qquad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad \frac{110}{3}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_5 = 0, \qquad \bar{\pi}_3 = \frac{37}{15}, \quad \bar{\pi}_4 = \frac{1}{15} \quad \text{of value} \quad \frac{110}{3}$$

If we increase the value of π_5, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 = \frac{37}{15} - \frac{7}{15}\pi_5 \\ \pi_4 = \frac{1}{15} - \frac{1}{15}\pi_5 \end{cases} \qquad \begin{cases} \underbrace{\frac{37}{15} - \frac{7}{15}\pi_5 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{1}{15} - \frac{1}{15}\pi_5 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_5 \leq \frac{37}{7} \\ \pi_5 \leq 1 \end{cases}$$

By increasing the value of π_5 to $\min\left\{\frac{37}{7}, 1\right\} = 1$, the value of π_4 decreases to 0. Then we have that:

π_5 becomes a dual basic variable and accordingly x_5 becomes a primal non basic variable

π_4 becomes a dual non-basic variable and accordingly x_4 becomes a primal basic variable

We have:

$$x_5 = -\frac{2}{3} + \frac{7}{15}x_3 + \frac{1}{15}x_4 \quad \Rightarrow \quad x_4 = 10 - 7x_3 + 15x_5.$$

Substituting in the other equations of the primal dictionary, we get:

$$x_1 = \frac{16}{3} - \frac{8}{15}x_3 + \frac{1}{15}(10 - 7x_3 + 15x_5) = 6 - x_3 + x_5$$

$$x_2 = \frac{10}{3} + \frac{1}{15}x_3 - \frac{2}{15}(10 - 7x_3 + 15x_5) = 2 + x_3 - 2x_5$$

$$z = \frac{110}{3} - \frac{37}{15}x_3 - \frac{1}{15}(10 - 7x_3 + 15x_5) = 36 - 2x_3 - x_5$$

Accordingly, we now have the following optimal primal dictionary:

$$\begin{cases} x_1 = 6 - x_3 + x_5 \\ x_2 = 2 + x_3 - 2x_5 \\ x_4 = 10 - 7x_3 + 15x_5 \\ z = 36 - 2x_3 - x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4\}$ and $\mathcal{N} = \{3, 5\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 6, \quad \tilde{x}_2 = 2, \quad \tilde{x}_4 = 10, \qquad \tilde{x}_3 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad 36$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_4 = 0, \qquad \tilde{\pi}_3 = 2, \quad \tilde{\pi}_5 = 1 \quad \text{of value} \quad 36$$

The optimal solution $\tilde{\boldsymbol{x}}(\, LP_2\,) = (6, 2)$ is **integer**. Therefore, it is also an optimal solution of the ILP problem.

$$z(\, ILP\,) = 36.$$

For the LP_2 problem in canonical form, we have:

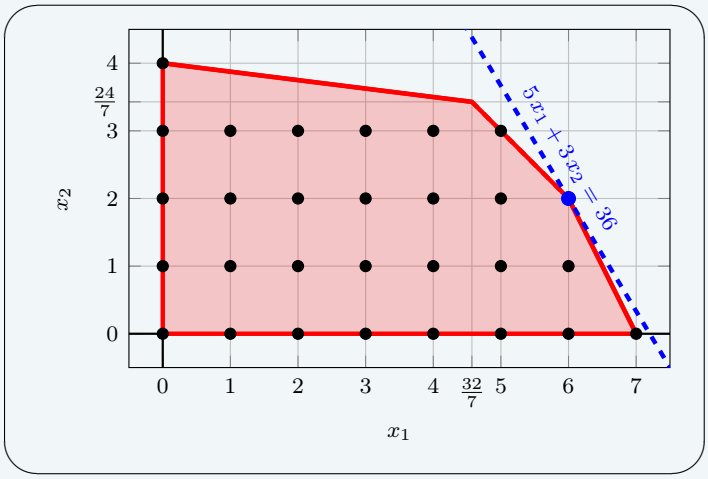
$$\begin{array}{rrcrcrl} \max & 5 & x_1 & + & 3 & x_2 & \\ & 2 & x_1 & + & & x_2 & \leq 14 \\ & & x_1 & + & 8 & x_2 & \leq 32 \\ & & x_1 & + & & x_2 & \leq 8 \\ & & x_1, & & & x_2 & \geq 0 \end{array}$$

An optimal solution $\tilde{\boldsymbol{x}}(\, LP_2\,)$ is:

$$\tilde{x}_1 = 6, \qquad \tilde{x}_2 = 2$$

The optimal value is:

$$z(\, LP_2\,) = 36$$



Exercise 6.5 — Cutting-plane algorithm ($P7$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ linear constraints:

$$\begin{array}{rrcrcr} \max & 3\,x_1 & + & x_2 & & \\ & x_1 & + & 4\,x_2 & \leq & 12 \\ & 2\,x_1 & + & x_2 & \leq & 5 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

and the following optimal simplex dictionary of its LP relaxation in standard form:

$$\begin{cases} x_1 = \frac{5}{2} - \frac{1}{2}\,x_2 - \frac{1}{2}\,x_4 \\ x_3 = \frac{19}{2} - \frac{7}{2}\,x_2 + \frac{1}{2}\,x_4 \\ z = \frac{15}{2} - \frac{1}{2}\,x_2 - \frac{3}{2}\,x_4 \end{cases}$$

Questions

1. Execute the cutting-plane algorithm based on Gomory fractional cuts using:
   (a) one Gomory fractional cut at each iteration;
   (b) the largest right-hand side as the cut selection strategy.

   Solve the LP problems after adding the Gomory fractional cut using the dual simplex algorithm. Graphically show the optimal solution of the LP problem at each iteration.

2. Graphically show the effect of each Gomory fractional cut on the LP relaxation of the ILP problem in canonical form.

Given a row $i \in \mathcal{B}$ of a simplex dictionary associated with a feasible basic solution such that $\bar{b}_i$ is fractional, the linear inequality

$$\sum_{j \in \mathcal{N}} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor)\, x_j \geq \bar{b}_i - \lfloor \bar{b}_i \rfloor$$

is called the **Gomory fractional cut** generated by row i.

Solution

Execution of the Cutting-Plane Algorithm to solve the ILP problem.

- **Initialization**

Let us consider the LP_1 problem in standard form:

$$\begin{array}{rrcrcrcrcl} \max & 3\,x_1 & + & x_2 & & & & & & \\ & x_1 & + & 4\,x_2 & + & x_3 & & & = & 12 \\ & 2\,x_1 & + & x_2 & & & + & x_4 & = & 5 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

and its optimal simplex dictionary:

$$\begin{cases} x_1 = \dfrac{5}{2} - \dfrac{1}{2}\,x_2 - \dfrac{1}{2}\,x_4 \\ x_3 = \dfrac{19}{2} - \dfrac{7}{2}\,x_2 + \dfrac{1}{2}\,x_4 \\ z = \dfrac{15}{2} - \dfrac{1}{2}\,x_2 - \dfrac{3}{2}\,x_4 \end{cases}$$

The optimal solution $\tilde{\boldsymbol{x}}(\,LP_1\,)$ is **fractional** and the LP_1 problem is **feasible**.

We have the upper bound on the optimal value of the ILP problem:

$$z(\,ILP\,) \leq \underbrace{\left\lfloor \frac{15}{2} \right\rfloor}_{=7}$$

For the LP_1 problem in canonical form, we have:

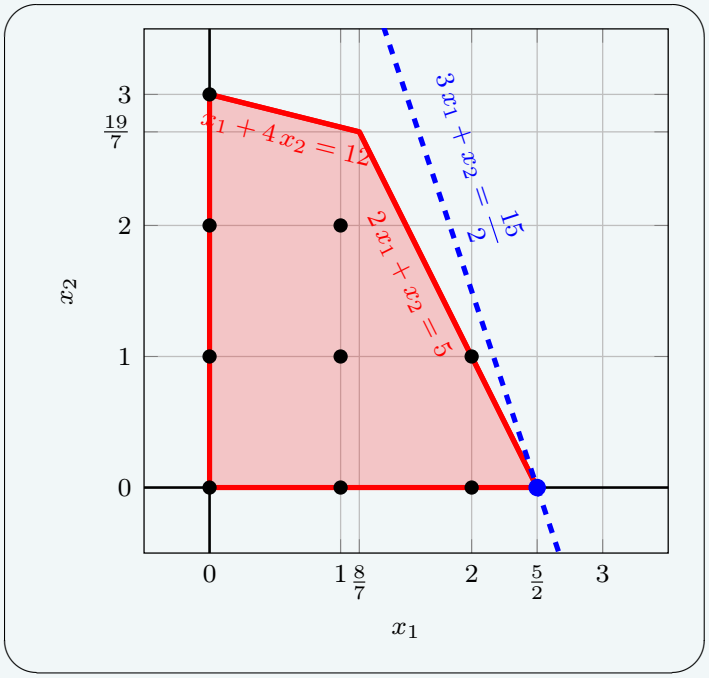
$$\begin{array}{rrcrcl} \max & 3\,x_1 & + & x_2 & & \\ & x_1 & + & 4\,x_2 & \leq & 12 \\ & 2\,x_1 & + & x_2 & \leq & 5 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

An optimal solution $\tilde{\boldsymbol{x}}(\,LP_1\,)$ is:

$$\tilde{x}_1 = \frac{5}{2}, \qquad \tilde{x}_2 = 0$$

The optimal value is:

$$z(\,LP_1\,) = \frac{15}{2}$$



- **Iteration 1**

a) **Cut generation phase**

We have the following simplex dictionary associated with a fractional basic feasible solution of the LP_1 problem in standard form:

$$\begin{cases} x_1 = \frac{5}{2} - \frac{1}{2}x_2 - \frac{1}{2}x_4 \\ x_3 = \frac{19}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_4 \\ z = \frac{15}{2} - \frac{1}{2}x_2 - \frac{3}{2}x_4 \end{cases}$$

where

$$\bar{\boldsymbol{A}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{7}{2} & -\frac{1}{2} \end{pmatrix} \qquad \bar{\boldsymbol{b}} = \begin{pmatrix} \frac{5}{2} \\ \frac{19}{2} \end{pmatrix}$$

– The Gomory fractional cut generated by the first row is:

$$\underbrace{\left(\frac{1}{2} - \left\lfloor \frac{1}{2} \right\rfloor\right)}_{\frac{1}{2}-0=\frac{1}{2}} x_2 + \underbrace{\left(\frac{1}{2} - \left\lfloor \frac{1}{2} \right\rfloor\right)}_{\frac{1}{2}-0=\frac{1}{2}} x_4 \geq \underbrace{\left(\frac{5}{2} - \left\lfloor \frac{5}{2} \right\rfloor\right)}_{\frac{5}{2}-2=\frac{1}{2}}$$

$$\frac{1}{2}\,x_2 + \frac{1}{2}\,x_4 \geq \frac{1}{2} \quad \Rightarrow \quad x_2 + x_4 \geq 1$$

By subtracting a surplus variable, we obtain the Gomory fractional cut in standard form:

$$\frac{1}{2}\,x_2 + \frac{1}{2}\,x_4 - x_5 = \frac{1}{2} \quad \Rightarrow \quad x_2 + x_4 - 2\,x_5 = 1$$

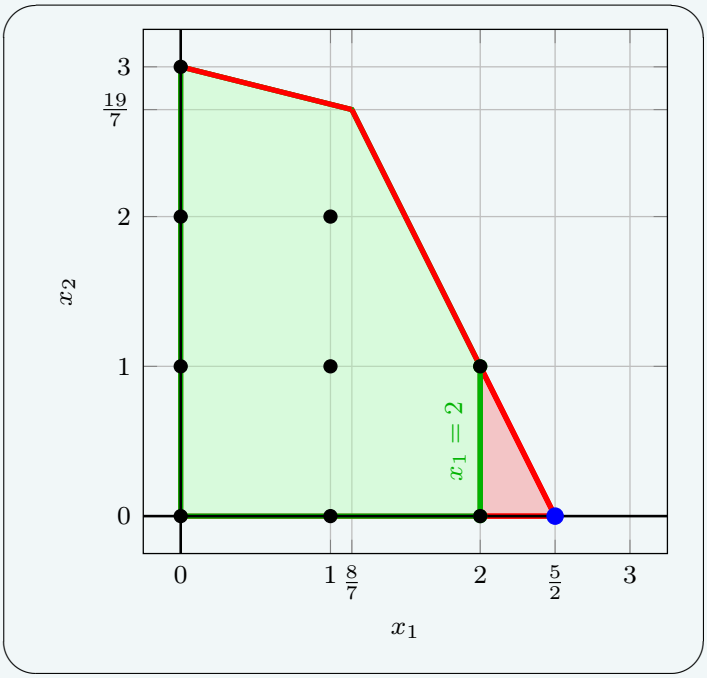
From the standard form of the LP_1 problem, we have:

$$x_4 = 5 - 2\,x_1 - x_2.$$

Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$x_2 + (5 - 2\,x_1 - x_2) \geq 1 \quad \Rightarrow \quad x_1 \leq 2$$

Effect of the Gomory fractional cut generated by the first row on the LP_1 problem in canonical form:



– The Gomory fractional cut generated by the second row is:

$$\underbrace{\left(\frac{7}{2} - \left\lfloor \frac{7}{2} \right\rfloor\right)}_{\frac{7}{2}-3=\frac{1}{2}}\,x_2 + \underbrace{\left(-\frac{1}{2} - \left\lfloor -\frac{1}{2} \right\rfloor\right)}_{-\frac{1}{2}-(-1)=\frac{1}{2}}\,x_4 \geq \underbrace{\left(\frac{19}{2} - \left\lfloor \frac{19}{2} \right\rfloor\right)}_{\frac{19}{2}-9=\frac{1}{2}}$$

$$\frac{1}{2}\,x_2 + \frac{1}{2}\,x_4 \geq \frac{1}{2} \quad \Rightarrow \quad x_2 + x_4 \geq 1$$

This Gomory fractional cut is the same as the one generated by the first row.

We select the Gomory fractional cut with the largest right-hand side and we create the LP_2 problem by adding it to the LP_1 problem.

b) **LP re-optimization phase**

Let us consider the LP_2 problem in standard form:

$$\begin{array}{lrcrcrcrcrcl}
\max & 3\,x_1 & + & x_2 & & & & & & & & \\
& x_1 & + & 4\,x_2 & + & x_3 & & & & & = & 12 \\
& 2\,x_1 & + & x_2 & & & + & x_4 & & & = & 5 \\
& x_1 & & & & & & & + & x_5 & = & 2 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
\end{array}$$

For the added Gomory fractional cut we have:

$$x_2 + x_4 - 2\,x_5 = 1 \quad \Rightarrow \quad x_5 = -\frac{1}{2} + \frac{1}{2}\,x_2 + \frac{1}{2}\,x_4$$

We then add this equation to the previous optimal dictionary and we obtain the following dictionary :

$$\begin{cases}
x_1 = \frac{5}{2} - \frac{1}{2}\,x_2 - \frac{1}{2}\,x_4 \\
x_3 = \frac{19}{2} - \frac{7}{2}\,x_2 + \frac{1}{2}\,x_4 \\
x_5 = -\frac{1}{2} + \frac{1}{2}\,x_2 + \frac{1}{2}\,x_4 \\
z = \frac{15}{2} - \frac{1}{2}\,x_2 - \frac{3}{2}\,x_4
\end{cases}$$

where $\mathcal{B} = \{1, 3, 5\}$ and $\mathcal{N} = \{2, 4\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{5}{2}, \quad \bar{x}_3 = \frac{19}{2}, \quad \bar{x}_5 = -\frac{1}{2}, \qquad \bar{x}_2 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad \frac{15}{2}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_5 = 0, \qquad \bar{\pi}_2 = \frac{1}{2}, \quad \bar{\pi}_4 = \frac{3}{2} \quad \text{of value} \quad \frac{15}{2}$$

If we increase the value of π_5, we can get a better dual solution. We have:

$$\begin{cases} \pi_2 = \frac{1}{2} - \frac{1}{2}\pi_5 \\ \pi_4 = \frac{3}{2} - \frac{1}{2}\pi_5 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{2} - \frac{1}{2}\pi_5 \geq 0}_{\pi_2 \geq 0} \\ \underbrace{\frac{3}{2} - \frac{1}{2}\pi_5 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_5 \leq 1 \\ \pi_5 \leq 3 \end{cases}$$

By increasing the value of π_5 to $\min\{1,3\} = 1$, the value of π_2 decreases to 0. Then we have that:

- π_5 becomes a dual basic variable and accordingly x_5 becomes a primal non basic variable
- π_2 becomes a dual non-basic variable and accordingly x_2 becomes a primal basic variable

We have:

$$x_5 = -\frac{1}{2} + \frac{1}{2}x_2 + \frac{1}{2}x_4 \quad \Longrightarrow \quad x_2 = 1 - x_4 + 2x_5$$

Substituting into the other equations of the primal dictionary, we get:

$$x_1 = \frac{5}{2} - \frac{1}{2}(1 - x_4 + 2x_5) - \frac{1}{2}x_4 = 2 - x_5$$

$$x_3 = \frac{19}{2} - \frac{7}{2}(1 - x_4 + 2x_5) + \frac{1}{2}x_4 = 6 + 4x_4 - 7x_5$$

$$z = \frac{15}{2} - \frac{1}{2}(1 - x_4 + 2x_5) - \frac{3}{2}x_4 = 7 - x_4 - x_5$$

Accordingly, we now have the following optimal primal dictionary:

$$\begin{cases} x_1 = 2 - x_5 \\ x_2 = 1 - x_4 + 2x_5 \\ x_3 = 6 + 4x_4 - 7x_5 \\ z = 7 - x_4 - x_5 \end{cases}$$

where $\mathcal{B} = \{1,2,3\}$ and $\mathcal{N} = \{4,5\}$.

- the basic primal optimal solution is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 1, \quad \tilde{x}_3 = 6, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \qquad \text{of value } 7$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 1, \quad \tilde{\pi}_5 = 1 \qquad \text{of value } 7$$

The optimal solution $\tilde{\boldsymbol{x}}(\, LP_2\,)$ is **integer**. Accordingly, an optimal solution $\tilde{\boldsymbol{x}}(\, ILP\,)$ is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 1$$

The optimal value $z(\, ILP\,)$ is 7.

For the LP_2 problem in canonical form, we have:

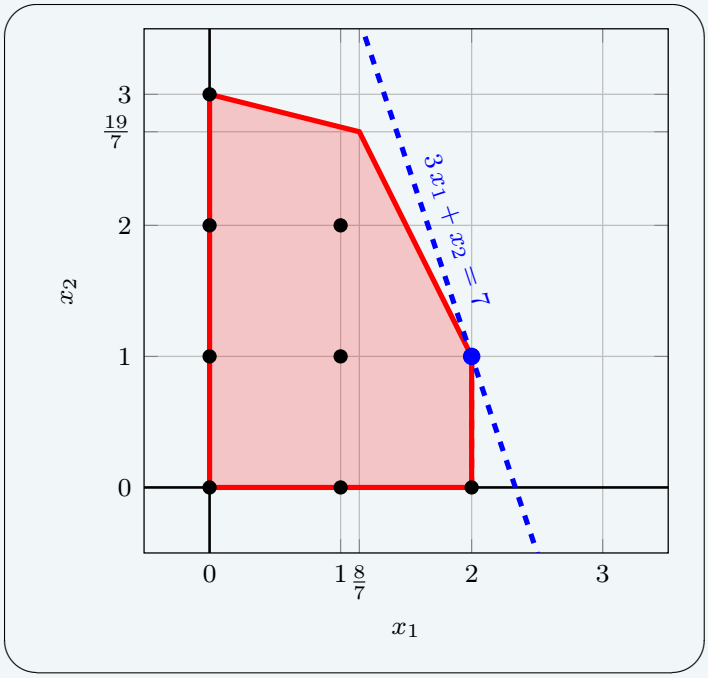
$$\begin{array}{rrrrrr} \max & 3\,x_1 & + & x_2 & & \\ & x_1 & + & 4\,x_2 & \leq & 12 \\ & 2\,x_1 & + & x_2 & \leq & 5 \\ & x_1 & & & \leq & 2 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

An optimal solution $\tilde{\boldsymbol{x}}(\, LP_2\,)$ is:

$$\tilde{x}_1 = 2, \qquad \tilde{x}_2 = 1$$

The optimal value is:

$$z(\, LP_2\,) = 7$$



Exercise 6.6 — Cutting-plane algorithm ($P8$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ linear constraints:

$$\begin{array}{lrcrcl} \max & x_1 & + & 2\,x_2 & & \\ & -x_1 & + & 2\,x_2 & \leq & 2 \\ & x_1 & + & x_2 & \leq & 3 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

and the following optimal simplex dictionary of its LP relaxation in standard form:

$$\begin{cases} x_1 = \frac{4}{3} + \frac{1}{3}\,x_3 - \frac{2}{3}\,x_4 \\ x_2 = \frac{5}{3} - \frac{1}{3}\,x_3 - \frac{1}{3}\,x_4 \\ z = \frac{14}{3} - \frac{1}{3}\,x_3 - \frac{4}{3}\,x_4 \end{cases}$$

Questions

1. Execute the cutting-plane algorithm based on Gomory fractional cuts using:
   (a) one Gomory fractional cut at each iteration;
   (b) the largest right-hand side as the cut selection strategy.

   Solve the LP problems after adding the Gomory fractional cut using the dual simplex algorithm. Graphically show the optimal solution of the LP problem at each iteration.

2. Graphically show the effect of each Gomory fractional cut on the LP relaxation of the ILP problem in canonical form.

Given a row $i \in \mathcal{B}$ of a simplex dictionary associated with a feasible basic solution such that $\bar{b}_i$ is fractional, the linear inequality

$$\sum_{j \in \mathcal{N}} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor)\, x_j \geq \bar{b}_i - \lfloor \bar{b}_i \rfloor$$

is called the **Gomory fractional cut** generated by row i.

Solution

Execution of the Cutting-Plane Algorithm to solve the ILP problem.

- **Initialization**

  Let us consider the LP_1 problem in standard form:

$$\begin{array}{rrrrrrrrrl} \max & x_1 & + & 2\,x_2 & & & & & & \\ & -x_1 & + & 2\,x_2 & + & x_3 & & & = & 2 \\ & x_1 & + & x_2 & & & + & x_4 & = & 3 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

  and its optimal simplex dictionary:

$$\begin{cases} x_1 = \frac{4}{3} + \frac{1}{3}\,x_3 - \frac{2}{3}\,x_4 \\ x_2 = \frac{5}{3} - \frac{1}{3}\,x_3 - \frac{1}{3}\,x_4 \\ z = \frac{14}{3} - \frac{1}{3}\,x_3 - \frac{4}{3}\,x_4 \end{cases}$$

  The optimal solution $\tilde{\boldsymbol{x}}(\,LP_1\,)$ is **fractional** and the LP_1 problem is **feasible**.

  We have the upper bound on the optimal value of the ILP problem:

$$z(\,ILP\,) \leq \underbrace{\left\lfloor \frac{14}{3} \right\rfloor}_{=4}$$

  For the LP_1 problem in canonical form, we have:

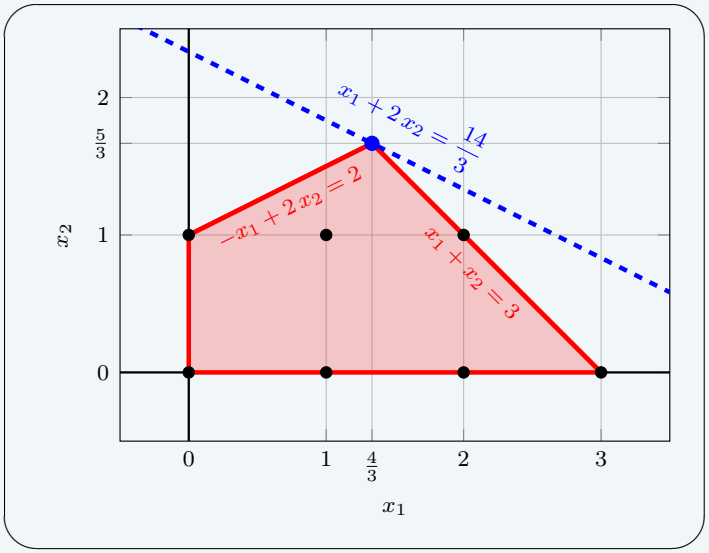
$$\begin{array}{rrrrrrl} \max & x_1 & + & 2\,x_2 & & & \\ & -x_1 & + & 2\,x_2 & \leq & 2 \\ & x_1 & + & x_2 & \leq & 3 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

  An optimal solution $\tilde{\boldsymbol{x}}(\,LP_1\,)$ is:

$$\tilde{x}_1 = \frac{4}{3}, \qquad \tilde{x}_2 = \frac{5}{3}$$

  The optimal value is:

$$z(\,LP_1\,) = \frac{14}{3}$$



- **Iteration 1**

a) **Cut generation phase**

We have the following simplex dictionary associated with a fractional basic feasible solution of the LP_1 problem in standard form:

$$\begin{cases} x_1 = \frac{4}{3} + \frac{1}{3}\,x_3 - \frac{2}{3}\,x_4 \\ x_2 = \frac{5}{3} - \frac{1}{3}\,x_3 - \frac{1}{3}\,x_4 \\ z = \frac{14}{3} - \frac{1}{3}\,x_3 - \frac{4}{3}\,x_4 \end{cases}$$

where

$$\bar{\boldsymbol{A}} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \qquad \bar{\boldsymbol{b}} = \begin{pmatrix} \frac{4}{3} \\ \frac{5}{3} \end{pmatrix}$$

– The Gomory fractional cut generated by the first row is:

$$\underbrace{\left(-\frac{1}{3} - \left\lfloor -\frac{1}{3} \right\rfloor\right)}_{-\frac{1}{3}-(-1)=\frac{2}{3}} x_3 + \underbrace{\left(\frac{2}{3} - \left\lfloor \frac{2}{3} \right\rfloor\right)}_{\frac{2}{3}-0=\frac{2}{3}} x_4 \geq \underbrace{\left(\frac{4}{3} - \left\lfloor \frac{4}{3} \right\rfloor\right)}_{\frac{4}{3}-1=\frac{1}{3}}$$

$$\frac{2}{3}\,x_3 + \frac{2}{3}\,x_4 \geq \frac{1}{3} \quad \Rightarrow \quad 2\,x_3 + 2\,x_4 \geq 1$$

From the standard form of the LP_1 problem, we have:

$$x_3 = 2 + x_1 - 2\,x_2, \qquad x_4 = 3 - x_1 - x_2.$$

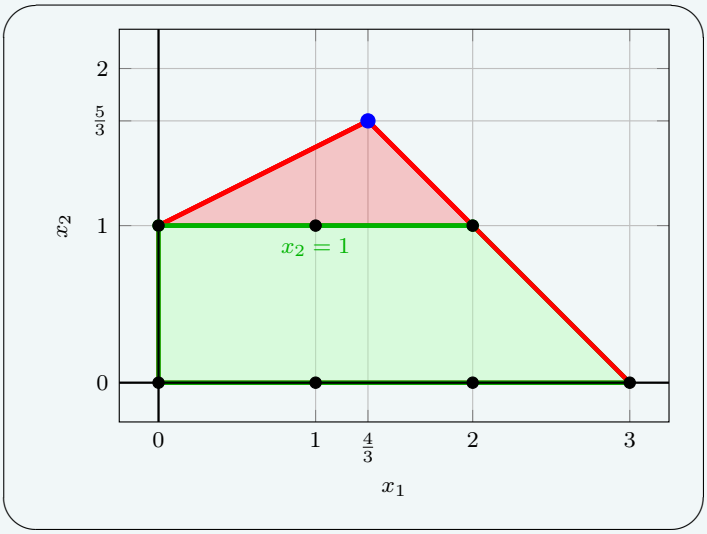
Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$2\,(2 + x_1 - 2\,x_2) + 2\,(3 - x_1 - x_2) \geq 1 \quad \Rightarrow \quad 6\,x_2 \leq 9$$

Dividing by the greatest common divisor 6 of the coefficients and rounding down the right-hand side, we obtain the cut in primitive form, the same inequality as the cut generated by the second row:

$$x_2 \leq \left\lfloor \frac{9}{6} \right\rfloor = 1$$

Effect of the Gomory fractional cut generated by the first row on the LP_1 problem in canonical form:



– The Gomory fractional cut generated by the second row is:

$$\underbrace{\left(\frac{1}{3} - \left\lfloor \frac{1}{3} \right\rfloor\right)}_{\frac{1}{3}-0=\frac{1}{3}} x_3 + \underbrace{\left(\frac{1}{3} - \left\lfloor \frac{1}{3} \right\rfloor\right)}_{\frac{1}{3}-0=\frac{1}{3}} x_4 \geq \underbrace{\left(\frac{5}{3} - \left\lfloor \frac{5}{3} \right\rfloor\right)}_{\frac{5}{3}-1=\frac{2}{3}}$$

$$\frac{1}{3}\,x_3 + \frac{1}{3}\,x_4 \geq \frac{2}{3} \quad \Rightarrow \quad x_3 + x_4 \geq 2$$

By subtracting a surplus variable, we obtain the Gomory fractional cut in standard form:

$$\frac{1}{3}\,x_3 + \frac{1}{3}\,x_4 - x_5 = \frac{2}{3} \quad \Rightarrow \quad x_3 + x_4 - 3\,x_5 = 2$$

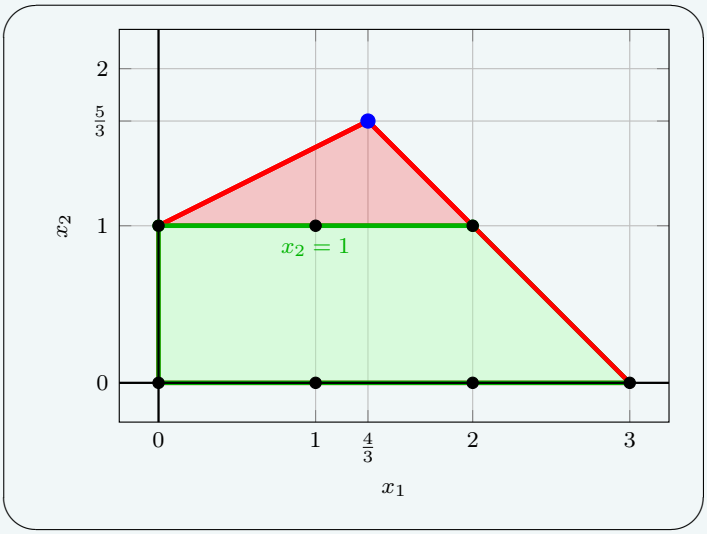
From the standard form of the LP_1 problem, we have:

$$x_3 = 2 + x_1 - 2\,x_2, \qquad x_4 = 3 - x_1 - x_2.$$

Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$(2 + x_1 - 2\,x_2) + (3 - x_1 - x_2) \geq 2 \quad \Rightarrow \quad x_2 \leq 1$$

Effect of the Gomory fractional cut generated by the second row on the LP_1 problem in canonical form:



We select the Gomory fractional cut with the largest right-hand side and we create the LP_2 problem by adding it to the LP_1 problem.

b) **LP re-optimization phase**

Let us consider the LP_2 problem in standard form:

$$\begin{array}{lrcrcrcrcrcl}
\max & x_1 & + & 2\,x_2 & & & & & & & & \\
 & -x_1 & + & 2\,x_2 & + & x_3 & & & & & = & 2 \\
 & x_1 & + & x_2 & & & + & x_4 & & & = & 3 \\
 & & & x_2 & & & & & + & x_5 & = & 1 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
\end{array}$$

For the added Gomory fractional cut we have:

$$x_3 + x_4 - 3\,x_5 = 2 \quad \Rightarrow \quad x_5 = -\frac{2}{3} + \frac{1}{3}\,x_3 + \frac{1}{3}\,x_4$$

We then add this equation to the previous optimal dictionary and we obtain the following dictionary :

$$\begin{cases} x_1 &= \frac{4}{3} + \frac{1}{3}\, x_3 - \frac{2}{3}\, x_4 \\ x_2 &= \frac{5}{3} - \frac{1}{3}\, x_3 - \frac{1}{3}\, x_4 \\ x_5 &= -\frac{2}{3} + \frac{1}{3}\, x_3 + \frac{1}{3}\, x_4 \\ z &= \frac{14}{3} - \frac{1}{3}\, x_3 - \frac{4}{3}\, x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{4}{3}, \quad \bar{x}_2 = \frac{5}{3}, \quad \bar{x}_5 = -\frac{2}{3}, \qquad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad \frac{14}{3}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_5 = 0, \qquad \bar{\pi}_3 = \frac{1}{3}, \quad \bar{\pi}_4 = \frac{4}{3} \quad \text{of value} \quad \frac{14}{3}$$

If we increase the value of π_5, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 &= \frac{1}{3} - \frac{1}{3}\, \pi_5 \\ \pi_4 &= \frac{4}{3} - \frac{1}{3}\, \pi_5 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{3} - \frac{1}{3}\, \pi_5 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{4}{3} - \frac{1}{3}\, \pi_5 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_5 &\leq 1 \\ \pi_5 &\leq 4 \end{cases}$$

By increasing the value of π_5 to $\min\{1, 4\} = 1$, the value of π_3 decreases to 0. Then we have that:

π_5 becomes a dual basic variable and accordingly x_5 becomes a primal non basic variable

π_3 becomes a dual non-basic variable and accordingly x_3 becomes a primal basic variable

We have:

$$x_5 = -\frac{2}{3} + \frac{1}{3}\, x_3 + \frac{1}{3}\, x_4 \quad \Longrightarrow \quad x_3 = 2 - x_4 + 3\, x_5$$

Substituting into the other equations of the primal dictionary, we get:

$$x_1 = \frac{4}{3} + \frac{1}{3}\,(2 - x_4 + 3\, x_5) - \frac{2}{3} x_4 = 2 - x_4 + x_5$$

$$x_2 = \frac{5}{3} - \frac{1}{3}\,(2 - x_4 + 3\, x_5) - \frac{1}{3} x_4 = 1 - x_5$$

$$z = \frac{14}{3} - \frac{1}{3}(2 - x_4 + 3\,x_5) - \frac{4}{3}x_4 = 4 - x_4 - x_5$$

Accordingly, we now have the following optimal primal dictionary:

$$\begin{cases} x_1 &= 2 - x_4 + x_5 \\ x_2 &= 1 - x_5 \\ x_3 &= 2 - x_4 + 3\,x_5 \\ z &= 4 - x_4 - x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$.

- the basic primal optimal solution is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 1, \quad \tilde{x}_3 = 2, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \qquad \text{of value } 4$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 1, \quad \tilde{\pi}_5 = 1 \qquad \text{of value } 4$$

The optimal solution $\tilde{\boldsymbol{x}}(\,LP_2\,)$ is **integer**. Accordingly, an optimal solution $\tilde{\boldsymbol{x}}(\,ILP\,)$ is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 1$$

The optimal value $z(\,ILP\,)$ is 4.

For the LP_2 problem in canonical form, we have:

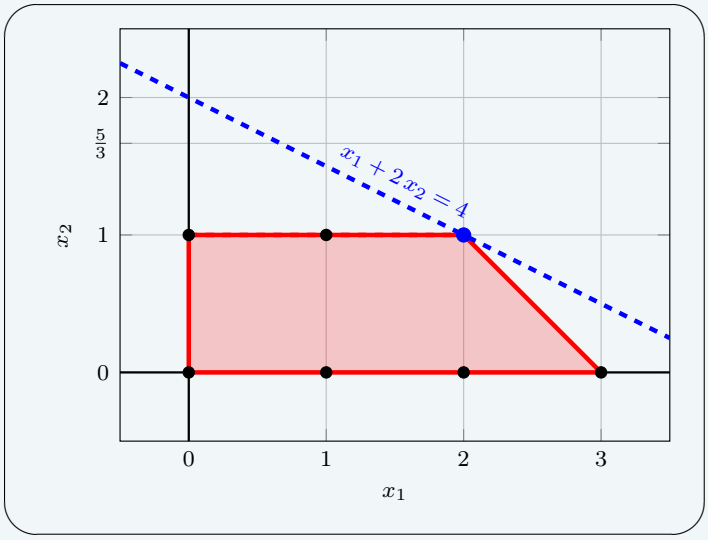
$$\begin{array}{lrcrcl} \max & x_1 & + & 2\,x_2 & & \\ & -x_1 & + & 2\,x_2 & \leq & 2 \\ & x_1 & + & x_2 & \leq & 3 \\ & & & x_2 & \leq & 1 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

An optimal solution $\tilde{\boldsymbol{x}}(\,LP_2\,)$ is:

$$\tilde{x}_1 = 2, \qquad \tilde{x}_2 = 1$$

The optimal value is:

$$z(\,LP_2\,) = 4$$



Exercise 6.7 — Cutting-plane algorithm ($P10$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 3$ linear constraints:

$$\begin{array}{rrcrcr} \max & 5\,x_1 & + & 2\,x_2 & & \\ & -x_1 & + & 3\,x_2 & \leq & 12 \\ & x_1 & + & 2\,x_2 & \leq & 12 \\ & 3\,x_1 & - & 2\,x_2 & \leq & 8 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

and the following optimal simplex dictionary of its LP relaxation in standard form:

$$\begin{cases} x_1 = 5 - \frac{1}{4}\,x_4 - \frac{1}{4}\,x_5 \\ x_2 = \frac{7}{2} - \frac{3}{8}\,x_4 + \frac{1}{8}\,x_5 \\ x_3 = \frac{13}{2} + \frac{7}{8}\,x_4 - \frac{5}{8}\,x_5 \\ z = 32 - 2\,x_4 - x_5 \end{cases}$$

Questions

1. Execute the cutting-plane algorithm based on Gomory fractional cuts using:
   (a) one Gomory fractional cut at each iteration;

(b) the largest right-hand side as the cut selection strategy.

Solve the LP problems after adding the Gomory fractional cut using the dual simplex algorithm. Graphically show the optimal solution of the LP problem at each iteration.

2. Graphically show the effect of each Gomory fractional cut on the LP relaxation of the ILP problem in canonical form.

Given a row $i \in \mathcal{B}$ of a simplex dictionary associated with a feasible basic solution such that $\bar{b}_i$ is fractional, the linear inequality

$$\sum_{j \in \mathcal{N}} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor)\, x_j \geq \bar{b}_i - \lfloor \bar{b}_i \rfloor$$

is called the **Gomory fractional cut** generated by row i.

Solution

Execution of the Cutting-Plane Algorithm to solve the ILP problem.

- **Initialization**

Let us consider the LP_1 problem in standard form:

$$\begin{array}{lrcrcrcrcrcl}
\max & 5\,x_1 & + & 2\,x_2 & & & & & & & & \\
 & -x_1 & + & 3\,x_2 & + & x_3 & & & & & = & 12 \\
 & x_1 & & & + & 2\,x_2 & + & x_4 & & & = & 12 \\
 & 3\,x_1 & - & 2\,x_2 & & & & & + & x_5 & = & 8 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
\end{array}$$

and its optimal simplex dictionary:

$$\begin{cases}
x_1 = 5 - \frac{1}{4}\, x_4 - \frac{1}{4}\, x_5 \\
x_2 = \frac{7}{2} - \frac{3}{8}\, x_4 + \frac{1}{8}\, x_5 \\
x_3 = \frac{13}{2} + \frac{7}{8}\, x_4 - \frac{5}{8}\, x_5 \\
z = 32 - 2\, x_4 - x_5
\end{cases}$$

The optimal solution $\tilde{\boldsymbol{x}}(\, LP_1\,)$ is **fractional** and the LP_1 problem is **feasible**.

We have the upper bound on the optimal value of the ILP problem:

$$z(\, ILP\,) \;\leq\; 32$$

For the LP_1 problem in canonical form, we have:

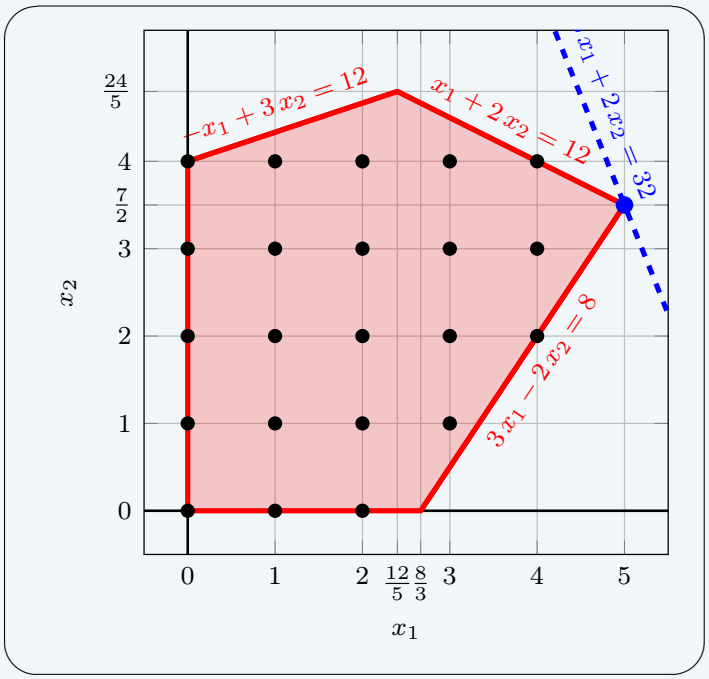
$$\begin{array}{lrcrcl} \max & 5\,x_1 & + & 2\,x_2 & & \\ & -x_1 & + & 3\,x_2 & \leq & 12 \\ & x_1 & + & 2\,x_2 & \leq & 12 \\ & 3\,x_1 & - & 2\,x_2 & \leq & 8 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

An optimal solution $\tilde{\boldsymbol{x}}(\, LP_1\,)$ is:

$$\tilde{x}_1 = 5, \qquad \tilde{x}_2 = \frac{7}{2}$$

The optimal value is:

$$z(\, LP_1\,) = 32$$



- **Iteration 1**

a) **Cut generation phase**

We have the following simplex dictionary associated with a fractional basic feasible solution of the LP_1 problem in standard form:

$$\begin{cases} x_1 = 5 - \frac{1}{4}\, x_4 - \frac{1}{4}\, x_5 \\ x_2 = \frac{7}{2} - \frac{3}{8}\, x_4 + \frac{1}{8}\, x_5 \\ x_3 = \frac{13}{2} + \frac{7}{8}\, x_4 - \frac{5}{8}\, x_5 \\ z = 32 - 2\, x_4 - x_5 \end{cases}$$

where

$$\bar{\boldsymbol{A}} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{8} & -\frac{1}{8} \\ -\frac{7}{8} & \frac{5}{8} \end{pmatrix} \qquad \bar{\boldsymbol{b}} = \begin{pmatrix} 5 \\ \frac{7}{2} \\ \frac{13}{2} \end{pmatrix}$$

– The Gomory fractional cut generated by the second row is:

$$\underbrace{\left(\frac{3}{8} - \left\lfloor \frac{3}{8} \right\rfloor\right)}_{\frac{3}{8}-0=\frac{3}{8}} x_4 + \underbrace{\left(-\frac{1}{8} - \left\lfloor -\frac{1}{8} \right\rfloor\right)}_{-\frac{1}{8}-(-1)=\frac{7}{8}} x_5 \geq \underbrace{\left(\frac{7}{2} - \left\lfloor \frac{7}{2} \right\rfloor\right)}_{\frac{7}{2}-3=\frac{1}{2}}$$

$$\frac{3}{8}\, x_4 + \frac{7}{8}\, x_5 \geq \frac{1}{2} \quad \Rightarrow \quad 3\, x_4 + 7\, x_5 \geq 4$$

By subtracting a surplus variable, we obtain the Gomory fractional cut in standard form:

$$\frac{3}{8}\, x_4 + \frac{7}{8}\, x_5 - x_6 = \frac{1}{2} \quad \Rightarrow \quad 3\, x_4 + 7\, x_5 - 8\, x_6 = 4$$

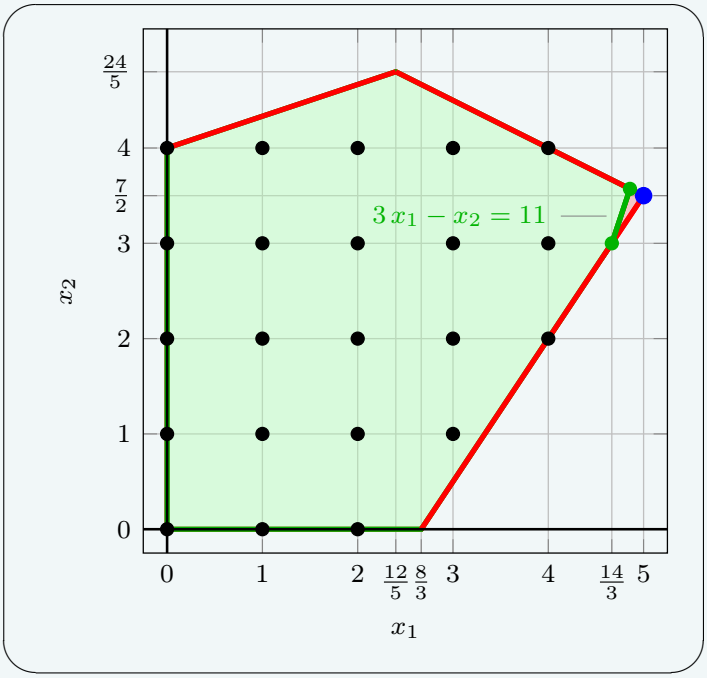
From the standard form of the LP_1 problem, we have:

$$x_4 = 12 - x_1 - 2\, x_2, \qquad x_5 = 8 - 3\, x_1 + 2\, x_2.$$

Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$3\,(12 - x_1 - 2\, x_2) + 7\,(8 - 3\, x_1 + 2\, x_2) \geq 4 \quad \Rightarrow \quad 3\, x_1 - x_2 \leq 11$$

Effect of the Gomory fractional cut generated by the second row on the LP_1 problem in canonical form:



– The Gomory fractional cut generated by the third row is:

$$\underbrace{\left(-\frac{7}{8} - \left\lfloor -\frac{7}{8} \right\rfloor\right)}_{-\frac{7}{8}-(-1)=\frac{1}{8}} x_4 + \underbrace{\left(\frac{5}{8} - \left\lfloor \frac{5}{8} \right\rfloor\right)}_{\frac{5}{8}-0=\frac{5}{8}} x_5 \geq \underbrace{\left(\frac{13}{2} - \left\lfloor \frac{13}{2} \right\rfloor\right)}_{\frac{13}{2}-6=\frac{1}{2}}$$

$$\frac{1}{8}\,x_4 + \frac{5}{8}\,x_5 \geq \frac{1}{2} \quad \Rightarrow \quad x_4 + 5\,x_5 \geq 4$$

By subtracting a surplus variable, we obtain the Gomory fractional cut in standard form:

$$\frac{1}{8}\,x_4 + \frac{5}{8}\,x_5 - x_6 = \frac{1}{2} \quad \Rightarrow \quad x_4 + 5\,x_5 - 8\,x_6 = 4$$

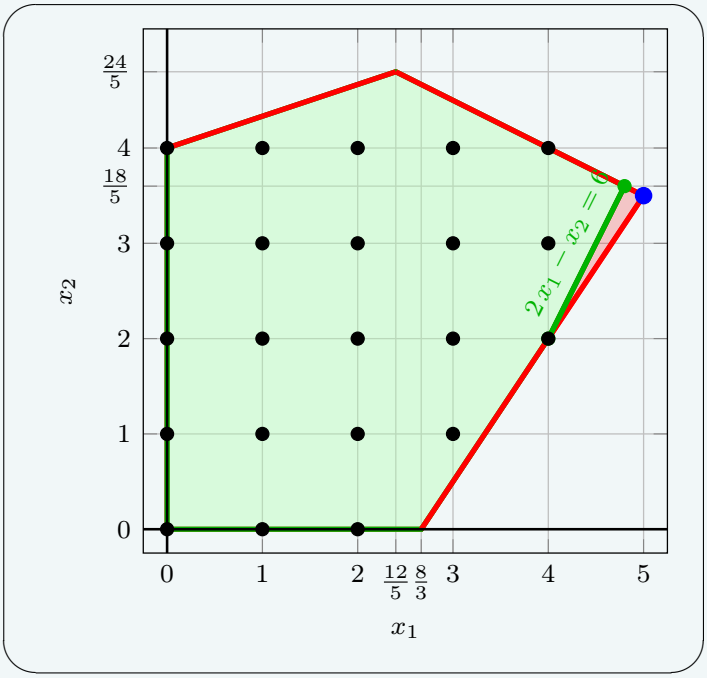
From the standard form of the LP_1 problem, we have:

$$x_4 = 12 - x_1 - 2\,x_2, \qquad x_5 = 8 - 3\,x_1 + 2\,x_2.$$

Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$(12 - x_1 - 2\,x_2) + 5\,(8 - 3\,x_1 + 2\,x_2) \geq 4 \quad \Rightarrow \quad 2\,x_1 - x_2 \leq 6$$

Effect of the Gomory fractional cut generated by the third row on the LP_1 problem in canonical form:



Both Gomory fractional cuts have the same right-hand side, equal to $\frac{1}{2}$. We select the Gomory fractional cut generated by the third row and we create the LP_2 problem by adding it to the LP_1 problem.

b) **LP re-optimization phase**

Let us consider the LP_2 problem in standard form:

$$
\begin{array}{lrcrcrcrcrcl}
\max & 5\,x_1 & + & 2\,x_2 & & & & & & & & \\
 & -x_1 & + & 3\,x_2 & + & x_3 & & & & & = & 12 \\
 & x_1 & & & + & 2\,x_2 & + & x_4 & & & = & 12 \\
 & 3\,x_1 & - & 2\,x_2 & & & & & + & x_5 & = & 8 \\
 & 2\,x_1 & - & x_2 & & & & & & + \; x_6 & = & 6 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, \quad x_6 & \geq & 0
\end{array}
$$

For the added Gomory fractional cut we have:

$$x_4 + 5\,x_5 - 8\,x_6 = 4 \quad \Rightarrow \quad x_6 = -\frac{1}{2} + \frac{1}{8}\,x_4 + \frac{5}{8}\,x_5$$

We then add this equation to the previous optimal dictionary and we obtain the following dictionary :

$$\begin{cases} x_1 = 5 - \frac{1}{4} x_4 - \frac{1}{4} x_5 \\ x_2 = \frac{7}{2} - \frac{3}{8} x_4 + \frac{1}{8} x_5 \\ x_3 = \frac{13}{2} + \frac{7}{8} x_4 - \frac{5}{8} x_5 \\ x_6 = -\frac{1}{2} + \frac{1}{8} x_4 + \frac{5}{8} x_5 \\ z = 32 - 2 x_4 - x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 6\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = 5, \quad \bar{x}_2 = \frac{7}{2}, \quad \bar{x}_3 = \frac{13}{2}, \quad \bar{x}_6 = -\frac{1}{2}, \qquad \bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad 32$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_6 = 0, \qquad \bar{\pi}_4 = 2, \quad \bar{\pi}_5 = 1 \quad \text{of value} \quad 32$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_4 = 2 - \frac{1}{8} \pi_6 \\ \pi_5 = 1 - \frac{5}{8} \pi_6 \end{cases} \qquad \begin{cases} \underbrace{2 - \frac{1}{8} \pi_6 \geq 0}_{\pi_4 \geq 0} \\ \underbrace{1 - \frac{5}{8} \pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 \leq 16 \\ \pi_6 \leq \frac{8}{5} \end{cases}$$

By increasing the value of π_6 to $\min\left\{16, \frac{8}{5}\right\} = \frac{8}{5}$, the value of π_5 decreases to 0. Then we have that:

π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non basic variable

π_5 becomes a dual non-basic variable and accordingly x_5 becomes a primal basic variable

We have:

$$x_6 = -\frac{1}{2} + \frac{1}{8} x_4 + \frac{5}{8} x_5 \quad \Longrightarrow \quad x_5 = \frac{4}{5} - \frac{1}{5} x_4 + \frac{8}{5} x_6$$

Substituting into the other equations of the primal dictionary, we get:

$$x_1 = 5 - \frac{1}{4} x_4 - \frac{1}{4} \left(\frac{4}{5} - \frac{1}{5} x_4 + \frac{8}{5} x_6 \right) = \frac{24}{5} - \frac{1}{5} x_4 - \frac{2}{5} x_6$$

$$x_2 = \frac{7}{2} - \frac{3}{8} x_4 + \frac{1}{8} \left(\frac{4}{5} - \frac{1}{5} x_4 + \frac{8}{5} x_6 \right) = \frac{18}{5} - \frac{2}{5} x_4 + \frac{1}{5} x_6$$

$$x_3 = \frac{13}{2} + \frac{7}{8}x_4 - \frac{5}{8}\left(\frac{4}{5} - \frac{1}{5}x_4 + \frac{8}{5}x_6\right) = 6 + x_4 - x_6$$

$$z = 32 - 2x_4 - \left(\frac{4}{5} - \frac{1}{5}x_4 + \frac{8}{5}x_6\right) = \frac{156}{5} - \frac{9}{5}x_4 - \frac{8}{5}x_6$$

Accordingly, we now have the following optimal primal dictionary:

$$\begin{cases} x_1 = \frac{24}{5} - \frac{1}{5}x_4 - \frac{2}{5}x_6 \\ x_2 = \frac{18}{5} - \frac{2}{5}x_4 + \frac{1}{5}x_6 \\ x_3 = 6 + x_4 - x_6 \\ x_5 = \frac{4}{5} - \frac{1}{5}x_4 + \frac{8}{5}x_6 \\ z = \frac{156}{5} - \frac{9}{5}x_4 - \frac{8}{5}x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 5\}$ and $\mathcal{N} = \{4, 6\}$.

- the basic primal optimal solution is:

$$\tilde{x}_1 = \frac{24}{5}, \quad \tilde{x}_2 = \frac{18}{5}, \quad \tilde{x}_3 = 6, \quad \tilde{x}_5 = \frac{4}{5}, \quad \tilde{x}_4 = 0, \quad \tilde{x}_6 = 0 \qquad \text{of value } \frac{156}{5}$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_5 = 0, \quad \tilde{\pi}_4 = \frac{9}{5}, \quad \tilde{\pi}_6 = \frac{8}{5} \qquad \text{of value } \frac{156}{5}$$

The optimal solution $\tilde{\boldsymbol{x}}(\, LP_2\,)$ is **fractional** and the LP_2 problem is **feasible**.

We have the upper bound on the optimal value of the ILP problem:

$$z(\, ILP\,) \leq \underbrace{\left\lfloor \frac{156}{5} \right\rfloor}_{=31}$$

For the LP_2 problem in canonical form, we have:

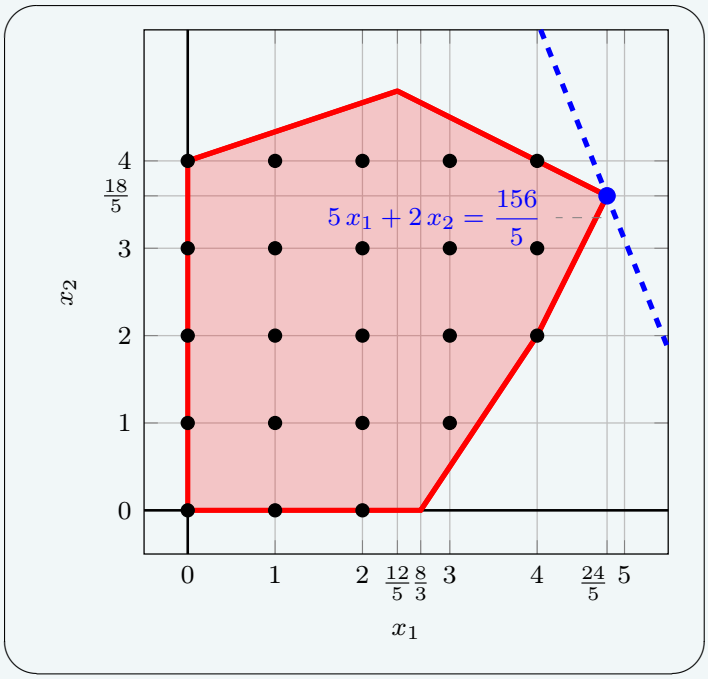
$$\begin{array}{rrrrrr} \max & 5\,x_1 & + & 2\,x_2 & & \\ & -x_1 & + & 3\,x_2 & \leq & 12 \\ & x_1 & + & 2\,x_2 & \leq & 12 \\ & 3\,x_1 & - & 2\,x_2 & \leq & 8 \\ & 2\,x_1 & - & x_2 & \leq & 6 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

An optimal solution $\tilde{\boldsymbol{x}}(\, LP_2 \,)$ is:

$$\tilde{x}_1 = \frac{24}{5}, \qquad \tilde{x}_2 = \frac{18}{5}$$

The optimal value is:

$$z(\, LP_2 \,) = \frac{156}{5}$$



- **Iteration 2**

a) **Cut generation phase**

We have the following simplex dictionary associated with a fractional basic feasible solution of the LP_2 problem in standard form:

$$\begin{cases} x_1 = \frac{24}{5} - \frac{1}{5}\,x_4 - \frac{2}{5}\,x_6 \\ x_2 = \frac{18}{5} - \frac{2}{5}\,x_4 + \frac{1}{5}\,x_6 \\ x_3 = 6 + x_4 - x_6 \\ x_5 = \frac{4}{5} - \frac{1}{5}\,x_4 + \frac{8}{5}\,x_6 \\ z = \frac{156}{5} - \frac{9}{5}\,x_4 - \frac{8}{5}\,x_6 \end{cases}$$

where

$$\bar{\boldsymbol{A}} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \\ -1 & 1 \\ \frac{1}{5} & -\frac{8}{5} \end{pmatrix} \qquad \bar{\boldsymbol{b}} = \begin{pmatrix} \frac{24}{5} \\ \frac{18}{5} \\ 6 \\ \frac{4}{5} \end{pmatrix}$$

– The Gomory fractional cut generated by the first row is:

$$\underbrace{\left(\frac{1}{5} - \left\lfloor \frac{1}{5} \right\rfloor\right)}_{\frac{1}{5}-0=\frac{1}{5}} x_4 + \underbrace{\left(\frac{2}{5} - \left\lfloor \frac{2}{5} \right\rfloor\right)}_{\frac{2}{5}-0=\frac{2}{5}} x_6 \geq \underbrace{\left(\frac{24}{5} - \left\lfloor \frac{24}{5} \right\rfloor\right)}_{\frac{24}{5}-4=\frac{4}{5}}$$

$$\frac{1}{5}\, x_4 + \frac{2}{5}\, x_6 \geq \frac{4}{5} \quad \Rightarrow \quad x_4 + 2\, x_6 \geq 4$$

By subtracting a surplus variable, we obtain the Gomory fractional cut in standard form:

$$\frac{1}{5}\, x_4 + \frac{2}{5}\, x_6 - x_7 = \frac{4}{5} \quad \Rightarrow \quad x_4 + 2\, x_6 - 5\, x_7 = 4$$

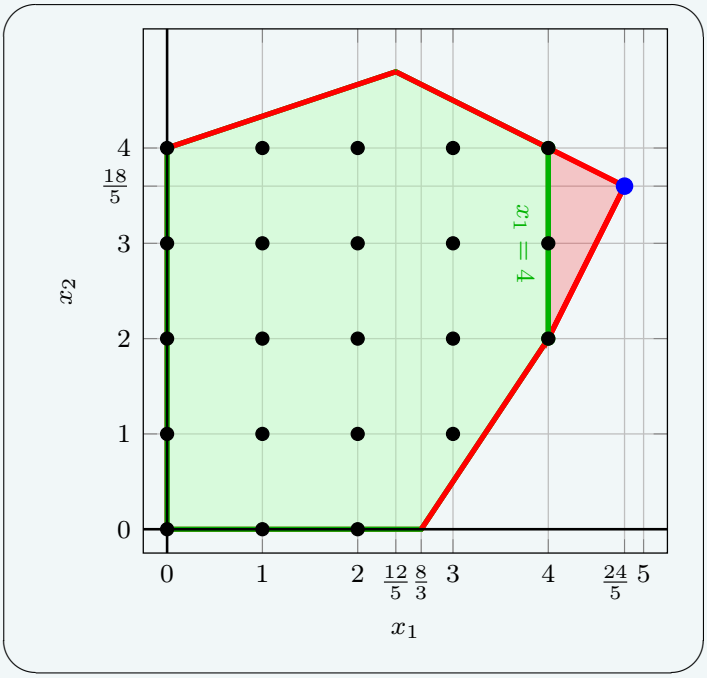
From the standard form of the LP_1 problem and the added cut, we have:

$$x_4 = 12 - x_1 - 2\, x_2, \qquad x_6 = 6 - 2\, x_1 + x_2.$$

Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$(12 - x_1 - 2\, x_2) + 2\,(6 - 2\, x_1 + x_2) \geq 4 \quad \Rightarrow \quad x_1 \leq 4$$

Effect of the Gomory fractional cut generated by the first row on the LP_2 problem in canonical form:



– The Gomory fractional cut generated by the second row is:

$$\underbrace{\left(\frac{2}{5} - \left\lfloor \frac{2}{5} \right\rfloor\right)}_{\frac{2}{5}-0=\frac{2}{5}} x_4 + \underbrace{\left(-\frac{1}{5} - \left\lfloor -\frac{1}{5} \right\rfloor\right)}_{-\frac{1}{5}-(-1)=\frac{4}{5}} x_6 \geq \underbrace{\left(\frac{18}{5} - \left\lfloor \frac{18}{5} \right\rfloor\right)}_{\frac{18}{5}-3=\frac{3}{5}}$$

$$\frac{2}{5}\, x_4 + \frac{4}{5}\, x_6 \geq \frac{3}{5} \quad \Rightarrow \quad 2\, x_4 + 4\, x_6 \geq 3$$

From the standard form of the LP_1 problem and the added cut, we have:

$$x_4 = 12 - x_1 - 2\, x_2, \qquad x_6 = 6 - 2\, x_1 + x_2.$$

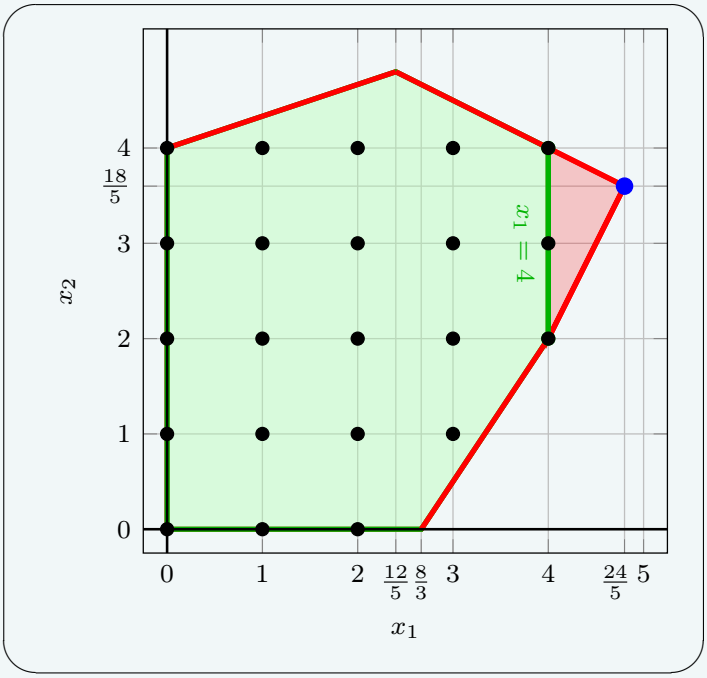
Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$2\,(12 - x_1 - 2\, x_2) + 4\,(6 - 2\, x_1 + x_2) \geq 3 \quad \Rightarrow \quad 10\, x_1 \leq 45$$

Dividing by the greatest common divisor 10 of the coefficients and rounding down the right-hand side, we obtain the cut in primitive form, the same inequality as the cut generated by the first row:

$$x_1 \leq \left\lfloor \frac{45}{10} \right\rfloor = 4$$

Effect of the Gomory fractional cut generated by the second row on the LP_2 problem in canonical form:



– The Gomory fractional cut generated by the fourth row is:

$$\underbrace{\left(\frac{1}{5} - \left\lfloor \frac{1}{5} \right\rfloor\right)}_{\frac{1}{5}-0=\frac{1}{5}} x_4 + \underbrace{\left(-\frac{8}{5} - \left\lfloor -\frac{8}{5} \right\rfloor\right)}_{-\frac{8}{5}-(-2)=\frac{2}{5}} x_6 \geq \underbrace{\left(\frac{4}{5} - \left\lfloor \frac{4}{5} \right\rfloor\right)}_{\frac{4}{5}-0=\frac{4}{5}}$$

$$\frac{1}{5}\, x_4 + \frac{2}{5}\, x_6 \geq \frac{4}{5} \quad \Rightarrow \quad x_4 + 2\, x_6 \geq 4$$

This Gomory fractional cut is the same as the one generated by the first row.

We select the Gomory fractional cut with the largest right-hand side and we create the LP_3 problem by adding it to the LP_2 problem.

b) **LP re-optimization phase**

Let us consider the LP_3 problem in standard form:

$$
\begin{array}{lrcrcrcrcrcrcl}
\max & 5\,x_1 & + & 2\,x_2 & & & & & & & & & & \\
 & -x_1 & + & 3\,x_2 & + & x_3 & & & & & & & = & 12 \\
 & x_1 & & & + & 2\,x_2 & + & x_4 & & & & & = & 12 \\
 & 3\,x_1 & - & 2\,x_2 & & & & & + & x_5 & & & = & 8 \\
 & 2\,x_1 & - & x_2 & & & & & & & + & x_6 & = & 6 \\
 & x_1 & & & & & & & & & & & + \; x_7 = & 4 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & \; x_7 \geq & 0
\end{array}
$$

For the added Gomory fractional cut we have:

$$x_4 + 2\,x_6 - 5\,x_7 = 4 \quad \Rightarrow \quad x_7 = -\frac{4}{5} + \frac{1}{5}\,x_4 + \frac{2}{5}\,x_6$$

We then add this equation to the previous optimal dictionary and we obtain the following dictionary :

$$
\begin{cases}
x_1 = \frac{24}{5} - \frac{1}{5}\,x_4 - \frac{2}{5}\,x_6 \\
x_2 = \frac{18}{5} - \frac{2}{5}\,x_4 + \frac{1}{5}\,x_6 \\
x_3 = 6 + x_4 - x_6 \\
x_5 = \frac{4}{5} - \frac{1}{5}\,x_4 + \frac{8}{5}\,x_6 \\
x_7 = -\frac{4}{5} + \frac{1}{5}\,x_4 + \frac{2}{5}\,x_6 \\
z = \frac{156}{5} - \frac{9}{5}\,x_4 - \frac{8}{5}\,x_6
\end{cases}
$$

where $\mathcal{B} = \{1, 2, 3, 5, 7\}$ and $\mathcal{N} = \{4, 6\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{24}{5}, \quad \bar{x}_2 = \frac{18}{5}, \quad \bar{x}_3 = 6, \quad \bar{x}_5 = \frac{4}{5}, \quad \bar{x}_7 = -\frac{4}{5},$$
$$\bar{x}_4 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad \frac{156}{5}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_5 = 0, \quad \bar{\pi}_7 = 0,$$
$$\bar{\pi}_4 = \frac{9}{5}, \quad \bar{\pi}_6 = \frac{8}{5} \quad \text{of value} \quad \frac{156}{5}$$

If we increase the value of π_7, we can get a better dual solution. We have:

$$\begin{cases} \pi_4 = \frac{9}{5} - \frac{1}{5}\pi_7 \\ \pi_6 = \frac{8}{5} - \frac{2}{5}\pi_7 \end{cases} \qquad \begin{cases} \underbrace{\frac{9}{5} - \frac{1}{5}\pi_7 \geq 0}_{\pi_4 \geq 0} \\ \underbrace{\frac{8}{5} - \frac{2}{5}\pi_7 \geq 0}_{\pi_6 \geq 0} \end{cases} \qquad \begin{cases} \pi_7 \leq 9 \\ \pi_7 \leq 4 \end{cases}$$

By increasing the value of π_7 to $\min\{9, 4\} = 4$, the value of π_6 decreases to 0. Then we have that:

- π_7 becomes a dual basic variable and accordingly x_7 becomes a primal non basic variable
- π_6 becomes a dual non-basic variable and accordingly x_6 becomes a primal basic variable

We have:

$$x_7 = -\frac{4}{5} + \frac{1}{5}x_4 + \frac{2}{5}x_6 \quad \Longrightarrow \quad x_6 = 2 - \frac{1}{2}x_4 + \frac{5}{2}x_7$$

Substituting into the other equations of the primal dictionary, we get:

$$x_1 = \frac{24}{5} - \frac{1}{5}x_4 - \frac{2}{5}\left(2 - \frac{1}{2}x_4 + \frac{5}{2}x_7\right) = 4 - x_7$$

$$x_2 = \frac{18}{5} - \frac{2}{5}x_4 + \frac{1}{5}\left(2 - \frac{1}{2}x_4 + \frac{5}{2}x_7\right) = 4 - \frac{1}{2}x_4 + \frac{1}{2}x_7$$

$$x_3 = 6 + x_4 - \left(2 - \frac{1}{2}x_4 + \frac{5}{2}x_7\right) = 4 + \frac{3}{2}x_4 - \frac{5}{2}x_7$$

$$x_5 = \frac{4}{5} - \frac{1}{5}x_4 + \frac{8}{5}\left(2 - \frac{1}{2}x_4 + \frac{5}{2}x_7\right) = 4 - x_4 + 4x_7$$

$$z = \frac{156}{5} - \frac{9}{5}x_4 - \frac{8}{5}\left(2 - \frac{1}{2}x_4 + \frac{5}{2}x_7\right) = 28 - x_4 - 4x_7$$

Accordingly, we now have the following optimal primal dictionary:

$$\begin{cases} x_1 = 4 - x_7 \\ x_2 = 4 - \frac{1}{2}x_4 + \frac{1}{2}x_7 \\ x_3 = 4 + \frac{3}{2}x_4 - \frac{5}{2}x_7 \\ x_5 = 4 - x_4 + 4x_7 \\ x_6 = 2 - \frac{1}{2}x_4 + \frac{5}{2}x_7 \\ z = 28 - x_4 - 4x_7 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 5, 6\}$ and $\mathcal{N} = \{4, 7\}$.

– the basic primal optimal solution is:

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = 4, \quad \tilde{x}_3 = 4, \quad \tilde{x}_5 = 4, \quad \tilde{x}_6 = 2, \quad \tilde{x}_4 = 0, \quad \tilde{x}_7 = 0$$

of value 28

– the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_5 = 0, \quad \tilde{\pi}_6 = 0, \quad \tilde{\pi}_4 = 1, \quad \tilde{\pi}_7 = 4$$

of value 28

The optimal solution $\tilde{\boldsymbol{x}}(\, LP_3\,)$ is **integer**. Accordingly, an optimal solution $\tilde{\boldsymbol{x}}(\, ILP\,)$ is:

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = 4$$

The optimal value $z(\, ILP\,)$ is 28.

For the LP_3 problem in canonical form, we have:

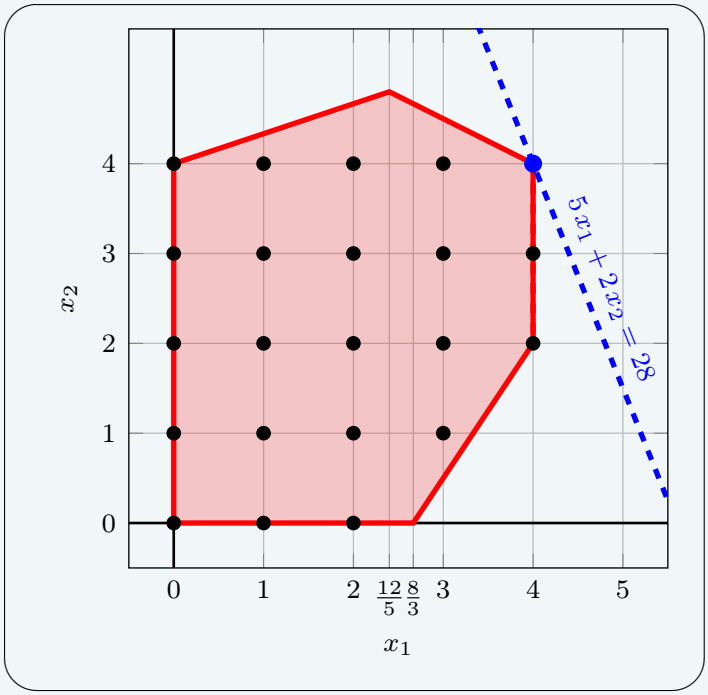
$$\begin{array}{lrcrcl} \max & 5\,x_1 & + & 2\,x_2 & & \\ & -x_1 & + & 3\,x_2 & \leq & 12 \\ & x_1 & + & 2\,x_2 & \leq & 12 \\ & 3\,x_1 & - & 2\,x_2 & \leq & 8 \\ & 2\,x_1 & - & x_2 & \leq & 6 \\ & x_1 & & & \leq & 4 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

An optimal solution $\tilde{\boldsymbol{x}}(\, LP_3\,)$ is:

$$\tilde{x}_1 = 4, \qquad \tilde{x}_2 = 4$$

The optimal value is:

$$z(\, LP_3\,) = 28$$



Exercise 6.8 — Cutting-plane algorithm ($P21$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 3$ linear constraints:

$$\begin{array}{rrrcr} \max & x_1 & + & 2\,x_2 & \\ & -x_1 & - & x_2 & \leq & -2 \\ & 4\,x_1 & - & 2\,x_2 & \leq & 7 \\ & x_1 & + & 5\,x_2 & \leq & 21 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

and the following optimal simplex dictionary of its LP relaxation in standard form:

$$\begin{cases} x_1 = \frac{7}{2} - \frac{5}{22}\,x_4 - \frac{1}{11}\,x_5 \\ x_2 = \frac{7}{2} + \frac{1}{22}\,x_4 - \frac{2}{11}\,x_5 \\ x_3 = 5 - \frac{2}{11}\,x_4 - \frac{3}{11}\,x_5 \\ z = \frac{21}{2} - \frac{3}{22}\,x_4 - \frac{5}{11}\,x_5 \end{cases}$$

Questions

1. Execute the cutting-plane algorithm based on Gomory fractional cuts using:
   (a) one Gomory fractional cut at each iteration;
   (b) the largest right-hand side as the cut selection strategy.

   Solve the LP problems after adding the Gomory fractional cut using the dual simplex algorithm. Graphically show the optimal solution of the LP problem at each iteration.

2. Graphically show the effect of each Gomory fractional cut on the LP relaxation of the ILP problem in canonical form.

Given a row $i \in \mathcal{B}$ of a simplex dictionary associated with a feasible basic solution such that $\bar{b}_i$ is fractional, the linear inequality

$$\sum_{j \in \mathcal{N}} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor)\, x_j \geq \bar{b}_i - \lfloor \bar{b}_i \rfloor$$

is called the **Gomory fractional cut** generated by row i.

Solution

Execution of the Cutting-Plane Algorithm to solve the ILP problem.

- **Initialization**

  Let us consider the LP_1 problem in standard form:

$$\begin{array}{lrcrcrcrcrcr}
\max & x_1 & + & 2\,x_2 & & & & & & & & \\
 & -x_1 & - & x_2 & + & x_3 & & & & & = & -2 \\
 & 4\,x_1 & - & 2\,x_2 & & & + & x_4 & & & = & 7 \\
 & x_1 & + & 5\,x_2 & & & & & + & x_5 & = & 21 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
\end{array}$$

  and its optimal simplex dictionary:

$$\begin{cases} x_1 = \frac{7}{2} - \frac{5}{22}\, x_4 - \frac{1}{11}\, x_5 \\ x_2 = \frac{7}{2} + \frac{1}{22}\, x_4 - \frac{2}{11}\, x_5 \\ x_3 = 5 - \frac{2}{11}\, x_4 - \frac{3}{11}\, x_5 \\ z = \frac{21}{2} - \frac{3}{22}\, x_4 - \frac{5}{11}\, x_5 \end{cases}$$

The optimal solution $\tilde{\boldsymbol{x}}(\, LP_1\,)$ is **fractional** and the LP_1 problem is **feasible**.

We have the upper bound on the optimal value of the ILP problem:

$$z(\, ILP\,) \leq \underbrace{\left\lfloor \frac{21}{2} \right\rfloor}_{=10}$$

For the LP_1 problem in canonical form, we have:

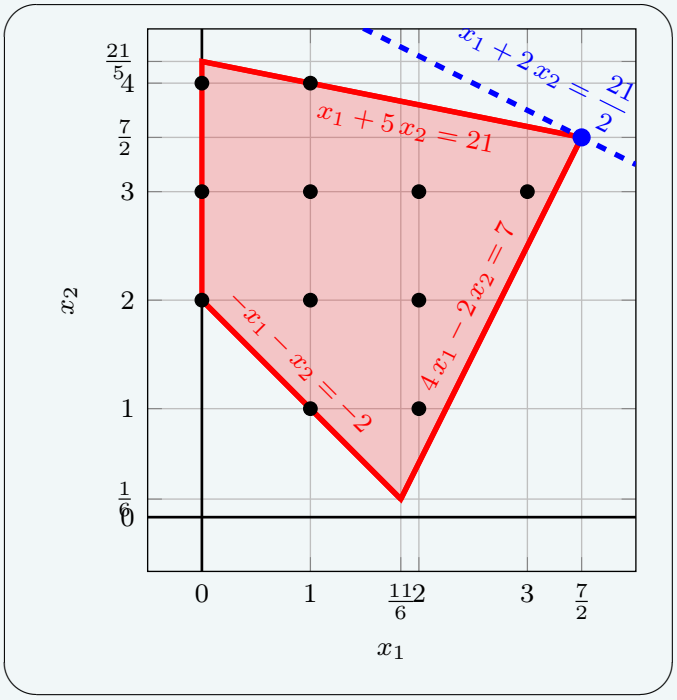
$$\begin{array}{lrcrcr} \max & x_1 & + & 2\,x_2 & & \\ & -x_1 & - & x_2 & \leq & -2 \\ & 4\,x_1 & - & 2\,x_2 & \leq & 7 \\ & x_1 & + & 5\,x_2 & \leq & 21 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

An optimal solution $\tilde{\boldsymbol{x}}(\, LP_1\,)$ is:

$$\tilde{x}_1 = \frac{7}{2}, \qquad \tilde{x}_2 = \frac{7}{2}$$

The optimal value is:

$$z(\, LP_1\,) = \frac{21}{2}$$



- **Iteration 1**

a) **Cut generation phase**

We have the following simplex dictionary associated with a fractional basic feasible solution of the LP_1 problem in standard form:

$$\begin{cases} x_1 = \frac{7}{2} - \frac{5}{22} x_4 - \frac{1}{11} x_5 \\ x_2 = \frac{7}{2} + \frac{1}{22} x_4 - \frac{2}{11} x_5 \\ x_3 = 5 - \frac{2}{11} x_4 - \frac{3}{11} x_5 \\ z = \frac{21}{2} - \frac{3}{22} x_4 - \frac{5}{11} x_5 \end{cases}$$

where

$$\bar{\boldsymbol{A}} = \begin{pmatrix} \frac{5}{22} & \frac{1}{11} \\ -\frac{1}{22} & \frac{2}{11} \\ \frac{2}{11} & \frac{3}{11} \end{pmatrix} \qquad \bar{\boldsymbol{b}} = \begin{pmatrix} \frac{7}{2} \\ \frac{7}{2} \\ 5 \end{pmatrix}$$

– The Gomory fractional cut generated by the first row is:

$$\underbrace{\left(\frac{5}{22} - \left\lfloor \frac{5}{22} \right\rfloor\right)}_{\frac{5}{22}-0=\frac{5}{22}} x_4 + \underbrace{\left(\frac{1}{11} - \left\lfloor \frac{1}{11} \right\rfloor\right)}_{\frac{1}{11}-0=\frac{1}{11}} x_5 \geq \underbrace{\left(\frac{7}{2} - \left\lfloor \frac{7}{2} \right\rfloor\right)}_{\frac{7}{2}-3=\frac{1}{2}}$$

$$\frac{5}{22}\, x_4 + \frac{1}{11}\, x_5 \geq \frac{1}{2} \quad \Rightarrow \quad 5\, x_4 + 2\, x_5 \geq 11$$

By subtracting a surplus variable, we obtain the Gomory fractional cut in standard form:

$$\frac{5}{22}\, x_4 + \frac{1}{11}\, x_5 - x_6 = \frac{1}{2} \quad \Rightarrow \quad 5\, x_4 + 2\, x_5 - 22\, x_6 = 11$$

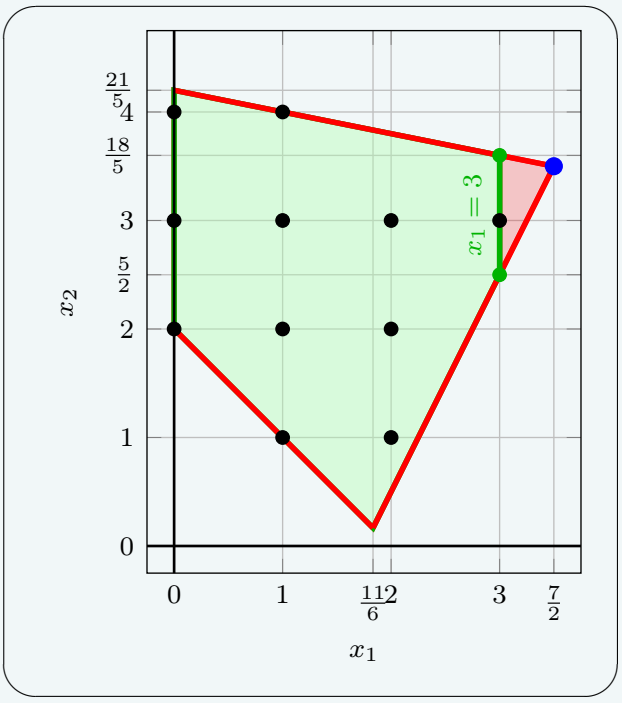
From the standard form of the LP_1 problem, we have:

$$x_4 = 7 - 4\, x_1 + 2\, x_2, \qquad x_5 = 21 - x_1 - 5\, x_2.$$

Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$5\,(7 - 4\, x_1 + 2\, x_2) + 2\,(21 - x_1 - 5\, x_2) \geq 11 \quad \Rightarrow \quad x_1 \leq 3$$

Effect of the Gomory fractional cut generated by the first row on the LP_1 problem in canonical form:



– The Gomory fractional cut generated by the second row is:

$$\underbrace{\left(-\frac{1}{22}-\left\lfloor-\frac{1}{22}\right\rfloor\right)}_{-\frac{1}{22}-(-1)=\frac{21}{22}}x_4+\underbrace{\left(\frac{2}{11}-\left\lfloor\frac{2}{11}\right\rfloor\right)}_{\frac{2}{11}-0=\frac{2}{11}}x_5\geq\underbrace{\left(\frac{7}{2}-\left\lfloor\frac{7}{2}\right\rfloor\right)}_{\frac{7}{2}-3=\frac{1}{2}}$$

$$\frac{21}{22}\,x_4+\frac{2}{11}\,x_5\geq\frac{1}{2}\quad\Rightarrow\quad 21\,x_4+4\,x_5\geq 11$$

By subtracting a surplus variable, we obtain the Gomory fractional cut in standard form:

$$\frac{21}{22}\,x_4+\frac{2}{11}\,x_5-x_6=\frac{1}{2}\quad\Rightarrow\quad 21\,x_4+4\,x_5-22\,x_6=11$$

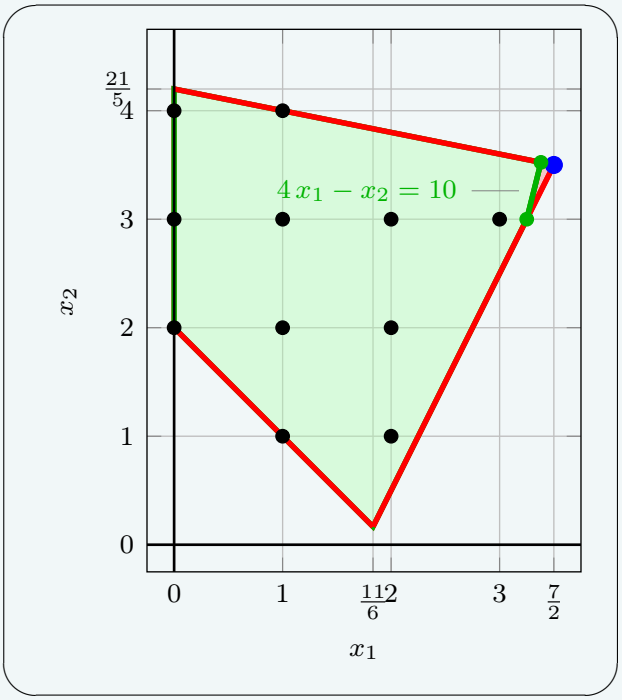
From the standard form of the LP_1 problem, we have:

$$x_4=7-4\,x_1+2\,x_2,\qquad x_5=21-x_1-5\,x_2.$$

Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$21\,(7-4\,x_1+2\,x_2)+4\,(21-x_1-5\,x_2)\geq 11\quad\Rightarrow\quad 4\,x_1-x_2\leq 10$$

Effect of the Gomory fractional cut generated by the second row on the LP_1 problem in canonical form:



Both Gomory fractional cuts have the same right-hand side, equal to $\frac{1}{2}$. We select the Gomory fractional cut generated by the first row and we create the LP_2 problem by adding it to the LP_1 problem.

b) **LP re-optimization phase**

Let us consider the LP_2 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrl}
\max & x_1 & + & 2\,x_2 & & & & & & & & & \\
 & -x_1 & - & x_2 & + & x_3 & & & & & & = & -2 \\
 & 4\,x_1 & - & 2\,x_2 & & & + & x_4 & & & & = & 7 \\
 & x_1 & + & 5\,x_2 & & & & & + & x_5 & & = & 21 \\
 & x_1 & & & & & & & & & + \; x_6 & = & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & x_6 & \geq & 0
\end{array}$$

For the added Gomory fractional cut we have:

$$5\,x_4 + 2\,x_5 - 22\,x_6 = 11 \quad \Rightarrow \quad x_6 = -\frac{1}{2} + \frac{5}{22}\,x_4 + \frac{1}{11}\,x_5$$

We then add this equation to the previous optimal dictionary and we obtain the following dictionary :

$$\begin{cases}
x_1 = \frac{7}{2} - \frac{5}{22}\,x_4 - \frac{1}{11}\,x_5 \\
x_2 = \frac{7}{2} + \frac{1}{22}\,x_4 - \frac{2}{11}\,x_5 \\
x_3 = 5 - \frac{2}{11}\,x_4 - \frac{3}{11}\,x_5 \\
x_6 = -\frac{1}{2} + \frac{5}{22}\,x_4 + \frac{1}{11}\,x_5 \\
z = \frac{21}{2} - \frac{3}{22}\,x_4 - \frac{5}{11}\,x_5
\end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 6\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{7}{2}, \quad \bar{x}_2 = \frac{7}{2}, \quad \bar{x}_3 = 5, \quad \bar{x}_6 = -\frac{1}{2}, \qquad \bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{21}{2}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_6 = 0, \qquad \bar{\pi}_4 = \frac{3}{22}, \quad \bar{\pi}_5 = \frac{5}{11} \quad \text{of value} \quad \frac{21}{2}$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_4 &= \frac{3}{22} - \frac{5}{22}\pi_6 \\ \pi_5 &= \frac{5}{11} - \frac{1}{11}\pi_6 \end{cases} \qquad \begin{cases} \underbrace{\frac{3}{22} - \frac{5}{22}\pi_6 \geq 0}_{\pi_4 \geq 0} \\ \underbrace{\frac{5}{11} - \frac{1}{11}\pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 &\leq \frac{3}{5} \\ \pi_6 &\leq 5 \end{cases}$$

By increasing the value of π_6 to $\min\left\{\frac{3}{5}, 5\right\} = \frac{3}{5}$, the value of π_4 decreases to 0. Then we have that:

π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non basic variable

π_4 becomes a dual non-basic variable and accordingly x_4 becomes a primal basic variable

We have:

$$x_6 = -\frac{1}{2} + \frac{5}{22}x_4 + \frac{1}{11}x_5 \quad \Longrightarrow \quad x_4 = \frac{11}{5} - \frac{2}{5}x_5 + \frac{22}{5}x_6$$

Substituting into the other equations of the primal dictionary, we get:

$$x_1 = \frac{7}{2} - \frac{5}{22}\left(\frac{11}{5} - \frac{2}{5}x_5 + \frac{22}{5}x_6\right) - \frac{1}{11}x_5 = 3 - x_6$$

$$x_2 = \frac{7}{2} + \frac{1}{22}\left(\frac{11}{5} - \frac{2}{5}x_5 + \frac{22}{5}x_6\right) - \frac{2}{11}x_5 = \frac{18}{5} - \frac{1}{5}x_5 + \frac{1}{5}x_6$$

$$x_3 = 5 - \frac{2}{11}\left(\frac{11}{5} - \frac{2}{5}x_5 + \frac{22}{5}x_6\right) - \frac{3}{11}x_5 = \frac{23}{5} - \frac{1}{5}x_5 - \frac{4}{5}x_6$$

$$z = \frac{21}{2} - \frac{3}{22}\left(\frac{11}{5} - \frac{2}{5}x_5 + \frac{22}{5}x_6\right) - \frac{5}{11}x_5 = \frac{51}{5} - \frac{2}{5}x_5 - \frac{3}{5}x_6$$

Accordingly, we now have the following optimal primal dictionary:

$$\begin{cases} x_1 &= 3 &&- x_6 \\ x_2 &= \frac{18}{5} - \frac{1}{5}x_5 &&+ \frac{1}{5}x_6 \\ x_3 &= \frac{23}{5} - \frac{1}{5}x_5 &&- \frac{4}{5}x_6 \\ x_4 &= \frac{11}{5} - \frac{2}{5}x_5 &&+ \frac{22}{5}x_6 \\ z &= \frac{51}{5} - \frac{2}{5}x_5 &&- \frac{3}{5}x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4\}$ and $\mathcal{N} = \{5, 6\}$.

– the basic primal optimal solution is:

$$\tilde{x}_1 = 3, \quad \tilde{x}_2 = \frac{18}{5}, \quad \tilde{x}_3 = \frac{23}{5}, \quad \tilde{x}_4 = \frac{11}{5}, \quad \tilde{x}_5 = 0, \quad \tilde{x}_6 = 0 \qquad \text{of value } \frac{51}{5}$$

– the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 0, \quad \tilde{\pi}_5 = \frac{2}{5}, \quad \tilde{\pi}_6 = \frac{3}{5} \qquad \text{of value } \frac{51}{5}$$

The optimal solution $\tilde{\boldsymbol{x}}(\, LP_2\,)$ is **fractional** and the LP_2 problem is **feasible**.

We have the upper bound on the optimal value of the ILP problem:

$$z(\, ILP\,) \;\leq\; \underbrace{\left\lfloor \frac{51}{5} \right\rfloor}_{=10}$$

For the LP_2 problem in canonical form, we have:

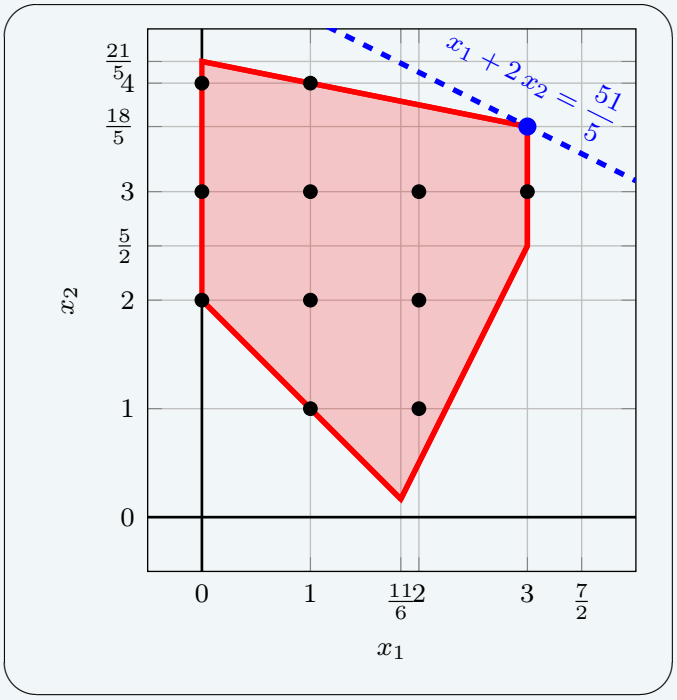
$$\begin{array}{lrcrcr} \max & x_1 & + & 2\,x_2 & & \\ & -x_1 & - & x_2 & \leq & -2 \\ & 4\,x_1 & - & 2\,x_2 & \leq & 7 \\ & x_1 & + & 5\,x_2 & \leq & 21 \\ & x_1 & & & \leq & 3 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

An optimal solution $\tilde{\boldsymbol{x}}(\, LP_2\,)$ is:

$$\tilde{x}_1 = 3, \qquad \tilde{x}_2 = \frac{18}{5}$$

The optimal value is:

$$z(\, LP_2\,) = \frac{51}{5}$$



- **Iteration 2**

a) **Cut generation phase**

We have the following simplex dictionary associated with a fractional basic feasible solution of the LP_2 problem in standard form:

$$\begin{cases} x_1 = 3 - x_6 \\ x_2 = \frac{18}{5} - \frac{1}{5}x_5 + \frac{1}{5}x_6 \\ x_3 = \frac{23}{5} - \frac{1}{5}x_5 - \frac{4}{5}x_6 \\ x_4 = \frac{11}{5} - \frac{2}{5}x_5 + \frac{22}{5}x_6 \\ z = \frac{51}{5} - \frac{2}{5}x_5 - \frac{3}{5}x_6 \end{cases}$$

where

$$\bar{\boldsymbol{A}} = \begin{pmatrix} 0 & 1 \\ \frac{1}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{4}{5} \\ \frac{2}{5} & -\frac{22}{5} \end{pmatrix} \qquad \bar{\boldsymbol{b}} = \begin{pmatrix} 3 \\ \frac{18}{5} \\ \frac{23}{5} \\ \frac{11}{5} \end{pmatrix}$$

– The Gomory fractional cut generated by the second row is:

$$\underbrace{\left(\frac{1}{5}-\left\lfloor\frac{1}{5}\right\rfloor\right)}_{\frac{1}{5}-0=\frac{1}{5}} x_5+\underbrace{\left(-\frac{1}{5}-\left\lfloor-\frac{1}{5}\right\rfloor\right)}_{-\frac{1}{5}-(-1)=\frac{4}{5}} x_6\geq\underbrace{\left(\frac{18}{5}-\left\lfloor\frac{18}{5}\right\rfloor\right)}_{\frac{18}{5}-3=\frac{3}{5}}$$

$$\frac{1}{5}\,x_5+\frac{4}{5}\,x_6\geq\frac{3}{5}\quad\Rightarrow\quad x_5+4\,x_6\geq 3$$

By subtracting a surplus variable, we obtain the Gomory fractional cut in standard form:

$$\frac{1}{5}\,x_5+\frac{4}{5}\,x_6-x_7=\frac{3}{5}\quad\Rightarrow\quad x_5+4\,x_6-5\,x_7=3$$

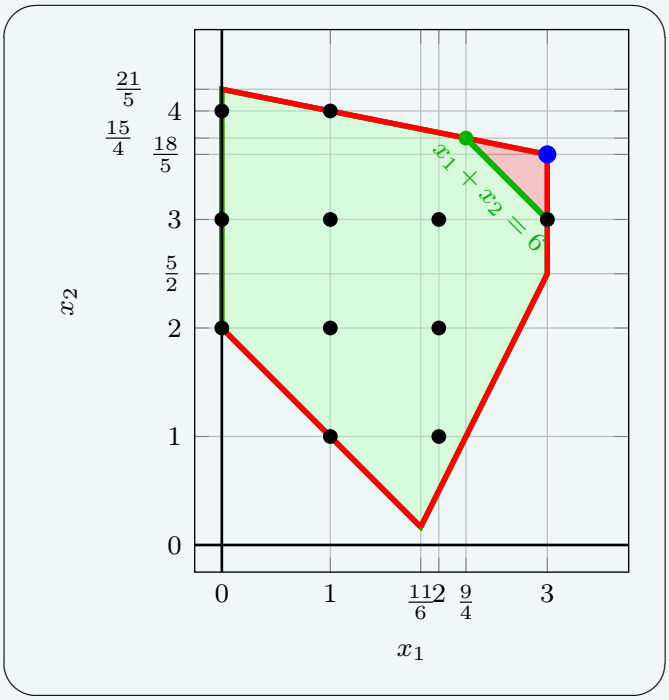
From the standard form of the LP_1 problem and the added cut, we have:

$$x_5=21-x_1-5\,x_2,\qquad x_6=3-x_1.$$

Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$(21-x_1-5\,x_2)+4\,(3-x_1)\geq 3\quad\Rightarrow\quad x_1+x_2\leq 6$$

Effect of the Gomory fractional cut generated by the second row on the LP_2 problem in canonical form:



– The Gomory fractional cut generated by the third row is:

$$\underbrace{\left(\frac{1}{5} - \left\lfloor \frac{1}{5} \right\rfloor\right)}_{\frac{1}{5}-0=\frac{1}{5}} x_5 + \underbrace{\left(\frac{4}{5} - \left\lfloor \frac{4}{5} \right\rfloor\right)}_{\frac{4}{5}-0=\frac{4}{5}} x_6 \geq \underbrace{\left(\frac{23}{5} - \left\lfloor \frac{23}{5} \right\rfloor\right)}_{\frac{23}{5}-4=\frac{3}{5}}$$

$$\frac{1}{5}\, x_5 + \frac{4}{5}\, x_6 \geq \frac{3}{5} \quad \Rightarrow \quad x_5 + 4\, x_6 \geq 3$$

This Gomory fractional cut is the same as the one generated by the second row.

– The Gomory fractional cut generated by the fourth row is:

$$\underbrace{\left(\frac{2}{5} - \left\lfloor \frac{2}{5} \right\rfloor\right)}_{\frac{2}{5}-0=\frac{2}{5}} x_5 + \underbrace{\left(-\frac{22}{5} - \left\lfloor -\frac{22}{5} \right\rfloor\right)}_{-\frac{22}{5}-(-5)=\frac{3}{5}} x_6 \geq \underbrace{\left(\frac{11}{5} - \left\lfloor \frac{11}{5} \right\rfloor\right)}_{\frac{11}{5}-2=\frac{1}{5}}$$

$$\frac{2}{5}\, x_5 + \frac{3}{5}\, x_6 \geq \frac{1}{5} \quad \Rightarrow \quad 2\, x_5 + 3\, x_6 \geq 1$$

By subtracting a surplus variable, we obtain the Gomory fractional cut in standard form:

$$\frac{2}{5}\, x_5 + \frac{3}{5}\, x_6 - x_7 = \frac{1}{5} \quad \Rightarrow \quad 2\, x_5 + 3\, x_6 - 5\, x_7 = 1$$

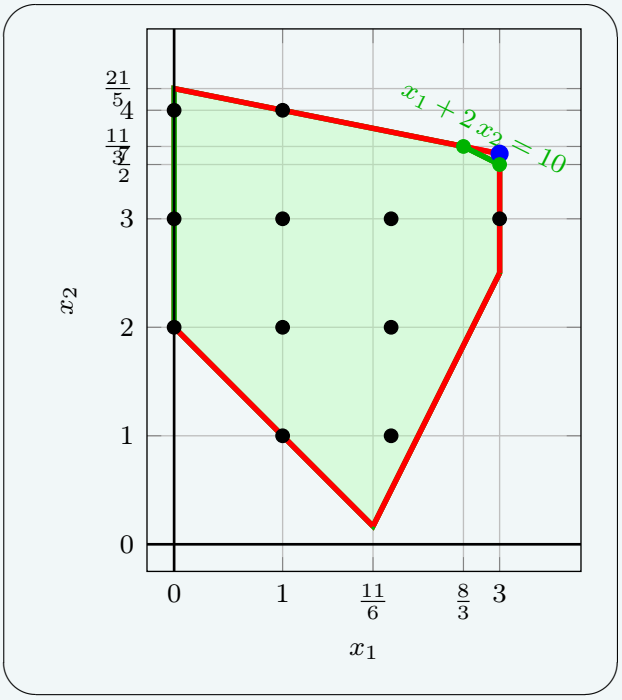
From the standard form of the LP_1 problem and the added cut, we have:

$$x_5 = 21 - x_1 - 5\, x_2, \qquad x_6 = 3 - x_1.$$

Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$2\,(21 - x_1 - 5\, x_2) + 3\,(3 - x_1) \geq 1 \quad \Rightarrow \quad x_1 + 2\, x_2 \leq 10$$

Effect of the Gomory fractional cut generated by the fourth row on the LP_2 problem in canonical form:



We select the Gomory fractional cut with the largest right-hand side and we create the LP_3 problem by adding it to the LP_2 problem.

b) **LP re-optimization phase**

Let us consider the LP_3 problem in standard form:

$$
\begin{array}{lrcrcrcrcrcrcl}
\max & x_1 & + & 2\,x_2 & & & & & & & & & & \\
 & -x_1 & - & x_2 & + & x_3 & & & & & & & = & -2 \\
 & 4\,x_1 & - & 2\,x_2 & & & + & x_4 & & & & & = & 7 \\
 & x_1 & + & 5\,x_2 & & & & & + & x_5 & & & = & 21 \\
 & x_1 & & & & & & & & & + & x_6 & = & 3 \\
 & x_1 & + & x_2 & & & & & & & & & + & x_7 & = & 6 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & & x_7 & \geq & 0
\end{array}
$$

For the added Gomory fractional cut we have:

$$x_5 + 4\,x_6 - 5\,x_7 = 3 \quad \Rightarrow \quad x_7 = -\frac{3}{5} + \frac{1}{5}\,x_5 + \frac{4}{5}\,x_6$$

We then add this equation to the previous optimal dictionary and we obtain the following dictionary :

$$\begin{cases} x_1 = 3 - x_6 \\ x_2 = \frac{18}{5} - \frac{1}{5}x_5 + \frac{1}{5}x_6 \\ x_3 = \frac{23}{5} - \frac{1}{5}x_5 - \frac{4}{5}x_6 \\ x_4 = \frac{11}{5} - \frac{2}{5}x_5 + \frac{22}{5}x_6 \\ x_7 = -\frac{3}{5} + \frac{1}{5}x_5 + \frac{4}{5}x_6 \\ z = \frac{51}{5} - \frac{2}{5}x_5 - \frac{3}{5}x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4, 7\}$ and $\mathcal{N} = \{5, 6\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = 3, \quad \bar{x}_2 = \frac{18}{5}, \quad \bar{x}_3 = \frac{23}{5}, \quad \bar{x}_4 = \frac{11}{5}, \quad \bar{x}_7 = -\frac{3}{5},$$
$$\bar{x}_5 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad \frac{51}{5}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_7 = 0,$$
$$\bar{\pi}_5 = \frac{2}{5}, \quad \bar{\pi}_6 = \frac{3}{5} \quad \text{of value} \quad \frac{51}{5}$$

If we increase the value of π_7, we can get a better dual solution. We have:

$$\begin{cases} \pi_5 = \frac{2}{5} - \frac{1}{5}\pi_7 \\ \pi_6 = \frac{3}{5} - \frac{4}{5}\pi_7 \end{cases} \qquad \begin{cases} \underbrace{\frac{2}{5} - \frac{1}{5}\pi_7 \geq 0}_{\pi_5 \geq 0} \\ \underbrace{\frac{3}{5} - \frac{4}{5}\pi_7 \geq 0}_{\pi_6 \geq 0} \end{cases} \qquad \begin{cases} \pi_7 \leq 2 \\ \pi_7 \leq \frac{3}{4} \end{cases}$$

By increasing the value of π_7 to $\min\left\{2, \frac{3}{4}\right\} = \frac{3}{4}$, the value of π_6 decreases to 0. Then we have that:

> π_7 becomes a dual basic variable and accordingly x_7 becomes a primal non basic variable
>
> π_6 becomes a dual non-basic variable and accordingly x_6 becomes a primal basic variable

We have:

$$x_7 = -\frac{3}{5} + \frac{1}{5}x_5 + \frac{4}{5}x_6 \quad \Longrightarrow \quad x_6 = \frac{3}{4} - \frac{1}{4}x_5 + \frac{5}{4}x_7$$

Substituting into the other equations of the primal dictionary, we get:

$$x_1 = 3 - \left(\frac{3}{4} - \frac{1}{4}x_5 + \frac{5}{4}x_7\right) = \frac{9}{4} + \frac{1}{4}x_5 - \frac{5}{4}x_7$$

$$x_2 = \frac{18}{5} - \frac{1}{5}x_5 + \frac{1}{5}\left(\frac{3}{4} - \frac{1}{4}x_5 + \frac{5}{4}x_7\right) = \frac{15}{4} - \frac{1}{4}x_5 + \frac{1}{4}x_7$$

$$x_3 = \frac{23}{5} - \frac{1}{5}x_5 - \frac{4}{5}\left(\frac{3}{4} - \frac{1}{4}x_5 + \frac{5}{4}x_7\right) = 4 - x_7$$

$$x_4 = \frac{11}{5} - \frac{2}{5}x_5 + \frac{22}{5}\left(\frac{3}{4} - \frac{1}{4}x_5 + \frac{5}{4}x_7\right) = \frac{11}{2} - \frac{3}{2}x_5 + \frac{11}{2}x_7$$

$$z = \frac{51}{5} - \frac{2}{5}x_5 - \frac{3}{5}\left(\frac{3}{4} - \frac{1}{4}x_5 + \frac{5}{4}x_7\right) = \frac{39}{4} - \frac{1}{4}x_5 - \frac{3}{4}x_7$$

Accordingly, we now have the following optimal primal dictionary:

$$\begin{cases} x_1 = \frac{9}{4} + \frac{1}{4}x_5 - \frac{5}{4}x_7 \\ x_2 = \frac{15}{4} - \frac{1}{4}x_5 + \frac{1}{4}x_7 \\ x_3 = 4 - x_7 \\ x_4 = \frac{11}{2} - \frac{3}{2}x_5 + \frac{11}{2}x_7 \\ x_6 = \frac{3}{4} - \frac{1}{4}x_5 + \frac{5}{4}x_7 \\ z = \frac{39}{4} - \frac{1}{4}x_5 - \frac{3}{4}x_7 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4, 6\}$ and $\mathcal{N} = \{5, 7\}$.

- the basic primal optimal solution is:

$$\tilde{x}_1 = \frac{9}{4}, \quad \tilde{x}_2 = \frac{15}{4}, \quad \tilde{x}_3 = 4, \quad \tilde{x}_4 = \frac{11}{2}, \quad \tilde{x}_6 = \frac{3}{4}, \quad \tilde{x}_5 = 0, \quad \tilde{x}_7 = 0$$

$$\text{of value } \frac{39}{4}$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 0, \quad \tilde{\pi}_6 = 0, \quad \tilde{\pi}_5 = \frac{1}{4}, \quad \tilde{\pi}_7 = \frac{3}{4}$$

$$\text{of value } \frac{39}{4}$$

The optimal solution $\tilde{\boldsymbol{x}}(\,LP_3\,)$ is **fractional** and the LP_3 problem is **feasible**.

We have the upper bound on the optimal value of the ILP problem:

$$z(\ ILP\ ) \quad \leq \quad \underbrace{\left\lfloor \frac{39}{4} \right\rfloor}_{=9}$$

For the LP_3 problem in canonical form, we have:

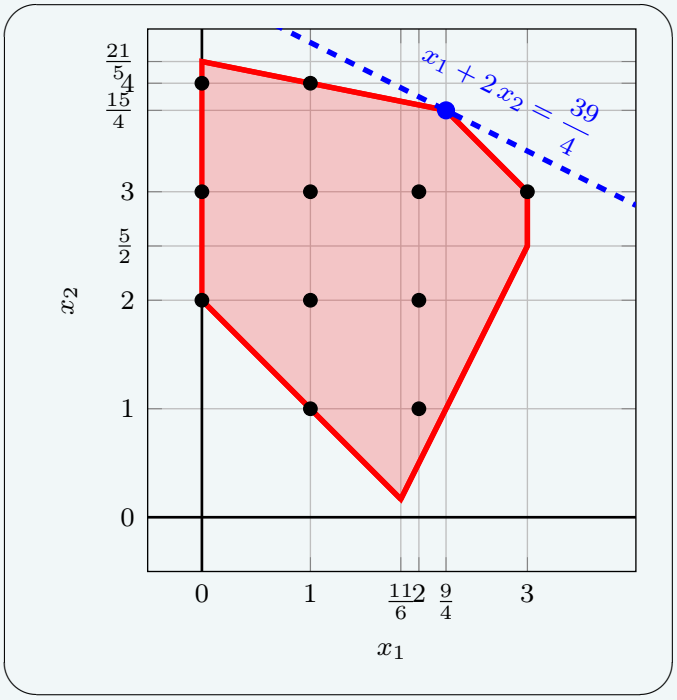
$$\begin{array}{lrcrcr} \max & x_1 & + & 2\,x_2 & & \\ & -x_1 & - & x_2 & \leq & -2 \\ & 4\,x_1 & - & 2\,x_2 & \leq & 7 \\ & x_1 & + & 5\,x_2 & \leq & 21 \\ & x_1 & & & \leq & 3 \\ & x_1 & + & x_2 & \leq & 6 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

An optimal solution $\tilde{\boldsymbol{x}}(\ LP_3\ )$ is:

$$\tilde{x}_1 = \frac{9}{4}, \qquad \tilde{x}_2 = \frac{15}{4}$$

The optimal value is:

$$z(\ LP_3\ ) = \frac{39}{4}$$



- **Iteration 3**

a) **Cut generation phase**

We have the following simplex dictionary associated with a fractional basic feasible solution of the LP_3 problem in standard form:

$$\begin{cases} x_1 = \frac{9}{4} + \frac{1}{4}\,x_5 - \frac{5}{4}\,x_7 \\ x_2 = \frac{15}{4} - \frac{1}{4}\,x_5 + \frac{1}{4}\,x_7 \\ x_3 = 4 - x_7 \\ x_4 = \frac{11}{2} - \frac{3}{2}\,x_5 + \frac{11}{2}\,x_7 \\ x_6 = \frac{3}{4} - \frac{1}{4}\,x_5 + \frac{5}{4}\,x_7 \\ z = \frac{39}{4} - \frac{1}{4}\,x_5 - \frac{3}{4}\,x_7 \end{cases}$$

where

$$\bar{\boldsymbol{A}} = \begin{pmatrix} -\frac{1}{4} & \frac{5}{4} \\ \frac{1}{4} & -\frac{1}{4} \\ 0 & 1 \\ \frac{3}{2} & -\frac{11}{2} \\ \frac{1}{4} & -\frac{5}{4} \end{pmatrix} \qquad \bar{\boldsymbol{b}} = \begin{pmatrix} \frac{9}{4} \\ \frac{15}{4} \\ 4 \\ \frac{11}{2} \\ \frac{3}{4} \end{pmatrix}$$

– The Gomory fractional cut generated by the first row is:

$$\underbrace{\left(-\frac{1}{4} - \left\lfloor -\frac{1}{4} \right\rfloor\right)}_{-\frac{1}{4}-(-1)=\frac{3}{4}} x_5 + \underbrace{\left(\frac{5}{4} - \left\lfloor \frac{5}{4} \right\rfloor\right)}_{\frac{5}{4}-1=\frac{1}{4}} x_7 \geq \underbrace{\left(\frac{9}{4} - \left\lfloor \frac{9}{4} \right\rfloor\right)}_{\frac{9}{4}-2=\frac{1}{4}}$$

$$\frac{3}{4}\,x_5 + \frac{1}{4}\,x_7 \geq \frac{1}{4} \quad \Rightarrow \quad 3\,x_5 + x_7 \geq 1$$

By subtracting a surplus variable, we obtain the Gomory fractional cut in standard form:

$$\frac{3}{4}\,x_5 + \frac{1}{4}\,x_7 - x_8 = \frac{1}{4} \quad \Rightarrow \quad 3\,x_5 + x_7 - 4\,x_8 = 1$$

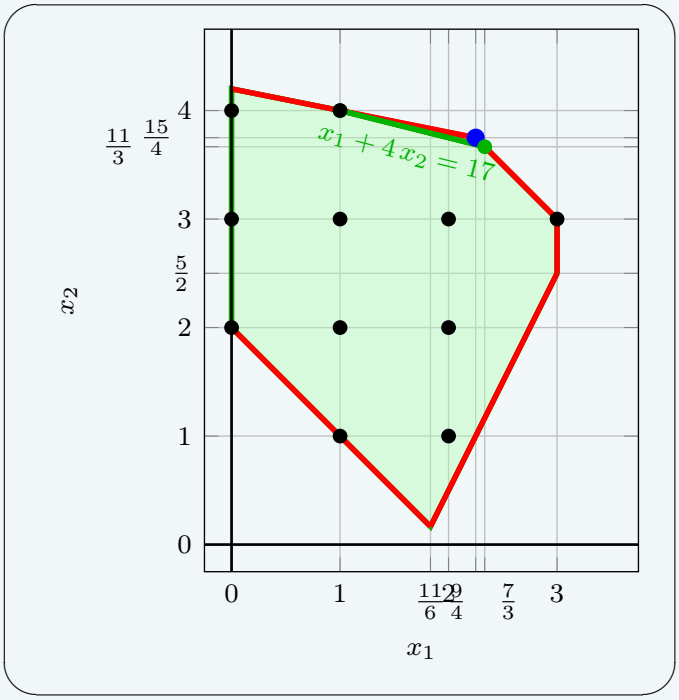
From the standard form of the LP_1 problem and the cut added at iteration 2 (the LP_3 problem), we have:

$$x_5 = 21 - x_1 - 5\,x_2, \qquad x_7 = 6 - x_1 - x_2.$$

Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$3\,(21 - x_1 - 5\,x_2) + (6 - x_1 - x_2) \geq 1 \quad \Rightarrow \quad x_1 + 4\,x_2 \leq 17$$

Effect of the Gomory fractional cut generated by the first row on the LP_3 problem in canonical form:



– The Gomory fractional cut generated by the second row is:

$$\underbrace{\left(\frac{1}{4} - \left\lfloor \frac{1}{4} \right\rfloor\right)}_{\frac{1}{4}-0=\frac{1}{4}} x_5 + \underbrace{\left(-\frac{1}{4} - \left\lfloor -\frac{1}{4} \right\rfloor\right)}_{-\frac{1}{4}-(-1)=\frac{3}{4}} x_7 \geq \underbrace{\left(\frac{15}{4} - \left\lfloor \frac{15}{4} \right\rfloor\right)}_{\frac{15}{4}-3=\frac{3}{4}}$$

$$\frac{1}{4}\,x_5 + \frac{3}{4}\,x_7 \geq \frac{3}{4} \quad \Rightarrow \quad x_5 + 3\,x_7 \geq 3$$

By subtracting a surplus variable, we obtain the Gomory fractional cut in standard form:

$$\frac{1}{4}\,x_5 + \frac{3}{4}\,x_7 - x_8 = \frac{3}{4} \quad \Rightarrow \quad x_5 + 3\,x_7 - 4\,x_8 = 3$$

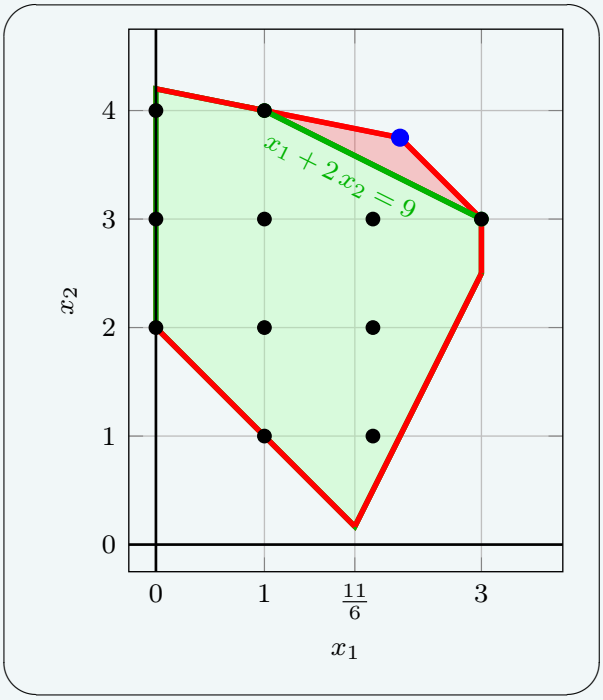
From the standard form of the LP_1 problem and the cut added at iteration 2 (the LP_3 problem), we have:

$$x_5 = 21 - x_1 - 5\,x_2, \qquad x_7 = 6 - x_1 - x_2.$$

Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$(21 - x_1 - 5\,x_2) + 3\,(6 - x_1 - x_2) \geq 3 \quad \Rightarrow \quad x_1 + 2\,x_2 \leq 9$$

Effect of the Gomory fractional cut generated by the second row on the LP_3 problem in canonical form:



– The Gomory fractional cut generated by the fourth row is:

$$\underbrace{\left(\frac{3}{2} - \left\lfloor \frac{3}{2} \right\rfloor\right)}_{\frac{3}{2}-1=\frac{1}{2}} x_5 + \underbrace{\left(-\frac{11}{2} - \left\lfloor -\frac{11}{2} \right\rfloor\right)}_{-\frac{11}{2}-(-6)=\frac{1}{2}} x_7 \geq \underbrace{\left(\frac{11}{2} - \left\lfloor \frac{11}{2} \right\rfloor\right)}_{\frac{11}{2}-5=\frac{1}{2}}$$

$$\frac{1}{2}\,x_5 + \frac{1}{2}\,x_7 \geq \frac{1}{2} \quad \Rightarrow \quad x_5 + x_7 \geq 1$$

By subtracting a surplus variable, we obtain the Gomory fractional cut in standard form:

$$\frac{1}{2}\,x_5 + \frac{1}{2}\,x_7 - x_8 = \frac{1}{2} \quad \Rightarrow \quad x_5 + x_7 - 2\,x_8 = 1$$

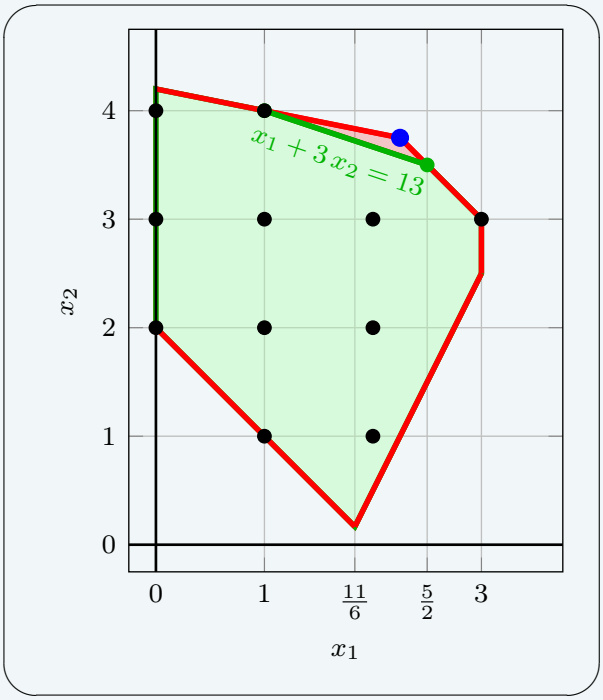
From the standard form of the LP_1 problem and the cut added at iteration 2 (the LP_3 problem), we have:

$$x_5 = 21 - x_1 - 5\,x_2, \qquad x_7 = 6 - x_1 - x_2.$$

Substituting in the Gomory fractional cut, we can express it in terms of the original variables of the ILP problem:

$$(21 - x_1 - 5\,x_2) + (6 - x_1 - x_2) \geq 1 \quad \Rightarrow \quad x_1 + 3\,x_2 \leq 13$$

Effect of the Gomory fractional cut generated by the fourth row on the LP_3 problem in canonical form:



– The Gomory fractional cut generated by the fifth row is:

$$\underbrace{\left(\frac{1}{4} - \left\lfloor \frac{1}{4} \right\rfloor\right)}_{\frac{1}{4}-0=\frac{1}{4}} x_5 + \underbrace{\left(-\frac{5}{4} - \left\lfloor -\frac{5}{4} \right\rfloor\right)}_{-\frac{5}{4}-(-2)=\frac{3}{4}} x_7 \geq \underbrace{\left(\frac{3}{4} - \left\lfloor \frac{3}{4} \right\rfloor\right)}_{\frac{3}{4}-0=\frac{3}{4}}$$

$$\frac{1}{4}\,x_5 + \frac{3}{4}\,x_7 \geq \frac{3}{4} \quad \Rightarrow \quad x_5 + 3\,x_7 \geq 3$$

This Gomory fractional cut is the same as the one generated by the second row.

We select the Gomory fractional cut with the largest right-hand side and we create the LP_4 problem by adding it to the LP_3 problem.

b) **LP re-optimization phase**

Let us consider the LP_4 problem in standard form:

$$
\begin{array}{lrcrcrcrcrcrcrcrcr}
\max & x_1 & + & 2\,x_2 & & & & & & & & & & & & \\
 & -x_1 & - & x_2 & + & x_3 & & & & & & & & & = & -2 \\
 & 4\,x_1 & - & 2\,x_2 & & & + & x_4 & & & & & & & = & 7 \\
 & x_1 & + & 5\,x_2 & & & & & + & x_5 & & & & & = & 21 \\
 & x_1 & & & & & & & & & + & x_6 & & & = & 3 \\
 & x_1 & + & x_2 & & & & & & & & & + & x_7 & = & 6 \\
 & x_1 & + & 2\,x_2 & & & & & & & & & & & + & x_8 & = & 9 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & & x_7, & & x_8 & \geq & 0
\end{array}
$$

For the added Gomory fractional cut we have:

$$x_5 + 3\,x_7 - 4\,x_8 = 3 \quad \Rightarrow \quad x_8 = -\frac{3}{4} + \frac{1}{4}\,x_5 + \frac{3}{4}\,x_7$$

We then add this equation to the previous optimal dictionary and we obtain the following dictionary :

$$
\begin{cases}
x_1 = \frac{9}{4} + \frac{1}{4}\,x_5 - \frac{5}{4}\,x_7 \\
x_2 = \frac{15}{4} - \frac{1}{4}\,x_5 + \frac{1}{4}\,x_7 \\
x_3 = 4 - x_7 \\
x_4 = \frac{11}{2} - \frac{3}{2}\,x_5 + \frac{11}{2}\,x_7 \\
x_6 = \frac{3}{4} - \frac{1}{4}\,x_5 + \frac{5}{4}\,x_7 \\
x_8 = -\frac{3}{4} + \frac{1}{4}\,x_5 + \frac{3}{4}\,x_7 \\
z = \frac{39}{4} - \frac{1}{4}\,x_5 - \frac{3}{4}\,x_7
\end{cases}
$$

where $\mathcal{B} = \{1, 2, 3, 4, 6, 8\}$ and $\mathcal{N} = \{5, 7\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{9}{4}, \quad \bar{x}_2 = \frac{15}{4}, \quad \bar{x}_3 = 4, \quad \bar{x}_4 = \frac{11}{2}, \quad \bar{x}_6 = \frac{3}{4}, \quad \bar{x}_8 = -\frac{3}{4},$$

$$\bar{x}_5 = 0, \quad \bar{x}_7 = 0 \quad \text{of value} \quad \frac{39}{4}$$

– the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_6 = 0, \quad \bar{\pi}_8 = 0,$$
$$\bar{\pi}_5 = \frac{1}{4}, \quad \bar{\pi}_7 = \frac{3}{4} \quad \text{of value} \quad \frac{39}{4}$$

If we increase the value of π_8, we can get a better dual solution. We have:

$$\begin{cases} \pi_5 &= \frac{1}{4} - \frac{1}{4}\pi_8 \\ \pi_7 &= \frac{3}{4} - \frac{3}{4}\pi_8 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{4} - \frac{1}{4}\pi_8 \geq 0}_{\pi_5 \geq 0} \\ \underbrace{\frac{3}{4} - \frac{3}{4}\pi_8 \geq 0}_{\pi_7 \geq 0} \end{cases} \qquad \begin{cases} \pi_8 &\leq 1 \\ \pi_8 &\leq 1 \end{cases}$$

By increasing the value of π_8 to $\min\{1, 1\} = 1$, the values of π_5 and π_7 decrease to 0 **at the same time**: the ratio test is tied. We choose to make x_5 enter the primal basis (the choice of x_7 instead would lead to the other optimal vertex $(1, 4)$, also with value 9: the ILP problem has two optimal solutions, $(3, 3)$ and $(1, 4)$). Then we have that:

π_8 becomes a dual basic variable and accordingly x_8 becomes a primal non basic variable

π_5 becomes a dual non-basic variable and accordingly x_5 becomes a primal basic variable

We have:

$$x_8 = -\frac{3}{4} + \frac{1}{4}x_5 + \frac{3}{4}x_7 \quad \Longrightarrow \quad x_5 = 3 - 3x_7 + 4x_8$$

Substituting into the other equations of the primal dictionary, we get:

$$x_1 = \frac{9}{4} + \frac{1}{4}(3 - 3x_7 + 4x_8) - \frac{5}{4}x_7 = 3 - 2x_7 + x_8$$
$$x_2 = \frac{15}{4} - \frac{1}{4}(3 - 3x_7 + 4x_8) + \frac{1}{4}x_7 = 3 + x_7 - x_8$$
$$x_3 = 4 - x_7$$
$$x_4 = \frac{11}{2} - \frac{3}{2}(3 - 3x_7 + 4x_8) + \frac{11}{2}x_7 = 1 + 10x_7 - 6x_8$$
$$x_6 = \frac{3}{4} - \frac{1}{4}(3 - 3x_7 + 4x_8) + \frac{5}{4}x_7 = 2x_7 - x_8$$
$$z = \frac{39}{4} - \frac{1}{4}(3 - 3x_7 + 4x_8) - \frac{3}{4}x_7 = 9 - x_8$$

Accordingly, we now have the following optimal primal dictionary:

$$\begin{cases} x_1 = 3 - 2\,x_7 + x_8 \\ x_2 = 3 + x_7 - x_8 \\ x_3 = 4 - x_7 \\ x_4 = 1 + 10\,x_7 - 6\,x_8 \\ x_5 = 3 - 3\,x_7 + 4\,x_8 \\ x_6 = 0 + 2\,x_7 - x_8 \\ z = 9 - x_8 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{N} = \{7, 8\}$.

- the basic primal optimal solution is:

$$\tilde{x}_1 = 3, \quad \tilde{x}_2 = 3, \quad \tilde{x}_3 = 4, \quad \tilde{x}_4 = 1, \quad \tilde{x}_5 = 3, \quad \tilde{x}_6 = 0, \quad \tilde{x}_7 = 0, \quad \tilde{x}_8 = 0$$

of value 9

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 0, \quad \tilde{\pi}_5 = 0, \quad \tilde{\pi}_6 = 0, \quad \tilde{\pi}_7 = 0, \quad \tilde{\pi}_8 = 1$$

of value 9

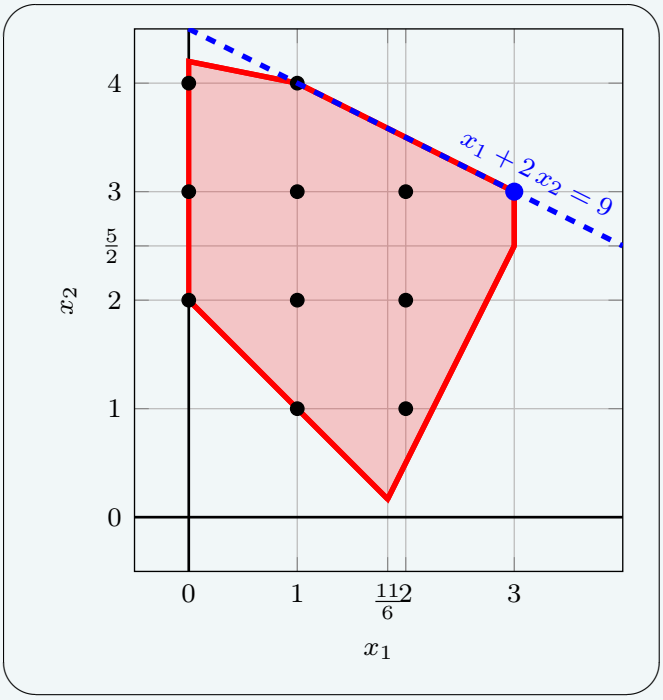
The optimal solution $\tilde{\boldsymbol{x}}(\,LP_4\,)$ is **integer**. Accordingly, an optimal solution $\tilde{\boldsymbol{x}}(\,ILP\,)$ is:

$$\tilde{x}_1 = 3, \quad \tilde{x}_2 = 3$$

The optimal value $z(\,ILP\,)$ is 9.

For the LP_4 problem in canonical form, we have:

$$\begin{array}{lrcrcl} \max & x_1 & + & 2\,x_2 & & \\ & -x_1 & - & x_2 & \leq & -2 \\ & 4\,x_1 & - & 2\,x_2 & \leq & 7 \\ & x_1 & + & 5\,x_2 & \leq & 21 \\ & x_1 & & & \leq & 3 \\ & x_1 & + & x_2 & \leq & 6 \\ & x_1 & + & 2\,x_2 & \leq & 9 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$


4
3
$\frac{5}{2}$
2
1
0
x_2
0
1
$\frac{11}{6}$2
3
x_1
$x_1 + 2x_2 = 9$

Chapter 7

LP-based branch-and-bound

7.1 Branch-and-bound algorithm

Exercise 7.1 — Branch-and-bound algorithm ($P1$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 1$ constraint:

$$\begin{array}{lrcrcrl} \max & x_1 & + & 8\ x_2 & & & \\ & 3\ x_1 & + & 6\ x_2 & \leq & 16 & \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} & \end{array}$$

and an optimal simplex dictionary of its LP relaxation in standard form:

$$\begin{cases} x_2 = \frac{8}{3} - \frac{1}{2}\ x_1 - \frac{1}{6}\ x_3 \\ z = \frac{64}{3} - 3\ x_1 - \frac{4}{3}\ x_3 \end{cases}$$

Questions

1. Execute the branch-and-bound algorithm using:
   (a) the smallest-index rule as branching-variable selection strategy;
   (b) the LIFO rule as branching-node selection strategy;
   (c) solve first the down-branching node and then the up-branching node;
   (d) set the lower bound LB to $-\infty$ (no heuristic algorithm is executed).

   Solve the LP problems of the branching nodes using the dual simplex algorithm. In the dual simplex algorithm, the leaving variable is the basic variable with negative value and smallest index, and ties in the ratio test are broken by the smallest index. Show graphically the optimal solution of the LP problems associated with the branching nodes.

2. Determine and comment the branch-and-bound tree, showing the reason why each branching node is pruned.

Solution

1. Execution of the branch-and-bound algorithm to solve the ILP problem.

   - **Initialization**

     Let us consider the LP_1 problem in standard form:

$$\begin{array}{lrcrcrcl} \max & x_1 & + & 8\ x_2 & & & & \\ & 3\ x_1 & + & 6\ x_2 & + & x_3 & = & 16 \\ & x_1, & & x_2, & & x_3 & \geq & 0 \end{array}$$

     and its optimal simplex dictionary:

$$\begin{cases} x_2 = \dfrac{8}{3} - \dfrac{1}{2}\ x_1 - \dfrac{1}{6}\ x_3 \\ z = \dfrac{64}{3} - 3\ x_1 - \dfrac{4}{3}\ x_3 \end{cases}$$

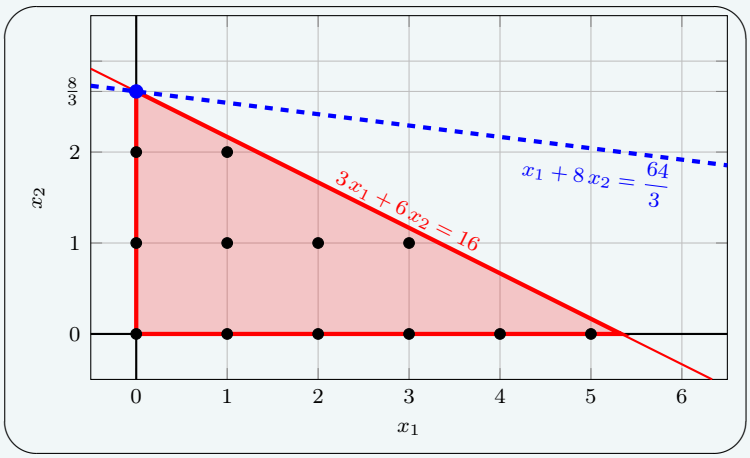


Figure 7.1: Node 1: feasible region of LP_1 (red), feasible points of the ILP problem (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_1)$ is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = \frac{8}{3}$$

the optimal value is:

$$z(\text{LP}_1) = \frac{64}{3}$$

The optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_1)$ is **fractional** and the LP_1 problem is **feasible**, so the LP_1 problem is inserted in the problem list L and the lower bound LB is set to $-\infty$.

Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \leq z(\text{ILP}) \leq \underbrace{\left\lfloor \frac{64}{3} \right\rfloor}_{=21}$$

- **Iteration 1**

The LP_1 problem is extracted and deleted from the problem list L.

The branching variable is x_2. The branching constraints are:

$$x_2 \leq \underbrace{\left\lfloor \frac{8}{3} \right\rfloor}_{=2} \quad \text{and} \quad x_2 \geq \underbrace{\left\lfloor \frac{8}{3} \right\rfloor + 1}_{=3}$$

Two LP_2 and LP_3 problems are created with these branching constraints.

(a) Solution of the LP_2 problem

Let us consider the LP_2 problem in standard form:

$$\begin{array}{lrcrcrcrcl} \max & x_1 & + & 8\,x_2 & & & & & & \\ & 3\,x_1 & + & 6\,x_2 & + & x_3 & & & = & 16 \\ & & & x_2 & & & + & x_4 & = & 2 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

For the branching constraint we have:

$$x_2 + x_4 = 2 \quad \Rightarrow \quad x_4 = 2 - x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_4 = 2 - \left(\frac{8}{3} - \frac{1}{2}\,x_1 - \frac{1}{6}\,x_3 \right) \quad \Rightarrow \quad x_4 = -\frac{2}{3} + \frac{1}{2}\,x_1 + \frac{1}{6}\,x_3$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_2 = \frac{8}{3} - \frac{1}{2}\,x_1 - \frac{1}{6}\,x_3 \\ x_4 = -\frac{2}{3} + \frac{1}{2}\,x_1 + \frac{1}{6}\,x_3 \\ z = \frac{64}{3} - 3\,x_1 - \frac{4}{3}\,x_3 \end{cases}$$

where $\mathcal{B} = \{2, 4\}$ and $\mathcal{N} = \{1, 3\}$ and

– the basic primal non-feasible solution is:

$$\bar{x}_2 = \frac{8}{3}, \quad \bar{x}_4 = -\frac{2}{3}, \qquad \bar{x}_1 = 0, \quad \bar{x}_3 = 0 \quad \text{of value} \quad \frac{64}{3}$$

– the basic dual feasible solution is:

$$\bar{\pi}_2 = 0, \quad \bar{\pi}_4 = 0, \qquad \bar{\pi}_1 = 3, \quad \bar{\pi}_3 = \frac{4}{3} \quad \text{of value} \quad \frac{64}{3}$$

If we increase the value of π_4, we can get a better dual solution. We have:

$$\begin{cases} \pi_1 &= 3 - \frac{1}{2}\pi_4 \\ \pi_3 &= \frac{4}{3} - \frac{1}{6}\pi_4 \end{cases} \qquad \begin{cases} \underbrace{3 - \frac{1}{2}\pi_4 \geq 0}_{\pi_1 \geq 0} \\ \underbrace{\frac{4}{3} - \frac{1}{6}\pi_4 \geq 0}_{\pi_3 \geq 0} \end{cases} \qquad \begin{cases} \pi_4 &\leq 6 \\ \pi_4 &\leq 8 \end{cases}$$

By increasing the value of π_4 to $\min\{6, 8\} = 6$, the value of π_1 decreases to 0. Then we have that:

π_4 becomes a dual basic variable and accordingly x_4 becomes a primal non-basic variable.

π_1 becomes a dual non-basic variable and accordingly x_1 becomes a primal basic variable.

We have:

$$x_4 = -\frac{2}{3} + \frac{1}{2}x_1 + \frac{1}{6}x_3 \quad \Rightarrow \quad x_1 = \frac{4}{3} - \frac{1}{3}x_3 + 2\,x_4$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_2 = \frac{8}{3} - \frac{1}{2}\left(\frac{4}{3} - \frac{1}{3}x_3 + 2\,x_4\right) - \frac{1}{6}x_3 = 2 - x_4$$

$$z = \frac{64}{3} - 3\left(\frac{4}{3} - \frac{1}{3}x_3 + 2\,x_4\right) - \frac{4}{3}x_3 = \frac{52}{3} - \frac{1}{3}x_3 - 6\,x_4$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 &= \frac{4}{3} - \frac{1}{3}x_3 + 2\,x_4 \\ x_2 &= 2 - x_4 \\ z &= \frac{52}{3} - \frac{1}{3}x_3 - 6\,x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$ and

– the basic primal optimal solution is:

$$\tilde{x}_1 = \frac{4}{3}, \quad \tilde{x}_2 = 2, \qquad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{of value} \quad \frac{52}{3}$$

– the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \qquad \tilde{\pi}_3 = \frac{1}{3}, \quad \tilde{\pi}_4 = 6 \quad \text{of value} \quad \frac{52}{3}$$

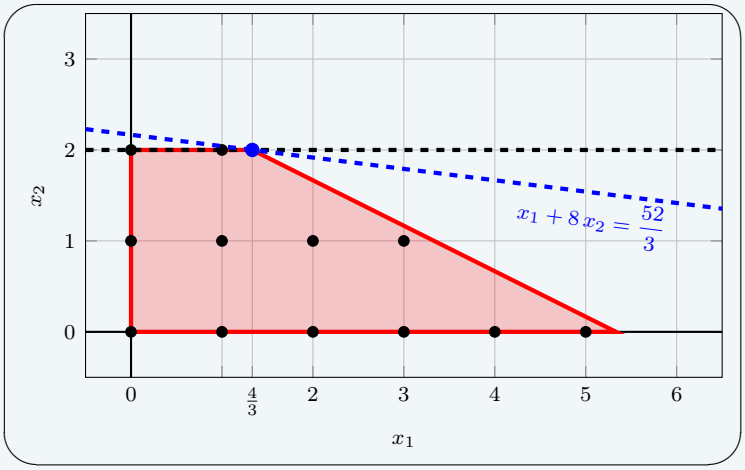


Figure 7.2: Node 2: feasible region of LP_2 (red), with the branching constraint $x_2 \leq 2$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is:

$$\tilde{x}_1 = \frac{4}{3}, \quad \tilde{x}_2 = 2$$

the optimal value is:

$$z(\mathrm{LP}_2) = \frac{52}{3}$$

The LP_2 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_2) \rfloor > \mathrm{LB} \qquad \text{since} \qquad \underbrace{\left\lfloor \frac{52}{3} \right\rfloor}_{=17} > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is **fractional**, accordingly, the LP_2 problem is **inserted in the problem list** L.

(b) Solution of the LP_3 problem

Let us consider the LP_3 problem in standard form:

$$\begin{array}{lrcrcrcrcl}
\max & x_1 & + & 8\,x_2 & & & & & & \\
& 3\,x_1 & + & 6\,x_2 & + & x_3 & & & = & 16 \\
& & & x_2 & & & - & x_4 & = & 3 \\
& x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_2 - x_4 = 3 \qquad \Rightarrow \qquad x_4 = -3 + x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_4 = -3 + \left(\frac{8}{3} - \frac{1}{2}\,x_1 - \frac{1}{6}\,x_3\right) \quad \Rightarrow \quad x_4 = -\frac{1}{3} - \frac{1}{2}\,x_1 - \frac{1}{6}\,x_3$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_2 = \frac{8}{3} - \frac{1}{2}\,x_1 - \frac{1}{6}\,x_3 \\ x_4 = -\frac{1}{3} - \frac{1}{2}\,x_1 - \frac{1}{6}\,x_3 \\ z = \frac{64}{3} - 3\,x_1 - \frac{4}{3}\,x_3 \end{cases}$$

where $\mathcal{B} = \{2, 4\}$ and $\mathcal{N} = \{1, 3\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_2 = \frac{8}{3}, \quad \bar{x}_4 = -\frac{1}{3}, \qquad \bar{x}_1 = 0, \quad \bar{x}_3 = 0 \quad \text{of value} \quad \frac{64}{3}$$

- the basic dual feasible solution is:

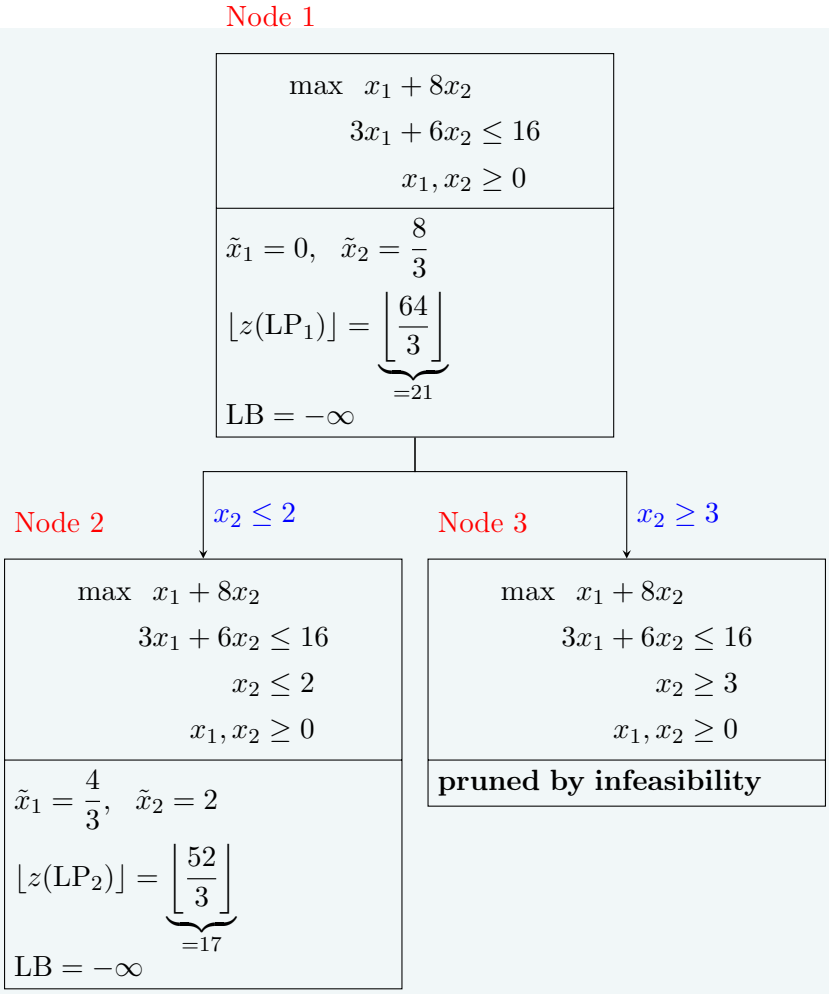
$$\bar{\pi}_2 = 0, \quad \bar{\pi}_4 = 0, \qquad \bar{\pi}_1 = 3, \quad \bar{\pi}_3 = \frac{4}{3} \quad \text{of value} \quad \frac{64}{3}$$

If we increase the value of π_4, we can get a better dual solution. We have:

$$\begin{cases} \pi_1 = 3 + \frac{1}{2}\,\pi_4 \\ \pi_3 = \frac{4}{3} + \frac{1}{6}\,\pi_4 \end{cases} \qquad \begin{cases} \underbrace{3 + \frac{1}{2}\,\pi_4 \geq 0}_{\pi_1 \geq 0} \\ \underbrace{\frac{4}{3} + \frac{1}{6}\,\pi_4 \geq 0}_{\pi_3 \geq 0} \end{cases} \qquad \begin{cases} \pi_4 \geq -6 \\ \pi_4 \geq -8 \end{cases}$$

Accordingly, we can increase the value of π_4 without violating dual feasibility, which implies that the dual problem is **unbounded**. As a result, the primal problem LP_3 is **infeasible**, and the corresponding branching node is pruned by infeasibility.

At the end of the first iteration, branching on x_2 from Node 1 has generated the down-branching node, Node 2, and the up-branching node, Node 3. Node 2 is inserted in the list L, while Node 3 is pruned by infeasibility. The corresponding branch-and-bound tree is:



Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \quad \leq \quad z(\text{ILP}) \quad \leq \quad \underbrace{\left\lfloor \frac{52}{3} \right\rfloor}_{=17}$$

- **Iteration 2**

The LP_2 problem is extracted and deleted from the problem list L.

The branching variable is x_1. The branching constraints are:

$$x_1 \leq \underbrace{\left\lfloor \frac{4}{3} \right\rfloor}_{=1} \quad \text{and} \quad x_1 \geq \underbrace{\left\lfloor \frac{4}{3} \right\rfloor + 1}_{=2}$$

The LP_4 problem is created with the first branching constraint (the second child is solved only if needed).

(a) Solution of the LP_4 problem

Let us consider the LP_4 problem in standard form:

$$\begin{array}{lrcrcrcrcl} \max & x_1 & + & 8\ x_2 & & & & & & \\ & 3\ x_1 & + & 6\ x_2 & + & x_3 & & & = & 16 \\ & & & x_2 & & & + & x_4 & = & 2 \\ & x_1 & & & & & & & + & x_5 & = & 1 \\ & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0 \end{array}$$

For the branching constraint we have:

$$x_1 + x_5 = 1 \qquad \Rightarrow \qquad x_5 = 1 - x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_5 = 1 - \left(\frac{4}{3} - \frac{1}{3}\, x_3 + 2\, x_4\right) \qquad \Rightarrow \qquad x_5 = -\frac{1}{3} + \frac{1}{3}\, x_3 - 2\, x_4$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = \frac{4}{3} - \frac{1}{3}\, x_3 + 2\, x_4 \\ x_2 = 2 - x_4 \\ x_5 = -\frac{1}{3} + \frac{1}{3}\, x_3 - 2\, x_4 \\ z = \frac{52}{3} - \frac{1}{3}\, x_3 - 6\, x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{4}{3}, \quad \bar{x}_2 = 2, \quad \bar{x}_5 = -\frac{1}{3}, \qquad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad \frac{52}{3}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_5 = 0, \qquad \bar{\pi}_3 = \frac{1}{3}, \quad \bar{\pi}_4 = 6 \quad \text{of value} \quad \frac{52}{3}$$

If we increase the value of π_5, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 = \frac{1}{3} - \frac{1}{3}\,\pi_5 \\ \pi_4 = 6 + 2\,\pi_5 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{3} - \frac{1}{3}\,\pi_5 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{6 + 2\,\pi_5 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_5 \leq 1 \\ \pi_5 \geq -3 \end{cases}$$

By increasing the value of π_5 to 1, the value of π_3 decreases to 0. Then we have that:

π_5 becomes a dual basic variable and accordingly x_5 becomes a primal non-basic variable.

π_3 becomes a dual non-basic variable and accordingly x_3 becomes a primal basic variable.

We have:

$$x_5 = -\frac{1}{3} + \frac{1}{3}\,x_3 - 2\,x_4 \quad \Rightarrow \quad x_3 = 1 + 6\,x_4 + 3\,x_5$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{4}{3} - \frac{1}{3}\,(1 + 6\,x_4 + 3\,x_5) + 2\,x_4 = 1 - x_5$$

$$z = \frac{52}{3} - \frac{1}{3}\,(1 + 6\,x_4 + 3\,x_5) - 6\,x_4 = 17 - 8\,x_4 - x_5$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 1 - x_5 \\ x_2 = 2 - x_4 \\ x_3 = 1 + 6\,x_4 + 3\,x_5 \\ z = 17 - 8\,x_4 - x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 2, \quad \tilde{x}_3 = 1, \qquad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad 17$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \qquad \tilde{\pi}_4 = 8, \quad \tilde{\pi}_5 = 1 \quad \text{of value} \quad 17$$

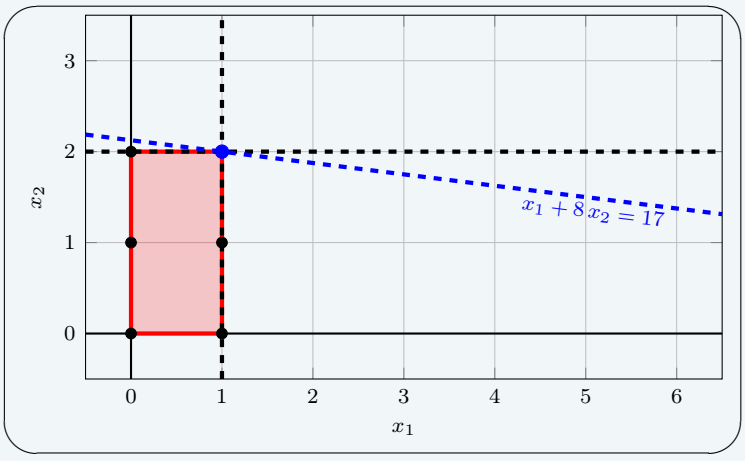


Figure 7.3: Node 4: feasible region of LP_4 (red), with the branching constraint $x_1 \leq 1$ (dashed), and its optimal solution (blue).
An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_4)$ is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 2$$

the optimal value is:

$$z(\mathrm{LP}_4) = 17$$

The LP_4 problem is **feasible** and we have

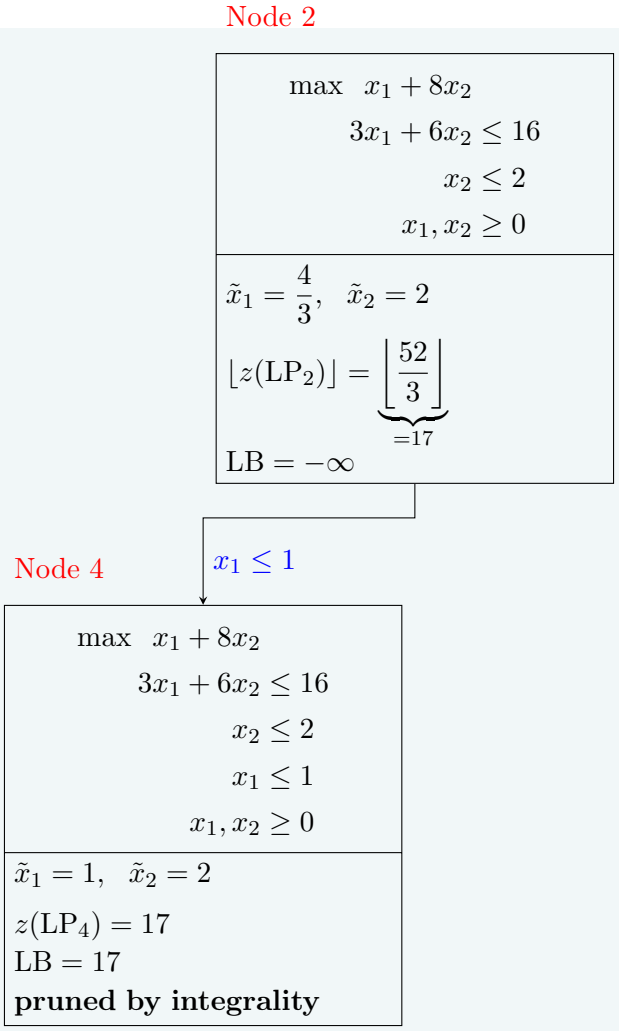
$$\lfloor z(\mathrm{LP}_4) \rfloor > \mathrm{LB} \qquad \text{since} \qquad 17 > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_4)$ is **integer**, accordingly, the lower bound LB becomes 17 and the incumbent solution $\bar{\boldsymbol{x}}(\mathrm{ILP})$ becomes $\bar{x}_1 = 1, \bar{x}_2 = 2$.
Accordingly, the branching node is pruned by integrality.

At the end of the second iteration, branching on x_1 from Node 2 has generated the down-branching node, Node 4. This node is pruned by integrality. The up-branching node is not solved, since after Node 4 the lower bound satisfies

$$\lfloor z(\mathrm{LP}_2) \rfloor \leq \mathrm{LB}.$$

This is the portion of the branch-and-bound tree showing the LP problems of the branching nodes that are solved:



Due to the current lower bound LB and upper bound UB, we have:

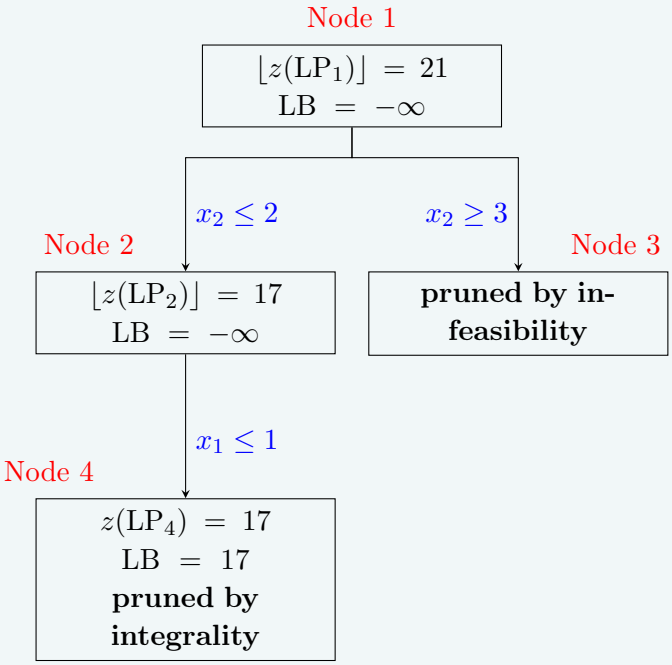
$$17 \quad \leq \quad z(\text{ILP}) \quad \leq \quad 17$$

- **Termination**

Since the list L is empty, the lower bound LB is equal to the upper bound UB and the ILP problem is solved to optimality.

The optimal value is $z(\text{ILP}) = 17$ (the value of the lower bound LB) and an optimal solution $\tilde{\boldsymbol{x}}(\text{ILP})$ is $\tilde{x}_1 = 1, \tilde{x}_2 = 2$ (the incumbent solution $\bar{\boldsymbol{x}}(\text{ILP})$).

2. The final branch-and-bound tree is:



The reasons why the branching nodes are pruned are the following.

- Node 1 has a fractional optimal solution with $\lfloor z(\text{LP}_1) \rfloor = 21$: it is branched on x_2.
- Node 2 has a fractional optimal solution with $\lfloor z(\text{LP}_2) \rfloor = 17$: it is branched on x_1.
- The up-branching node of Node 2 is not solved, and it is not drawn, since $\lfloor z(\text{LP}_2) \rfloor = 17 \leq \text{LB} = 17$ after its first child.
- Node 3 is pruned by infeasibility: the LP_3 problem has no feasible solution.
- Node 4 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_4) = (1, 2)$ is integer, with value 17, which becomes the incumbent solution.

The optimal value is $z(\text{ILP}) = 17$ and the branch-and-bound algorithm returns the optimal solution $\tilde{\boldsymbol{x}}(\text{ILP}) = (1, 2)$.

Exercise 7.2 — Branch-and-bound algorithm ($P2$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{array}{lrcrcl} \max & x_1 & + & 3\ x_2 & & \\ & -4\ x_1 & + & 10\ x_2 & \leq & 35 \\ & 3\ x_1 & + & x_2 & \leq & 12 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

and an optimal simplex dictionary of its LP relaxation in standard form:

$$\begin{cases} x_1 = \frac{5}{2} + \frac{1}{34}\ x_3 - \frac{5}{17}\ x_4 \\ x_2 = \frac{9}{2} - \frac{3}{34}\ x_3 - \frac{2}{17}\ x_4 \\ z = 16 - \frac{4}{17}\ x_3 - \frac{11}{17}\ x_4 \end{cases}$$

Questions

1. Execute the branch-and-bound algorithm using:
   (a) the smallest-index rule as branching-variable selection strategy;
   (b) the FIFO rule as branching-node selection strategy;
   (c) solve first the down-branching node and then the up-branching node;
   (d) set the lower bound LB to $-\infty$ (no heuristic algorithm is executed).

   Solve the LP problems of the branching nodes using the dual simplex algorithm. In the dual simplex algorithm, the leaving variable is the basic variable with negative value and smallest index, and ties in the ratio test are broken by the smallest index. Show graphically the optimal solution of the LP problems associated with the branching nodes.
2. Determine and comment the branch-and-bound tree, showing the reason why each branching node is pruned.

Solution

1. Execution of the branch-and-bound algorithm to solve the ILP problem.
   - **Initialization**

     Let us consider the LP_1 problem in standard form:

$$
\begin{array}{lrcrcrcrcl}
\max & x_1 & + & 3\ x_2 & & & & & & \\
& -4\ x_1 & + & 10\ x_2 & + & x_3 & & & = & 35 \\
& 3\ x_1 & + & x_2 & & & + & x_4 & = & 12 \\
& x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}
$$

and its optimal simplex dictionary:

$$
\begin{cases}
x_1 = \frac{5}{2} + \frac{1}{34} x_3 - \frac{5}{17} x_4 \\
x_2 = \frac{9}{2} - \frac{3}{34} x_3 - \frac{2}{17} x_4 \\
z = 16 - \frac{4}{17} x_3 - \frac{11}{17} x_4
\end{cases}
$$

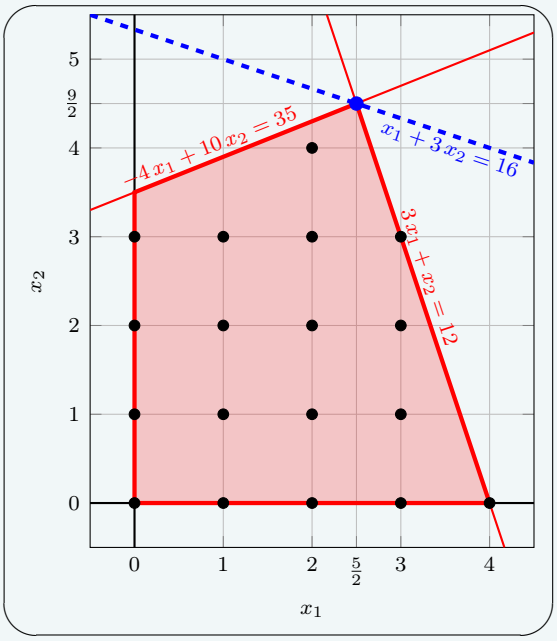


Figure 7.4: Node 1: feasible region of LP_1 (red), feasible points of the ILP problem (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is:

$$\tilde{x}_1 = \frac{5}{2}, \quad \tilde{x}_2 = \frac{9}{2}$$

the optimal value is:

$$z(\mathrm{LP}_1) = 16$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is **fractional** and the LP_1 problem is **feasible**, so the LP_1 problem is inserted in the problem list L and the lower bound LB is set to $-\infty$.

Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \leq z(\mathrm{ILP}) \leq 16$$

- **Iteration 1**

The LP_1 problem is extracted and deleted from the problem list L.

The branching variable is x_1. The branching constraints are:

$$x_1 \leq \underbrace{\left\lfloor \frac{5}{2} \right\rfloor}_{=2} \quad \text{and} \quad x_1 \geq \underbrace{\left\lfloor \frac{5}{2} \right\rfloor + 1}_{=3}$$

Two LP_2 and LP_3 problems are created with these branching constraints.

(a) Solution of the LP_2 problem

Let us consider the LP_2 problem in standard form:

$$\begin{array}{lrcrcrcrcrcr} \max & x_1 & + & 3\ x_2 & & & & & & & & \\ & -4\ x_1 & + & 10\ x_2 & + & x_3 & & & & & = & 35 \\ & 3\ x_1 & + & x_2 & & & + & x_4 & & & = & 12 \\ & x_1 & & & & & & & + & x_5 & = & 2 \\ & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0 \end{array}$$

For the branching constraint we have:

$$x_1 + x_5 = 2 \quad \Rightarrow \quad x_5 = 2 - x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_5 = 2 - \left(\frac{5}{2} + \frac{1}{34}\, x_3 - \frac{5}{17}\, x_4 \right) \quad \Rightarrow \quad x_5 = -\frac{1}{2} - \frac{1}{34}\, x_3 + \frac{5}{17}\, x_4$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = \frac{5}{2} + \frac{1}{34}\, x_3 - \frac{5}{17}\, x_4 \\ x_2 = \frac{9}{2} - \frac{3}{34}\, x_3 - \frac{2}{17}\, x_4 \\ x_5 = -\frac{1}{2} - \frac{1}{34}\, x_3 + \frac{5}{17}\, x_4 \\ z = 16 - \frac{4}{17}\, x_3 - \frac{11}{17}\, x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$ and

– the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{5}{2}, \quad \bar{x}_2 = \frac{9}{2}, \quad \bar{x}_5 = -\frac{1}{2}, \qquad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad 16$$

– the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_5 = 0, \qquad \bar{\pi}_3 = \frac{4}{17}, \quad \bar{\pi}_4 = \frac{11}{17} \quad \text{of value} \quad 16$$

If we increase the value of π_5, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 &= \dfrac{4}{17} + \dfrac{1}{34}\,\pi_5 \\ \pi_4 &= \dfrac{11}{17} - \dfrac{5}{17}\,\pi_5 \end{cases} \qquad \begin{cases} \underbrace{\dfrac{4}{17} + \dfrac{1}{34}\,\pi_5 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\dfrac{11}{17} - \dfrac{5}{17}\,\pi_5 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_5 &\geq -8 \\ \pi_5 &\leq \dfrac{11}{5} \end{cases}$$

By increasing the value of π_5 to $\dfrac{11}{5}$, the value of π_4 decreases to 0. Then we have that:

π_5 becomes a dual basic variable and accordingly x_5 becomes a primal non-basic variable.

π_4 becomes a dual non-basic variable and accordingly x_4 becomes a primal basic variable.

We have:

$$x_5 = -\frac{1}{2} - \frac{1}{34}\,x_3 + \frac{5}{17}\,x_4 \quad \Rightarrow \quad x_4 = \frac{17}{10} + \frac{1}{10}\,x_3 + \frac{17}{5}\,x_5$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{5}{2} + \frac{1}{34}\,x_3 - \frac{5}{17}\left(\frac{17}{10} + \frac{1}{10}\,x_3 + \frac{17}{5}\,x_5\right) = 2 - x_5$$

$$x_2 = \frac{9}{2} - \frac{3}{34}\,x_3 - \frac{2}{17}\left(\frac{17}{10} + \frac{1}{10}\,x_3 + \frac{17}{5}\,x_5\right) = \frac{43}{10} - \frac{1}{10}\,x_3 - \frac{2}{5}\,x_5$$

$$z = 16 - \frac{4}{17}\,x_3 - \frac{11}{17}\left(\frac{17}{10} + \frac{1}{10}\,x_3 + \frac{17}{5}\,x_5\right) = \frac{149}{10} - \frac{3}{10}\,x_3 - \frac{11}{5}\,x_5$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 &= 2 &&- x_5 \\ x_2 &= \dfrac{43}{10} - \dfrac{1}{10}\,x_3 &&- \dfrac{2}{5}\,x_5 \\ x_4 &= \dfrac{17}{10} + \dfrac{1}{10}\,x_3 &&+ \dfrac{17}{5}\,x_5 \\ z &= \dfrac{149}{10} - \dfrac{3}{10}\,x_3 &&- \dfrac{11}{5}\,x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4\}$ and $\mathcal{N} = \{3, 5\}$ and

– the basic primal optimal solution is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = \frac{43}{10}, \quad \tilde{x}_4 = \frac{17}{10}, \qquad \tilde{x}_3 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad \frac{149}{10}$$

– the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_4 = 0, \qquad \tilde{\pi}_3 = \frac{3}{10}, \quad \tilde{\pi}_5 = \frac{11}{5} \quad \text{of value} \quad \frac{149}{10}$$

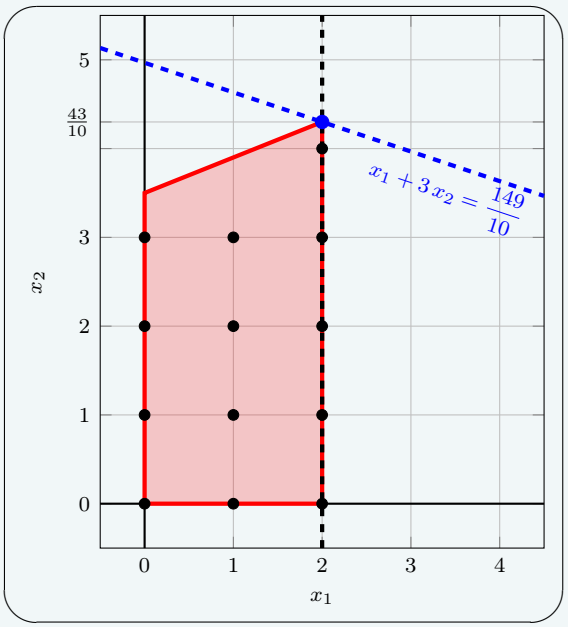


Figure 7.5: Node 2: feasible region of LP_2 (red), with the branching constraint $x_1 \leq 2$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = \frac{43}{10}$$

the optimal value is:

$$z(\mathrm{LP}_2) = \frac{149}{10}$$

The LP_2 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_2) \rfloor > \mathrm{LB} \qquad \text{since} \qquad \underbrace{\left\lfloor \frac{149}{10} \right\rfloor}_{=14} > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is **fractional**, accordingly, the LP_2 problem is **inserted in the problem list** L.

(b) Solution of the LP_3 problem

Let us consider the LP_3 problem in standard form:

$$\begin{array}{lrcrcrcrcrcl}
\max & x_1 & + & 3\,x_2 & & & & & & & & \\
& -4\,x_1 & + & 10\,x_2 & + & x_3 & & & & & = & 35 \\
& 3\,x_1 & + & x_2 & & & + & x_4 & & & = & 12 \\
& x_1 & & & & & & & - & x_5 & = & 3 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_1 - x_5 = 3 \quad \Rightarrow \quad x_5 = -3 + x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_5 = -3 + \left(\frac{5}{2} + \frac{1}{34}\,x_3 - \frac{5}{17}\,x_4\right) \quad \Rightarrow \quad x_5 = -\frac{1}{2} + \frac{1}{34}\,x_3 - \frac{5}{17}\,x_4$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases}
x_1 = \frac{5}{2} + \frac{1}{34}\,x_3 - \frac{5}{17}\,x_4 \\
x_2 = \frac{9}{2} - \frac{3}{34}\,x_3 - \frac{2}{17}\,x_4 \\
x_5 = -\frac{1}{2} + \frac{1}{34}\,x_3 - \frac{5}{17}\,x_4 \\
z = 16 - \frac{4}{17}\,x_3 - \frac{11}{17}\,x_4
\end{cases}$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{5}{2}, \quad \bar{x}_2 = \frac{9}{2}, \quad \bar{x}_5 = -\frac{1}{2}, \qquad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad 16$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_5 = 0, \qquad \bar{\pi}_3 = \frac{4}{17}, \quad \bar{\pi}_4 = \frac{11}{17} \quad \text{of value} \quad 16$$

If we increase the value of π_5, we can get a better dual solution. We have:

$$\begin{cases}
\pi_3 = \frac{4}{17} - \frac{1}{34}\,\pi_5 \\
\pi_4 = \frac{11}{17} + \frac{5}{17}\,\pi_5
\end{cases}
\qquad
\begin{cases}
\underbrace{\frac{4}{17} - \frac{1}{34}\,\pi_5 \geq 0}_{\pi_3 \geq 0} \\
\underbrace{\frac{11}{17} + \frac{5}{17}\,\pi_5 \geq 0}_{\pi_4 \geq 0}
\end{cases}
\qquad
\begin{cases}
\pi_5 \leq 8 \\
\pi_5 \geq -\frac{11}{5}
\end{cases}$$

By increasing the value of π_5 to 8, the value of π_3 decreases to 0. Then we have that:

π_5 becomes a dual basic variable and accordingly x_5 becomes a primal non-basic variable.

π_3 becomes a dual non-basic variable and accordingly x_3 becomes a primal basic variable.

We have:

$$x_5 = -\frac{1}{2} + \frac{1}{34}\,x_3 - \frac{5}{17}\,x_4 \quad \Rightarrow \quad x_3 = 17 + 10\,x_4 + 34\,x_5$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{5}{2} + \frac{1}{34}\,(17 + 10\,x_4 + 34\,x_5) - \frac{5}{17}\,x_4 = 3 + x_5$$

$$x_2 = \frac{9}{2} - \frac{3}{34}\,(17 + 10\,x_4 + 34\,x_5) - \frac{2}{17}\,x_4 = 3 - x_4 - 3\,x_5$$

$$z = 16 - \frac{4}{17}\,(17 + 10\,x_4 + 34\,x_5) - \frac{11}{17}\,x_4 = 12 - 3\,x_4 - 8\,x_5$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 3 + x_5 \\ x_2 = 3 - x_4 - 3\,x_5 \\ x_3 = 17 + 10\,x_4 + 34\,x_5 \\ z = 12 - 3\,x_4 - 8\,x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 3, \quad \tilde{x}_2 = 3, \quad \tilde{x}_3 = 17, \qquad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad 12$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \qquad \tilde{\pi}_4 = 3, \quad \tilde{\pi}_5 = 8 \quad \text{of value} \quad 12$$

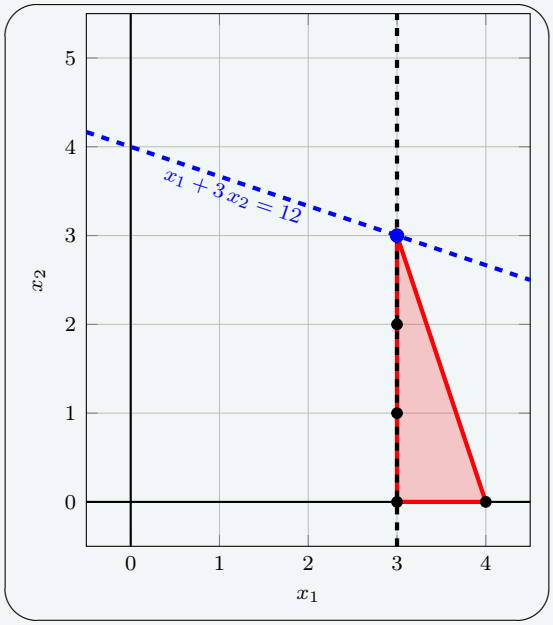


Figure 7.6: Node 3: feasible region of LP_3 (red), with the branching constraint $x_1 \geq 3$ (dashed), and its optimal solution (blue).
An optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_3)$ is:

$$\tilde{x}_1 = 3, \quad \tilde{x}_2 = 3$$

the optimal value is:

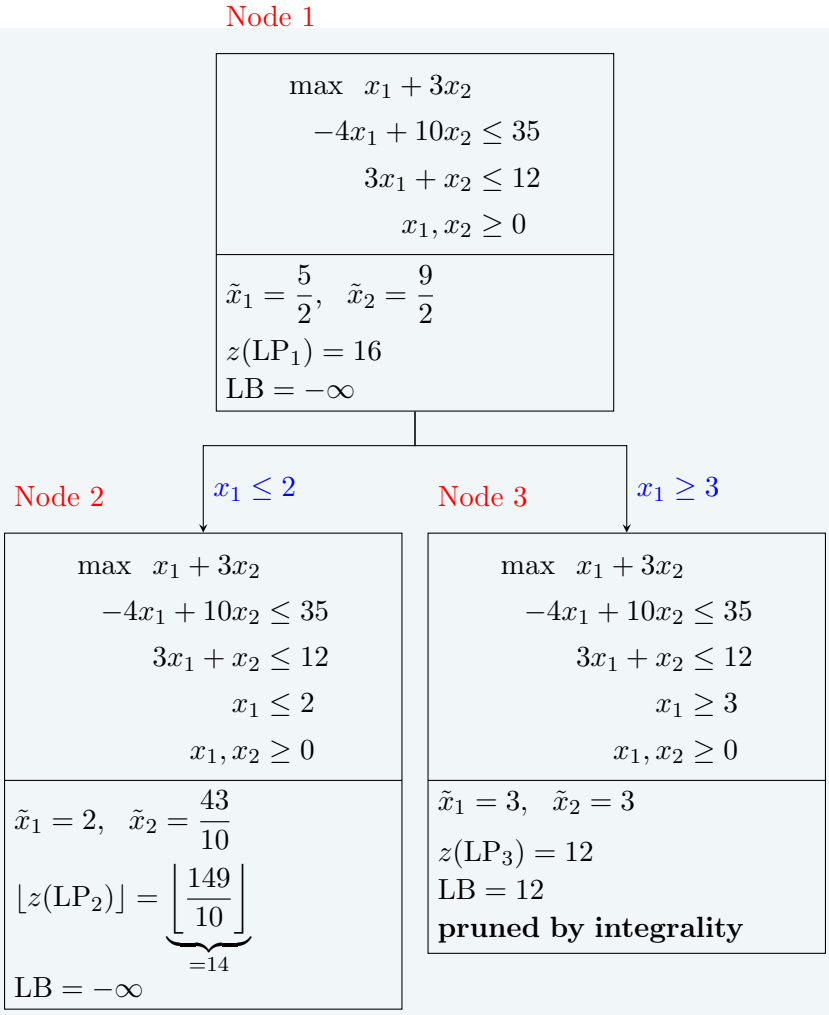
$$z(\text{LP}_3) = 12$$

The LP_3 problem is **feasible** and we have

$$\lfloor z(\text{LP}_3) \rfloor > \text{LB} \qquad \text{since} \qquad 12 > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_3)$ is **integer**, accordingly, the lower bound LB becomes 12 and the incumbent solution $\bar{\boldsymbol{x}}(\text{ILP})$ becomes $\bar{x}_1 = 3, \bar{x}_2 = 3$.
Accordingly, the branching node is pruned by integrality.

At the end of the first iteration, branching on x_1 from Node 1 has generated the down-branching node, Node 2, and the up-branching node, Node 3. Node 2 is inserted in the list L, while Node 3 is pruned by integrality. The corresponding branch-and-bound tree is:



Due to the current lower bound LB and upper bound UB, we have:

$$12 \quad \leq \quad z(\text{ILP}) \quad \leq \quad \underbrace{\left\lfloor \frac{149}{10} \right\rfloor}_{=14}$$

- **Iteration 2**

The LP_2 problem is extracted and deleted from the problem list L.

The branching variable is x_2. The branching constraints are:

$$x_2 \leq \underbrace{\left\lfloor \frac{43}{10} \right\rfloor}_{=4} \quad \text{and} \quad x_2 \geq \underbrace{\left\lfloor \frac{43}{10} \right\rfloor + 1}_{=5}$$

The LP_4 problem is created with the first branching constraint (the second child is solved only if needed).

(a) Solution of the LP_4 problem

Let us consider the LP_4 problem in standard form:

$$
\begin{array}{lrcrcrcrcrcrcr}
\max & x_1 & + & 3\ x_2 & & & & & & & & & \\
& -4\ x_1 & + & 10\ x_2 & + & x_3 & & & & & & = & 35 \\
& 3\ x_1 & + & x_2 & & & + & x_4 & & & & = & 12 \\
& x_1 & & & & & & & + & x_5 & & = & 2 \\
& & & x_2 & & & & & & & + \ x_6 & = & 4 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0
\end{array}
$$

For the branching constraint we have:

$$x_2 + x_6 = 4 \qquad \Rightarrow \quad x_6 = 4 - x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_6 = 4 - \left(\frac{43}{10} - \frac{1}{10}\, x_3 - \frac{2}{5}\, x_5\right) \qquad \Rightarrow \quad x_6 = -\frac{3}{10} + \frac{1}{10}\, x_3 + \frac{2}{5}\, x_5$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$
\begin{cases}
x_1 = 2 - x_5 \\
x_2 = \frac{43}{10} - \frac{1}{10}\, x_3 - \frac{2}{5}\, x_5 \\
x_4 = \frac{17}{10} + \frac{1}{10}\, x_3 + \frac{17}{5}\, x_5 \\
x_6 = -\frac{3}{10} + \frac{1}{10}\, x_3 + \frac{2}{5}\, x_5 \\
z = \frac{149}{10} - \frac{3}{10}\, x_3 - \frac{11}{5}\, x_5
\end{cases}
$$

where $\mathcal{B} = \{1, 2, 4, 6\}$ and $\mathcal{N} = \{3, 5\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = 2, \quad \bar{x}_2 = \frac{43}{10}, \quad \bar{x}_4 = \frac{17}{10}, \quad \bar{x}_6 = -\frac{3}{10},$$
$$\bar{x}_3 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{149}{10}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_6 = 0,$$
$$\bar{\pi}_3 = \frac{3}{10}, \quad \bar{\pi}_5 = \frac{11}{5} \quad \text{of value} \quad \frac{149}{10}$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 &= \dfrac{3}{10} - \dfrac{1}{10}\,\pi_6 \\ \pi_5 &= \dfrac{11}{5} - \dfrac{2}{5}\,\pi_6 \end{cases} \qquad \begin{cases} \underbrace{\dfrac{3}{10} - \dfrac{1}{10}\,\pi_6 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\dfrac{11}{5} - \dfrac{2}{5}\,\pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 &\leq 3 \\ \pi_6 &\leq \dfrac{11}{2} \end{cases}$$

By increasing the value of π_6 to $\min\left\{3, \dfrac{11}{2}\right\} = 3$, the value of π_3 decreases to 0. Then we have that:

π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non-basic variable.

π_3 becomes a dual non-basic variable and accordingly x_3 becomes a primal basic variable.

We have:

$$x_6 = -\frac{3}{10} + \frac{1}{10}\,x_3 + \frac{2}{5}\,x_5 \quad \Rightarrow \quad x_3 = 3 - 4\,x_5 + 10\,x_6$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_2 = \frac{43}{10} - \frac{1}{10}\,(3 - 4\,x_5 + 10\,x_6) - \frac{2}{5}\,x_5 = 4 - x_6$$

$$x_4 = \frac{17}{10} + \frac{1}{10}\,(3 - 4\,x_5 + 10\,x_6) + \frac{17}{5}\,x_5 = 2 + 3\,x_5 + x_6$$

$$z = \frac{149}{10} - \frac{3}{10}\,(3 - 4\,x_5 + 10\,x_6) - \frac{11}{5}\,x_5 = 14 - x_5 - 3\,x_6$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 &= 2 &- &x_5 & & \\ x_2 &= 4 & & & - &x_6 \\ x_3 &= 3 &- &4\,x_5 &+ &10\,x_6 \\ x_4 &= 2 &+ &3\,x_5 &+ &x_6 \\ z &= 14 &- &x_5 &- &3\,x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4\}$ and $\mathcal{N} = \{5, 6\}$ and

– the basic primal optimal solution is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 4, \quad \tilde{x}_3 = 3, \quad \tilde{x}_4 = 2,$$
$$\tilde{x}_5 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad 14$$

– the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 0,$$
$$\tilde{\pi}_5 = 1, \quad \tilde{\pi}_6 = 3 \quad \text{of value} \quad 14$$

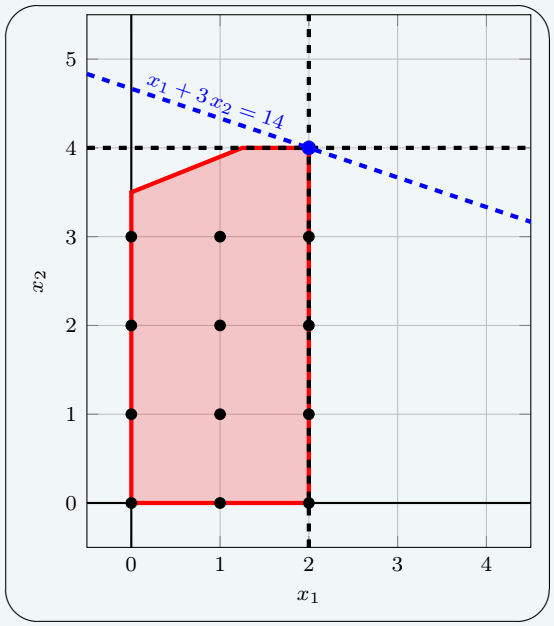


Figure 7.7: Node 4: feasible region of LP_4 (red), with the branching constraint $x_2 \leq 4$ (dashed), and its optimal solution (blue).
An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_4)$ is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 4$$

the optimal value is:

$$z(\mathrm{LP}_4) = 14$$

The LP_4 problem is **feasible** and we have

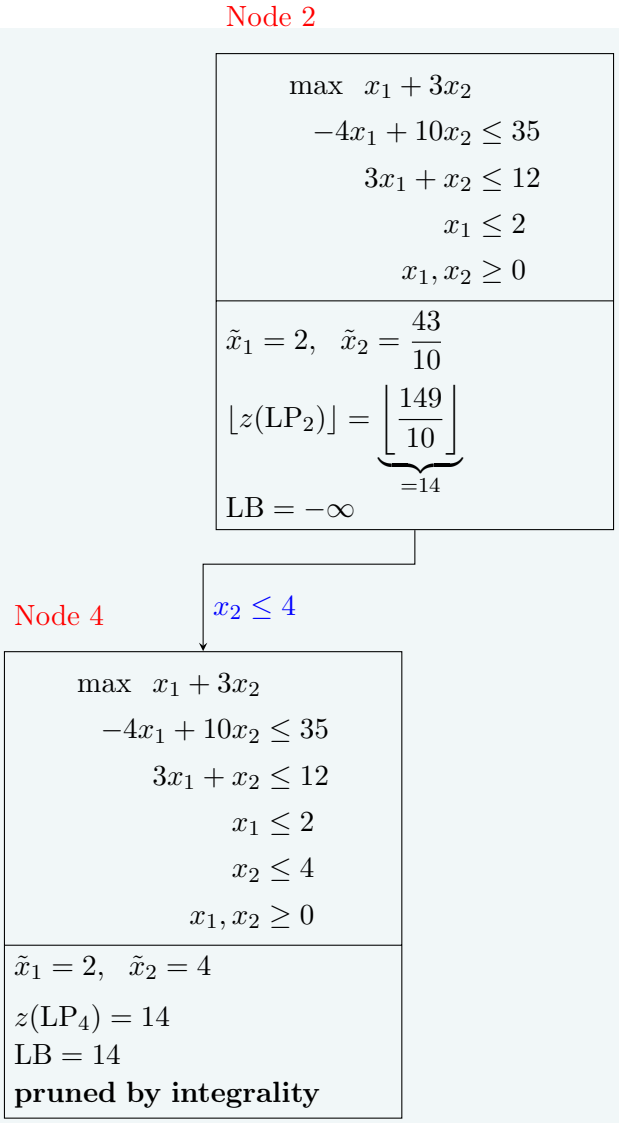
$$\lfloor z(\mathrm{LP}_4) \rfloor > \mathrm{LB} \qquad \text{since} \qquad 14 > 12$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_4)$ is **integer**, accordingly, the lower bound LB becomes 14 and the incumbent solution $\bar{\boldsymbol{x}}(\mathrm{ILP})$ becomes $\bar{x}_1 = 2, \bar{x}_2 = 4$.
Accordingly, the branching node is pruned by integrality.

At the end of the second iteration, branching on x_2 from Node 2 has generated the down-branching node, Node 4. This node is pruned by integrality. The up-branching node is not solved, since after Node 4 the lower bound satisfies

$$\lfloor z(\mathrm{LP}_2) \rfloor \leq \mathrm{LB}.$$

This is the portion of the branch-and-bound tree showing the LP problems of the branching nodes that are solved:



Due to the current lower bound LB and upper bound UB, we have:

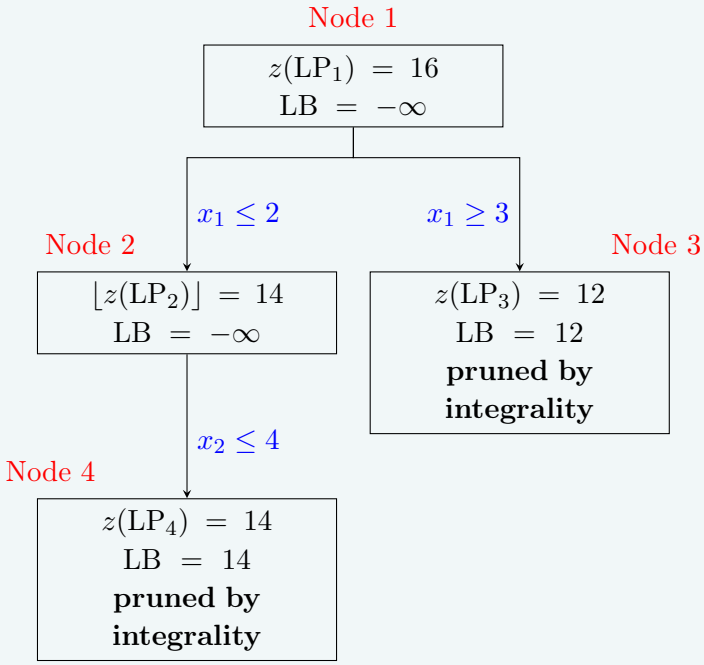
$$14 \quad \leq \quad z(\text{ILP}) \quad \leq \quad 14$$

- **Termination**

Since the list L is empty, the lower bound LB is equal to the upper bound UB and the ILP problem is solved to optimality.

The optimal value is $z(\text{ILP}) = 14$ (the value of the lower bound LB) and an optimal solution $\tilde{\boldsymbol{x}}(\text{ILP})$ is $\tilde{x}_1 = 2, \tilde{x}_2 = 4$ (the incumbent solution $\bar{\boldsymbol{x}}(\text{ILP})$).

2. The final branch-and-bound tree is:



The reasons why the branching nodes are pruned are the following.

- Node 1 has a fractional optimal solution with $z(\text{LP}_1) = 16$: it is branched on x_1.
- Node 2 has a fractional optimal solution with $\lfloor z(\text{LP}_2) \rfloor = 14$: it is branched on x_2.
- The up-branching node of Node 2 is not solved, and it is not drawn, since $\lfloor z(\text{LP}_2) \rfloor = 14 \leq \text{LB} = 14$ after its first child.
- Node 3 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_3) = (3, 3)$ is integer, with value 12, which becomes the incumbent solution.
- Node 4 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_4) = (2, 4)$ is integer, with value 14, which becomes the incumbent solution.

The optimal value is $z(\text{ILP}) = 14$ and the branch-and-bound algorithm returns the optimal solution $\tilde{\boldsymbol{x}}(\text{ILP}) = (2, 4)$.

Exercise 7.3 — Branch-and-bound algorithm ($P3$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{array}{rrcrcrl} \max & 2\,x_1 & + & 4\,x_2 & & & \\ & -x_1 & + & x_2 & \leq & 2 & \\ & 8\,x_1 & + & 2\,x_2 & \leq & 19 & \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} & \end{array}$$

and an optimal simplex dictionary of its LP relaxation in standard form:

$$\begin{cases} x_1 = \frac{3}{2} + \frac{1}{5}\,x_3 - \frac{1}{10}\,x_4 \\ x_2 = \frac{7}{2} - \frac{4}{5}\,x_3 - \frac{1}{10}\,x_4 \\ z = 17 - \frac{14}{5}\,x_3 - \frac{3}{5}\,x_4 \end{cases}$$

Questions

1. Execute the branch-and-bound algorithm using:
   (a) the largest-index rule as branching-variable selection strategy;
   (b) the LIFO rule as branching-node selection strategy;
   (c) solve first the down-branching node and then the up-branching node;
   (d) set the lower bound LB to $-\infty$ (no heuristic algorithm is executed).

   Solve the LP problems of the branching nodes using the dual simplex algorithm. In the dual simplex algorithm, the leaving variable is the basic variable with negative value and smallest index, and ties in the ratio test are broken by the smallest index. Show graphically the optimal solution of the LP problems associated with the branching nodes.
2. Determine and comment the branch-and-bound tree, showing the reason why each branching node is pruned.

Solution

1. Execution of the branch-and-bound algorithm to solve the ILP problem.
   - **Initialization**

     Let us consider the LP_1 problem in standard form:

$$\begin{array}{lrcrcrcrcl} \max & 2\,x_1 & + & 4\,x_2 & & & & & & \\ & -x_1 & + & x_2 & + & x_3 & & & = & 2 \\ & 8\,x_1 & + & 2\,x_2 & & & + & x_4 & = & 19 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

and its optimal simplex dictionary:

$$\begin{cases} x_1 = \frac{3}{2} + \frac{1}{5}\,x_3 - \frac{1}{10}\,x_4 \\ x_2 = \frac{7}{2} - \frac{4}{5}\,x_3 - \frac{1}{10}\,x_4 \\ z = 17 - \frac{14}{5}\,x_3 - \frac{3}{5}\,x_4 \end{cases}$$

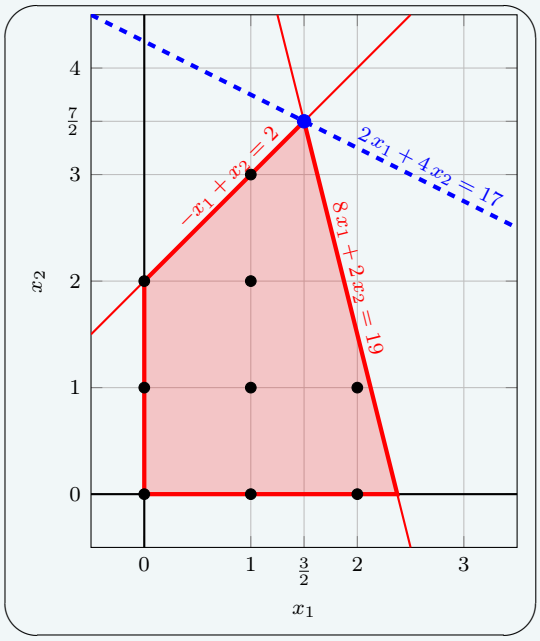


Figure 7.8: Node 1: feasible region of LP_1 (red), feasible points of the ILP problem (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is:

$$\tilde{x}_1 = \frac{3}{2}, \quad \tilde{x}_2 = \frac{7}{2}$$

the optimal value is:

$$z(\mathrm{LP}_1) = 17$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is **fractional** and the LP_1 problem is **feasible**, so the LP_1 problem is inserted in the problem list L and the lower bound LB is set to $-\infty$.

Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \leq z(\mathrm{ILP}) \leq 17$$

- **Iteration 1**

The LP_1 problem is extracted and deleted from the problem list L.

The branching variable is x_2. The branching constraints are:

$$x_2 \leq \underbrace{\left\lfloor \frac{7}{2} \right\rfloor}_{=3} \quad \text{and} \quad x_2 \geq \underbrace{\left\lfloor \frac{7}{2} \right\rfloor + 1}_{=4}$$

Two LP_2 and LP_3 problems are created with these branching constraints.

(a) Solution of the LP_2 problem

Let us consider the LP_2 problem in standard form:

$$\begin{array}{lrcrcrcrcr} \max & 2\,x_1 & + & 4\,x_2 & & & & & & \\ & -x_1 & + & x_2 & + & x_3 & & & = & 2 \\ & 8\,x_1 & + & 2\,x_2 & & & + & x_4 & = & 19 \\ & & & x_2 & & & & & + \; x_5 \; = & 3 \\ & x_1, & & x_2, & & x_3, & & x_4, & & x_5 \geq 0 \end{array}$$

For the branching constraint we have:

$$x_2 + x_5 = 3 \quad \Rightarrow \quad x_5 = 3 - x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_5 = 3 - \left(\frac{7}{2} - \frac{4}{5}\, x_3 - \frac{1}{10}\, x_4 \right) \quad \Rightarrow \quad x_5 = -\frac{1}{2} + \frac{4}{5}\, x_3 + \frac{1}{10}\, x_4$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = \frac{3}{2} + \frac{1}{5}\, x_3 - \frac{1}{10}\, x_4 \\ x_2 = \frac{7}{2} - \frac{4}{5}\, x_3 - \frac{1}{10}\, x_4 \\ x_5 = -\frac{1}{2} + \frac{4}{5}\, x_3 + \frac{1}{10}\, x_4 \\ z = 17 - \frac{14}{5}\, x_3 - \frac{3}{5}\, x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$ and

– the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{3}{2}, \quad \bar{x}_2 = \frac{7}{2}, \quad \bar{x}_5 = -\frac{1}{2}, \qquad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad 17$$

– the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_5 = 0, \qquad \bar{\pi}_3 = \frac{14}{5}, \quad \bar{\pi}_4 = \frac{3}{5} \quad \text{of value} \quad 17$$

If we increase the value of π_5, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 = \dfrac{14}{5} - \dfrac{4}{5}\pi_5 \\ \pi_4 = \dfrac{3}{5} - \dfrac{1}{10}\pi_5 \end{cases} \qquad \begin{cases} \underbrace{\dfrac{14}{5} - \dfrac{4}{5}\pi_5 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\dfrac{3}{5} - \dfrac{1}{10}\pi_5 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_5 \leq \dfrac{7}{2} \\ \pi_5 \leq 6 \end{cases}$$

By increasing the value of π_5 to $\min\left\{\frac{7}{2}, 6\right\} = \frac{7}{2}$, the value of π_3 decreases to 0. Then we have that:

π_5 becomes a dual basic variable and accordingly x_5 becomes a primal non-basic variable.

π_3 becomes a dual non-basic variable and accordingly x_3 becomes a primal basic variable.

We have:

$$x_5 = -\frac{1}{2} + \frac{4}{5}x_3 + \frac{1}{10}x_4 \quad \Rightarrow \quad x_3 = \frac{5}{8} - \frac{1}{8}x_4 + \frac{5}{4}x_5$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{3}{2} + \frac{1}{5}\left(\frac{5}{8} - \frac{1}{8}x_4 + \frac{5}{4}x_5\right) - \frac{1}{10}x_4 = \frac{13}{8} - \frac{1}{8}x_4 + \frac{1}{4}x_5$$

$$x_2 = \frac{7}{2} - \frac{4}{5}\left(\frac{5}{8} - \frac{1}{8}x_4 + \frac{5}{4}x_5\right) - \frac{1}{10}x_4 = 3 - x_5$$

$$z = 17 - \frac{14}{5}\left(\frac{5}{8} - \frac{1}{8}x_4 + \frac{5}{4}x_5\right) - \frac{3}{5}x_4 = \frac{61}{4} - \frac{1}{4}x_4 - \frac{7}{2}x_5$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = \dfrac{13}{8} - \dfrac{1}{8}x_4 + \dfrac{1}{4}x_5 \\ x_2 = 3 - x_5 \\ x_3 = \dfrac{5}{8} - \dfrac{1}{8}x_4 + \dfrac{5}{4}x_5 \\ z = \dfrac{61}{4} - \dfrac{1}{4}x_4 - \dfrac{7}{2}x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$ and

– the basic primal optimal solution is:

$$\tilde{x}_1 = \frac{13}{8}, \quad \tilde{x}_2 = 3, \quad \tilde{x}_3 = \frac{5}{8}, \qquad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad \frac{61}{4}$$

– the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \qquad \tilde{\pi}_4 = \frac{1}{4}, \quad \tilde{\pi}_5 = \frac{7}{2} \quad \text{of value} \quad \frac{61}{4}$$

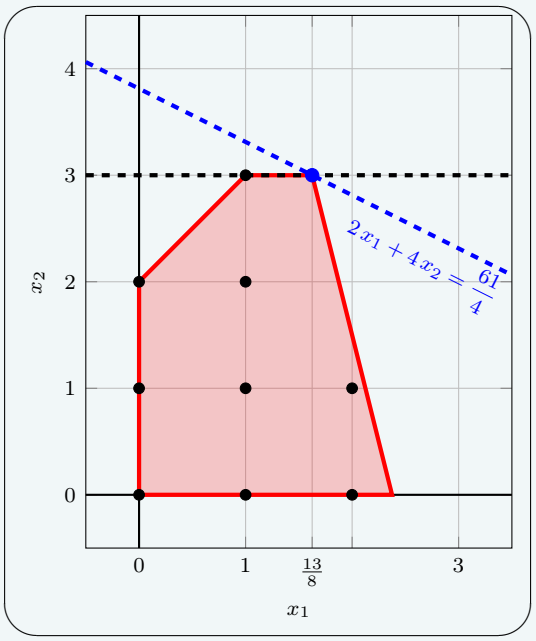


Figure 7.9: Node 2: feasible region of LP_2 (red), with the branching constraint $x_2 \leq 3$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is:

$$\tilde{x}_1 = \frac{13}{8}, \quad \tilde{x}_2 = 3$$

the optimal value is:

$$z(\mathrm{LP}_2) = \frac{61}{4}$$

The LP_2 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_2) \rfloor > \mathrm{LB} \qquad \text{since} \qquad \underbrace{\left\lfloor \frac{61}{4} \right\rfloor}_{=15} > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is **fractional**, accordingly, the LP_2 problem is **inserted in the problem list** L.

(b) Solution of the LP_3 problem

Let us consider the LP_3 problem in standard form:

$$
\begin{array}{lrcrcrcrcl}
\max & 2\,x_1 & + & 4\,x_2 & & & & & & \\
& -x_1 & + & x_2 & + & x_3 & & & = & 2 \\
& 8\,x_1 & + & 2\,x_2 & & & + & x_4 & = & 19 \\
& & & x_2 & & & & - & x_5 = & 4 \\
& x_1, & & x_2, & & x_3, & & x_4, & x_5 \geq & 0
\end{array}
$$

For the branching constraint we have:

$$x_2 - x_5 = 4 \quad \Rightarrow \quad x_5 = -4 + x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_5 = -4 + \left(\frac{7}{2} - \frac{4}{5}\,x_3 - \frac{1}{10}\,x_4\right) \quad \Rightarrow \quad x_5 = -\frac{1}{2} - \frac{4}{5}\,x_3 - \frac{1}{10}\,x_4$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$
\begin{cases}
x_1 = \frac{3}{2} + \frac{1}{5}\,x_3 - \frac{1}{10}\,x_4 \\
x_2 = \frac{7}{2} - \frac{4}{5}\,x_3 - \frac{1}{10}\,x_4 \\
x_5 = -\frac{1}{2} - \frac{4}{5}\,x_3 - \frac{1}{10}\,x_4 \\
z = 17 - \frac{14}{5}\,x_3 - \frac{3}{5}\,x_4
\end{cases}
$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{3}{2}, \quad \bar{x}_2 = \frac{7}{2}, \quad \bar{x}_5 = -\frac{1}{2}, \qquad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad 17$$

- the basic dual feasible solution is:

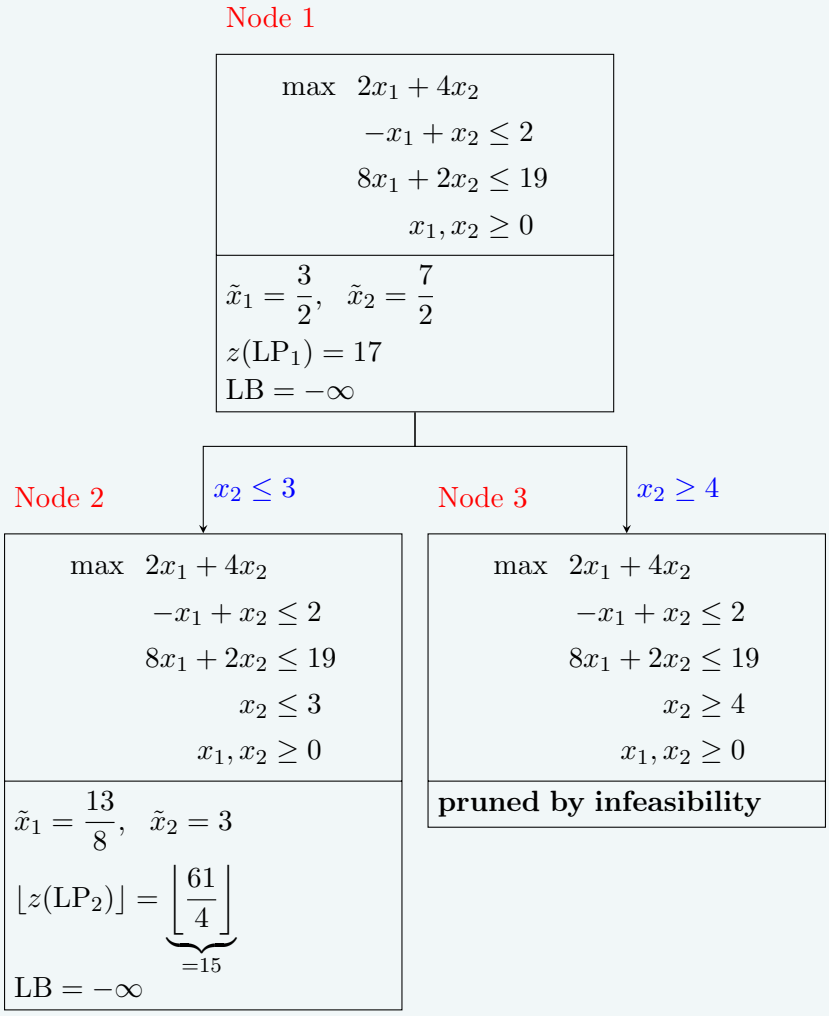
$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_5 = 0, \qquad \bar{\pi}_3 = \frac{14}{5}, \quad \bar{\pi}_4 = \frac{3}{5} \quad \text{of value} \quad 17$$

If we increase the value of π_5, we can get a better dual solution. We have:

$$
\begin{cases}
\pi_3 = \frac{14}{5} + \frac{4}{5}\,\pi_5 \\
\pi_4 = \frac{3}{5} + \frac{1}{10}\,\pi_5
\end{cases}
\qquad
\begin{cases}
\underbrace{\frac{14}{5} + \frac{4}{5}\,\pi_5 \geq 0}_{\pi_3 \geq 0} \\
\underbrace{\frac{3}{5} + \frac{1}{10}\,\pi_5 \geq 0}_{\pi_4 \geq 0}
\end{cases}
\qquad
\begin{cases}
\pi_5 \geq -\frac{7}{2} \\
\pi_5 \geq -6
\end{cases}
$$

Accordingly, we can increase the value of π_5 without violating dual feasibility, which implies that the dual problem is **unbounded**. As a result, the primal problem LP_3 is **infeasible**, and the corresponding branching node is pruned by infeasibility.

At the end of the first iteration, branching on x_2 from Node 1 has generated the down-branching node, Node 2, and the up-branching node, Node 3. Node 2 is inserted in the list L, while Node 3 is pruned by infeasibility. The corresponding branch-and-bound tree is:



Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \quad \leq \quad z(\text{ILP}) \quad \leq \quad \underbrace{\left\lfloor \frac{61}{4} \right\rfloor}_{=15}$$

- **Iteration 2**

The LP_2 problem is extracted and deleted from the problem list L.

The branching variable is x_1. The branching constraints are:

$$x_1 \leq \underbrace{\left\lfloor \frac{13}{8} \right\rfloor}_{=1} \quad \text{and} \quad x_1 \geq \underbrace{\left\lfloor \frac{13}{8} \right\rfloor}_{=2} + 1$$

Two LP_4 and LP_5 problems are created with these branching constraints.

(a) Solution of the LP_4 problem

Let us consider the LP_4 problem in standard form:

$$\begin{array}{lrcrcrcrcrcl} \max & 2\,x_1 & + & 4\,x_2 & & & & & & & & \\ & -x_1 & + & x_2 & + & x_3 & & & & & = & 2 \\ & 8\,x_1 & + & 2\,x_2 & & & + & x_4 & & & = & 19 \\ & & & x_2 & & & & & + & x_5 & = & 3 \\ & x_1 & & & & & & & & & = & 1 \\ & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & \geq & 0 \end{array}$$

For the branching constraint we have:

$$x_1 + x_6 = 1 \qquad \Rightarrow \quad x_6 = 1 - x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_6 = 1 - \left(\frac{13}{8} - \frac{1}{8}\,x_4 + \frac{1}{4}\,x_5\right) \qquad \Rightarrow \quad x_6 = -\frac{5}{8} + \frac{1}{8}\,x_4 - \frac{1}{4}\,x_5$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = \frac{13}{8} - \frac{1}{8}\,x_4 + \frac{1}{4}\,x_5 \\ x_2 = 3 - x_5 \\ x_3 = \frac{5}{8} - \frac{1}{8}\,x_4 + \frac{5}{4}\,x_5 \\ x_6 = -\frac{5}{8} + \frac{1}{8}\,x_4 - \frac{1}{4}\,x_5 \\ z = \frac{61}{4} - \frac{1}{4}\,x_4 - \frac{7}{2}\,x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 6\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{13}{8}, \quad \bar{x}_2 = 3, \quad \bar{x}_3 = \frac{5}{8}, \quad \bar{x}_6 = -\frac{5}{8},$$
$$\bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{61}{4}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_6 = 0,$$
$$\bar{\pi}_4 = \frac{1}{4}, \quad \bar{\pi}_5 = \frac{7}{2} \quad \text{of value} \quad \frac{61}{4}$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_4 &= \frac{1}{4} - \frac{1}{8}\,\pi_6 \\ \pi_5 &= \frac{7}{2} + \frac{1}{4}\,\pi_6 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{4} - \frac{1}{8}\,\pi_6 \geq 0}_{\pi_4 \geq 0} \\ \underbrace{\frac{7}{2} + \frac{1}{4}\,\pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 &\leq 2 \\ \pi_6 &\geq -14 \end{cases}$$

By increasing the value of π_6 to 2, the value of π_4 decreases to 0. Then we have that:

π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non-basic variable.

π_4 becomes a dual non-basic variable and accordingly x_4 becomes a primal basic variable.

We have:

$$x_6 = -\frac{5}{8} + \frac{1}{8}\,x_4 - \frac{1}{4}\,x_5 \quad \Rightarrow \quad x_4 = 5 + 2\,x_5 + 8\,x_6$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{13}{8} - \frac{1}{8}\,(5 + 2\,x_5 + 8\,x_6) + \frac{1}{4}\,x_5 = 1 - x_6$$

$$x_3 = \frac{5}{8} - \frac{1}{8}\,(5 + 2\,x_5 + 8\,x_6) + \frac{5}{4}\,x_5 = 0 + x_5 - x_6$$

$$z = \frac{61}{4} - \frac{1}{4}\,(5 + 2\,x_5 + 8\,x_6) - \frac{7}{2}\,x_5 = 14 - 4\,x_5 - 2\,x_6$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 &= 1 &&&&&- &x_6 \\ x_2 &= 3 &- &x_5 \\ x_3 &= 0 &+ &x_5 &&&- &x_6 \\ x_4 &= 5 &+ &2\,x_5 &&&+ &8\,x_6 \\ z &= 14 &- &4\,x_5 &&&- &2\,x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4\}$ and $\mathcal{N} = \{5, 6\}$ and

– the basic primal optimal solution is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 3, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 5,$$
$$\tilde{x}_5 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad 14$$

– the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 0,$$
$$\tilde{\pi}_5 = 4, \quad \tilde{\pi}_6 = 2 \quad \text{of value} \quad 14$$

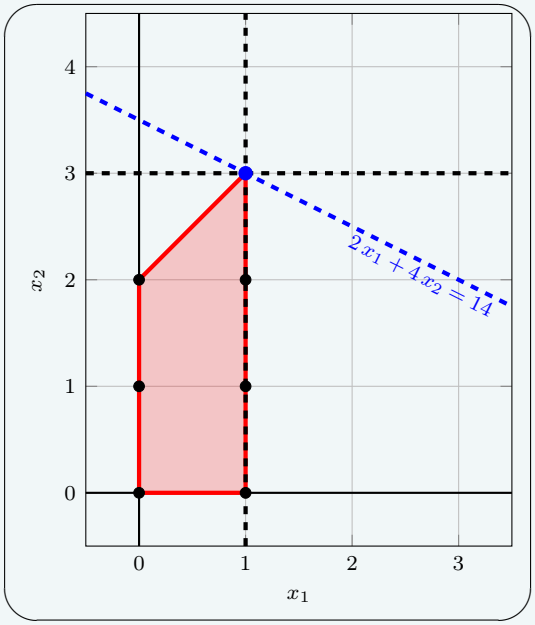


Figure 7.10: Node 4: feasible region of LP_4 (red), with the branching constraint $x_1 \leq 1$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_4)$ is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 3$$

the optimal value is:

$$z(\mathrm{LP}_4) = 14$$

The LP_4 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_4) \rfloor > \mathrm{LB} \qquad \text{since} \qquad 14 > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_4)$ is **integer**, accordingly, the lower bound LB becomes 14 and the incumbent solution $\bar{\boldsymbol{x}}(\mathrm{ILP})$ becomes $\bar{x}_1 = 1, \bar{x}_2 = 3$.

Accordingly, the branching node is pruned by integrality.

(b) Solution of the LP_5 problem

Let us consider the LP_5 problem in standard form:

$$\begin{array}{rrrrrrrrrrrrr}
\max & 2\,x_1 & + & 4\,x_2 & & & & & & & & & \\
& -x_1 & + & x_2 & + & x_3 & & & & & & = & 2 \\
& 8\,x_1 & + & 2\,x_2 & & & + & x_4 & & & & = & 19 \\
& & & x_2 & & & & & + & x_5 & & = & 3 \\
& x_1 & & & & & & & & & - & x_6 & = & 2 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_1 - x_6 = 2 \qquad \Rightarrow \quad x_6 = -2 + x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_6 = -2 + \left(\frac{13}{8} - \frac{1}{8}\,x_4 + \frac{1}{4}\,x_5\right) \qquad \Rightarrow \quad x_6 = -\frac{3}{8} - \frac{1}{8}\,x_4 + \frac{1}{4}\,x_5$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = \frac{13}{8} - \frac{1}{8}\,x_4 + \frac{1}{4}\,x_5 \\ x_2 = 3 - x_5 \\ x_3 = \frac{5}{8} - \frac{1}{8}\,x_4 + \frac{5}{4}\,x_5 \\ x_6 = -\frac{3}{8} - \frac{1}{8}\,x_4 + \frac{1}{4}\,x_5 \\ z = \frac{61}{4} - \frac{1}{4}\,x_4 - \frac{7}{2}\,x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 6\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{13}{8}, \quad \bar{x}_2 = 3, \quad \bar{x}_3 = \frac{5}{8}, \quad \bar{x}_6 = -\frac{3}{8},$$
$$\bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{61}{4}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_6 = 0,$$
$$\bar{\pi}_4 = \frac{1}{4}, \quad \bar{\pi}_5 = \frac{7}{2} \quad \text{of value} \quad \frac{61}{4}$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_4 = \frac{1}{4} + \frac{1}{8}\,\pi_6 \\ \pi_5 = \frac{7}{2} - \frac{1}{4}\,\pi_6 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{4} + \frac{1}{8}\,\pi_6 \geq 0}_{\pi_4 \geq 0} \\ \underbrace{\frac{7}{2} - \frac{1}{4}\,\pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 \geq -2 \\ \pi_6 \leq 14 \end{cases}$$

By increasing the value of π_6 to 14, the value of π_5 decreases to 0. Then we have that:

π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non-basic variable.

π_5 becomes a dual non-basic variable and accordingly x_5 becomes a primal basic variable.

We have:

$$x_6 = -\frac{3}{8} - \frac{1}{8}\,x_4 + \frac{1}{4}\,x_5 \quad \Rightarrow \quad x_5 = \frac{3}{2} + \frac{1}{2}\,x_4 + 4\,x_6$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{13}{8} - \frac{1}{8}\,x_4 + \frac{1}{4}\left(\frac{3}{2} + \frac{1}{2}\,x_4 + 4\,x_6\right) = 2 + x_6$$

$$x_2 = 3 - \left(\frac{3}{2} + \frac{1}{2}\,x_4 + 4\,x_6\right) = \frac{3}{2} - \frac{1}{2}\,x_4 - 4\,x_6$$

$$x_3 = \frac{5}{8} - \frac{1}{8}\,x_4 + \frac{5}{4}\left(\frac{3}{2} + \frac{1}{2}\,x_4 + 4\,x_6\right) = \frac{5}{2} + \frac{1}{2}\,x_4 + 5\,x_6$$

$$z = \frac{61}{4} - \frac{1}{4}\,x_4 - \frac{7}{2}\left(\frac{3}{2} + \frac{1}{2}\,x_4 + 4\,x_6\right) = 10 - 2\,x_4 - 14\,x_6$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 2 + x_6 \\ x_2 = \frac{3}{2} - \frac{1}{2}\,x_4 - 4\,x_6 \\ x_3 = \frac{5}{2} + \frac{1}{2}\,x_4 + 5\,x_6 \\ x_5 = \frac{3}{2} + \frac{1}{2}\,x_4 + 4\,x_6 \\ z = 10 - 2\,x_4 - 14\,x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 5\}$ and $\mathcal{N} = \{4, 6\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = \frac{3}{2}, \quad \tilde{x}_3 = \frac{5}{2}, \quad \tilde{x}_5 = \frac{3}{2},$$
$$\tilde{x}_4 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad 10$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_5 = 0,$$
$$\tilde{\pi}_4 = 2, \quad \tilde{\pi}_6 = 14 \quad \text{of value} \quad 10$$

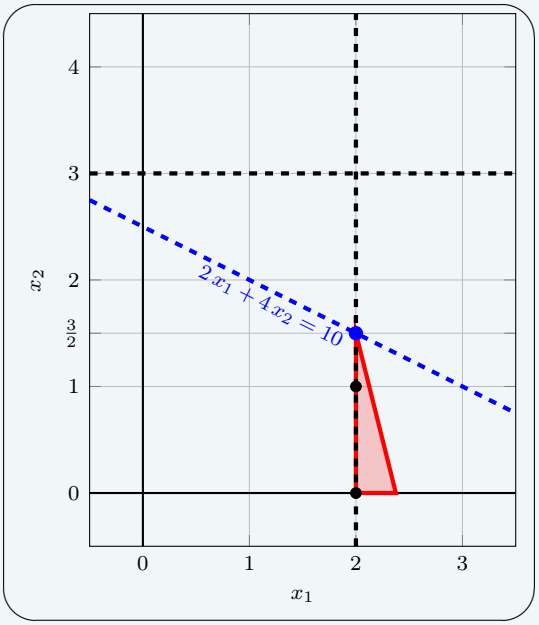


Figure 7.11: Node 5: feasible region of LP_5 (red), with the branching constraint $x_1 \geq 2$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_5)$ is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = \frac{3}{2}$$

the optimal value is:

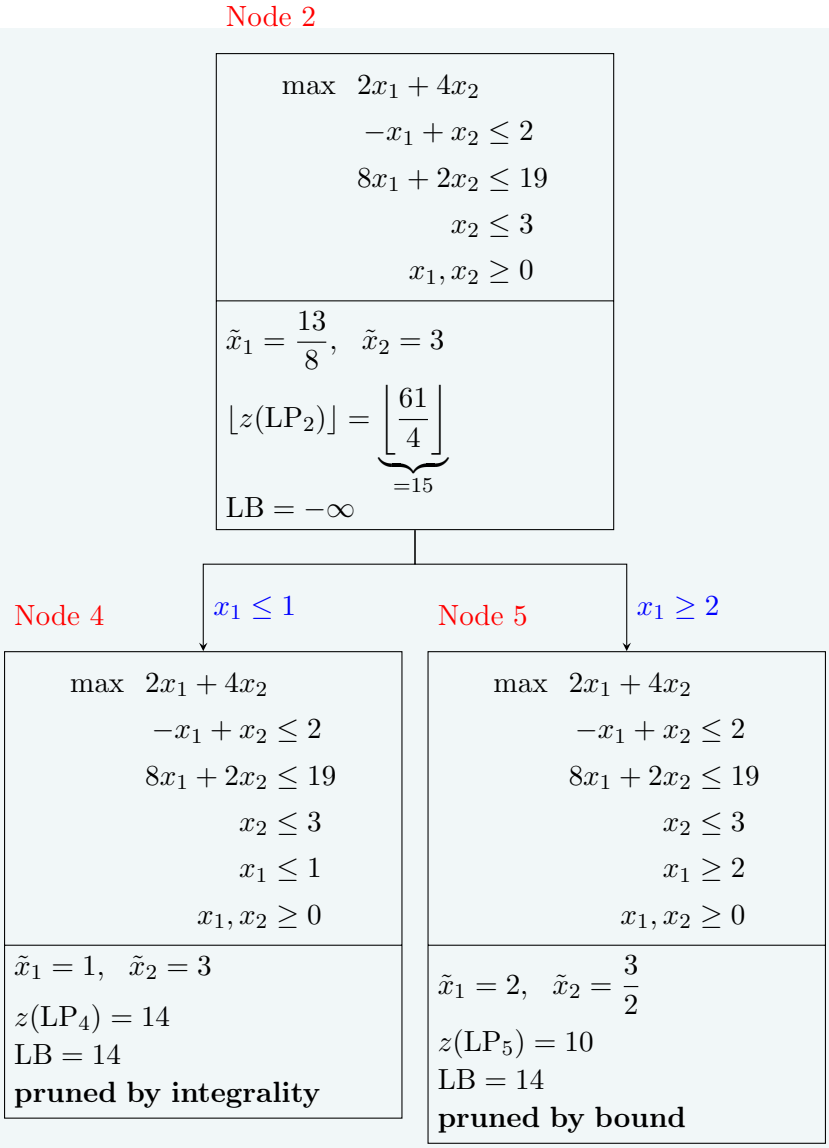
$$z(\text{LP}_5) = 10$$

The LP_5 problem is **feasible** and we have

$$\lfloor z(\text{LP}_5) \rfloor \leq \text{LB} \qquad \text{since} \qquad 10 \leq 14$$

Accordingly, the branching node is pruned by bound.

At the end of the second iteration, branching on x_1 from Node 2 has generated the down-branching node, Node 4, and the up-branching node, Node 5. Node 4 is pruned by integrality, while Node 5 is pruned by bound. The corresponding branch-and-bound tree is:



Due to the current lower bound LB and upper bound UB, we have:

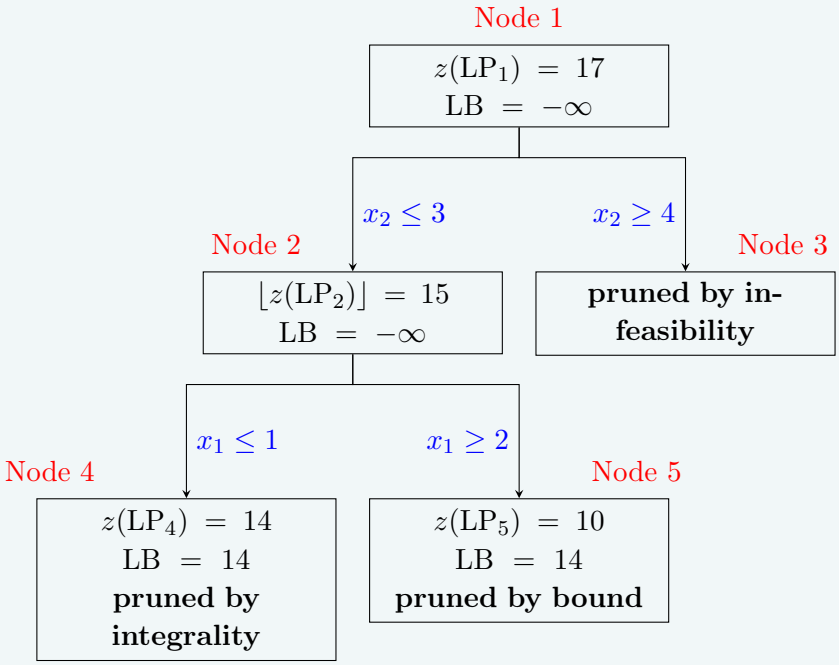
$$14 \quad \leq \quad z(\mathrm{ILP}) \quad \leq \quad 14$$

- **Termination**

Since the list L is empty, the lower bound LB is equal to the upper bound UB and the ILP problem is solved to optimality.

The optimal value is $z(\mathrm{ILP}) = 14$ (the value of the lower bound LB) and an optimal solution $\tilde{\boldsymbol{x}}(\mathrm{ILP})$ is $\tilde{x}_1 = 1, \tilde{x}_2 = 3$ (the incumbent solution $\bar{\boldsymbol{x}}(\mathrm{ILP})$).

2. The final branch-and-bound tree is:



The reasons why the branching nodes are pruned are the following.

- Node 1 has a fractional optimal solution with $z(\text{LP}_1) = 17$: it is branched on x_2.
- Node 2 has a fractional optimal solution with $\lfloor z(\text{LP}_2) \rfloor = 15$: it is branched on x_1.
- Node 3 is pruned by infeasibility: the LP_3 problem has no feasible solution.
- Node 4 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_4) = (1, 3)$ is integer, with value 14, which becomes the incumbent solution.
- Node 5 is pruned by bound: $z(\text{LP}_5) = 10 \leq \text{LB} = 14$.

The optimal value is $z(\text{ILP}) = 14$ and the branch-and-bound algorithm returns the optimal solution $\tilde{\boldsymbol{x}}(\text{ILP}) = (1, 3)$.

Exercise 7.4 — Branch-and-bound algorithm ($P4$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{array}{rrcrcrl} \max & 5\,x_1 & + & 3\,x_2 & & & \\ & 2\,x_1 & + & x_2 & \leq & 14 & \\ & x_1 & + & 8\,x_2 & \leq & 32 & \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} & \end{array}$$

and an optimal simplex dictionary of its LP relaxation in standard form:

$$\begin{cases} x_1 = \dfrac{16}{3} - \dfrac{8}{15}\,x_3 + \dfrac{1}{15}\,x_4 \\ x_2 = \dfrac{10}{3} + \dfrac{1}{15}\,x_3 - \dfrac{2}{15}\,x_4 \\ z = \dfrac{110}{3} - \dfrac{37}{15}\,x_3 - \dfrac{1}{15}\,x_4 \end{cases}$$

Questions

1. Execute the branch-and-bound algorithm using:
   (a) the largest-index rule as branching-variable selection strategy;
   (b) the LIFO rule as branching-node selection strategy;
   (c) solve first the down-branching node and then the up-branching node;
   (d) set the lower bound LB to $-\infty$ (no heuristic algorithm is executed).

   Solve the LP problems of the branching nodes using the dual simplex algorithm. In the dual simplex algorithm, the leaving variable is the basic variable with negative value and smallest index, and ties in the ratio test are broken by the smallest index. Show graphically the optimal solution of the LP problems associated with the branching nodes.
2. Determine and comment the branch-and-bound tree, showing the reason why each branching node is pruned.

Solution

1. Execution of the branch-and-bound algorithm to solve the ILP problem.
   - **Initialization**

     Let us consider the LP_1 problem in standard form:

$$\begin{array}{lrcrcrcrcl} \max & 5\,x_1 & + & 3\,x_2 & & & & & & \\ & 2\,x_1 & + & x_2 & + & x_3 & & & = & 14 \\ & x_1 & + & 8\,x_2 & & & + & x_4 & = & 32 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

and its optimal simplex dictionary:

$$\begin{cases} x_1 = \dfrac{16}{3} - \dfrac{8}{15}\,x_3 + \dfrac{1}{15}\,x_4 \\ x_2 = \dfrac{10}{3} + \dfrac{1}{15}\,x_3 - \dfrac{2}{15}\,x_4 \\ z = \dfrac{110}{3} - \dfrac{37}{15}\,x_3 - \dfrac{1}{15}\,x_4 \end{cases}$$

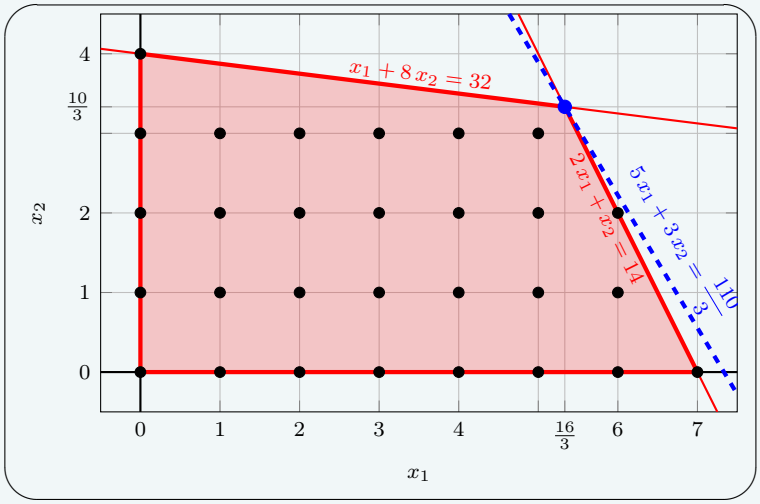


Figure 7.12: Node 1: feasible region of LP_1 (red), feasible points of the ILP problem (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is:

$$\tilde{x}_1 = \frac{16}{3}, \quad \tilde{x}_2 = \frac{10}{3}$$

the optimal value is:

$$z(\mathrm{LP}_1) = \frac{110}{3}$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is **fractional** and the LP_1 problem is **feasible**, so the LP_1 problem is inserted in the problem list L and the lower bound LB is set to $-\infty$.

Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \leq z(\mathrm{ILP}) \leq \underbrace{\left\lfloor \frac{110}{3} \right\rfloor}_{=36}$$

- **Iteration 1**

The LP_1 problem is extracted and deleted from the problem list L.

The branching variable is x_2. The branching constraints are:

$$x_2 \leq \underbrace{\left\lfloor \frac{10}{3} \right\rfloor}_{=3} \quad \text{and} \quad x_2 \geq \underbrace{\left\lfloor \frac{10}{3} \right\rfloor + 1}_{=4}$$

Two LP_2 and LP_3 problems are created with these branching constraints.

(a) Solution of the LP_2 problem

Let us consider the LP_2 problem in standard form:

$$\begin{array}{lrcrcrcrcrcl}
\max & 5\,x_1 & + & 3\,x_2 & & & & & & & & \\
& 2\,x_1 & + & x_2 & + & x_3 & & & & & = & 14 \\
& x_1 & + & 8\,x_2 & & & + & x_4 & & & = & 32 \\
& & & x_2 & & & & & + & x_5 & = & 3 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_2 + x_5 = 3 \quad \Rightarrow \quad x_5 = 3 - x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_5 = 3 - \left(\frac{10}{3} + \frac{1}{15}\,x_3 - \frac{2}{15}\,x_4 \right) \quad \Rightarrow \quad x_5 = -\frac{1}{3} - \frac{1}{15}\,x_3 + \frac{2}{15}\,x_4$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases}
x_1 = \dfrac{16}{3} - \dfrac{8}{15}\,x_3 + \dfrac{1}{15}\,x_4 \\
x_2 = \dfrac{10}{3} + \dfrac{1}{15}\,x_3 - \dfrac{2}{15}\,x_4 \\
x_5 = -\dfrac{1}{3} - \dfrac{1}{15}\,x_3 + \dfrac{2}{15}\,x_4 \\
z = \dfrac{110}{3} - \dfrac{37}{15}\,x_3 - \dfrac{1}{15}\,x_4
\end{cases}$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$ and

– the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{16}{3}, \quad \bar{x}_2 = \frac{10}{3}, \quad \bar{x}_5 = -\frac{1}{3}, \qquad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad \frac{110}{3}$$

– the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_5 = 0, \qquad \bar{\pi}_3 = \frac{37}{15}, \quad \bar{\pi}_4 = \frac{1}{15} \quad \text{of value} \quad \frac{110}{3}$$

If we increase the value of π_5, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 = \frac{37}{15} + \frac{1}{15}\pi_5 \\ \pi_4 = \frac{1}{15} - \frac{2}{15}\pi_5 \end{cases} \qquad \begin{cases} \underbrace{\frac{37}{15} + \frac{1}{15}\pi_5 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{1}{15} - \frac{2}{15}\pi_5 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_5 \geq -37 \\ \pi_5 \leq \frac{1}{2} \end{cases}$$

By increasing the value of π_5 to $\frac{1}{2}$, the value of π_4 decreases to 0. Then we have that:

π_5 becomes a dual basic variable and accordingly x_5 becomes a primal non-basic variable.

π_4 becomes a dual non-basic variable and accordingly x_4 becomes a primal basic variable.

We have:

$$x_5 = -\frac{1}{3} - \frac{1}{15}x_3 + \frac{2}{15}x_4 \quad \Rightarrow \quad x_4 = \frac{5}{2} + \frac{1}{2}x_3 + \frac{15}{2}x_5$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{16}{3} - \frac{8}{15}x_3 + \frac{1}{15}\left(\frac{5}{2} + \frac{1}{2}x_3 + \frac{15}{2}x_5\right) = \frac{11}{2} - \frac{1}{2}x_3 + \frac{1}{2}x_5$$

$$x_2 = \frac{10}{3} + \frac{1}{15}x_3 - \frac{2}{15}\left(\frac{5}{2} + \frac{1}{2}x_3 + \frac{15}{2}x_5\right) = 3 - x_5$$

$$z = \frac{110}{3} - \frac{37}{15}x_3 - \frac{1}{15}\left(\frac{5}{2} + \frac{1}{2}x_3 + \frac{15}{2}x_5\right) = \frac{73}{2} - \frac{5}{2}x_3 - \frac{1}{2}x_5$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = \frac{11}{2} - \frac{1}{2}x_3 + \frac{1}{2}x_5 \\ x_2 = 3 - x_5 \\ x_4 = \frac{5}{2} + \frac{1}{2}x_3 + \frac{15}{2}x_5 \\ z = \frac{73}{2} - \frac{5}{2}x_3 - \frac{1}{2}x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4\}$ and $\mathcal{N} = \{3, 5\}$ and

– the basic primal optimal solution is:

$$\tilde{x}_1 = \frac{11}{2}, \quad \tilde{x}_2 = 3, \quad \tilde{x}_4 = \frac{5}{2}, \qquad \tilde{x}_3 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad \frac{73}{2}$$

– the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_4 = 0, \qquad \tilde{\pi}_3 = \frac{5}{2}, \quad \tilde{\pi}_5 = \frac{1}{2} \quad \text{of value} \quad \frac{73}{2}$$

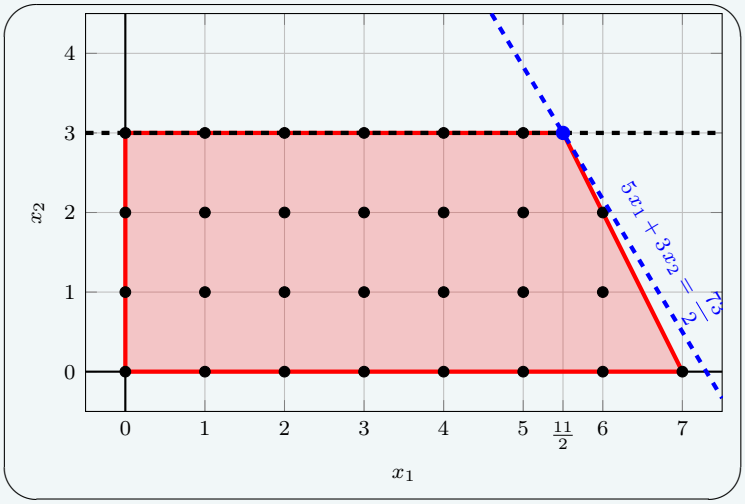


Figure 7.13: Node 2: feasible region of LP_2 (red), with the branching constraint $x_2 \leq 3$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is:

$$\tilde{x}_1 = \frac{11}{2}, \quad \tilde{x}_2 = 3$$

the optimal value is:

$$z(\mathrm{LP}_2) = \frac{73}{2}$$

The LP_2 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_2) \rfloor > \mathrm{LB} \qquad \text{since} \qquad \underbrace{\left\lfloor \frac{73}{2} \right\rfloor}_{=36} > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is **fractional**, accordingly, the LP_2 problem is **inserted in the problem list** L.

(b) Solution of the LP_3 problem

Let us consider the LP_3 problem in standard form:

$$\begin{array}{lrcrcrcrcl}
\max & 5\,x_1 & + & 3\,x_2 & & & & & & \\
& 2\,x_1 & + & x_2 & + & x_3 & & & = & 14 \\
& x_1 & + & 8\,x_2 & & & + & x_4 & = & 32 \\
& & & x_2 & & & & - & x_5 = & 4 \\
& x_1, & & x_2, & & x_3, & & x_4, & x_5 \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_2 - x_5 = 4 \qquad \Rightarrow \qquad x_5 = -4 + x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_5 = -4 + \left(\frac{10}{3} + \frac{1}{15}\,x_3 - \frac{2}{15}\,x_4\right) \qquad \Rightarrow \qquad x_5 = -\frac{2}{3} + \frac{1}{15}\,x_3 - \frac{2}{15}\,x_4$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 &= \frac{16}{3} - \frac{8}{15}\,x_3 + \frac{1}{15}\,x_4 \\ x_2 &= \frac{10}{3} + \frac{1}{15}\,x_3 - \frac{2}{15}\,x_4 \\ x_5 &= -\frac{2}{3} + \frac{1}{15}\,x_3 - \frac{2}{15}\,x_4 \\ z &= \frac{110}{3} - \frac{37}{15}\,x_3 - \frac{1}{15}\,x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{16}{3}, \quad \bar{x}_2 = \frac{10}{3}, \quad \bar{x}_5 = -\frac{2}{3}, \qquad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad \frac{110}{3}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_5 = 0, \qquad \bar{\pi}_3 = \frac{37}{15}, \quad \bar{\pi}_4 = \frac{1}{15} \quad \text{of value} \quad \frac{110}{3}$$

If we increase the value of π_5, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 &= \frac{37}{15} - \frac{1}{15}\,\pi_5 \\ \pi_4 &= \frac{1}{15} + \frac{2}{15}\,\pi_5 \end{cases} \qquad \begin{cases} \underbrace{\frac{37}{15} - \frac{1}{15}\,\pi_5 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{1}{15} + \frac{2}{15}\,\pi_5 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_5 &\leq 37 \\ \pi_5 &\geq -\frac{1}{2} \end{cases}$$

By increasing the value of π_5 to 37, the value of π_3 decreases to 0. Then we have that:

π_5 becomes a dual basic variable and accordingly x_5 becomes a primal non-basic variable.

π_3 becomes a dual non-basic variable and accordingly x_3 becomes a primal basic variable.

We have:

$$x_5 = -\frac{2}{3} + \frac{1}{15}\,x_3 - \frac{2}{15}\,x_4 \quad \Rightarrow \quad x_3 = 10 + 2\,x_4 + 15\,x_5$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{16}{3} - \frac{8}{15}\,(10 + 2\,x_4 + 15\,x_5) + \frac{1}{15}\,x_4 = 0 - x_4 - 8\,x_5$$

$$x_2 = \frac{10}{3} + \frac{1}{15}\,(10 + 2\,x_4 + 15\,x_5) - \frac{2}{15}\,x_4 = 4 + x_5$$

$$z = \frac{110}{3} - \frac{37}{15}\,(10 + 2\,x_4 + 15\,x_5) - \frac{1}{15}\,x_4 = 12 - 5\,x_4 - 37\,x_5$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 0 - x_4 - 8\,x_5 \\ x_2 = 4 + x_5 \\ x_3 = 10 + 2\,x_4 + 15\,x_5 \\ z = 12 - 5\,x_4 - 37\,x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = 4, \quad \tilde{x}_3 = 10, \qquad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad 12$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \qquad \tilde{\pi}_4 = 5, \quad \tilde{\pi}_5 = 37 \quad \text{of value} \quad 12$$

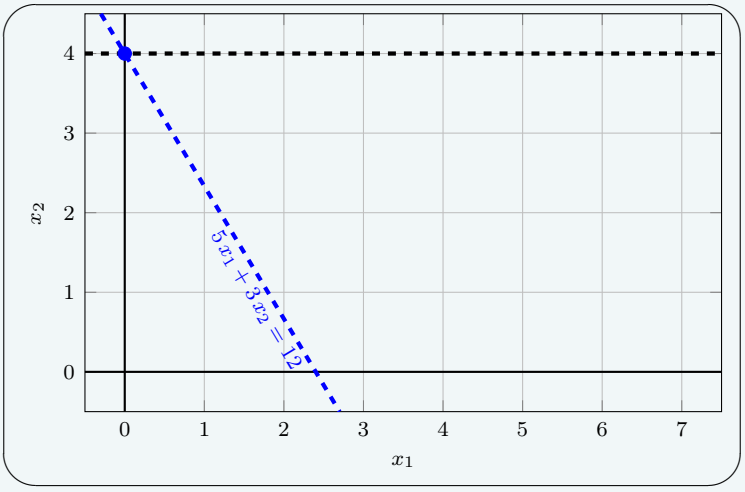


Figure 7.14: Node 3: feasible region of LP_3 (red), with the branching constraint $x_2 \geq 4$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_3)$ is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = 4$$

the optimal value is:

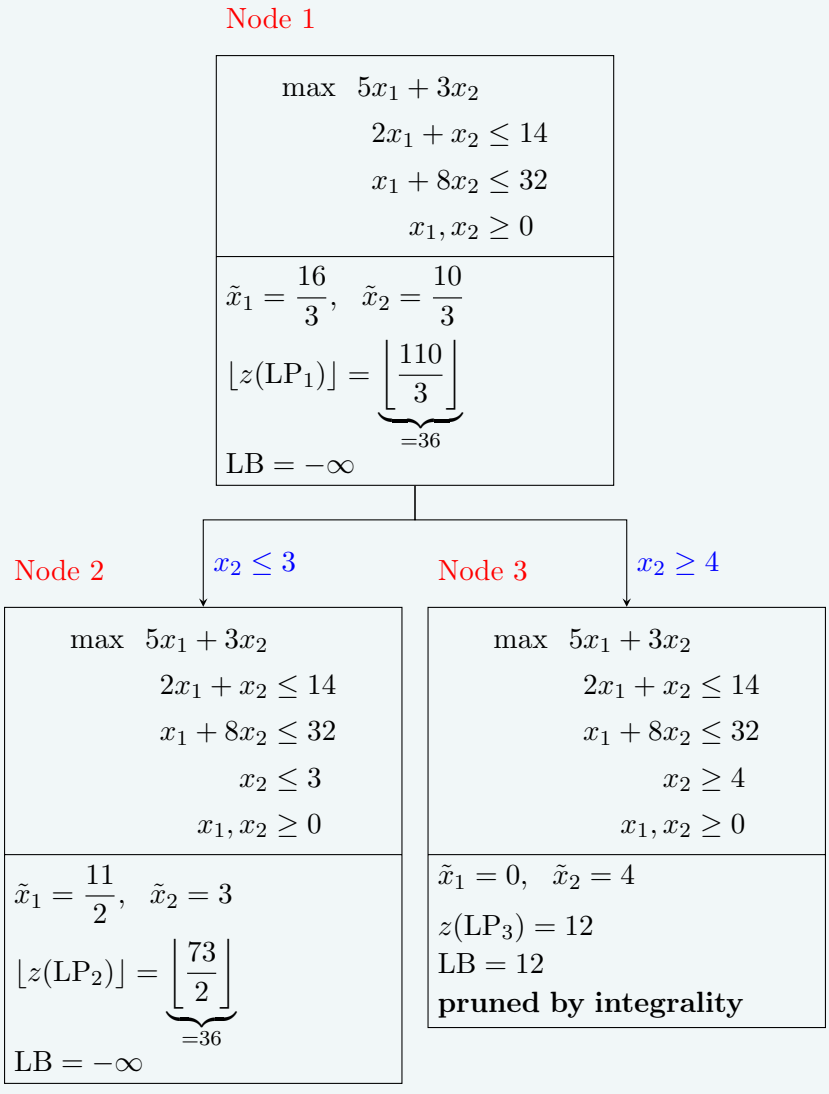
$$z(\text{LP}_3) = 12$$

The LP_3 problem is **feasible** and we have

$$\lfloor z(\text{LP}_3) \rfloor > \text{LB} \qquad \text{since} \qquad 12 > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_3)$ is **integer**, accordingly, the lower bound LB becomes 12 and the incumbent solution $\bar{\boldsymbol{x}}(\text{ILP})$ becomes $\bar{x}_1 = 0, \bar{x}_2 = 4$.
Accordingly, the branching node is pruned by integrality.

At the end of the first iteration, branching on x_2 from Node 1 has generated the down-branching node, Node 2, and the up-branching node, Node 3. Node 2 is inserted in the list L, while Node 3 is pruned by integrality. The corresponding branch-and-bound tree is:



Due to the current lower bound LB and upper bound UB, we have:

$$12 \quad \leq \quad z(\text{ILP}) \quad \leq \quad \underbrace{\left\lfloor \frac{73}{2} \right\rfloor}_{=36}$$

- **Iteration 2**

The LP_2 problem is extracted and deleted from the problem list L.

The branching variable is x_1. The branching constraints are:

$$x_1 \leq \underbrace{\left\lfloor \frac{11}{2} \right\rfloor}_{=5} \quad \text{and} \quad x_1 \geq \underbrace{\left\lfloor \frac{11}{2} \right\rfloor + 1}_{=6}$$

Two LP_4 and LP_5 problems are created with these branching constraints.

(a) Solution of the LP_4 problem

Let us consider the LP_4 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcr}
\max & 5\,x_1 & + & 3\,x_2 & & & & & & & & & & \\
& 2\,x_1 & + & x_2 & + & x_3 & & & & & & & = & 14 \\
& x_1 & + & 8\,x_2 & & & + & x_4 & & & & & = & 32 \\
& & & x_2 & & & & & + & x_5 & & & = & 3 \\
& x_1 & & & & & & & & & + & x_6 & = & 5 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_1 + x_6 = 5 \qquad \Rightarrow \qquad x_6 = 5 - x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_6 = 5 - \left(\frac{11}{2} - \frac{1}{2}\,x_3 + \frac{1}{2}\,x_5 \right) \qquad \Rightarrow \qquad x_6 = -\frac{1}{2} + \frac{1}{2}\,x_3 - \frac{1}{2}\,x_5$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = \frac{11}{2} - \frac{1}{2}x_3 + \frac{1}{2}x_5 \\ x_2 = 3 - x_5 \\ x_4 = \frac{5}{2} + \frac{1}{2}x_3 + \frac{15}{2}x_5 \\ x_6 = -\frac{1}{2} + \frac{1}{2}x_3 - \frac{1}{2}x_5 \\ z = \frac{73}{2} - \frac{5}{2}x_3 - \frac{1}{2}x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4, 6\}$ and $\mathcal{N} = \{3, 5\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{11}{2}, \quad \bar{x}_2 = 3, \quad \bar{x}_4 = \frac{5}{2}, \quad \bar{x}_6 = -\frac{1}{2}, \\ \bar{x}_3 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{73}{2}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_6 = 0, \\ \bar{\pi}_3 = \frac{5}{2}, \quad \bar{\pi}_5 = \frac{1}{2} \quad \text{of value} \quad \frac{73}{2}$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 = \frac{5}{2} - \frac{1}{2}\pi_6 \\ \pi_5 = \frac{1}{2} + \frac{1}{2}\pi_6 \end{cases} \qquad \begin{cases} \underbrace{\frac{5}{2} - \frac{1}{2}\pi_6 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{1}{2} + \frac{1}{2}\pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 \leq 5 \\ \pi_6 \geq -1 \end{cases}$$

By increasing the value of π_6 to 5, the value of π_3 decreases to 0. Then we have that:

> π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non-basic variable.
>
> π_3 becomes a dual non-basic variable and accordingly x_3 becomes a primal basic variable.

We have:

$$x_6 = -\frac{1}{2} + \frac{1}{2}x_3 - \frac{1}{2}x_5 \quad \Rightarrow \quad x_3 = 1 + x_5 + 2x_6$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{11}{2} - \frac{1}{2}(1 + x_5 + 2x_6) + \frac{1}{2}x_5 = 5 - x_6$$

$$x_4 = \frac{5}{2} + \frac{1}{2}(1 + x_5 + 2\,x_6) + \frac{15}{2}\,x_5 = 3 + 8\,x_5 + x_6$$
$$z = \frac{73}{2} - \frac{5}{2}(1 + x_5 + 2\,x_6) - \frac{1}{2}\,x_5 = 34 - 3\,x_5 - 5\,x_6$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 5 - x_6 \\ x_2 = 3 - x_5 \\ x_3 = 1 + x_5 + 2\,x_6 \\ x_4 = 3 + 8\,x_5 + x_6 \\ z = 34 - 3\,x_5 - 5\,x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4\}$ and $\mathcal{N} = \{5, 6\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 5, \quad \tilde{x}_2 = 3, \quad \tilde{x}_3 = 1, \quad \tilde{x}_4 = 3,$$
$$\tilde{x}_5 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad 34$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 0,$$
$$\tilde{\pi}_5 = 3, \quad \tilde{\pi}_6 = 5 \quad \text{of value} \quad 34$$

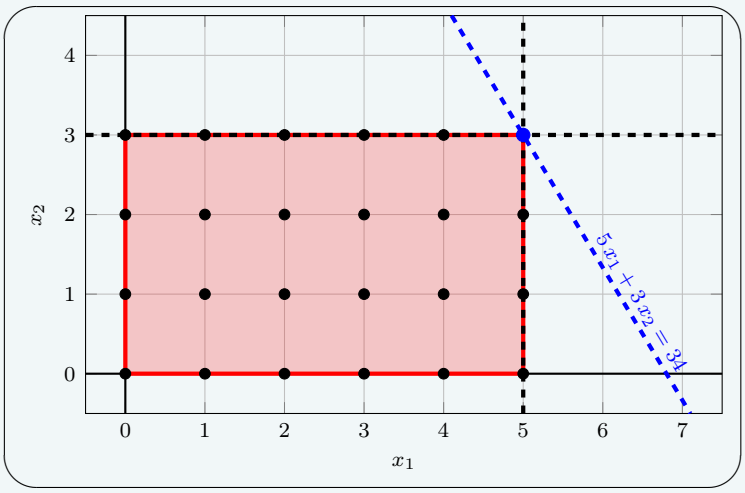


Figure 7.15: Node 4: feasible region of LP_4 (red), with the branching constraint $x_1 \leq 5$ (dashed), and its optimal solution (blue).
An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_4)$ is:

$$\tilde{x}_1 = 5, \quad \tilde{x}_2 = 3$$

the optimal value is:

$$z(\mathrm{LP}_4) = 34$$

The LP_4 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_4) \rfloor > \mathrm{LB} \qquad \text{since} \qquad 34 > 12$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_4)$ is **integer**, accordingly, the lower bound LB becomes 34 and the incumbent solution $\bar{\boldsymbol{x}}(\mathrm{ILP})$ becomes $\bar{x}_1 = 5, \bar{x}_2 = 3$.
Accordingly, the branching node is pruned by integrality.

(b) Solution of the LP_5 problem

Let us consider the LP_5 problem in standard form:

$$\begin{array}{rrrrrrrrrrrrrr}
\max & 5\,x_1 & + & 3\,x_2 & & & & & & & & & & \\
 & 2\,x_1 & + & x_2 & + & x_3 & & & & & & & = & 14 \\
 & x_1 & + & 8\,x_2 & & & + & x_4 & & & & & = & 32 \\
 & & & x_2 & & & & & + & x_5 & & & = & 3 \\
 & x_1 & & & & & & & & & - & x_6 & = & 6 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_1 - x_6 = 6 \qquad \Rightarrow \qquad x_6 = -6 + x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_6 = -6 + \left(\frac{11}{2} - \frac{1}{2}\,x_3 + \frac{1}{2}\,x_5\right) \qquad \Rightarrow \qquad x_6 = -\frac{1}{2} - \frac{1}{2}\,x_3 + \frac{1}{2}\,x_5$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases}
x_1 = \dfrac{11}{2} - \dfrac{1}{2}\,x_3 + \dfrac{1}{2}\,x_5 \\
x_2 = 3 - x_5 \\
x_4 = \dfrac{5}{2} + \dfrac{1}{2}\,x_3 + \dfrac{15}{2}\,x_5 \\
x_6 = -\dfrac{1}{2} - \dfrac{1}{2}\,x_3 + \dfrac{1}{2}\,x_5 \\
z = \dfrac{73}{2} - \dfrac{5}{2}\,x_3 - \dfrac{1}{2}\,x_5
\end{cases}$$

where $\mathcal{B} = \{1, 2, 4, 6\}$ and $\mathcal{N} = \{3, 5\}$ and

– the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{11}{2}, \quad \bar{x}_2 = 3, \quad \bar{x}_4 = \frac{5}{2}, \quad \bar{x}_6 = -\frac{1}{2},$$
$$\bar{x}_3 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{73}{2}$$

– the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_6 = 0,$$
$$\bar{\pi}_3 = \frac{5}{2}, \quad \bar{\pi}_5 = \frac{1}{2} \quad \text{of value} \quad \frac{73}{2}$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 = \frac{5}{2} + \frac{1}{2}\pi_6 \\ \pi_5 = \frac{1}{2} - \frac{1}{2}\pi_6 \end{cases} \qquad \begin{cases} \underbrace{\frac{5}{2} + \frac{1}{2}\pi_6 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{1}{2} - \frac{1}{2}\pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 \geq -5 \\ \pi_6 \leq 1 \end{cases}$$

By increasing the value of π_6 to 1, the value of π_5 decreases to 0. Then we have that:

π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non-basic variable.

π_5 becomes a dual non-basic variable and accordingly x_5 becomes a primal basic variable.

We have:

$$x_6 = -\frac{1}{2} - \frac{1}{2}x_3 + \frac{1}{2}x_5 \quad \Rightarrow \quad x_5 = 1 + x_3 + 2x_6$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{11}{2} - \frac{1}{2}x_3 + \frac{1}{2}(1 + x_3 + 2x_6) = 6 + x_6$$
$$x_2 = 3 - (1 + x_3 + 2x_6) = 2 - x_3 - 2x_6$$
$$x_4 = \frac{5}{2} + \frac{1}{2}x_3 + \frac{15}{2}(1 + x_3 + 2x_6) = 10 + 8x_3 + 15x_6$$
$$z = \frac{73}{2} - \frac{5}{2}x_3 - \frac{1}{2}(1 + x_3 + 2x_6) = 36 - 3x_3 - x_6$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 6 + x_6 \\ x_2 = 2 - x_3 - 2x_6 \\ x_4 = 10 + 8x_3 + 15x_6 \\ x_5 = 1 + x_3 + 2x_6 \\ z = 36 - 3x_3 - x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4, 5\}$ and $\mathcal{N} = \{3, 6\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 6, \quad \tilde{x}_2 = 2, \quad \tilde{x}_4 = 10, \quad \tilde{x}_5 = 1,$$
$$\tilde{x}_3 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad 36$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_4 = 0, \quad \tilde{\pi}_5 = 0,$$
$$\tilde{\pi}_3 = 3, \quad \tilde{\pi}_6 = 1 \quad \text{of value} \quad 36$$

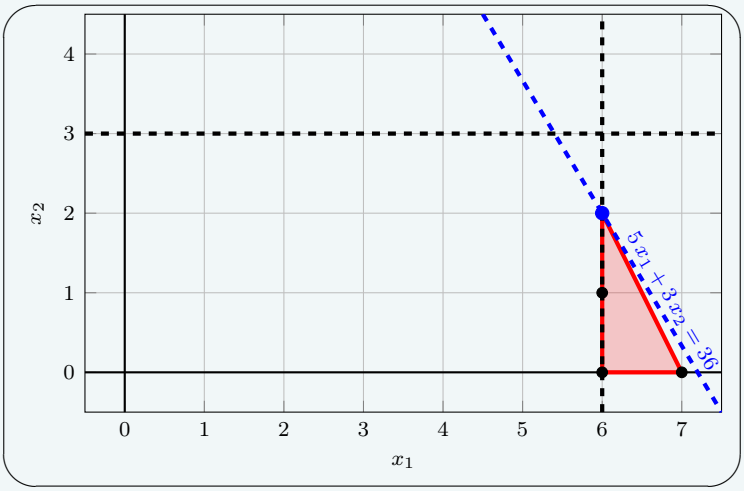


Figure 7.16: Node 5: feasible region of LP_5 (red), with the branching constraint $x_1 \geq 6$ (dashed), and its optimal solution (blue).
An optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_5)$ is:

$$\tilde{x}_1 = 6, \quad \tilde{x}_2 = 2$$

the optimal value is:

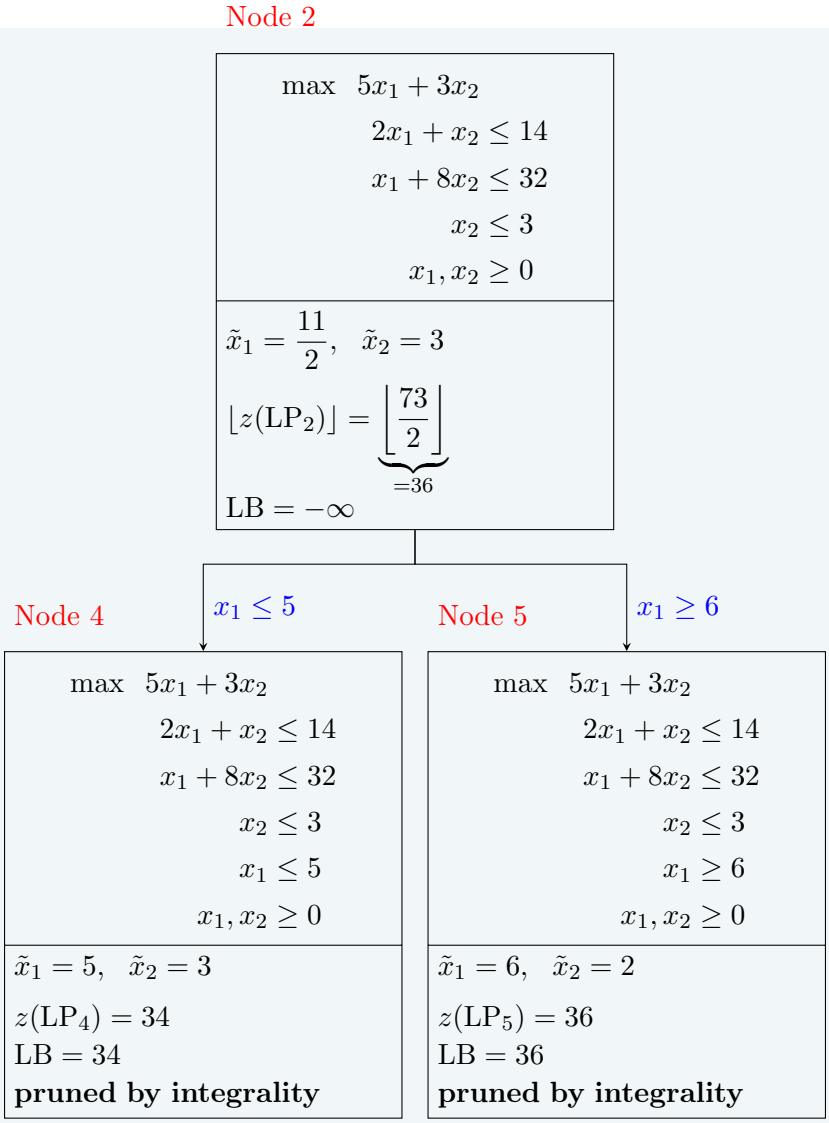
$$z(\text{LP}_5) = 36$$

The LP_5 problem is **feasible** and we have

$$\lfloor z(\text{LP}_5) \rfloor > \text{LB} \qquad \text{since} \qquad 36 > 34$$

The optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_5)$ is **integer**, accordingly, the lower bound LB becomes 36 and the incumbent solution $\bar{\boldsymbol{x}}(\text{ILP})$ becomes $\bar{x}_1 = 6, \bar{x}_2 = 2$.
Accordingly, the branching node is pruned by integrality.

At the end of the second iteration, branching on x_1 from Node 2 has generated the down-branching node, Node 4, and the up-branching node, Node 5. Both nodes are pruned by integrality. The corresponding branch-and-bound tree is:



Due to the current lower bound LB and upper bound UB, we have:

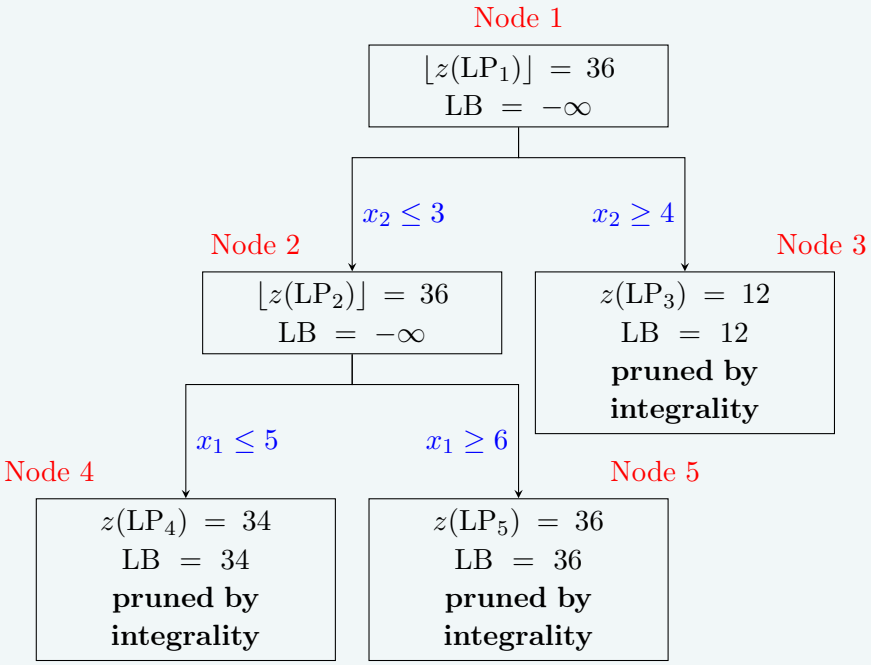
$$36 \quad \leq \quad z(\text{ILP}) \quad \leq \quad 36$$

- **Termination**

Since the list L is empty, the lower bound LB is equal to the upper bound UB and the ILP problem is solved to optimality.

The optimal value is $z(\text{ILP}) = 36$ (the value of the lower bound LB) and an optimal solution $\tilde{\boldsymbol{x}}(\text{ILP})$ is $\tilde{x}_1 = 6, \tilde{x}_2 = 2$ (the incumbent solution $\bar{\boldsymbol{x}}(\text{ILP})$).

2. The final branch-and-bound tree is:



The reasons why the branching nodes are pruned are the following.

- Node 1 has a fractional optimal solution with $\lfloor z(\text{LP}_1) \rfloor = 36$: it is branched on x_2.
- Node 2 has a fractional optimal solution with $\lfloor z(\text{LP}_2) \rfloor = 36$: it is branched on x_1.
- Node 3 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_3) = (0, 4)$ is integer, with value 12, which becomes the incumbent solution.
- Node 4 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_4) = (5, 3)$ is integer, with value 34, which becomes the incumbent solution.
- Node 5 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_5) = (6, 2)$ is integer, with value 36, which becomes the incumbent solution.

The optimal value is $z(\text{ILP}) = 36$ and the branch-and-bound algorithm returns the optimal solution $\tilde{\boldsymbol{x}}(\text{ILP}) = (6, 2)$.

Exercise 7.5 — Branch-and-bound algorithm ($P5$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{array}{rrcrcr} \max & 2\ x_1 & + & 3\ x_2 & & \\ & 6\ x_1 & + & 21\ x_2 & \leq & 77 \\ & 8\ x_1 & + & 3\ x_2 & \leq & 36 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

and an optimal simplex dictionary of its LP relaxation in standard form:

$$\begin{cases} x_1 = \frac{7}{2} + \frac{1}{50}\ x_3 - \frac{7}{50}\ x_4 \\ x_2 = \frac{8}{3} - \frac{4}{75}\ x_3 + \frac{1}{25}\ x_4 \\ z = 15 - \frac{3}{25}\ x_3 - \frac{4}{25}\ x_4 \end{cases}$$

Questions

1. Execute the branch-and-bound algorithm using:
   (a) the smallest-index rule as branching-variable selection strategy;
   (b) the LIFO rule as branching-node selection strategy;
   (c) solve first the down-branching node and then the up-branching node;
   (d) set the lower bound LB to $-\infty$ (no heuristic algorithm is executed).

   Solve the LP problems of the branching nodes using the dual simplex algorithm. In the dual simplex algorithm, the leaving variable is the basic variable with negative value and smallest index, and ties in the ratio test are broken by the smallest index. Show graphically the optimal solution of the LP problems associated with the branching nodes.
2. Determine and comment the branch-and-bound tree, showing the reason why each branching node is pruned.

Solution

1. Execution of the branch-and-bound algorithm to solve the ILP problem.
   - **Initialization**

     Let us consider the LP_1 problem in standard form:

$$\begin{array}{lrcrcrcrcl} \max & 2\,x_1 & + & 3\,x_2 & & & & & & \\ & 6\,x_1 & + & 21\,x_2 & + & x_3 & & & = & 77 \\ & 8\,x_1 & + & 3\,x_2 & & & + & x_4 & = & 36 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

and its optimal simplex dictionary:

$$\begin{cases} x_1 = \frac{7}{2} + \frac{1}{50}\,x_3 - \frac{7}{50}\,x_4 \\ x_2 = \frac{8}{3} - \frac{4}{75}\,x_3 + \frac{1}{25}\,x_4 \\ z = 15 - \frac{3}{25}\,x_3 - \frac{4}{25}\,x_4 \end{cases}$$

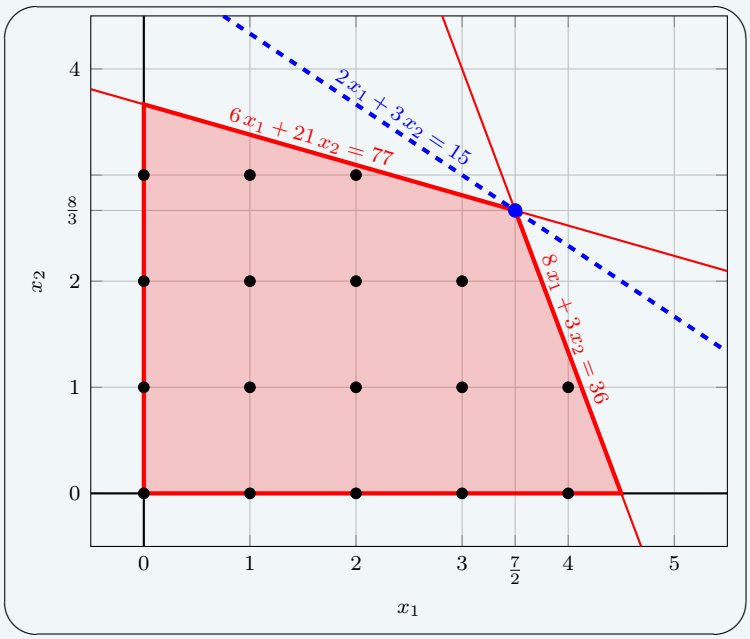


Figure 7.17: Node 1: feasible region of LP_1 (red), feasible points of the ILP problem (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is:

$$\tilde{x}_1 = \frac{7}{2}, \quad \tilde{x}_2 = \frac{8}{3}$$

the optimal value is:

$$z(\mathrm{LP}_1) = 15$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is **fractional** and the LP_1 problem is **feasible**, so the LP_1 problem is inserted in the problem list L and the lower bound LB is set to $-\infty$.

Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \leq z(\mathrm{ILP}) \leq 15$$

- **Iteration 1**

The LP_1 problem is extracted and deleted from the problem list L.

The branching variable is x_1. The branching constraints are:

$$x_1 \leq \underbrace{\left\lfloor \frac{7}{2} \right\rfloor}_{=3} \quad \text{and} \quad x_1 \geq \underbrace{\left\lfloor \frac{7}{2} \right\rfloor + 1}_{=4}$$

Two LP_2 and LP_3 problems are created with these branching constraints.

(a) Solution of the LP_2 problem

Let us consider the LP_2 problem in standard form:

$$\begin{array}{lrcrcrcrcrcl} \max & 2\,x_1 & + & 3\,x_2 & & & & & & & & \\ & 6\,x_1 & + & 21\,x_2 & + & x_3 & & & & & = & 77 \\ & 8\,x_1 & + & 3\,x_2 & & & + & x_4 & & & = & 36 \\ & x_1 & & & & & & & + & x_5 & = & 3 \\ & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0 \end{array}$$

For the branching constraint we have:

$$x_1 + x_5 = 3 \quad \Rightarrow \quad x_5 = 3 - x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_5 = 3 - \left(\frac{7}{2} + \frac{1}{50}\, x_3 - \frac{7}{50}\, x_4 \right) \quad \Rightarrow \quad x_5 = -\frac{1}{2} - \frac{1}{50}\, x_3 + \frac{7}{50}\, x_4$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = \frac{7}{2} + \frac{1}{50}\, x_3 - \frac{7}{50}\, x_4 \\ x_2 = \frac{8}{3} - \frac{4}{75}\, x_3 + \frac{1}{25}\, x_4 \\ x_5 = -\frac{1}{2} - \frac{1}{50}\, x_3 + \frac{7}{50}\, x_4 \\ z = 15 - \frac{3}{25}\, x_3 - \frac{4}{25}\, x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$ and

– the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{7}{2}, \quad \bar{x}_2 = \frac{8}{3}, \quad \bar{x}_5 = -\frac{1}{2}, \qquad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad 15$$

– the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_5 = 0, \qquad \bar{\pi}_3 = \frac{3}{25}, \quad \bar{\pi}_4 = \frac{4}{25} \quad \text{of value} \quad 15$$

If we increase the value of π_5, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 &= \dfrac{3}{25} + \dfrac{1}{50}\,\pi_5 \\ \pi_4 &= \dfrac{4}{25} - \dfrac{7}{50}\,\pi_5 \end{cases} \qquad \begin{cases} \underbrace{\dfrac{3}{25} + \dfrac{1}{50}\,\pi_5 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\dfrac{4}{25} - \dfrac{7}{50}\,\pi_5 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_5 &\geq -6 \\ \pi_5 &\leq \dfrac{8}{7} \end{cases}$$

By increasing the value of π_5 to $\frac{8}{7}$, the value of π_4 decreases to 0. Then we have that:

π_5 becomes a dual basic variable and accordingly x_5 becomes a primal non-basic variable.

π_4 becomes a dual non-basic variable and accordingly x_4 becomes a primal basic variable.

We have:

$$x_5 = -\frac{1}{2} - \frac{1}{50}\,x_3 + \frac{7}{50}\,x_4 \quad \Rightarrow \quad x_4 = \frac{25}{7} + \frac{1}{7}\,x_3 + \frac{50}{7}\,x_5$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{7}{2} + \frac{1}{50}\,x_3 - \frac{7}{50}\left(\frac{25}{7} + \frac{1}{7}\,x_3 + \frac{50}{7}\,x_5\right) = 3 - x_5$$

$$x_2 = \frac{8}{3} - \frac{4}{75}\,x_3 + \frac{1}{25}\left(\frac{25}{7} + \frac{1}{7}\,x_3 + \frac{50}{7}\,x_5\right) = \frac{59}{21} - \frac{1}{21}\,x_3 + \frac{2}{7}\,x_5$$

$$z = 15 - \frac{3}{25}\,x_3 - \frac{4}{25}\left(\frac{25}{7} + \frac{1}{7}\,x_3 + \frac{50}{7}\,x_5\right) = \frac{101}{7} - \frac{1}{7}\,x_3 - \frac{8}{7}\,x_5$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 &= 3 &&- x_5 \\ x_2 &= \dfrac{59}{21} - \dfrac{1}{21}\,x_3 &&+ \dfrac{2}{7}\,x_5 \\ x_4 &= \dfrac{25}{7} + \dfrac{1}{7}\,x_3 &&+ \dfrac{50}{7}\,x_5 \\ z &= \dfrac{101}{7} - \dfrac{1}{7}\,x_3 &&- \dfrac{8}{7}\,x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4\}$ and $\mathcal{N} = \{3, 5\}$ and

– the basic primal optimal solution is:

$$\tilde{x}_1 = 3, \quad \tilde{x}_2 = \frac{59}{21}, \quad \tilde{x}_4 = \frac{25}{7}, \qquad \tilde{x}_3 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad \frac{101}{7}$$

– the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_4 = 0, \qquad \tilde{\pi}_3 = \frac{1}{7}, \quad \tilde{\pi}_5 = \frac{8}{7} \quad \text{of value} \quad \frac{101}{7}$$

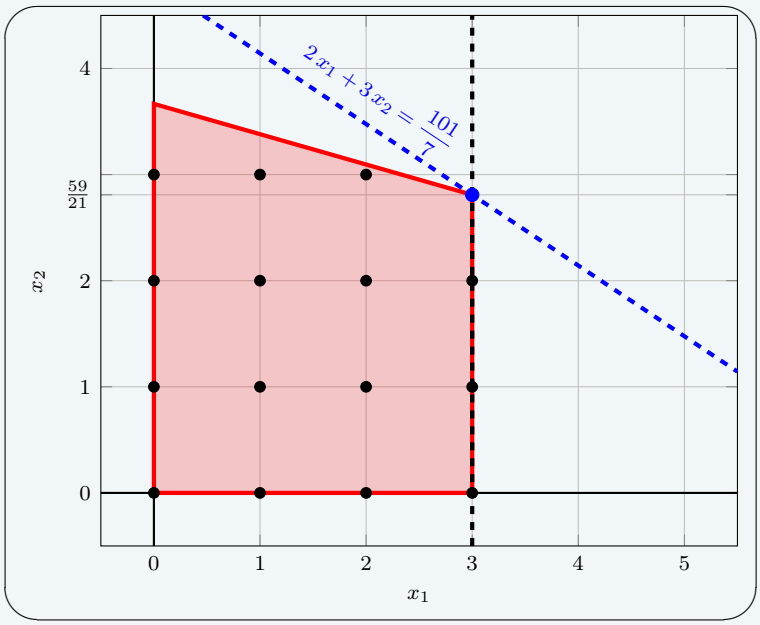


Figure 7.18: Node 2: feasible region of LP_2 (red), with the branching constraint $x_1 \leq 3$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is:

$$\tilde{x}_1 = 3, \quad \tilde{x}_2 = \frac{59}{21}$$

the optimal value is:

$$z(\mathrm{LP}_2) = \frac{101}{7}$$

The LP_2 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_2) \rfloor > \mathrm{LB} \qquad \text{since} \qquad \underbrace{\left\lfloor \frac{101}{7} \right\rfloor}_{=14} > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is **fractional**, accordingly, the LP_2 problem is **inserted in the problem list** L.

(b) Solution of the LP_3 problem

Let us consider the LP_3 problem in standard form:

$$
\begin{array}{lrcrcrcrcl}
\max & 2\,x_1 & + & 3\,x_2 & & & & & & \\
& 6\,x_1 & + & 21\,x_2 & + & x_3 & & & = & 77 \\
& 8\,x_1 & + & 3\,x_2 & & & + & x_4 & = & 36 \\
& x_1 & & & & & & - & x_5 = & 4 \\
& x_1, & & x_2, & & x_3, & & x_4, & x_5 \geq & 0
\end{array}
$$

For the branching constraint we have:

$$x_1 - x_5 = 4 \quad \Rightarrow \quad x_5 = -4 + x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_5 = -4 + \left(\frac{7}{2} + \frac{1}{50}\,x_3 - \frac{7}{50}\,x_4\right) \quad \Rightarrow \quad x_5 = -\frac{1}{2} + \frac{1}{50}\,x_3 - \frac{7}{50}\,x_4$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$
\begin{cases}
x_1 = \frac{7}{2} + \frac{1}{50}\,x_3 - \frac{7}{50}\,x_4 \\
x_2 = \frac{8}{3} - \frac{4}{75}\,x_3 + \frac{1}{25}\,x_4 \\
x_5 = -\frac{1}{2} + \frac{1}{50}\,x_3 - \frac{7}{50}\,x_4 \\
z = 15 - \frac{3}{25}\,x_3 - \frac{4}{25}\,x_4
\end{cases}
$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{7}{2}, \quad \bar{x}_2 = \frac{8}{3}, \quad \bar{x}_5 = -\frac{1}{2}, \qquad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad 15$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_5 = 0, \qquad \bar{\pi}_3 = \frac{3}{25}, \quad \bar{\pi}_4 = \frac{4}{25} \quad \text{of value} \quad 15$$

If we increase the value of π_5, we can get a better dual solution. We have:

$$
\begin{cases}
\pi_3 = \frac{3}{25} - \frac{1}{50}\,\pi_5 \\
\pi_4 = \frac{4}{25} + \frac{7}{50}\,\pi_5
\end{cases}
\qquad
\begin{cases}
\underbrace{\frac{3}{25} - \frac{1}{50}\,\pi_5 \geq 0}_{\pi_3 \geq 0} \\
\underbrace{\frac{4}{25} + \frac{7}{50}\,\pi_5 \geq 0}_{\pi_4 \geq 0}
\end{cases}
\qquad
\begin{cases}
\pi_5 \leq 6 \\
\pi_5 \geq -\frac{8}{7}
\end{cases}
$$

By increasing the value of π_5 to 6, the value of π_3 decreases to 0. Then we have that:

π_5 becomes a dual basic variable and accordingly x_5 becomes a primal non-basic variable.

π_3 becomes a dual non-basic variable and accordingly x_3 becomes a primal basic variable.

We have:

$$x_5 = -\frac{1}{2} + \frac{1}{50}\,x_3 - \frac{7}{50}\,x_4 \quad \Rightarrow \quad x_3 = 25 + 7\,x_4 + 50\,x_5$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{7}{2} + \frac{1}{50}\,(25 + 7\,x_4 + 50\,x_5) - \frac{7}{50}\,x_4 = 4 + x_5$$

$$x_2 = \frac{8}{3} - \frac{4}{75}\,(25 + 7\,x_4 + 50\,x_5) + \frac{1}{25}\,x_4 = \frac{4}{3} - \frac{1}{3}\,x_4 - \frac{8}{3}\,x_5$$

$$z = 15 - \frac{3}{25}\,(25 + 7\,x_4 + 50\,x_5) - \frac{4}{25}\,x_4 = 12 - x_4 - 6\,x_5$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 4 + x_5 \\ x_2 = \frac{4}{3} - \frac{1}{3}\,x_4 - \frac{8}{3}\,x_5 \\ x_3 = 25 + 7\,x_4 + 50\,x_5 \\ z = 12 - x_4 - 6\,x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = \frac{4}{3}, \quad \tilde{x}_3 = 25, \qquad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad 12$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \qquad \tilde{\pi}_4 = 1, \quad \tilde{\pi}_5 = 6 \quad \text{of value} \quad 12$$

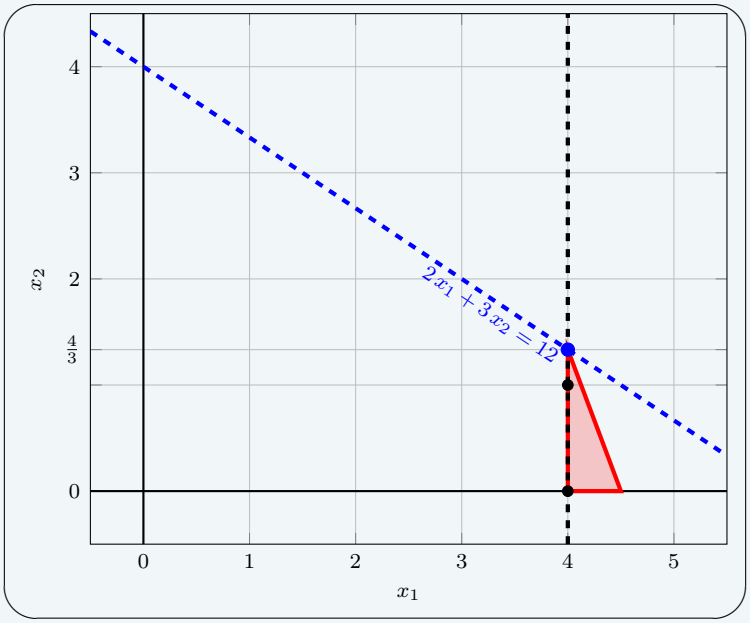


Figure 7.19: Node 3: feasible region of LP_3 (red), with the branching constraint $x_1 \geq 4$ (dashed), and its optimal solution (blue).
An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_3)$ is:

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = \frac{4}{3}$$

the optimal value is:

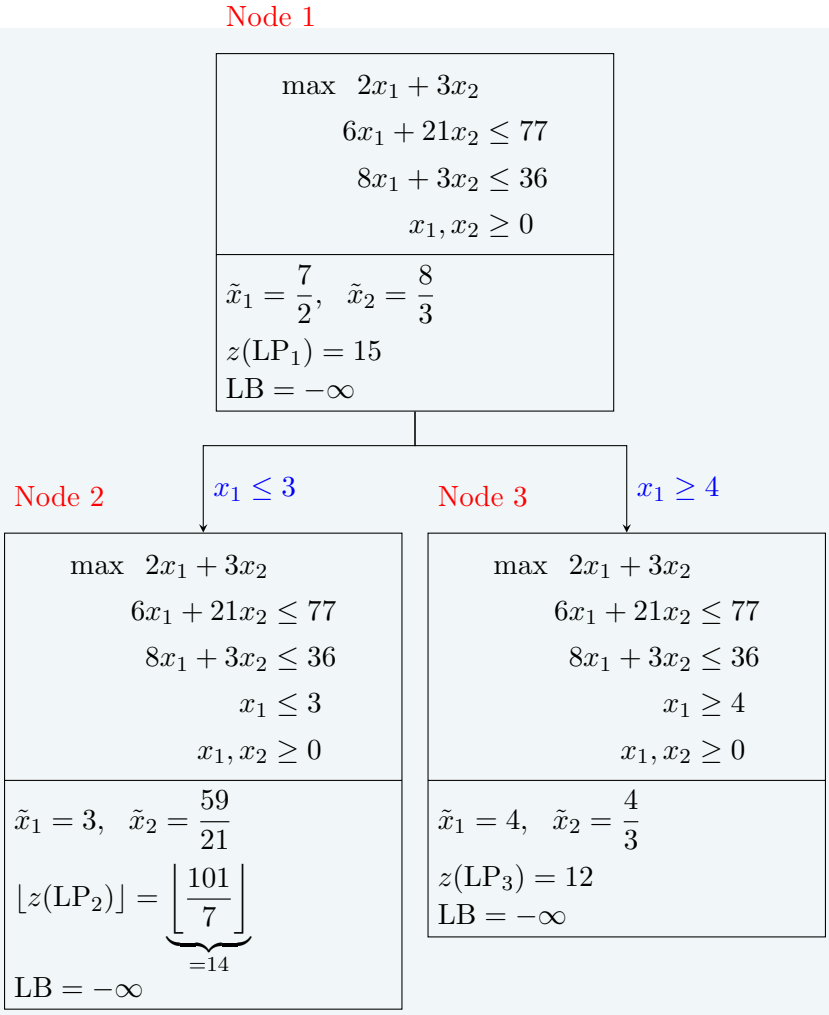
$$z(\mathrm{LP}_3) = 12$$

The LP_3 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_3) \rfloor > \mathrm{LB} \qquad \text{since} \qquad 12 > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_3)$ is **fractional**, accordingly, the LP_3 problem is **inserted in the problem list** L.

At the end of the first iteration, branching on x_1 from Node 1 has generated the down-branching node, Node 2, and the up-branching node, Node 3. Both nodes are inserted in the list L. The corresponding branch-and-bound tree is:



Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \quad \leq \quad z(\text{ILP}) \quad \leq \quad \underbrace{\left\lfloor \frac{101}{7} \right\rfloor}_{=14}$$

- **Iteration 2**

The LP_3 problem is extracted and deleted from the problem list L.

The branching variable is x_2. The branching constraints are:

$$x_2 \leq \underbrace{\left\lfloor \frac{4}{3} \right\rfloor}_{=1} \quad \text{and} \quad x_2 \geq \underbrace{\left\lfloor \frac{4}{3} \right\rfloor + 1}_{=2}$$

Two LP_4 and LP_5 problems are created with these branching constraints.

(a) Solution of the LP_4 problem

Let us consider the LP_4 problem in standard form:

$$
\begin{array}{lrcrcrcrcrcl}
\max & 2\,x_1 & + & 3\,x_2 & & & & & & & & \\
 & 6\,x_1 & + & 21\,x_2 & + & x_3 & & & & & = & 77 \\
 & 8\,x_1 & + & 3\,x_2 & & & + & x_4 & & & = & 36 \\
 & x_1 & & & & & & & - & x_5 & = & 4 \\
 & & & x_2 & & & & & & & + \; x_6 \; = & 1 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & x_6 \; \geq & 0
\end{array}
$$

For the branching constraint we have:

$$x_2 + x_6 = 1 \qquad \Rightarrow \qquad x_6 = 1 - x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_6 = 1 - \left(\frac{4}{3} - \frac{1}{3}\, x_4 - \frac{8}{3}\, x_5 \right) \qquad \Rightarrow \qquad x_6 = -\frac{1}{3} + \frac{1}{3}\, x_4 + \frac{8}{3}\, x_5$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$
\begin{cases}
x_1 = 4 + x_5 \\
x_2 = \frac{4}{3} - \frac{1}{3}\, x_4 - \frac{8}{3}\, x_5 \\
x_3 = 25 + 7\, x_4 + 50\, x_5 \\
x_6 = -\frac{1}{3} + \frac{1}{3}\, x_4 + \frac{8}{3}\, x_5 \\
z = 12 - x_4 - 6\, x_5
\end{cases}
$$

where $\mathcal{B} = \{1, 2, 3, 6\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = 4, \quad \bar{x}_2 = \frac{4}{3}, \quad \bar{x}_3 = 25, \quad \bar{x}_6 = -\frac{1}{3},$$
$$\bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad 12$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_6 = 0,$$
$$\bar{\pi}_4 = 1, \quad \bar{\pi}_5 = 6 \quad \text{of value} \quad 12$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_4 &= 1 - \frac{1}{3}\pi_6 \\ \pi_5 &= 6 - \frac{8}{3}\pi_6 \end{cases} \qquad \begin{cases} \underbrace{1 - \frac{1}{3}\pi_6 \geq 0}_{\pi_4 \geq 0} \\ \underbrace{6 - \frac{8}{3}\pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 &\leq 3 \\ \pi_6 &\leq \frac{9}{4} \end{cases}$$

By increasing the value of π_6 to $\min\left\{3, \frac{9}{4}\right\} = \frac{9}{4}$, the value of π_5 decreases to 0. Then we have that:

π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non-basic variable.

π_5 becomes a dual non-basic variable and accordingly x_5 becomes a primal basic variable.

We have:

$$x_6 = -\frac{1}{3} + \frac{1}{3}x_4 + \frac{8}{3}x_5 \quad \Rightarrow \quad x_5 = \frac{1}{8} - \frac{1}{8}x_4 + \frac{3}{8}x_6$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = 4 + \left(\frac{1}{8} - \frac{1}{8}x_4 + \frac{3}{8}x_6\right) = \frac{33}{8} - \frac{1}{8}x_4 + \frac{3}{8}x_6$$

$$x_2 = \frac{4}{3} - \frac{1}{3}x_4 - \frac{8}{3}\left(\frac{1}{8} - \frac{1}{8}x_4 + \frac{3}{8}x_6\right) = 1 - x_6$$

$$x_3 = 25 + 7\,x_4 + 50\left(\frac{1}{8} - \frac{1}{8}x_4 + \frac{3}{8}x_6\right) = \frac{125}{4} + \frac{3}{4}x_4 + \frac{75}{4}x_6$$

$$z = 12 - x_4 - 6\left(\frac{1}{8} - \frac{1}{8}x_4 + \frac{3}{8}x_6\right) = \frac{45}{4} - \frac{1}{4}x_4 - \frac{9}{4}x_6$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 &= \frac{33}{8} - \frac{1}{8}x_4 + \frac{3}{8}x_6 \\ x_2 &= 1 - x_6 \\ x_3 &= \frac{125}{4} + \frac{3}{4}x_4 + \frac{75}{4}x_6 \\ x_5 &= \frac{1}{8} - \frac{1}{8}x_4 + \frac{3}{8}x_6 \\ z &= \frac{45}{4} - \frac{1}{4}x_4 - \frac{9}{4}x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 5\}$ and $\mathcal{N} = \{4, 6\}$ and

– the basic primal optimal solution is:

$$\tilde{x}_1 = \frac{33}{8}, \quad \tilde{x}_2 = 1, \quad \tilde{x}_3 = \frac{125}{4}, \quad \tilde{x}_5 = \frac{1}{8},$$
$$\tilde{x}_4 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad \frac{45}{4}$$

– the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_5 = 0,$$
$$\tilde{\pi}_4 = \frac{1}{4}, \quad \tilde{\pi}_6 = \frac{9}{4} \quad \text{of value} \quad \frac{45}{4}$$

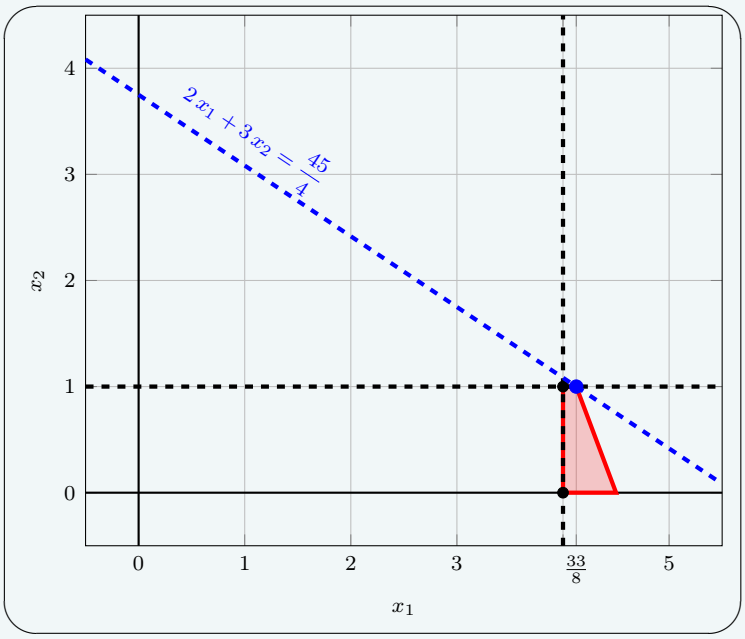


Figure 7.20: Node 4: feasible region of LP_4 (red), with the branching constraint $x_2 \leq 1$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_4)$ is:

$$\tilde{x}_1 = \frac{33}{8}, \quad \tilde{x}_2 = 1$$

the optimal value is:

$$z(\mathrm{LP}_4) = \frac{45}{4}$$

The LP_4 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_4) \rfloor > \mathrm{LB} \qquad \text{since} \qquad \underbrace{\left\lfloor \frac{45}{4} \right\rfloor}_{=11} > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_4)$ is **fractional**, accordingly, the LP_4 problem is **inserted in the problem list** L.

(b) Solution of the LP_5 problem

Let us consider the LP_5 problem in standard form:

$$
\begin{array}{lrcrcrcrcrcl}
\max & 2\,x_1 & + & 3\,x_2 & & & & & & & & \\
 & 6\,x_1 & + & 21\,x_2 & + & x_3 & & & & & = & 77 \\
 & 8\,x_1 & + & 3\,x_2 & & & + & x_4 & & & = & 36 \\
 & x_1 & & & & & & & - & x_5 & = & 4 \\
 & & & x_2 & & & & & & x_6 & = & 2 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, \quad x_6 & \geq & 0
\end{array}
$$

For the branching constraint we have:

$$x_2 - x_6 = 2 \quad \Rightarrow \quad x_6 = -2 + x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_6 = -2 + \left(\frac{4}{3} - \frac{1}{3}\,x_4 - \frac{8}{3}\,x_5\right) \quad \Rightarrow \quad x_6 = -\frac{2}{3} - \frac{1}{3}\,x_4 - \frac{8}{3}\,x_5$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$
\begin{cases}
x_1 = 4 + x_5 \\
x_2 = \frac{4}{3} - \frac{1}{3}\,x_4 - \frac{8}{3}\,x_5 \\
x_3 = 25 + 7\,x_4 + 50\,x_5 \\
x_6 = -\frac{2}{3} - \frac{1}{3}\,x_4 - \frac{8}{3}\,x_5 \\
z = 12 - x_4 - 6\,x_5
\end{cases}
$$

where $\mathcal{B} = \{1, 2, 3, 6\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = 4, \quad \bar{x}_2 = \frac{4}{3}, \quad \bar{x}_3 = 25, \quad \bar{x}_6 = -\frac{2}{3},$$
$$\bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad 12$$

- the basic dual feasible solution is:

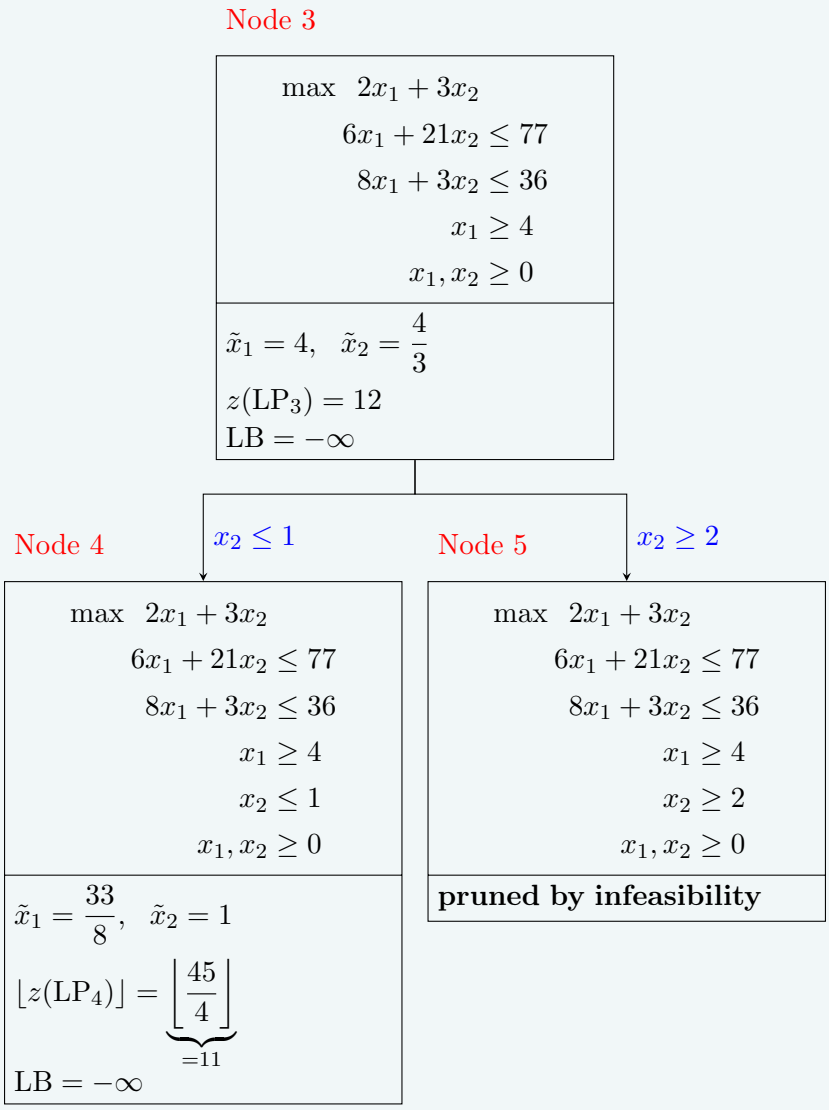
$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_6 = 0,$$
$$\bar{\pi}_4 = 1, \quad \bar{\pi}_5 = 6 \quad \text{of value} \quad 12$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_4 &= 1 + \frac{1}{3}\pi_6 \\ \pi_5 &= 6 + \frac{8}{3}\pi_6 \end{cases} \qquad \begin{cases} \underbrace{1 + \frac{1}{3}\pi_6 \geq 0}_{\pi_4 \geq 0} \\ \underbrace{6 + \frac{8}{3}\pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 &\geq -3 \\ \pi_6 &\geq -\frac{9}{4} \end{cases}$$

Accordingly, we can increase the value of π_6 without violating dual feasibility, which implies that the dual problem is **unbounded**. As a result, the primal problem LP_5 is **infeasible**, and the corresponding branching node is pruned by infeasibility.

At the end of the second iteration, branching on x_2 from Node 3 has generated the down-branching node, Node 4, and the up-branching node, Node 5. Node 4 is inserted in the list L, while Node 5 is pruned by infeasibility. The corresponding branch-and-bound tree is:



Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \;\leq\; z(\text{ILP}) \;\leq\; \underbrace{\left\lfloor \frac{101}{7} \right\rfloor}_{=14}$$

- **Iteration 3**

The LP_4 problem is extracted and deleted from the problem list L.

The branching variable is x_1. The branching constraints are:

$$x_1 \leq \underbrace{\left\lfloor \frac{33}{8} \right\rfloor}_{=4} \quad \text{and} \quad x_1 \geq \underbrace{\left\lfloor \frac{33}{8} \right\rfloor + 1}_{=5}$$

The LP_6 problem is created with the first branching constraint (the second child is solved only if needed).

(a) Solution of the LP_6 problem

Let us consider the LP_6 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcr}
\max & 2\,x_1 & + & 3\,x_2 & & & & & & & & & & \\
 & 6\,x_1 & + & 21\,x_2 & + & x_3 & & & & & & & = & 77 \\
 & 8\,x_1 & + & 3\,x_2 & & & + & x_4 & & & & & = & 36 \\
 & x_1 & & & & & & & - & x_5 & & & = & 4 \\
 & & & x_2 & & & & & & & + & x_6 & = & 1 \\
 & x_1 & & & & & & & & & & & + & x_7 & = & 4 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & & x_7 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_1 + x_7 = 4 \qquad \Rightarrow \qquad x_7 = 4 - x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_7 = 4 - \left(\frac{33}{8} - \frac{1}{8}\,x_4 + \frac{3}{8}\,x_6 \right) \qquad \Rightarrow \qquad x_7 = -\frac{1}{8} + \frac{1}{8}\,x_4 - \frac{3}{8}\,x_6$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = \frac{33}{8} - \frac{1}{8}\,x_4 + \frac{3}{8}\,x_6 \\ x_2 = 1 - x_6 \\ x_3 = \frac{125}{4} + \frac{3}{4}\,x_4 + \frac{75}{4}\,x_6 \\ x_5 = \frac{1}{8} - \frac{1}{8}\,x_4 + \frac{3}{8}\,x_6 \\ x_7 = -\frac{1}{8} + \frac{1}{8}\,x_4 - \frac{3}{8}\,x_6 \\ z = \frac{45}{4} - \frac{1}{4}\,x_4 - \frac{9}{4}\,x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 5, 7\}$ and $\mathcal{N} = \{4, 6\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{33}{8}, \quad \bar{x}_2 = 1, \quad \bar{x}_3 = \frac{125}{4}, \quad \bar{x}_5 = \frac{1}{8}, \quad \bar{x}_7 = -\frac{1}{8},$$
$$\bar{x}_4 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad \frac{45}{4}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_5 = 0, \quad \bar{\pi}_7 = 0,$$
$$\bar{\pi}_4 = \frac{1}{4}, \quad \bar{\pi}_6 = \frac{9}{4} \quad \text{of value} \quad \frac{45}{4}$$

If we increase the value of π_7, we can get a better dual solution. We have:

$$\begin{cases} \pi_4 = \frac{1}{4} - \frac{1}{8}\,\pi_7 \\ \pi_6 = \frac{9}{4} + \frac{3}{8}\,\pi_7 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{4} - \frac{1}{8}\,\pi_7 \geq 0}_{\pi_4 \geq 0} \\ \underbrace{\frac{9}{4} + \frac{3}{8}\,\pi_7 \geq 0}_{\pi_6 \geq 0} \end{cases} \qquad \begin{cases} \pi_7 \leq 2 \\ \pi_7 \geq -6 \end{cases}$$

By increasing the value of π_7 to 2, the value of π_4 decreases to 0. Then we have that:

π_7 becomes a dual basic variable and accordingly x_7 becomes a primal non-basic variable.

π_4 becomes a dual non-basic variable and accordingly x_4 becomes a primal basic variable.

We have:

$$x_7 = -\frac{1}{8} + \frac{1}{8}\,x_4 - \frac{3}{8}\,x_6 \quad \Rightarrow \quad x_4 = 1 + 3\,x_6 + 8\,x_7$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{33}{8} - \frac{1}{8}(1 + 3\,x_6 + 8\,x_7) + \frac{3}{8}\,x_6 = 4 - x_7$$

$$x_3 = \frac{125}{4} + \frac{3}{4}(1 + 3\,x_6 + 8\,x_7) + \frac{75}{4}\,x_6 = 32 + 21\,x_6 + 6\,x_7$$

$$x_5 = \frac{1}{8} - \frac{1}{8}(1 + 3\,x_6 + 8\,x_7) + \frac{3}{8}\,x_6 = 0 - x_7$$

$$z = \frac{45}{4} - \frac{1}{4}(1 + 3\,x_6 + 8\,x_7) - \frac{9}{4}\,x_6 = 11 - 3\,x_6 - 2\,x_7$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 4 - x_7 \\ x_2 = 1 - x_6 \\ x_3 = 32 + 21\,x_6 + 6\,x_7 \\ x_4 = 1 + 3\,x_6 + 8\,x_7 \\ x_5 = 0 - x_7 \\ z = 11 - 3\,x_6 - 2\,x_7 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4, 5\}$ and $\mathcal{N} = \{6, 7\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = 1, \quad \tilde{x}_3 = 32, \quad \tilde{x}_4 = 1, \quad \tilde{x}_5 = 0,$$
$$\tilde{x}_6 = 0, \quad \tilde{x}_7 = 0 \quad \text{of value} \quad 11$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 0, \quad \tilde{\pi}_5 = 0,$$
$$\tilde{\pi}_6 = 3, \quad \tilde{\pi}_7 = 2 \quad \text{of value} \quad 11$$

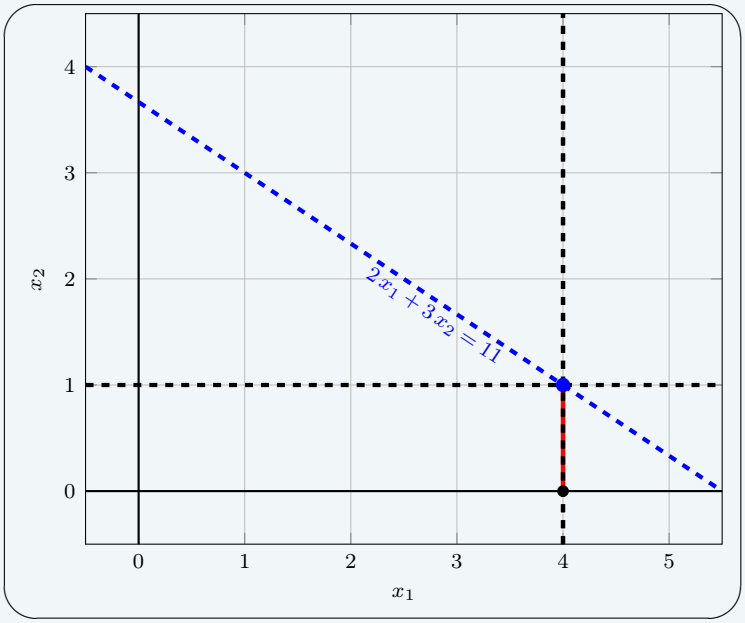


Figure 7.21: Node 6: feasible region of LP_6 (red), with the branching constraint $x_1 \leq 4$ (dashed), and its optimal solution (blue).
An optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_6)$ is:

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = 1$$

the optimal value is:

$$z(\text{LP}_6) = 11$$

The LP_6 problem is **feasible** and we have

$$\lfloor z(\text{LP}_6) \rfloor > \text{LB} \qquad \text{since} \qquad 11 > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_6)$ is **integer**, accordingly, the lower bound LB becomes 11 and the incumbent solution $\bar{\boldsymbol{x}}(\text{ILP})$ becomes $\bar{x}_1 = 4, \bar{x}_2 = 1$.
Accordingly, the branching node is pruned by integrality.

At the end of the third iteration, branching on x_1 from Node 4 has generated the down-branching node, Node 6. This node is pruned by integrality. The up-branching node is not solved, since after Node 6 the lower bound satisfies

$$\lfloor z(\text{LP}_4) \rfloor \leq \text{LB}.$$

This is the portion of the branch-and-bound tree showing the LP problems of the branching nodes that are solved:

Node 4

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ & 6x_1 + 21x_2 \leq 77 \\ & 8x_1 + 3x_2 \leq 36 \\ & x_1 \geq 4 \\ & x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\tilde{x}_1 = \frac{33}{8}, \quad \tilde{x}_2 = 1$

$\lfloor z(\text{LP}_4) \rfloor = \underbrace{\left\lfloor \frac{45}{4} \right\rfloor}_{=11}$

$\text{LB} = -\infty$

$x_1 \leq 4$

Node 6

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ & 6x_1 + 21x_2 \leq 77 \\ & 8x_1 + 3x_2 \leq 36 \\ & x_1 \geq 4 \\ & x_2 \leq 1 \\ & x_1 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\tilde{x}_1 = 4, \quad \tilde{x}_2 = 1$

$z(\text{LP}_6) = 11$

$\text{LB} = 11$

pruned by integrality

Due to the current lower bound LB and upper bound UB, we have:

$$11 \quad \leq \quad z(\text{ILP}) \quad \leq \quad \underbrace{\left\lfloor \frac{101}{7} \right\rfloor}_{=14}$$

- **Iteration 4**

The LP_2 problem is extracted and deleted from the problem list L.

The branching variable is x_2. The branching constraints are:

$$x_2 \leq \underbrace{\left\lfloor \frac{59}{21} \right\rfloor}_{=2} \quad \text{and} \quad x_2 \geq \underbrace{\left\lfloor \frac{59}{21} \right\rfloor + 1}_{=3}$$

Two LP_7 and LP_8 problems are created with these branching constraints.

(a) Solution of the LP_7 problem

Let us consider the LP_7 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcr}
\max & 2\,x_1 & + & 3\,x_2 & & & & & & & & & \\
& 6\,x_1 & + & 21\,x_2 & + & x_3 & & & & & & = & 77 \\
& 8\,x_1 & + & 3\,x_2 & & & + & x_4 & & & & = & 36 \\
& x_1 & & & & & & & + & x_5 & & = & 3 \\
& & & x_2 & & & & & & & + & x_6 & = & 2 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_2 + x_6 = 2 \qquad \Rightarrow \qquad x_6 = 2 - x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_6 = 2 - \left(\frac{59}{21} - \frac{1}{21}\,x_3 + \frac{2}{7}\,x_5\right) \qquad \Rightarrow \qquad x_6 = -\frac{17}{21} + \frac{1}{21}\,x_3 - \frac{2}{7}\,x_5$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases}
x_1 = 3 - x_5 \\
x_2 = \frac{59}{21} - \frac{1}{21}\,x_3 + \frac{2}{7}\,x_5 \\
x_4 = \frac{25}{7} + \frac{1}{7}\,x_3 + \frac{50}{7}\,x_5 \\
x_6 = -\frac{17}{21} + \frac{1}{21}\,x_3 - \frac{2}{7}\,x_5 \\
z = \frac{101}{7} - \frac{1}{7}\,x_3 - \frac{8}{7}\,x_5
\end{cases}$$

where $\mathcal{B} = \{1, 2, 4, 6\}$ and $\mathcal{N} = \{3, 5\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = 3, \quad \bar{x}_2 = \frac{59}{21}, \quad \bar{x}_4 = \frac{25}{7}, \quad \bar{x}_6 = -\frac{17}{21},$$
$$\bar{x}_3 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{101}{7}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_6 = 0,$$
$$\bar{\pi}_3 = \frac{1}{7}, \quad \bar{\pi}_5 = \frac{8}{7} \quad \text{of value} \quad \frac{101}{7}$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 &= \frac{1}{7} - \frac{1}{21}\pi_6 \\ \pi_5 &= \frac{8}{7} + \frac{2}{7}\pi_6 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{7} - \frac{1}{21}\pi_6 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{8}{7} + \frac{2}{7}\pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 &\leq 3 \\ \pi_6 &\geq -4 \end{cases}$$

By increasing the value of π_6 to 3, the value of π_3 decreases to 0. Then we have that:

π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non-basic variable.

π_3 becomes a dual non-basic variable and accordingly x_3 becomes a primal basic variable.

We have:

$$x_6 = -\frac{17}{21} + \frac{1}{21}x_3 - \frac{2}{7}x_5 \quad \Rightarrow \quad x_3 = 17 + 6\,x_5 + 21\,x_6$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_2 = \frac{59}{21} - \frac{1}{21}(17 + 6\,x_5 + 21\,x_6) + \frac{2}{7}x_5 = 2 - x_6$$

$$x_4 = \frac{25}{7} + \frac{1}{7}(17 + 6\,x_5 + 21\,x_6) + \frac{50}{7}x_5 = 6 + 8\,x_5 + 3\,x_6$$

$$z = \frac{101}{7} - \frac{1}{7}(17 + 6\,x_5 + 21\,x_6) - \frac{8}{7}x_5 = 12 - 2\,x_5 - 3\,x_6$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 &= 3 &- x_5 \\ x_2 &= 2 & &- x_6 \\ x_3 &= 17 &+ 6\,x_5 &+ 21\,x_6 \\ x_4 &= 6 &+ 8\,x_5 &+ 3\,x_6 \\ z &= 12 &- 2\,x_5 &- 3\,x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4\}$ and $\mathcal{N} = \{5, 6\}$ and

– the basic primal optimal solution is:

$$\tilde{x}_1 = 3, \quad \tilde{x}_2 = 2, \quad \tilde{x}_3 = 17, \quad \tilde{x}_4 = 6,$$
$$\tilde{x}_5 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad 12$$

– the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 0,$$
$$\tilde{\pi}_5 = 2, \quad \tilde{\pi}_6 = 3 \quad \text{of value} \quad 12$$

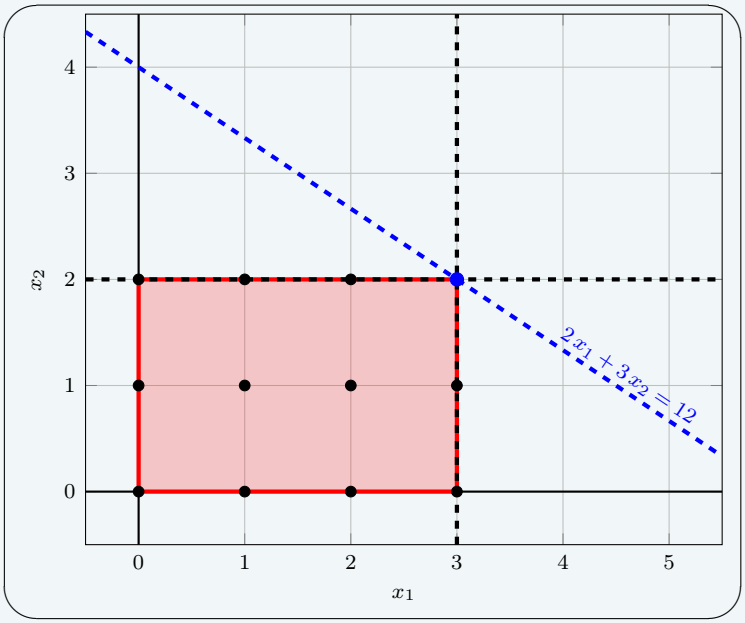


Figure 7.22: Node 7: feasible region of LP_7 (red), with the branching constraint $x_2 \leq 2$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_7)$ is:

$$\tilde{x}_1 = 3, \quad \tilde{x}_2 = 2$$

the optimal value is:

$$z(\mathrm{LP}_7) = 12$$

The LP_7 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_7) \rfloor > \mathrm{LB} \qquad \text{since} \qquad 12 > 11$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_7)$ is **integer**, accordingly, the lower bound LB becomes 12 and the incumbent solution $\bar{\boldsymbol{x}}(\mathrm{ILP})$ becomes $\bar{x}_1 = 3, \bar{x}_2 = 2$.

Accordingly, the branching node is pruned by integrality.

(b) Solution of the LP_8 problem

Let us consider the LP_8 problem in standard form:

$$\begin{array}{lrcrcrcrcrcr}
\max & 2\,x_1 & + & 3\,x_2 & & & & & & & & \\
& 6\,x_1 & + & 21\,x_2 & + & x_3 & & & & & = & 77 \\
& 8\,x_1 & + & 3\,x_2 & & & + & x_4 & & & = & 36 \\
& x_1 & & & & & & & + & x_5 & = & 3 \\
& & & x_2 & & & & & & - \; x_6 & = & 3 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5, \quad x_6 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_2 - x_6 = 3 \qquad \Rightarrow \qquad x_6 = -3 + x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_6 = -3 + \left(\frac{59}{21} - \frac{1}{21}\, x_3 + \frac{2}{7}\, x_5\right) \qquad \Rightarrow \qquad x_6 = -\frac{4}{21} - \frac{1}{21}\, x_3 + \frac{2}{7}\, x_5$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = 3 - x_5 \\ x_2 = \frac{59}{21} - \frac{1}{21}\, x_3 + \frac{2}{7}\, x_5 \\ x_4 = \frac{25}{7} + \frac{1}{7}\, x_3 + \frac{50}{7}\, x_5 \\ x_6 = -\frac{4}{21} - \frac{1}{21}\, x_3 + \frac{2}{7}\, x_5 \\ z = \frac{101}{7} - \frac{1}{7}\, x_3 - \frac{8}{7}\, x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4, 6\}$ and $\mathcal{N} = \{3, 5\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = 3, \quad \bar{x}_2 = \frac{59}{21}, \quad \bar{x}_4 = \frac{25}{7}, \quad \bar{x}_6 = -\frac{4}{21}, \quad \bar{x}_3 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{101}{7}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_6 = 0, \quad \bar{\pi}_3 = \frac{1}{7}, \quad \bar{\pi}_5 = \frac{8}{7} \quad \text{of value} \quad \frac{101}{7}$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 = \frac{1}{7} + \frac{1}{21}\, \pi_6 \\ \pi_5 = \frac{8}{7} - \frac{2}{7}\, \pi_6 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{7} + \frac{1}{21}\, \pi_6 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{8}{7} - \frac{2}{7}\, \pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 \geq -3 \\ \pi_6 \leq 4 \end{cases}$$

By increasing the value of π_6 to 4, the value of π_5 decreases to 0. Then we have that:

π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non-basic variable.

π_5 becomes a dual non-basic variable and accordingly x_5 becomes a primal basic variable.

We have:

$$x_6 = -\frac{4}{21} - \frac{1}{21}\,x_3 + \frac{2}{7}\,x_5 \quad \Rightarrow \quad x_5 = \frac{2}{3} + \frac{1}{6}\,x_3 + \frac{7}{2}\,x_6$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = 3 - \left(\frac{2}{3} + \frac{1}{6}\,x_3 + \frac{7}{2}\,x_6\right) = \frac{7}{3} - \frac{1}{6}\,x_3 - \frac{7}{2}\,x_6$$

$$x_2 = \frac{59}{21} - \frac{1}{21}\,x_3 + \frac{2}{7}\left(\frac{2}{3} + \frac{1}{6}\,x_3 + \frac{7}{2}\,x_6\right) = 3 + x_6$$

$$x_4 = \frac{25}{7} + \frac{1}{7}\,x_3 + \frac{50}{7}\left(\frac{2}{3} + \frac{1}{6}\,x_3 + \frac{7}{2}\,x_6\right) = \frac{25}{3} + \frac{4}{3}\,x_3 + 25\,x_6$$

$$z = \frac{101}{7} - \frac{1}{7}\,x_3 - \frac{8}{7}\left(\frac{2}{3} + \frac{1}{6}\,x_3 + \frac{7}{2}\,x_6\right) = \frac{41}{3} - \frac{1}{3}\,x_3 - 4\,x_6$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = \frac{7}{3} - \frac{1}{6}\,x_3 - \frac{7}{2}\,x_6 \\ x_2 = 3 + x_6 \\ x_4 = \frac{25}{3} + \frac{4}{3}\,x_3 + 25\,x_6 \\ x_5 = \frac{2}{3} + \frac{1}{6}\,x_3 + \frac{7}{2}\,x_6 \\ z = \frac{41}{3} - \frac{1}{3}\,x_3 - 4\,x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4, 5\}$ and $\mathcal{N} = \{3, 6\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = \frac{7}{3}, \quad \tilde{x}_2 = 3, \quad \tilde{x}_4 = \frac{25}{3}, \quad \tilde{x}_5 = \frac{2}{3},$$
$$\tilde{x}_3 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad \frac{41}{3}$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_4 = 0, \quad \tilde{\pi}_5 = 0,$$
$$\tilde{\pi}_3 = \frac{1}{3}, \quad \tilde{\pi}_6 = 4 \quad \text{of value} \quad \frac{41}{3}$$

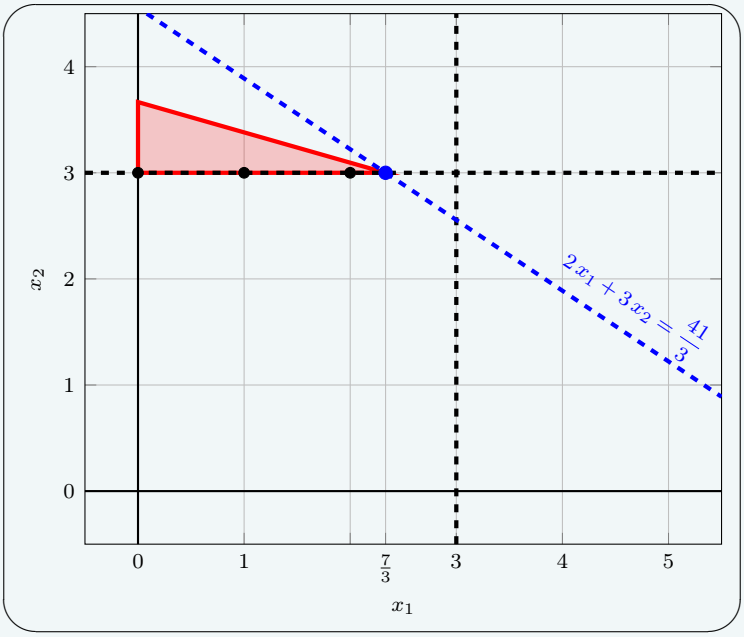


Figure 7.23: Node 8: feasible region of LP_8 (red), with the branching constraint $x_2 \geq 3$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_8)$ is:

$$\tilde{x}_1 = \frac{7}{3}, \quad \tilde{x}_2 = 3$$

the optimal value is:

$$z(\mathrm{LP}_8) = \frac{41}{3}$$

The LP_8 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_8) \rfloor > \mathrm{LB} \qquad \text{since} \qquad \underbrace{\left\lfloor \frac{41}{3} \right\rfloor}_{=13} > 12$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_8)$ is **fractional**, accordingly, the LP_8 problem is **inserted in the problem list** L.

At the end of the fourth iteration, branching on x_2 from Node 2 has generated the down-branching node, Node 7, and the up-branching node, Node 8. Node 7 is pruned by integrality, while Node 8 is inserted in the list L. The corresponding branch-and-bound tree is:

Node 2

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ & 6x_1 + 21x_2 \leq 77 \\ & 8x_1 + 3x_2 \leq 36 \\ & x_1 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\tilde{x}_1 = 3, \quad \tilde{x}_2 = \dfrac{59}{21}$

$\lfloor z(\text{LP}_2) \rfloor = \underbrace{\left\lfloor \dfrac{101}{7} \right\rfloor}_{=14}$

$\text{LB} = -\infty$

$x_2 \leq 2$ | $x_2 \geq 3$

Node 7

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ & 6x_1 + 21x_2 \leq 77 \\ & 8x_1 + 3x_2 \leq 36 \\ & x_1 \leq 3 \\ & x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\tilde{x}_1 = 3, \quad \tilde{x}_2 = 2$

$z(\text{LP}_7) = 12$

$\text{LB} = 12$

pruned by integrality

Node 8

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ & 6x_1 + 21x_2 \leq 77 \\ & 8x_1 + 3x_2 \leq 36 \\ & x_1 \leq 3 \\ & x_2 \geq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\tilde{x}_1 = \dfrac{7}{3}, \quad \tilde{x}_2 = 3$

$\lfloor z(\text{LP}_8) \rfloor = \underbrace{\left\lfloor \dfrac{41}{3} \right\rfloor}_{=13}$

$\text{LB} = 12$

Due to the current lower bound LB and upper bound UB, we have:

$$12 \quad \leq \quad z(\text{ILP}) \quad \leq \quad \underbrace{\left\lfloor \frac{41}{3} \right\rfloor}_{=13}$$

- **Iteration 5**

The LP_8 problem is extracted and deleted from the problem list L.

The branching variable is x_1. The branching constraints are:

$$x_1 \leq \underbrace{\left\lfloor \frac{7}{3} \right\rfloor}_{=2} \quad \text{and} \quad x_1 \geq \underbrace{\left\lfloor \frac{7}{3} \right\rfloor + 1}_{=3}$$

Two LP_9 and LP_{10} problems are created with these branching constraints.

(a) Solution of the LP_9 problem

Let us consider the LP_9 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcr}
\max & 2\,x_1 & + & 3\,x_2 & & & & & & & & & & \\
 & 6\,x_1 & + & 21\,x_2 & + & x_3 & & & & & & & = & 77 \\
 & 8\,x_1 & + & 3\,x_2 & & & + & x_4 & & & & & = & 36 \\
 & x_1 & & & & & & & + & x_5 & & & = & 3 \\
 & & & x_2 & & & & & & & - & x_6 & = & 3 \\
 & x_1 & & & & & & & & & & & + \; x_7 = & 2 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & & x_7 \geq 0
\end{array}$$

For the branching constraint we have:

$$x_1 + x_7 = 2 \qquad \Rightarrow \quad x_7 = 2 - x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_7 = 2 - \left(\frac{7}{3} - \frac{1}{6}\,x_3 - \frac{7}{2}\,x_6\right) \qquad \Rightarrow \quad x_7 = -\frac{1}{3} + \frac{1}{6}\,x_3 + \frac{7}{2}\,x_6$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases}
x_1 = \frac{7}{3} - \frac{1}{6}\,x_3 - \frac{7}{2}\,x_6 \\
x_2 = 3 + x_6 \\
x_4 = \frac{25}{3} + \frac{4}{3}\,x_3 + 25\,x_6 \\
x_5 = \frac{2}{3} + \frac{1}{6}\,x_3 + \frac{7}{2}\,x_6 \\
x_7 = -\frac{1}{3} + \frac{1}{6}\,x_3 + \frac{7}{2}\,x_6 \\
z = \frac{41}{3} - \frac{1}{3}\,x_3 - 4\,x_6
\end{cases}$$

where $\mathcal{B} = \{1, 2, 4, 5, 7\}$ and $\mathcal{N} = \{3, 6\}$ and

– the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{7}{3}, \quad \bar{x}_2 = 3, \quad \bar{x}_4 = \frac{25}{3}, \quad \bar{x}_5 = \frac{2}{3}, \quad \bar{x}_7 = -\frac{1}{3},$$
$$\bar{x}_3 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad \frac{41}{3}$$

– the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_5 = 0, \quad \bar{\pi}_7 = 0,$$
$$\bar{\pi}_3 = \frac{1}{3}, \quad \bar{\pi}_6 = 4 \quad \text{of value} \quad \frac{41}{3}$$

If we increase the value of π_7, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 &= \frac{1}{3} - \frac{1}{6}\pi_7 \\ \pi_6 &= 4 - \frac{7}{2}\pi_7 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{3} - \frac{1}{6}\pi_7 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{4 - \frac{7}{2}\pi_7 \geq 0}_{\pi_6 \geq 0} \end{cases} \qquad \begin{cases} \pi_7 &\leq 2 \\ \pi_7 &\leq \frac{8}{7} \end{cases}$$

By increasing the value of π_7 to $\min\left\{2, \frac{8}{7}\right\} = \frac{8}{7}$, the value of π_6 decreases to 0. Then we have that:

π_7 becomes a dual basic variable and accordingly x_7 becomes a primal non-basic variable.

π_6 becomes a dual non-basic variable and accordingly x_6 becomes a primal basic variable.

We have:

$$x_7 = -\frac{1}{3} + \frac{1}{6}x_3 + \frac{7}{2}x_6 \quad \Rightarrow \quad x_6 = \frac{2}{21} - \frac{1}{21}x_3 + \frac{2}{7}x_7$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{7}{3} - \frac{1}{6}x_3 - \frac{7}{2}\left(\frac{2}{21} - \frac{1}{21}x_3 + \frac{2}{7}x_7\right) = 2 - x_7$$
$$x_2 = 3 + \left(\frac{2}{21} - \frac{1}{21}x_3 + \frac{2}{7}x_7\right) = \frac{65}{21} - \frac{1}{21}x_3 + \frac{2}{7}x_7$$
$$x_4 = \frac{25}{3} + \frac{4}{3}x_3 + 25\left(\frac{2}{21} - \frac{1}{21}x_3 + \frac{2}{7}x_7\right) = \frac{75}{7} + \frac{1}{7}x_3 + \frac{50}{7}x_7$$
$$x_5 = \frac{2}{3} + \frac{1}{6}x_3 + \frac{7}{2}\left(\frac{2}{21} - \frac{1}{21}x_3 + \frac{2}{7}x_7\right) = 1 + x_7$$
$$z = \frac{41}{3} - \frac{1}{3}x_3 - 4\left(\frac{2}{21} - \frac{1}{21}x_3 + \frac{2}{7}x_7\right) = \frac{93}{7} - \frac{1}{7}x_3 - \frac{8}{7}x_7$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 2 - x_7 \\ x_2 = \frac{65}{21} - \frac{1}{21} x_3 + \frac{2}{7} x_7 \\ x_4 = \frac{75}{7} + \frac{1}{7} x_3 + \frac{50}{7} x_7 \\ x_5 = 1 + x_7 \\ x_6 = \frac{2}{21} - \frac{1}{21} x_3 + \frac{2}{7} x_7 \\ z = \frac{93}{7} - \frac{1}{7} x_3 - \frac{8}{7} x_7 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4, 5, 6\}$ and $\mathcal{N} = \{3, 7\}$ and

– the basic primal optimal solution is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = \frac{65}{21}, \quad \tilde{x}_4 = \frac{75}{7}, \quad \tilde{x}_5 = 1, \quad \tilde{x}_6 = \frac{2}{21},$$
$$\tilde{x}_3 = 0, \quad \tilde{x}_7 = 0 \quad \text{of value} \quad \frac{93}{7}$$

– the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_4 = 0, \quad \tilde{\pi}_5 = 0, \quad \tilde{\pi}_6 = 0,$$
$$\tilde{\pi}_3 = \frac{1}{7}, \quad \tilde{\pi}_7 = \frac{8}{7} \quad \text{of value} \quad \frac{93}{7}$$

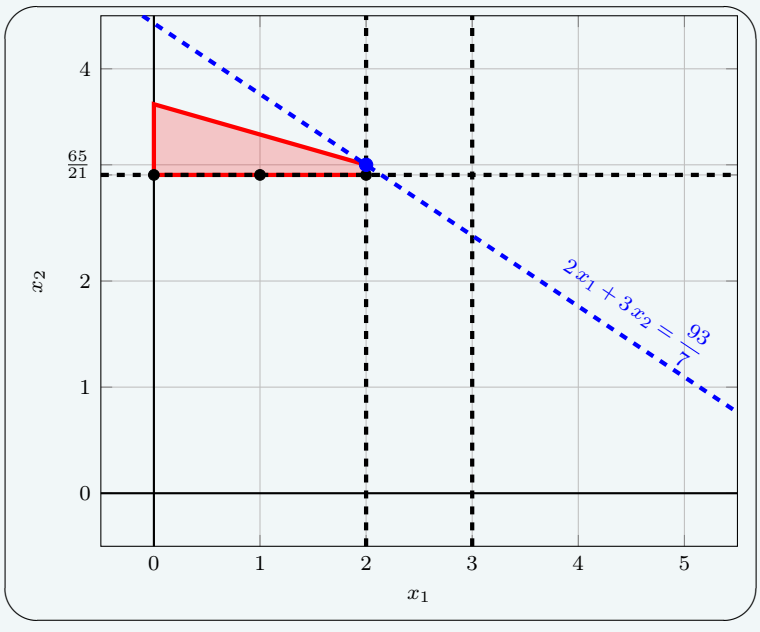


Figure 7.24: Node 9: feasible region of LP_9 (red), with the branching constraint $x_1 \leq 2$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_9)$ is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = \frac{65}{21}$$

the optimal value is:

$$z(\mathrm{LP}_9) = \frac{93}{7}$$

The LP_9 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_9) \rfloor > \mathrm{LB} \qquad \text{since} \qquad \underbrace{\left\lfloor \frac{93}{7} \right\rfloor}_{=13} > 12$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_9)$ is **fractional**, accordingly, the LP_9 problem is **inserted in the problem list** L.

(b) Solution of the LP_{10} problem

Let us consider the LP_{10} problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcl}
\max & 2\,x_1 & + & 3\,x_2 & & & & & & & & & & \\
& 6\,x_1 & + & 21\,x_2 & + & x_3 & & & & & & & = & 77 \\
& 8\,x_1 & + & 3\,x_2 & & & + & x_4 & & & & & = & 36 \\
& x_1 & & & & & & & + & x_5 & & & = & 3 \\
& & & x_2 & & & & & & & - & x_6 & = & 3 \\
& x_1 & & & & & & & & & & x_7 & = & 3 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & x_7 \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_1 - x_7 = 3 \qquad \Rightarrow \qquad x_7 = -3 + x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_7 = -3 + \left(\frac{7}{3} - \frac{1}{6}\,x_3 - \frac{7}{2}\,x_6\right) \qquad \Rightarrow \qquad x_7 = -\frac{2}{3} - \frac{1}{6}\,x_3 - \frac{7}{2}\,x_6$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = \frac{7}{3} - \frac{1}{6}\, x_3 - \frac{7}{2}\, x_6 \\ x_2 = 3 + x_6 \\ x_4 = \frac{25}{3} + \frac{4}{3}\, x_3 + 25\, x_6 \\ x_5 = \frac{2}{3} + \frac{1}{6}\, x_3 + \frac{7}{2}\, x_6 \\ x_7 = -\frac{2}{3} - \frac{1}{6}\, x_3 - \frac{7}{2}\, x_6 \\ z = \frac{41}{3} - \frac{1}{3}\, x_3 - 4\, x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4, 5, 7\}$ and $\mathcal{N} = \{3, 6\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{7}{3}, \quad \bar{x}_2 = 3, \quad \bar{x}_4 = \frac{25}{3}, \quad \bar{x}_5 = \frac{2}{3}, \quad \bar{x}_7 = -\frac{2}{3},$$
$$\bar{x}_3 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad \frac{41}{3}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_5 = 0, \quad \bar{\pi}_7 = 0,$$
$$\bar{\pi}_3 = \frac{1}{3}, \quad \bar{\pi}_6 = 4 \quad \text{of value} \quad \frac{41}{3}$$

If we increase the value of π_7, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 = \frac{1}{3} + \frac{1}{6}\,\pi_7 \\ \pi_6 = 4 + \frac{7}{2}\,\pi_7 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{3} + \frac{1}{6}\,\pi_7 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{4 + \frac{7}{2}\,\pi_7 \geq 0}_{\pi_6 \geq 0} \end{cases} \qquad \begin{cases} \pi_7 \geq -2 \\ \pi_7 \geq -\frac{8}{7} \end{cases}$$

Accordingly, we can increase the value of π_7 without violating dual feasibility, which implies that the dual problem is **unbounded**. As a result, the primal problem LP_{10} is **infeasible**, and the corresponding branching node is pruned by infeasibility.

At the end of the fifth iteration, branching on x_1 from Node 8 has generated the down-branching node, Node 9, and the up-branching node, Node 10. Node 9 is inserted in the list L, while Node 10 is pruned by infeasibility. The corresponding branch-and-bound tree is:

Node 8

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ & 6x_1 + 21x_2 \leq 77 \\ & 8x_1 + 3x_2 \leq 36 \\ & x_1 \leq 3 \\ & x_2 \geq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\tilde{x}_1 = \frac{7}{3}, \quad \tilde{x}_2 = 3$

$\lfloor z(\text{LP}_8) \rfloor = \underbrace{\left\lfloor \frac{41}{3} \right\rfloor}_{=13}$

$\text{LB} = 12$

Node 9 — $x_1 \leq 2$

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ & 6x_1 + 21x_2 \leq 77 \\ & 8x_1 + 3x_2 \leq 36 \\ & x_1 \leq 3 \\ & x_2 \geq 3 \\ & x_1 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\tilde{x}_1 = 2, \quad \tilde{x}_2 = \frac{65}{21}$

$\lfloor z(\text{LP}_9) \rfloor = \underbrace{\left\lfloor \frac{93}{7} \right\rfloor}_{=13}$

$\text{LB} = 12$

Node 10 — $x_1 \geq 3$

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ & 6x_1 + 21x_2 \leq 77 \\ & 8x_1 + 3x_2 \leq 36 \\ & x_1 \leq 3 \\ & x_2 \geq 3 \\ & x_1 \geq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

pruned by infeasibility

Due to the current lower bound LB and upper bound UB, we have:

$$12 \quad \leq \quad z(\text{ILP}) \quad \leq \quad \underbrace{\left\lfloor \frac{93}{7} \right\rfloor}_{=13}$$

- **Iteration 6**

The LP_9 problem is extracted and deleted from the problem list L.

The branching variable is x_2. The branching constraints are:

$$x_2 \leq \underbrace{\left\lfloor \frac{65}{21} \right\rfloor}_{=3} \quad \text{and} \quad x_2 \geq \underbrace{\left\lfloor \frac{65}{21} \right\rfloor + 1}_{=4}$$

The LP_{11} problem is created with the first branching constraint (the second child is solved only if needed).

(a) Solution of the LP_{11} problem

Let us consider the LP_{11} problem in standard form:

$$
\begin{array}{rrrrrrrrrrrrrrrrrrrrr}
\max & 2\,x_1 & + & 3\,x_2 & & & & & & & & & & & & & & & & & \\
 & 6\,x_1 & + & 21\,x_2 & + & x_3 & & & & & & & & & & & & & = & 77 \\
 & 8\,x_1 & + & 3\,x_2 & & & + & x_4 & & & & & & & & & & & = & 36 \\
 & x_1 & & & & & & & + & x_5 & & & & & & & & & = & 3 \\
 & & & x_2 & & & & & & & - & x_6 & & & & & & & = & 3 \\
 & x_1 & & & & & & & & & & & + & x_7 & & & & & = & 2 \\
 & & & x_2 & & & & & & & & & & & + & x_8 & = & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & & x_7, & & x_8 & \geq & 0
\end{array}
$$

For the branching constraint we have:

$$x_2 + x_8 = 3 \qquad \Rightarrow \qquad x_8 = 3 - x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_8 = 3 - \left(\frac{65}{21} - \frac{1}{21}\,x_3 + \frac{2}{7}\,x_7\right) \qquad \Rightarrow \qquad x_8 = -\frac{2}{21} + \frac{1}{21}\,x_3 - \frac{2}{7}\,x_7$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$
\begin{cases}
x_1 = 2 - x_7 \\
x_2 = \frac{65}{21} - \frac{1}{21}\,x_3 + \frac{2}{7}\,x_7 \\
x_4 = \frac{75}{7} + \frac{1}{7}\,x_3 + \frac{50}{7}\,x_7 \\
x_5 = 1 + x_7 \\
x_6 = \frac{2}{21} - \frac{1}{21}\,x_3 + \frac{2}{7}\,x_7 \\
x_8 = -\frac{2}{21} + \frac{1}{21}\,x_3 - \frac{2}{7}\,x_7 \\
z = \frac{93}{7} - \frac{1}{7}\,x_3 - \frac{8}{7}\,x_7
\end{cases}
$$

where $\mathcal{B} = \{1, 2, 4, 5, 6, 8\}$ and $\mathcal{N} = \{3, 7\}$ and

– the basic primal non-feasible solution is:

$$\bar{x}_1 = 2, \quad \bar{x}_2 = \frac{65}{21}, \quad \bar{x}_4 = \frac{75}{7},$$
$$\bar{x}_5 = 1, \quad \bar{x}_6 = \frac{2}{21}, \quad \bar{x}_8 = -\frac{2}{21},$$
$$\bar{x}_3 = 0, \quad \bar{x}_7 = 0 \quad \text{of value} \quad \frac{93}{7}$$

– the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_4 = 0,$$
$$\bar{\pi}_5 = 0, \quad \bar{\pi}_6 = 0, \quad \bar{\pi}_8 = 0,$$
$$\bar{\pi}_3 = \frac{1}{7}, \quad \bar{\pi}_7 = \frac{8}{7} \quad \text{of value} \quad \frac{93}{7}$$

If we increase the value of π_8, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 = \frac{1}{7} - \frac{1}{21}\pi_8 \\ \pi_7 = \frac{8}{7} + \frac{2}{7}\pi_8 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{7} - \frac{1}{21}\pi_8 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{8}{7} + \frac{2}{7}\pi_8 \geq 0}_{\pi_7 \geq 0} \end{cases} \qquad \begin{cases} \pi_8 \leq 3 \\ \pi_8 \geq -4 \end{cases}$$

By increasing the value of π_8 to 3, the value of π_3 decreases to 0. Then we have that:

π_8 becomes a dual basic variable and accordingly x_8 becomes a primal non-basic variable.

π_3 becomes a dual non-basic variable and accordingly x_3 becomes a primal basic variable.

We have:

$$x_8 = -\frac{2}{21} + \frac{1}{21}x_3 - \frac{2}{7}x_7 \quad \Rightarrow \quad x_3 = 2 + 6\,x_7 + 21\,x_8$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_2 = \frac{65}{21} - \frac{1}{21}(2 + 6\,x_7 + 21\,x_8) + \frac{2}{7}x_7 = 3 - x_8$$
$$x_4 = \frac{75}{7} + \frac{1}{7}(2 + 6\,x_7 + 21\,x_8) + \frac{50}{7}x_7 = 11 + 8\,x_7 + 3\,x_8$$
$$x_6 = \frac{2}{21} - \frac{1}{21}(2 + 6\,x_7 + 21\,x_8) + \frac{2}{7}x_7 = 0 - x_8$$
$$z = \frac{93}{7} - \frac{1}{7}(2 + 6\,x_7 + 21\,x_8) - \frac{8}{7}x_7 = 13 - 2\,x_7 - 3\,x_8$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 2 - x_7 \\ x_2 = 3 - x_8 \\ x_3 = 2 + 6\,x_7 + 21\,x_8 \\ x_4 = 11 + 8\,x_7 + 3\,x_8 \\ x_5 = 1 + x_7 \\ x_6 = 0 - x_8 \\ z = 13 - 2\,x_7 - 3\,x_8 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{N} = \{7, 8\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 3, \quad \tilde{x}_3 = 2,$$
$$\tilde{x}_4 = 11, \quad \tilde{x}_5 = 1, \quad \tilde{x}_6 = 0,$$
$$\tilde{x}_7 = 0, \quad \tilde{x}_8 = 0 \quad \text{of value} \quad 13$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0,$$
$$\tilde{\pi}_4 = 0, \quad \tilde{\pi}_5 = 0, \quad \tilde{\pi}_6 = 0,$$
$$\tilde{\pi}_7 = 2, \quad \tilde{\pi}_8 = 3 \quad \text{of value} \quad 13$$

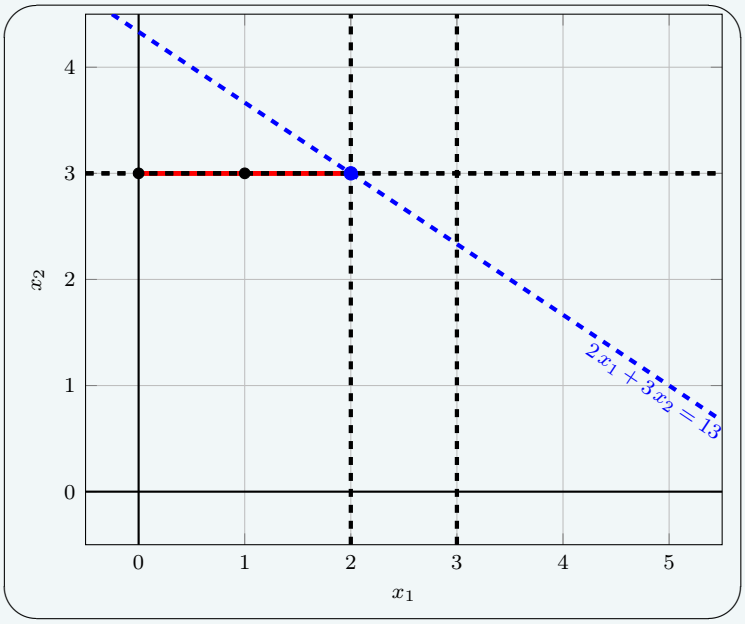


Figure 7.25: Node 11: feasible region of LP_{11} (red), with the branching constraint $x_2 \leq 3$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_{11})$ is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 3$$

the optimal value is:

$$z(\mathrm{LP}_{11}) = 13$$

The LP_{11} problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_{11}) \rfloor > \mathrm{LB} \qquad \text{since} \qquad 13 > 12$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_{11})$ is **integer**, accordingly, the lower bound LB becomes 13 and the incumbent solution $\bar{\boldsymbol{x}}(\mathrm{ILP})$ becomes $\bar{x}_1 = 2, \bar{x}_2 = 3$.
Accordingly, the branching node is pruned by integrality.

At the end of the sixth iteration, branching on x_2 from Node 9 has generated the down-branching node, Node 11. This node is pruned by integrality. The up-branching node is not solved, since after Node 11 the lower bound satisfies

$$\lfloor z(\mathrm{LP}_9) \rfloor \leq \mathrm{LB}.$$

This is the portion of the branch-and-bound tree showing the LP problems of the branching nodes that are solved:

Node 9

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ & 6x_1 + 21x_2 \leq 77 \\ & 8x_1 + 3x_2 \leq 36 \\ & x_1 \leq 3 \\ & x_2 \geq 3 \\ & x_1 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\tilde{x}_1 = 2, \quad \tilde{x}_2 = \dfrac{65}{21}$

$\lfloor z(\text{LP}_9) \rfloor = \underbrace{\left\lfloor \dfrac{93}{7} \right\rfloor}_{=13}$

$\text{LB} = 12$

$x_2 \leq 3$

Node 11

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ & 6x_1 + 21x_2 \leq 77 \\ & 8x_1 + 3x_2 \leq 36 \\ & x_1 \leq 3 \\ & x_2 \geq 3 \\ & x_1 \leq 2 \\ & x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 3$

$z(\text{LP}_{11}) = 13$

$\text{LB} = 13$

pruned by integrality

Due to the current lower bound LB and upper bound UB, we have:

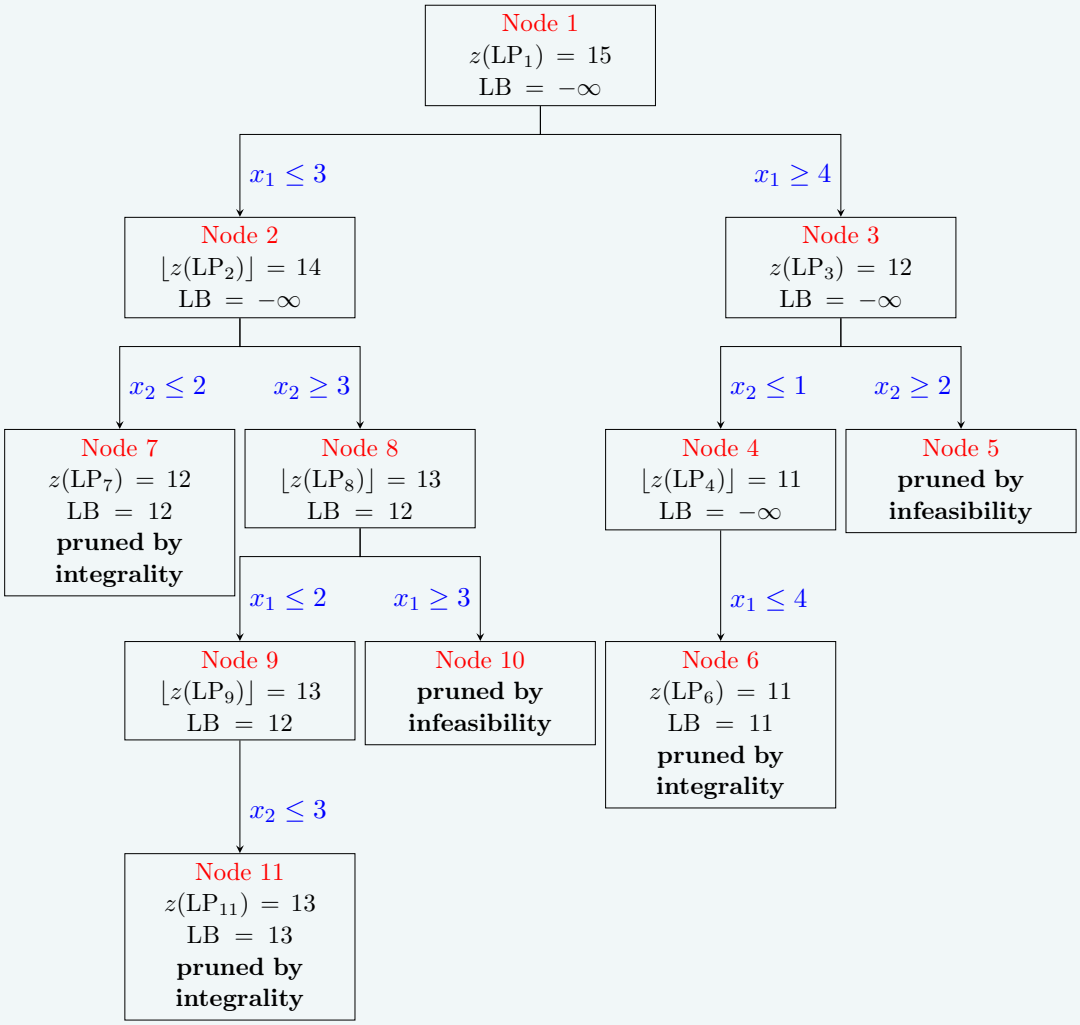
$$13 \quad \leq \quad z(\text{ILP}) \quad \leq \quad 13$$

- **Termination**

Since the list L is empty, the lower bound LB is equal to the upper bound UB and the ILP problem is solved to optimality.

The optimal value is $z(\text{ILP}) = 13$ (the value of the lower bound LB) and an optimal solution $\tilde{\boldsymbol{x}}(\text{ILP})$ is $\tilde{x}_1 = 2, \tilde{x}_2 = 3$ (the incumbent solution $\bar{\boldsymbol{x}}(\text{ILP})$).

2. The final branch-and-bound tree is:



The reasons why the branching nodes are pruned are the following.

- Node 1 has a fractional optimal solution with $z(\text{LP}_1) = 15$: it is branched on x_1.
- Node 2 has a fractional optimal solution with $\lfloor z(\text{LP}_2) \rfloor = 14$: it is branched on x_2.
- Node 3 has a fractional optimal solution with $z(\text{LP}_3) = 12$: it is branched on x_2.
- Node 4 has a fractional optimal solution with $\lfloor z(\text{LP}_4) \rfloor = 11$: it is branched on x_1.
- The up-branching node of Node 4 is not solved, and it is not drawn, since $\lfloor z(\text{LP}_4) \rfloor = 11 \leq \text{LB} = 11$ after its first child.
- Node 5 is pruned by infeasibility: the LP_5 problem has no feasible solution.
- Node 6 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_6) = (4, 1)$ is integer, with value 11, which becomes the incumbent solution.
- Node 7 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_7) = (3, 2)$ is integer, with value 12, which becomes the incumbent solution.

- Node 8 has a fractional optimal solution with $\lfloor z(\text{LP}_8) \rfloor = 13$: it is branched on x_1.
- Node 9 has a fractional optimal solution with $\lfloor z(\text{LP}_9) \rfloor = 13$: it is branched on x_2.
- The up-branching node of Node 9 is not solved, and it is not drawn, since $\lfloor z(\text{LP}_9) \rfloor = 13 \leq \text{LB} = 13$ after its first child.
- Node 10 is pruned by infeasibility: the LP_{10} problem has no feasible solution.
- Node 11 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_{11}) = (2, 3)$ is integer, with value 13, which becomes the incumbent solution.

The optimal value is $z(\text{ILP}) = 13$ and the branch-and-bound algorithm returns the optimal solution $\tilde{\boldsymbol{x}}(\text{ILP}) = (2, 3)$.

Exercise 7.6 — Branch-and-bound algorithm ($P6$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{array}{lrcrcrl} \max & x_1 & + & & x_2 & & \\ & 4\ x_1 & + & 15\ & x_2 & \leq & 70 \\ & 12\ x_1 & + & 3\ & x_2 & \leq & 52 \\ & x_1, & & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

and an optimal simplex dictionary of its LP relaxation in standard form:

$$\begin{cases} x_1 = \dfrac{95}{28} + \dfrac{1}{56}\ x_3 - \dfrac{5}{56}\ x_4 \\ x_2 = \dfrac{79}{21} - \dfrac{1}{14}\ x_3 + \dfrac{1}{42}\ x_4 \\ z = \dfrac{601}{84} - \dfrac{3}{56}\ x_3 - \dfrac{11}{168}\ x_4 \end{cases}$$

Questions

1. Execute the branch-and-bound algorithm using:
   (a) the smallest-index rule as branching-variable selection strategy;
   (b) the LIFO rule as branching-node selection strategy;
   (c) solve first the down-branching node and then the up-branching node;
   (d) set the lower bound LB to $-\infty$ (no heuristic algorithm is executed).

Solve the LP problems of the branching nodes using the dual simplex algorithm. In the dual simplex algorithm, the leaving variable is the basic variable with negative value and smallest index, and ties in the ratio test are broken by the smallest index. Show graphically the optimal solution of the LP problems associated with the branching nodes.

2. Determine and comment the branch-and-bound tree, showing the reason why each branching node is pruned.

Solution

1. Execution of the branch-and-bound algorithm to solve the ILP problem.

   - **Initialization**

     Let us consider the LP_1 problem in standard form:

$$
\begin{array}{lrcrcrcrcl}
\max & x_1 & + & & x_2 & & & & & \\
 & 4\,x_1 & + & 15\,x_2 & + & x_3 & & & = & 70 \\
 & 12\,x_1 & + & 3\,x_2 & & & + & x_4 & = & 52 \\
 & x_1, & & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}
$$

     and its optimal simplex dictionary:

$$
\begin{cases}
x_1 = \dfrac{95}{28} + \dfrac{1}{56}\,x_3 - \dfrac{5}{56}\,x_4 \\
x_2 = \dfrac{79}{21} - \dfrac{1}{14}\,x_3 + \dfrac{1}{42}\,x_4 \\
z = \dfrac{601}{84} - \dfrac{3}{56}\,x_3 - \dfrac{11}{168}\,x_4
\end{cases}
$$

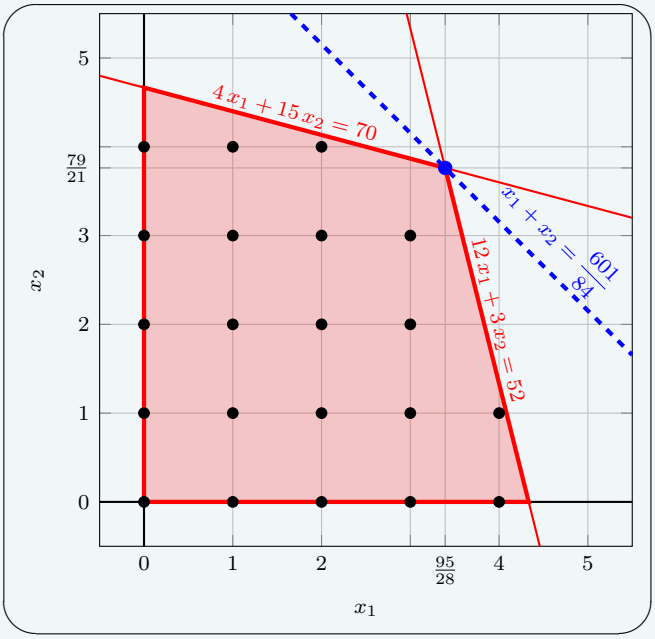


Figure 7.26: Node 1: feasible region of LP_1 (red), feasible points of the ILP problem (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is:

$$\tilde{x}_1 = \frac{95}{28}, \quad \tilde{x}_2 = \frac{79}{21}$$

the optimal value is:

$$z(\mathrm{LP}_1) = \frac{601}{84}$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is **fractional** and the LP_1 problem is **feasible**, so the LP_1 problem is inserted in the problem list L and the lower bound LB is set to $-\infty$.

Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \quad \leq \quad z(\mathrm{ILP}) \quad \leq \quad \underbrace{\left\lfloor \frac{601}{84} \right\rfloor}_{=7}$$

- **Iteration 1**

The LP_1 problem is extracted and deleted from the problem list L.

The branching variable is x_1. The branching constraints are:

$$x_1 \leq \underbrace{\left\lfloor \frac{95}{28} \right\rfloor}_{=3} \quad \text{and} \quad x_1 \geq \underbrace{\left\lfloor \frac{95}{28} \right\rfloor + 1}_{=4}$$

Two LP_2 and LP_3 problems are created with these branching constraints.

(a) Solution of the LP_2 problem

Let us consider the LP_2 problem in standard form:

$$
\begin{array}{lrcrcrcrcrcl}
\max & x_1 & + & x_2 & & & & & & & & \\
& 4\,x_1 & + & 15\,x_2 & + & x_3 & & & & & = & 70 \\
& 12\,x_1 & + & 3\,x_2 & & & + & x_4 & & & = & 52 \\
& x_1 & & & & & & & + & x_5 & = & 3 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
\end{array}
$$

For the branching constraint we have:

$$x_1 + x_5 = 3 \qquad \Rightarrow \qquad x_5 = 3 - x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_5 = 3 - \left(\frac{95}{28} + \frac{1}{56}\,x_3 - \frac{5}{56}\,x_4\right) \qquad \Rightarrow \qquad x_5 = -\frac{11}{28} - \frac{1}{56}\,x_3 + \frac{5}{56}\,x_4$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$
\begin{cases}
x_1 = \dfrac{95}{28} + \dfrac{1}{56}\,x_3 - \dfrac{5}{56}\,x_4 \\
x_2 = \dfrac{79}{21} - \dfrac{1}{14}\,x_3 + \dfrac{1}{42}\,x_4 \\
x_5 = -\dfrac{11}{28} - \dfrac{1}{56}\,x_3 + \dfrac{5}{56}\,x_4 \\
z = \dfrac{601}{84} - \dfrac{3}{56}\,x_3 - \dfrac{11}{168}\,x_4
\end{cases}
$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{95}{28}, \quad \bar{x}_2 = \frac{79}{21}, \quad \bar{x}_5 = -\frac{11}{28}, \qquad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad \frac{601}{84}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_5 = 0, \qquad \bar{\pi}_3 = \frac{3}{56}, \quad \bar{\pi}_4 = \frac{11}{168} \quad \text{of value} \quad \frac{601}{84}$$

If we increase the value of π_5, we can get a better dual solution. We have:

$$
\begin{cases}
\pi_3 = \dfrac{3}{56} + \dfrac{1}{56}\,\pi_5 \\
\pi_4 = \dfrac{11}{168} - \dfrac{5}{56}\,\pi_5
\end{cases}
\qquad
\begin{cases}
\underbrace{\dfrac{3}{56} + \dfrac{1}{56}\,\pi_5 \geq 0}_{\pi_3 \geq 0} \\
\underbrace{\dfrac{11}{168} - \dfrac{5}{56}\,\pi_5 \geq 0}_{\pi_4 \geq 0}
\end{cases}
\qquad
\begin{cases}
\pi_5 \geq -3 \\
\pi_5 \leq \dfrac{11}{15}
\end{cases}
$$

By increasing the value of π_5 to $\frac{11}{15}$, the value of π_4 decreases to 0. Then we have that:

π_5 becomes a dual basic variable and accordingly x_5 becomes a primal non-basic variable.

π_4 becomes a dual non-basic variable and accordingly x_4 becomes a primal basic variable.

We have:

$$x_5 = -\frac{11}{28} - \frac{1}{56}x_3 + \frac{5}{56}x_4 \quad \Rightarrow \quad x_4 = \frac{22}{5} + \frac{1}{5}x_3 + \frac{56}{5}x_5$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{95}{28} + \frac{1}{56}x_3 - \frac{5}{56}\left(\frac{22}{5} + \frac{1}{5}x_3 + \frac{56}{5}x_5\right) = 3 - x_5$$

$$x_2 = \frac{79}{21} - \frac{1}{14}x_3 + \frac{1}{42}\left(\frac{22}{5} + \frac{1}{5}x_3 + \frac{56}{5}x_5\right) = \frac{58}{15} - \frac{1}{15}x_3 + \frac{4}{15}x_5$$

$$z = \frac{601}{84} - \frac{3}{56}x_3 - \frac{11}{168}\left(\frac{22}{5} + \frac{1}{5}x_3 + \frac{56}{5}x_5\right) = \frac{103}{15} - \frac{1}{15}x_3 - \frac{11}{15}x_5$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 3 - x_5 \\ x_2 = \frac{58}{15} - \frac{1}{15}x_3 + \frac{4}{15}x_5 \\ x_4 = \frac{22}{5} + \frac{1}{5}x_3 + \frac{56}{5}x_5 \\ z = \frac{103}{15} - \frac{1}{15}x_3 - \frac{11}{15}x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4\}$ and $\mathcal{N} = \{3, 5\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 3, \quad \tilde{x}_2 = \frac{58}{15}, \quad \tilde{x}_4 = \frac{22}{5}, \qquad \tilde{x}_3 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad \frac{103}{15}$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_4 = 0, \qquad \tilde{\pi}_3 = \frac{1}{15}, \quad \tilde{\pi}_5 = \frac{11}{15} \quad \text{of value} \quad \frac{103}{15}$$

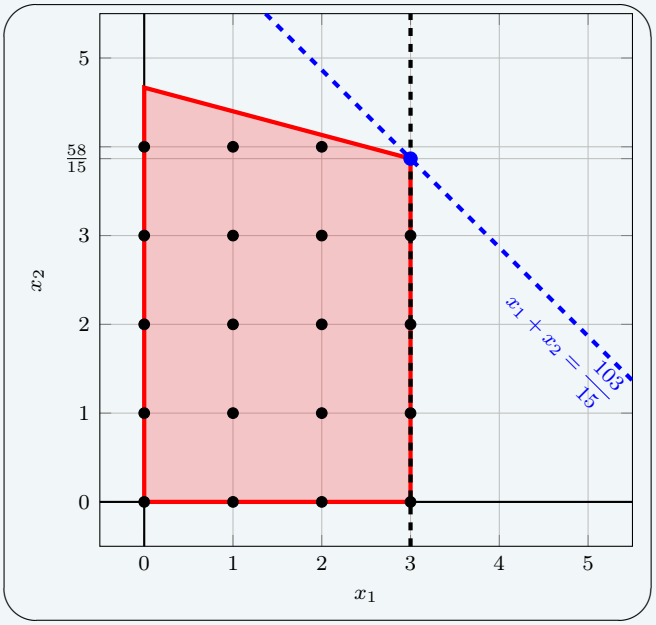


Figure 7.27: Node 2: feasible region of LP_2 (red), with the branching constraint $x_1 \leq 3$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is:

$$\tilde{x}_1 = 3, \quad \tilde{x}_2 = \frac{58}{15}$$

the optimal value is:

$$z(\mathrm{LP}_2) = \frac{103}{15}$$

The LP_2 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_2) \rfloor > \mathrm{LB} \qquad \text{since} \qquad \underbrace{\left\lfloor \frac{103}{15} \right\rfloor}_{=6} > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is **fractional**, accordingly, the LP_2 problem is **inserted in the problem list** L.

(b) Solution of the LP_3 problem

Let us consider the LP_3 problem in standard form:

$$\begin{array}{lrcrcrcrcrcl}
\max & x_1 & + & x_2 & & & & & & & & \\
& 4\ x_1 & + & 15\ x_2 & + & x_3 & & & & & = & 70 \\
& 12\ x_1 & + & 3\ x_2 & & & + & x_4 & & & = & 52 \\
& x_1 & & & & & & & - & x_5 & = & 4 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_1 - x_5 = 4 \qquad \Rightarrow \qquad x_5 = -4 + x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_5 = -4 + \left(\frac{95}{28} + \frac{1}{56}\,x_3 - \frac{5}{56}\,x_4\right) \quad \Rightarrow \quad x_5 = -\frac{17}{28} + \frac{1}{56}\,x_3 - \frac{5}{56}\,x_4$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = \frac{95}{28} + \frac{1}{56}\,x_3 - \frac{5}{56}\,x_4 \\ x_2 = \frac{79}{21} - \frac{1}{14}\,x_3 + \frac{1}{42}\,x_4 \\ x_5 = -\frac{17}{28} + \frac{1}{56}\,x_3 - \frac{5}{56}\,x_4 \\ z = \frac{601}{84} - \frac{3}{56}\,x_3 - \frac{11}{168}\,x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{95}{28}, \quad \bar{x}_2 = \frac{79}{21}, \quad \bar{x}_5 = -\frac{17}{28}, \qquad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad \frac{601}{84}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_5 = 0, \qquad \bar{\pi}_3 = \frac{3}{56}, \quad \bar{\pi}_4 = \frac{11}{168} \quad \text{of value} \quad \frac{601}{84}$$

If we increase the value of π_5, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 = \frac{3}{56} - \frac{1}{56}\,\pi_5 \\ \pi_4 = \frac{11}{168} + \frac{5}{56}\,\pi_5 \end{cases} \qquad \begin{cases} \underbrace{\frac{3}{56} - \frac{1}{56}\,\pi_5 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{11}{168} + \frac{5}{56}\,\pi_5 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_5 \leq 3 \\ \pi_5 \geq -\frac{11}{15} \end{cases}$$

By increasing the value of π_5 to 3, the value of π_3 decreases to 0. Then we have that:

π_5 becomes a dual basic variable and accordingly x_5 becomes a primal non-basic variable.

π_3 becomes a dual non-basic variable and accordingly x_3 becomes a primal basic variable.

We have:

$$x_5 = -\frac{17}{28} + \frac{1}{56}\,x_3 - \frac{5}{56}\,x_4 \quad \Rightarrow \quad x_3 = 34 + 5\,x_4 + 56\,x_5$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{95}{28} + \frac{1}{56}\,(34 + 5\,x_4 + 56\,x_5) - \frac{5}{56}\,x_4 = 4 + x_5$$

$$x_2 = \frac{79}{21} - \frac{1}{14}(34 + 5\,x_4 + 56\,x_5) + \frac{1}{42}\,x_4 = \frac{4}{3} - \frac{1}{3}\,x_4 - 4\,x_5$$
$$z = \frac{601}{84} - \frac{3}{56}(34 + 5\,x_4 + 56\,x_5) - \frac{11}{168}\,x_4 = \frac{16}{3} - \frac{1}{3}\,x_4 - 3\,x_5$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 4 + x_5 \\ x_2 = \frac{4}{3} - \frac{1}{3}\,x_4 - 4\,x_5 \\ x_3 = 34 + 5\,x_4 + 56\,x_5 \\ z = \frac{16}{3} - \frac{1}{3}\,x_4 - 3\,x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = \frac{4}{3}, \quad \tilde{x}_3 = 34, \qquad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad \frac{16}{3}$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \qquad \tilde{\pi}_4 = \frac{1}{3}, \quad \tilde{\pi}_5 = 3 \quad \text{of value} \quad \frac{16}{3}$$

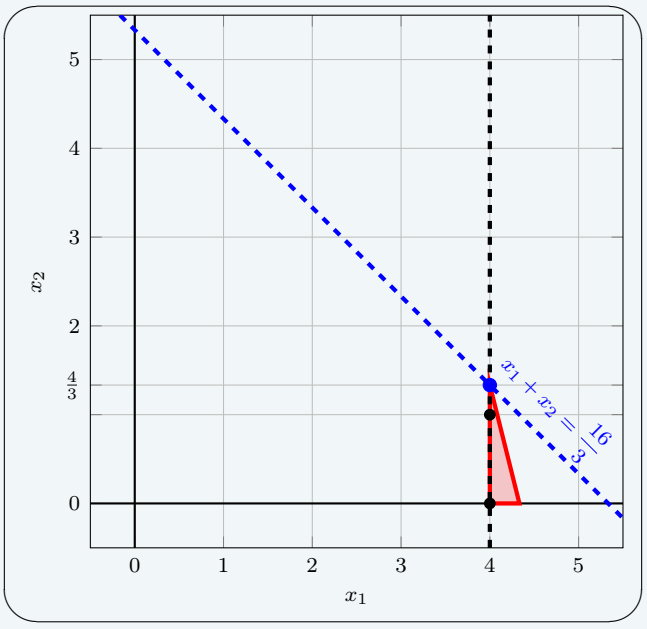


Figure 7.28: Node 3: feasible region of LP_3 (red), with the branching constraint $x_1 \geq 4$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_3)$ is:

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = \frac{4}{3}$$

the optimal value is:

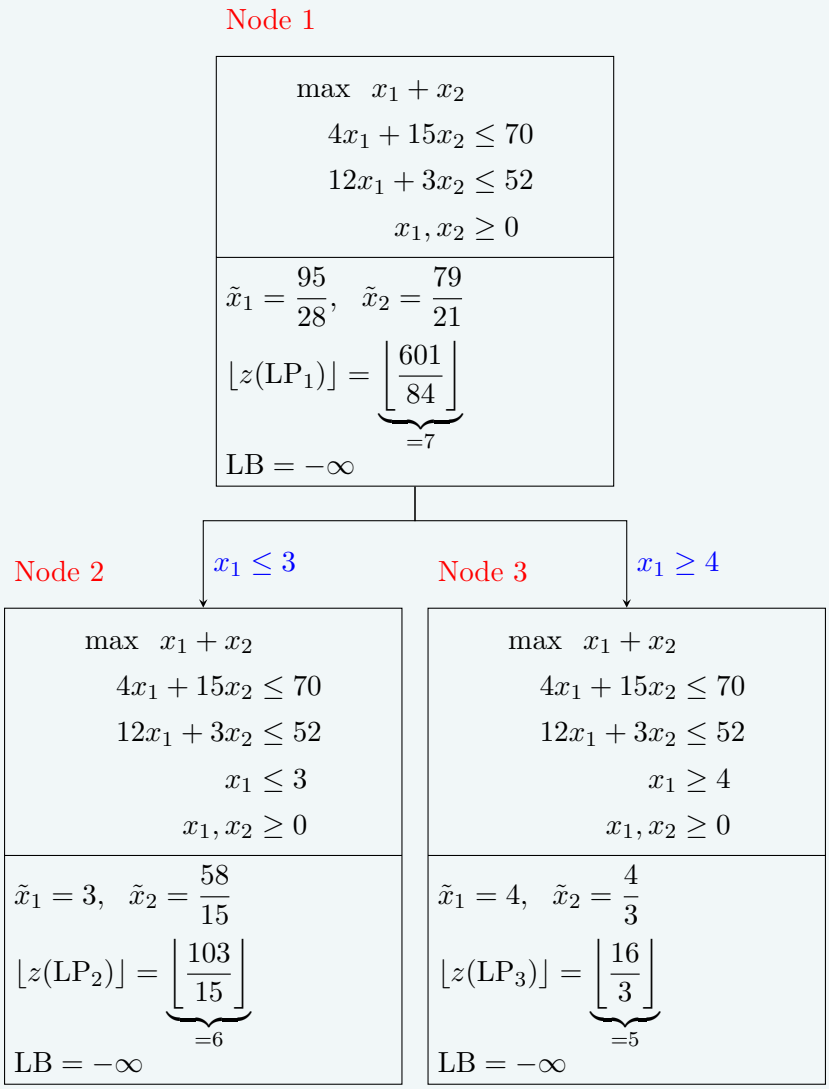
$$z(\mathrm{LP}_3) = \frac{16}{3}$$

The LP_3 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_3) \rfloor > \mathrm{LB} \qquad \text{since} \qquad \underbrace{\left\lfloor \frac{16}{3} \right\rfloor}_{=5} > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_3)$ is **fractional**, accordingly, the LP_3 problem is **inserted in the problem list** L.

At the end of the first iteration, branching on x_1 from Node 1 has generated the down-branching node, Node 2, and the up-branching node, Node 3. Both nodes are inserted in the list L. The corresponding branch-and-bound tree is:



Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \;\leq\; z(\text{ILP}) \;\leq\; \underbrace{\left\lfloor \frac{103}{15} \right\rfloor}_{=6}$$

- **Iteration 2**

The LP_3 problem is extracted and deleted from the problem list L.

The branching variable is x_2. The branching constraints are:

$$x_2 \leq \underbrace{\left\lfloor \frac{4}{3} \right\rfloor}_{=1} \quad \text{and} \quad x_2 \geq \underbrace{\left\lfloor \frac{4}{3} \right\rfloor + 1}_{=2}$$

Two LP_4 and LP_5 problems are created with these branching constraints.

(a) Solution of the LP_4 problem

Let us consider the LP_4 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcr}
\max & x_1 & + & x_2 & & & & & & & & & \\
& 4\,x_1 & + & 15\,x_2 & + & x_3 & & & & & & = & 70 \\
& 12\,x_1 & + & 3\,x_2 & & & + & x_4 & & & & = & 52 \\
& x_1 & & & & & & & - & x_5 & & = & 4 \\
& & & x_2 & & & & & & & + \; x_6 & = & 1 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_2 + x_6 = 1 \qquad \Rightarrow \qquad x_6 = 1 - x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_6 = 1 - \left(\frac{4}{3} - \frac{1}{3}\, x_4 - 4\, x_5 \right) \qquad \Rightarrow \qquad x_6 = -\frac{1}{3} + \frac{1}{3}\, x_4 + 4\, x_5$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 &= 4 &&&& + & x_5 \\ x_2 &= \frac{4}{3} & - & \frac{1}{3} & x_4 & - & 4\ x_5 \\ x_3 &= 34 & + & 5 & x_4 & + & 56\ x_5 \\ x_6 &= -\frac{1}{3} & + & \frac{1}{3} & x_4 & + & 4\ x_5 \\ z &= \frac{16}{3} & - & \frac{1}{3} & x_4 & - & 3\ x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 6\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = 4, \quad \bar{x}_2 = \frac{4}{3}, \quad \bar{x}_3 = 34, \quad \bar{x}_6 = -\frac{1}{3},$$
$$\bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{16}{3}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_6 = 0,$$
$$\bar{\pi}_4 = \frac{1}{3}, \quad \bar{\pi}_5 = 3 \quad \text{of value} \quad \frac{16}{3}$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_4 &= \frac{1}{3} - \frac{1}{3}\,\pi_6 \\ \pi_5 &= 3 - 4\,\pi_6 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{3} - \frac{1}{3}\,\pi_6 \geq 0}_{\pi_4 \geq 0} \\ \underbrace{3 - 4\,\pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 &\leq 1 \\ \pi_6 &\leq \frac{3}{4} \end{cases}$$

By increasing the value of π_6 to $\min\left\{1, \frac{3}{4}\right\} = \frac{3}{4}$, the value of π_5 decreases to 0. Then we have that:

π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non-basic variable.

π_5 becomes a dual non-basic variable and accordingly x_5 becomes a primal basic variable.

We have:

$$x_6 = -\frac{1}{3} + \frac{1}{3}\,x_4 + 4\,x_5 \quad \Rightarrow \quad x_5 = \frac{1}{12} - \frac{1}{12}\,x_4 + \frac{1}{4}\,x_6$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = 4 + \left(\frac{1}{12} - \frac{1}{12}\,x_4 + \frac{1}{4}\,x_6\right) = \frac{49}{12} - \frac{1}{12}\,x_4 + \frac{1}{4}\,x_6$$

$$x_2 = \frac{4}{3} - \frac{1}{3}\,x_4 - 4\left(\frac{1}{12} - \frac{1}{12}\,x_4 + \frac{1}{4}\,x_6\right) = 1 - x_6$$

$$x_3 = 34 + 5\,x_4 + 56\left(\frac{1}{12} - \frac{1}{12}\,x_4 + \frac{1}{4}\,x_6\right) = \frac{116}{3} + \frac{1}{3}\,x_4 + 14\,x_6$$

$$z = \frac{16}{3} - \frac{1}{3}\,x_4 - 3\left(\frac{1}{12} - \frac{1}{12}\,x_4 + \frac{1}{4}\,x_6\right) = \frac{61}{12} - \frac{1}{12}\,x_4 - \frac{3}{4}\,x_6$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = \frac{49}{12} - \frac{1}{12}\,x_4 + \frac{1}{4}\,x_6 \\ x_2 = 1 - x_6 \\ x_3 = \frac{116}{3} + \frac{1}{3}\,x_4 + 14\,x_6 \\ x_5 = \frac{1}{12} - \frac{1}{12}\,x_4 + \frac{1}{4}\,x_6 \\ z = \frac{61}{12} - \frac{1}{12}\,x_4 - \frac{3}{4}\,x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 5\}$ and $\mathcal{N} = \{4, 6\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = \frac{49}{12}, \quad \tilde{x}_2 = 1, \quad \tilde{x}_3 = \frac{116}{3}, \quad \tilde{x}_5 = \frac{1}{12},$$
$$\tilde{x}_4 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad \frac{61}{12}$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_5 = 0,$$
$$\tilde{\pi}_4 = \frac{1}{12}, \quad \tilde{\pi}_6 = \frac{3}{4} \quad \text{of value} \quad \frac{61}{12}$$

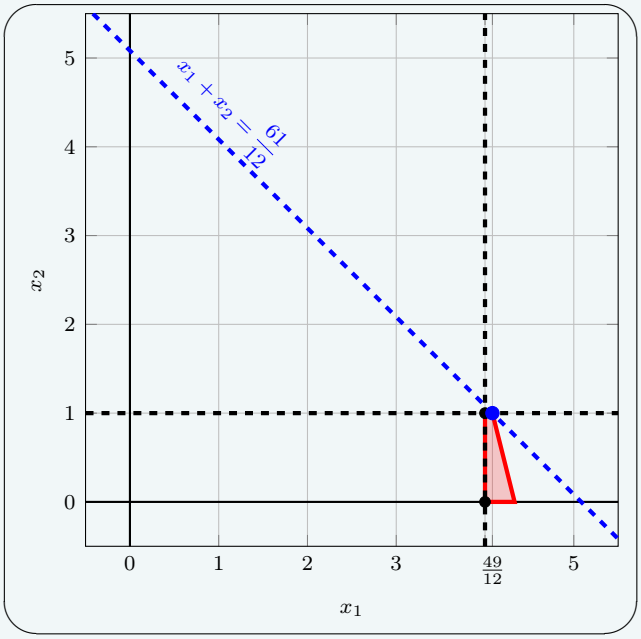


Figure 7.29: Node 4: feasible region of LP_4 (red), with the branching constraint $x_2 \leq 1$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_4)$ is:

$$\tilde{x}_1 = \frac{49}{12}, \quad \tilde{x}_2 = 1$$

the optimal value is:

$$z(\mathrm{LP}_4) = \frac{61}{12}$$

The LP_4 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_4) \rfloor > \mathrm{LB} \qquad \text{since} \qquad \underbrace{\left\lfloor \frac{61}{12} \right\rfloor}_{=5} > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_4)$ is **fractional**, accordingly, the LP_4 problem is **inserted in the problem list** L.

(b) Solution of the LP_5 problem

Let us consider the LP_5 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcr}
\max & x_1 & + & x_2 & & & & & & & & & \\
& 4\,x_1 & + & 15\,x_2 & + & x_3 & & & & & & = & 70 \\
& 12\,x_1 & + & 3\,x_2 & & & + & x_4 & & & & = & 52 \\
& x_1 & & & & & & & - & x_5 & & = & 4 \\
& & & x_2 & & & & & & & - & x_6 = & 2 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_2 - x_6 = 2 \qquad \Rightarrow \qquad x_6 = -2 + x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_6 = -2 + \left(\frac{4}{3} - \frac{1}{3}\,x_4 - 4\,x_5\right) \qquad \Rightarrow \qquad x_6 = -\frac{2}{3} - \frac{1}{3}\,x_4 - 4\,x_5$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = 4 + x_5 \\ x_2 = \frac{4}{3} - \frac{1}{3}\,x_4 - 4\,x_5 \\ x_3 = 34 + 5\,x_4 + 56\,x_5 \\ x_6 = -\frac{2}{3} - \frac{1}{3}\,x_4 - 4\,x_5 \\ z = \frac{16}{3} - \frac{1}{3}\,x_4 - 3\,x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 6\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = 4, \quad \bar{x}_2 = \frac{4}{3}, \quad \bar{x}_3 = 34, \quad \bar{x}_6 = -\frac{2}{3},$$
$$\bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{16}{3}$$

- the basic dual feasible solution is:

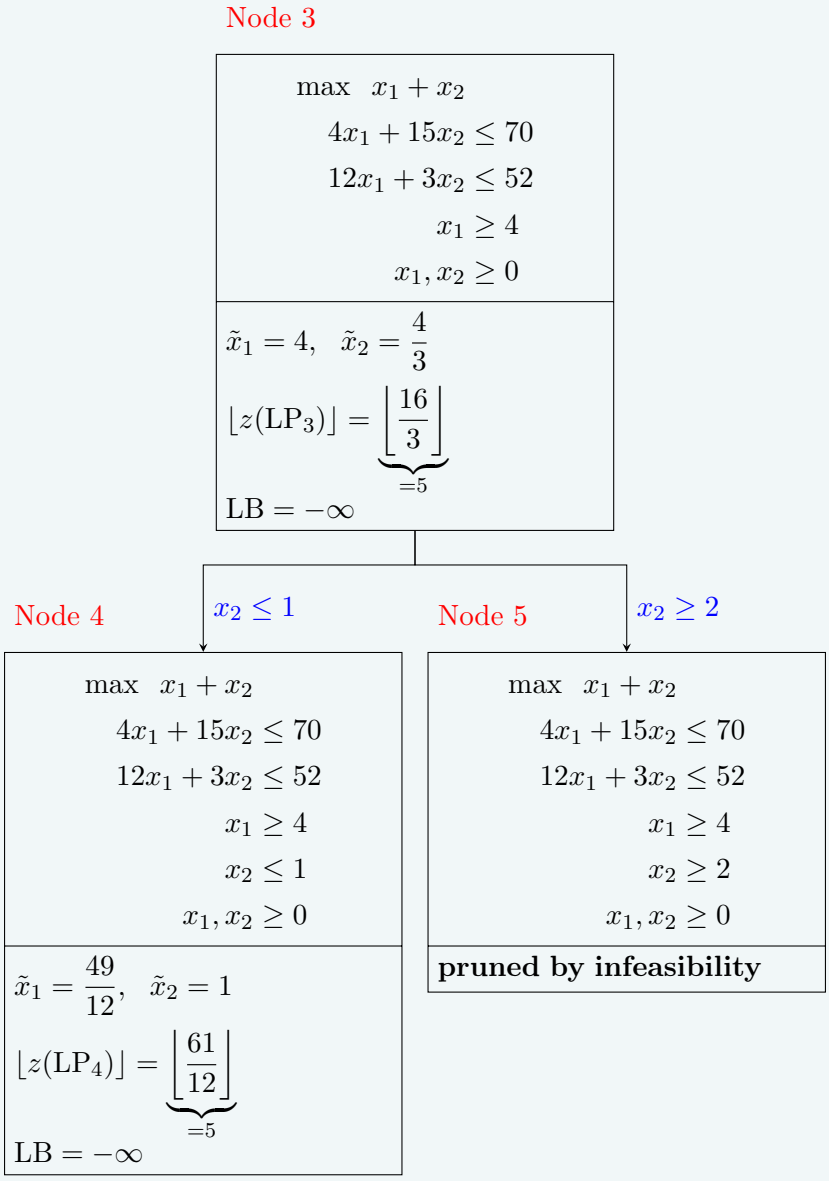
$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_6 = 0,$$
$$\bar{\pi}_4 = \frac{1}{3}, \quad \bar{\pi}_5 = 3 \quad \text{of value} \quad \frac{16}{3}$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_4 = \frac{1}{3} + \frac{1}{3}\,\pi_6 \\ \pi_5 = 3 + 4\,\pi_6 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{3} + \frac{1}{3}\,\pi_6 \geq 0}_{\pi_4 \geq 0} \\ \underbrace{3 + 4\,\pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 \geq -1 \\ \pi_6 \geq -\frac{3}{4} \end{cases}$$

Accordingly, we can increase the value of π_6 without violating dual feasibility, which implies that the dual problem is **unbounded**. As a result, the primal problem LP_5 is **infeasible**, and the corresponding branching node is pruned by infeasibility.

At the end of the second iteration, branching on x_2 from Node 3 has generated the down-branching node, Node 4, and the up-branching node, Node 5. Node 4 is inserted in the list L, while Node 5 is pruned by infeasibility. The corresponding branch-and-bound tree is:



Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \quad \leq \quad z(\text{ILP}) \quad \leq \quad \underbrace{\left\lfloor \frac{103}{15} \right\rfloor}_{=6}$$

- **Iteration 3**

The LP_4 problem is extracted and deleted from the problem list L.

The branching variable is x_1. The branching constraints are:

$$x_1 \leq \underbrace{\left\lfloor \frac{49}{12} \right\rfloor}_{=4} \quad \text{and} \quad x_1 \geq \underbrace{\left\lfloor \frac{49}{12} \right\rfloor + 1}_{=5}$$

The LP_6 problem is created with the first branching constraint (the second child is solved only if needed).

(a) Solution of the LP_6 problem

Let us consider the LP_6 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcl}
\max & x_1 & + & x_2 & & & & & & & & & & \\
& 4\,x_1 & + & 15\,x_2 & + & x_3 & & & & & & & = & 70 \\
& 12\,x_1 & + & 3\,x_2 & & & + & x_4 & & & & & = & 52 \\
& x_1 & & & & & & & - & x_5 & & & = & 4 \\
& & & x_2 & & & & & & & + & x_6 & = & 1 \\
& x_1 & & & & & & & & & & & + & x_7 & = & 4 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & & x_7 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_1 + x_7 = 4 \qquad \Rightarrow \qquad x_7 = 4 - x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_7 = 4 - \left(\frac{49}{12} - \frac{1}{12}\, x_4 + \frac{1}{4}\, x_6 \right) \qquad \Rightarrow \qquad x_7 = -\frac{1}{12} + \frac{1}{12}\, x_4 - \frac{1}{4}\, x_6$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases}
x_1 = \frac{49}{12} - \frac{1}{12}\, x_4 + \frac{1}{4}\, x_6 \\
x_2 = 1 - x_6 \\
x_3 = \frac{116}{3} + \frac{1}{3}\, x_4 + 14\, x_6 \\
x_5 = \frac{1}{12} - \frac{1}{12}\, x_4 + \frac{1}{4}\, x_6 \\
x_7 = -\frac{1}{12} + \frac{1}{12}\, x_4 - \frac{1}{4}\, x_6 \\
z = \frac{61}{12} - \frac{1}{12}\, x_4 - \frac{3}{4}\, x_6
\end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 5, 7\}$ and $\mathcal{N} = \{4, 6\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{49}{12}, \quad \bar{x}_2 = 1, \quad \bar{x}_3 = \frac{116}{3}, \quad \bar{x}_5 = \frac{1}{12}, \quad \bar{x}_7 = -\frac{1}{12},$$
$$\bar{x}_4 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad \frac{61}{12}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_5 = 0, \quad \bar{\pi}_7 = 0,$$
$$\bar{\pi}_4 = \frac{1}{12}, \quad \bar{\pi}_6 = \frac{3}{4} \quad \text{of value} \quad \frac{61}{12}$$

If we increase the value of π_7, we can get a better dual solution. We have:

$$\begin{cases} \pi_4 = \frac{1}{12} - \frac{1}{12}\pi_7 \\ \pi_6 = \frac{3}{4} + \frac{1}{4}\pi_7 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{12} - \frac{1}{12}\pi_7 \geq 0}_{\pi_4 \geq 0} \\ \underbrace{\frac{3}{4} + \frac{1}{4}\pi_7 \geq 0}_{\pi_6 \geq 0} \end{cases} \qquad \begin{cases} \pi_7 \leq 1 \\ \pi_7 \geq -3 \end{cases}$$

By increasing the value of π_7 to 1, the value of π_4 decreases to 0. Then we have that:

π_7 becomes a dual basic variable and accordingly x_7 becomes a primal non-basic variable.

π_4 becomes a dual non-basic variable and accordingly x_4 becomes a primal basic variable.

We have:

$$x_7 = -\frac{1}{12} + \frac{1}{12}x_4 - \frac{1}{4}x_6 \quad \Rightarrow \quad x_4 = 1 + 3x_6 + 12x_7$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{49}{12} - \frac{1}{12}(1 + 3x_6 + 12x_7) + \frac{1}{4}x_6 = 4 - x_7$$
$$x_3 = \frac{116}{3} + \frac{1}{3}(1 + 3x_6 + 12x_7) + 14x_6 = 39 + 15x_6 + 4x_7$$
$$x_5 = \frac{1}{12} - \frac{1}{12}(1 + 3x_6 + 12x_7) + \frac{1}{4}x_6 = 0 - x_7$$
$$z = \frac{61}{12} - \frac{1}{12}(1 + 3x_6 + 12x_7) - \frac{3}{4}x_6 = 5 - x_6 - x_7$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 4 - x_7 \\ x_2 = 1 - x_6 \\ x_3 = 39 + 15\,x_6 + 4\,x_7 \\ x_4 = 1 + 3\,x_6 + 12\,x_7 \\ x_5 = 0 - x_7 \\ z = 5 - x_6 - x_7 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4, 5\}$ and $\mathcal{N} = \{6, 7\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = 1, \quad \tilde{x}_3 = 39, \quad \tilde{x}_4 = 1, \quad \tilde{x}_5 = 0,$$
$$\tilde{x}_6 = 0, \quad \tilde{x}_7 = 0 \quad \text{of value} \quad 5$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 0, \quad \tilde{\pi}_5 = 0,$$
$$\tilde{\pi}_6 = 1, \quad \tilde{\pi}_7 = 1 \quad \text{of value} \quad 5$$

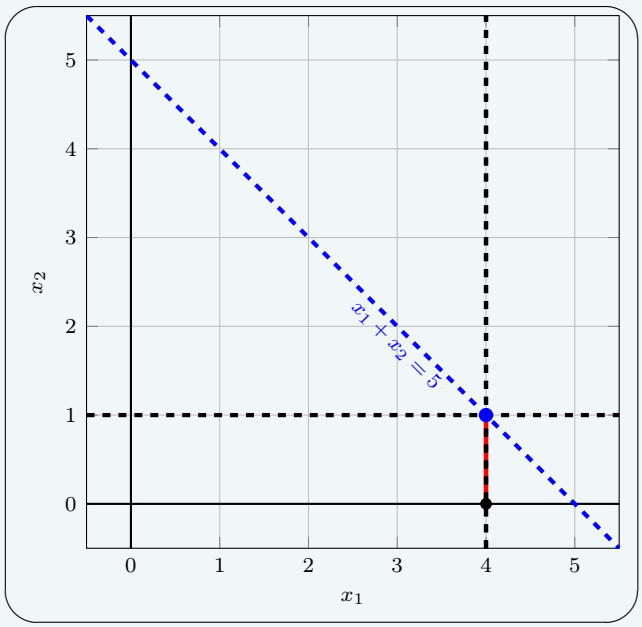


Figure 7.30: Node 6: feasible region of LP_6 (red), with the branching constraint $x_1 \leq 4$ (dashed), and its optimal solution (blue).
An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_6)$ is:

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = 1$$

the optimal value is:

$$z(\mathrm{LP}_6) = 5$$

The LP_6 problem is **feasible** and we have

$$\lfloor z(\text{LP}_6) \rfloor > \text{LB} \qquad \text{since} \qquad 5 > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_6)$ is **integer**, accordingly, the lower bound LB becomes 5 and the incumbent solution $\bar{\boldsymbol{x}}(\text{ILP})$ becomes $\bar{x}_1 = 4, \bar{x}_2 = 1$.
Accordingly, the branching node is pruned by integrality.

At the end of the third iteration, branching on x_1 from Node 4 has generated the down-branching node, Node 6. This node is pruned by integrality. The up-branching node is not solved, since after Node 6 the lower bound satisfies

$$\lfloor z(\text{LP}_4) \rfloor \leq \text{LB}.$$

This is the portion of the branch-and-bound tree showing the LP problems of the branching nodes that are solved:

Node 4

$$\begin{aligned} \max \quad & x_1 + x_2 \\ & 4x_1 + 15x_2 \leq 70 \\ & 12x_1 + 3x_2 \leq 52 \\ & x_1 \geq 4 \\ & x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\tilde{x}_1 = \frac{49}{12}, \quad \tilde{x}_2 = 1$$

$$\lfloor z(\text{LP}_4) \rfloor = \underbrace{\left\lfloor \frac{61}{12} \right\rfloor}_{=5}$$

$$\text{LB} = -\infty$$

$x_1 \leq 4$

Node 6

$$\begin{aligned} \max \quad & x_1 + x_2 \\ & 4x_1 + 15x_2 \leq 70 \\ & 12x_1 + 3x_2 \leq 52 \\ & x_1 \geq 4 \\ & x_2 \leq 1 \\ & x_1 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = 1$$

$$z(\text{LP}_6) = 5$$

$$\text{LB} = 5$$

pruned by integrality

Due to the current lower bound LB and upper bound UB, we have:

$$5 \leq z(\text{ILP}) \leq \underbrace{\left\lfloor \frac{103}{15} \right\rfloor}_{=6}$$

- **Iteration 4**

The LP_2 problem is extracted and deleted from the problem list L.

The branching variable is x_2. The branching constraints are:

$$x_2 \leq \underbrace{\left\lfloor \frac{58}{15} \right\rfloor}_{=3} \quad \text{and} \quad x_2 \geq \underbrace{\left\lfloor \frac{58}{15} \right\rfloor + 1}_{=4}$$

The LP_7 problem is created with the first branching constraint (the second child is solved only if needed).

(a) Solution of the LP_7 problem

Let us consider the LP_7 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcr}
\max & x_1 & + & x_2 & & & & & & & & & \\
& 4\,x_1 & + & 15\,x_2 & + & x_3 & & & & & & = & 70 \\
& 12\,x_1 & + & 3\,x_2 & & & + & x_4 & & & & = & 52 \\
& x_1 & & & & & & & + & x_5 & & = & 3 \\
& & & x_2 & & & & & & & + & x_6 & = & 3 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_2 + x_6 = 3 \quad \Rightarrow \quad x_6 = 3 - x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_6 = 3 - \left(\frac{58}{15} - \frac{1}{15}\, x_3 + \frac{4}{15}\, x_5 \right) \quad \Rightarrow \quad x_6 = -\frac{13}{15} + \frac{1}{15}\, x_3 - \frac{4}{15}\, x_5$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = 3 - x_5 \\ x_2 = \frac{58}{15} - \frac{1}{15}x_3 + \frac{4}{15}x_5 \\ x_4 = \frac{22}{5} + \frac{1}{5}x_3 + \frac{56}{5}x_5 \\ x_6 = -\frac{13}{15} + \frac{1}{15}x_3 - \frac{4}{15}x_5 \\ z = \frac{103}{15} - \frac{1}{15}x_3 - \frac{11}{15}x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4, 6\}$ and $\mathcal{N} = \{3, 5\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = 3, \quad \bar{x}_2 = \frac{58}{15}, \quad \bar{x}_4 = \frac{22}{5}, \quad \bar{x}_6 = -\frac{13}{15},$$
$$\bar{x}_3 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{103}{15}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_6 = 0,$$
$$\bar{\pi}_3 = \frac{1}{15}, \quad \bar{\pi}_5 = \frac{11}{15} \quad \text{of value} \quad \frac{103}{15}$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 = \frac{1}{15} - \frac{1}{15}\pi_6 \\ \pi_5 = \frac{11}{15} + \frac{4}{15}\pi_6 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{15} - \frac{1}{15}\pi_6 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{11}{15} + \frac{4}{15}\pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 \leq 1 \\ \pi_6 \geq -\frac{11}{4} \end{cases}$$

By increasing the value of π_6 to 1, the value of π_3 decreases to 0. Then we have that:

> π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non-basic variable.
>
> π_3 becomes a dual non-basic variable and accordingly x_3 becomes a primal basic variable.

We have:

$$x_6 = -\frac{13}{15} + \frac{1}{15}x_3 - \frac{4}{15}x_5 \quad \Rightarrow \quad x_3 = 13 + 4x_5 + 15x_6$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_2 = \frac{58}{15} - \frac{1}{15}(13 + 4x_5 + 15x_6) + \frac{4}{15}x_5 = 3 - x_6$$

$$x_4 = \frac{22}{5} + \frac{1}{5}(13 + 4\,x_5 + 15\,x_6) + \frac{56}{5}\,x_5 = 7 + 12\,x_5 + 3\,x_6$$
$$z = \frac{103}{15} - \frac{1}{15}(13 + 4\,x_5 + 15\,x_6) - \frac{11}{15}\,x_5 = 6 - x_5 - x_6$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 3 - x_5 \\ x_2 = 3 - x_6 \\ x_3 = 13 + 4\,x_5 + 15\,x_6 \\ x_4 = 7 + 12\,x_5 + 3\,x_6 \\ z = 6 - x_5 - x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4\}$ and $\mathcal{N} = \{5, 6\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 3, \quad \tilde{x}_2 = 3, \quad \tilde{x}_3 = 13, \quad \tilde{x}_4 = 7,$$
$$\tilde{x}_5 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad 6$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 0,$$
$$\tilde{\pi}_5 = 1, \quad \tilde{\pi}_6 = 1 \quad \text{of value} \quad 6$$

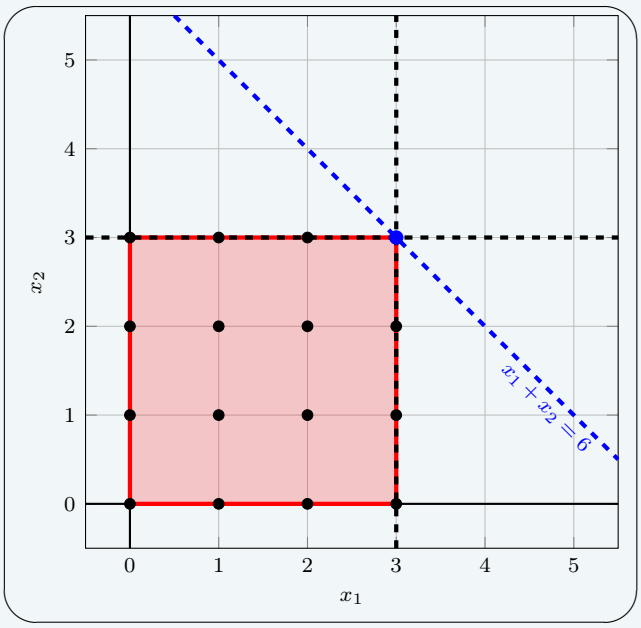


Figure 7.31: Node 7: feasible region of LP_7 (red), with the branching constraint $x_2 \leq 3$ (dashed), and its optimal solution (blue).
An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_7)$ is:

$$\tilde{x}_1 = 3, \quad \tilde{x}_2 = 3$$

the optimal value is:

$$z(\mathrm{LP}_7) = 6$$

The LP_7 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_7) \rfloor > \mathrm{LB} \qquad \text{since} \qquad 6 > 5$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_7)$ is **integer**, accordingly, the lower bound LB becomes 6 and the incumbent solution $\bar{\boldsymbol{x}}(\mathrm{ILP})$ becomes $\bar{x}_1 = 3, \bar{x}_2 = 3$.
Accordingly, the branching node is pruned by integrality.

At the end of the fourth iteration, branching on x_2 from Node 2 has generated the down-branching node, Node 7. This node is pruned by integrality. The up-branching node is not solved, since after Node 7 the lower bound satisfies

$$\lfloor z(\mathrm{LP}_2) \rfloor \leq \mathrm{LB}.$$

This is the portion of the branch-and-bound tree showing the LP problems of the branching nodes that are solved:

Node 2

$$\begin{aligned} \max \quad & x_1 + x_2 \\ & 4x_1 + 15x_2 \leq 70 \\ & 12x_1 + 3x_2 \leq 52 \\ & x_1 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\tilde{x}_1 = 3, \quad \tilde{x}_2 = \dfrac{58}{15}$

$\lfloor z(\mathrm{LP}_2) \rfloor = \underbrace{\left\lfloor \dfrac{103}{15} \right\rfloor}_{=6}$

$\mathrm{LB} = -\infty$

$x_2 \leq 3$

Node 7

$$\begin{aligned} \max \quad & x_1 + x_2 \\ & 4x_1 + 15x_2 \leq 70 \\ & 12x_1 + 3x_2 \leq 52 \\ & x_1 \leq 3 \\ & x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\tilde{x}_1 = 3, \quad \tilde{x}_2 = 3$

$z(\mathrm{LP}_7) = 6$

$\mathrm{LB} = 6$

pruned by integrality

Due to the current lower bound LB and upper bound UB, we have:

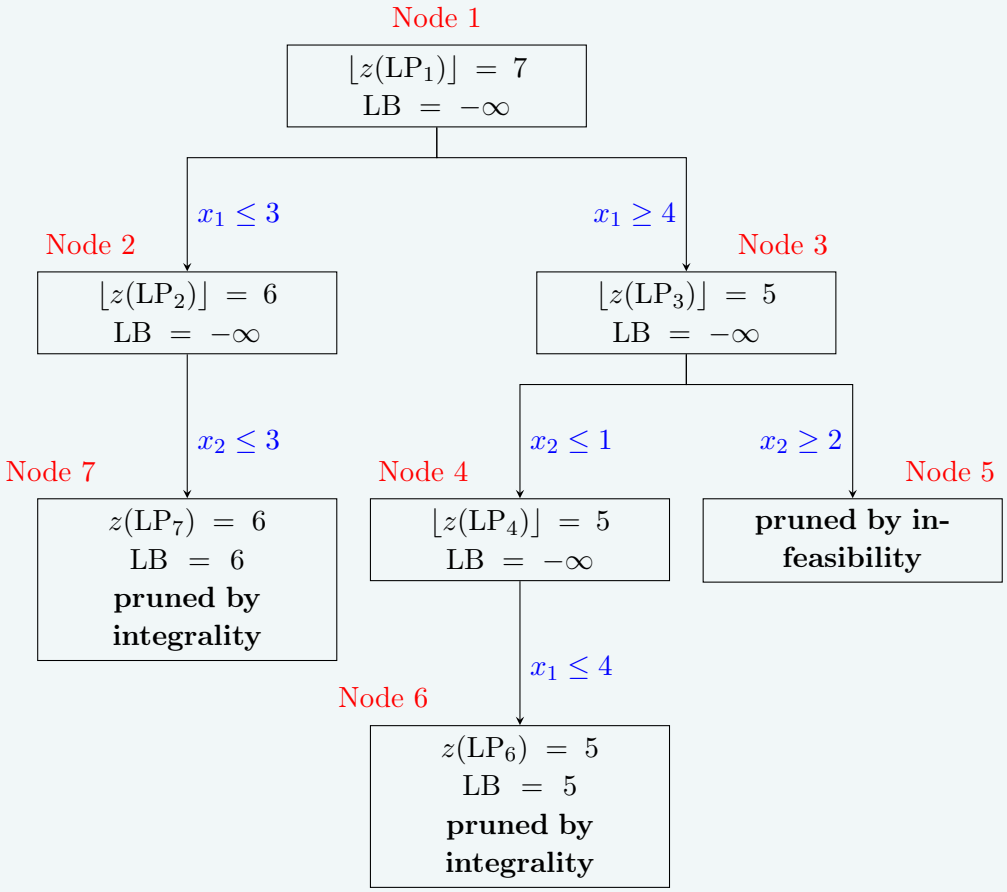
$$6 \quad \leq \quad z(\mathrm{ILP}) \quad \leq \quad 6$$

- **Termination**

Since the list L is empty, the lower bound LB is equal to the upper bound UB and the ILP problem is solved to optimality.

The optimal value is $z(\mathrm{ILP}) = 6$ (the value of the lower bound LB) and an optimal solution $\tilde{\boldsymbol{x}}(\mathrm{ILP})$ is $\tilde{x}_1 = 3, \tilde{x}_2 = 3$ (the incumbent solution $\bar{\boldsymbol{x}}(\mathrm{ILP})$).

2. The final branch-and-bound tree is:



The reasons why the branching nodes are pruned are the following.

- Node 1 has a fractional optimal solution with $\lfloor z(\text{LP}_1) \rfloor = 7$: it is branched on x_1.
- Node 2 has a fractional optimal solution with $\lfloor z(\text{LP}_2) \rfloor = 6$: it is branched on x_2.
- The up-branching node of Node 2 is not solved, and it is not drawn, since $\lfloor z(\text{LP}_2) \rfloor = 6 \leq \text{LB} = 6$ after its first child.
- Node 3 has a fractional optimal solution with $\lfloor z(\text{LP}_3) \rfloor = 5$: it is branched on x_2.
- Node 4 has a fractional optimal solution with $\lfloor z(\text{LP}_4) \rfloor = 5$: it is branched on x_1.
- The up-branching node of Node 4 is not solved, and it is not drawn, since $\lfloor z(\text{LP}_4) \rfloor = 5 \leq \text{LB} = 5$ after its first child.
- Node 5 is pruned by infeasibility: the LP_5 problem has no feasible solution.
- Node 6 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_6) = (4, 1)$ is integer, with value 5, which becomes the incumbent solution.
- Node 7 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_7) = (3, 3)$ is integer, with value 6, which becomes the incumbent solution.

The optimal value is $z(\text{ILP}) = 6$ and the branch-and-bound algorithm returns the optimal solution $\tilde{\boldsymbol{x}}(\text{ILP}) = (3, 3)$. The optimal solution is **not unique**: also $(2, 4)$ has value 6. With the rules stated in the question, the algorithm finds $(3, 3)$.

Exercise 7.7 — Branch-and-bound algorithm ($P7$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{array}{lrcrcrl} \max & 3\,x_1 & + & x_2 & & & \\ & x_1 & + & 4\,x_2 & \leq & 12 & \\ & 2\,x_1 & + & x_2 & \leq & 5 & \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} & \end{array}$$

and an optimal simplex dictionary of its LP relaxation in standard form:

$$\begin{cases} x_1 = \dfrac{5}{2} - \dfrac{1}{2}\,x_2 - \dfrac{1}{2}\,x_4 \\ x_3 = \dfrac{19}{2} - \dfrac{7}{2}\,x_2 + \dfrac{1}{2}\,x_4 \\ z = \dfrac{15}{2} - \dfrac{1}{2}\,x_2 - \dfrac{3}{2}\,x_4 \end{cases}$$

Questions

1. Execute the branch-and-bound algorithm using:
   (a) the smallest-index rule as branching-variable selection strategy;
   (b) the FIFO rule as branching-node selection strategy;
   (c) solve first the down-branching node and then the up-branching node;
   (d) set the lower bound LB to $-\infty$ (no heuristic algorithm is executed).

   Solve the LP problems of the branching nodes using the dual simplex algorithm. In the dual simplex algorithm, the leaving variable is the basic variable with negative value and smallest index, and ties in the ratio test are broken by the smallest index. Show graphically the optimal solution of the LP problems associated with the branching nodes.
2. Determine and comment the branch-and-bound tree, showing the reason why each branching node is pruned.

Solution

1. Execution of the branch-and-bound algorithm to solve the ILP problem.

   - **Initialization**

     Let us consider the LP_1 problem in standard form:

$$\begin{array}{lrcrcrcrcl} \max & 3\,x_1 & + & & x_2 & & & & & \\ & x_1 & + & 4\,x_2 & + & x_3 & & & = & 12 \\ & 2\,x_1 & + & & x_2 & & & + \; x_4 & = & 5 \\ & x_1, & & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

     and its optimal simplex dictionary:

$$\begin{cases} x_1 = \dfrac{5}{2} - \dfrac{1}{2}\,x_2 - \dfrac{1}{2}\,x_4 \\ x_3 = \dfrac{19}{2} - \dfrac{7}{2}\,x_2 + \dfrac{1}{2}\,x_4 \\ z = \dfrac{15}{2} - \dfrac{1}{2}\,x_2 - \dfrac{3}{2}\,x_4 \end{cases}$$

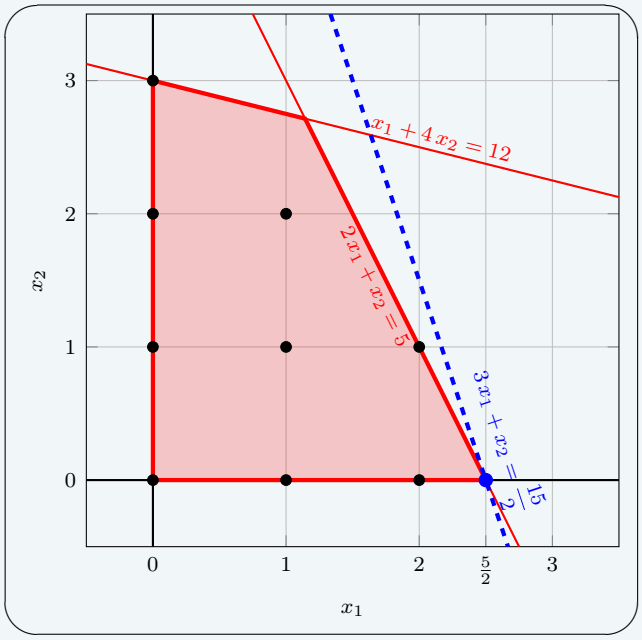


Figure 7.32: Node 1: feasible region of LP_1 (red), feasible points of the ILP problem (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_1)$ is:

$$\tilde{x}_1 = \frac{5}{2}, \quad \tilde{x}_2 = 0$$

the optimal value is:

$$z(\mathrm{LP}_1) = \frac{15}{2}$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is **fractional** and the LP_1 problem is **feasible**, so the LP_1 problem is inserted in the problem list L and the lower bound LB is set to $-\infty$.
Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \quad \leq \quad z(\mathrm{ILP}) \quad \leq \quad \underbrace{\left\lfloor \frac{15}{2} \right\rfloor}_{=7}$$

- **Iteration 1**

The LP_1 problem is extracted and deleted from the problem list L.

The branching variable is x_1. The branching constraints are:

$$x_1 \leq \underbrace{\left\lfloor \frac{5}{2} \right\rfloor}_{=2} \quad \text{and} \quad x_1 \geq \underbrace{\left\lfloor \frac{5}{2} \right\rfloor + 1}_{=3}$$

The LP_2 problem is created with the first branching constraint (the second child is solved only if needed).

(a) Solution of the LP_2 problem

Let us consider the LP_2 problem in standard form:

$$\begin{array}{lrcrcrcrcrcr}
\max & 3\,x_1 & + & x_2 & & & & & & & & \\
 & x_1 & + & 4\,x_2 & + & x_3 & & & & & = & 12 \\
 & 2\,x_1 & + & x_2 & & & + & x_4 & & & = & 5 \\
 & x_1 & & & & & & & + & x_5 & = & 2 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_1 + x_5 = 2 \qquad \Rightarrow \quad x_5 = 2 - x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_5 = 2 - \left(\frac{5}{2} - \frac{1}{2}\, x_2 - \frac{1}{2}\, x_4 \right) \qquad \Rightarrow \quad x_5 = -\frac{1}{2} + \frac{1}{2}\, x_2 + \frac{1}{2}\, x_4$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = \frac{5}{2} - \frac{1}{2}x_2 - \frac{1}{2}x_4 \\ x_3 = \frac{19}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_4 \\ x_5 = -\frac{1}{2} + \frac{1}{2}x_2 + \frac{1}{2}x_4 \\ z = \frac{15}{2} - \frac{1}{2}x_2 - \frac{3}{2}x_4 \end{cases}$$

where $\mathcal{B} = \{1, 3, 5\}$ and $\mathcal{N} = \{2, 4\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{5}{2}, \quad \bar{x}_3 = \frac{19}{2}, \quad \bar{x}_5 = -\frac{1}{2}, \qquad \bar{x}_2 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad \frac{15}{2}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_5 = 0, \qquad \bar{\pi}_2 = \frac{1}{2}, \quad \bar{\pi}_4 = \frac{3}{2} \quad \text{of value} \quad \frac{15}{2}$$

If we increase the value of π_5, we can get a better dual solution. We have:

$$\begin{cases} \pi_2 = \frac{1}{2} - \frac{1}{2}\pi_5 \\ \pi_4 = \frac{3}{2} - \frac{1}{2}\pi_5 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{2} - \frac{1}{2}\pi_5 \geq 0}_{\pi_2 \geq 0} \\ \underbrace{\frac{3}{2} - \frac{1}{2}\pi_5 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_5 \leq 1 \\ \pi_5 \leq 3 \end{cases}$$

By increasing the value of π_5 to $\min\{1, 3\} = 1$, the value of π_2 decreases to 0. Then we have that:

π_5 becomes a dual basic variable and accordingly x_5 becomes a primal non-basic variable.

π_2 becomes a dual non-basic variable and accordingly x_2 becomes a primal basic variable.

We have:

$$x_5 = -\frac{1}{2} + \frac{1}{2}x_2 + \frac{1}{2}x_4 \quad \Rightarrow \quad x_2 = 1 - x_4 + 2x_5$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{5}{2} - \frac{1}{2}(1 - x_4 + 2x_5) - \frac{1}{2}x_4 = 2 - x_5$$

$$x_3 = \frac{19}{2} - \frac{7}{2}(1 - x_4 + 2x_5) + \frac{1}{2}x_4 = 6 + 4x_4 - 7x_5$$

$$z = \frac{15}{2} - \frac{1}{2}(1 - x_4 + 2x_5) - \frac{3}{2}x_4 = 7 - x_4 - x_5$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 2 - x_5 \\ x_2 = 1 - x_4 + 2\,x_5 \\ x_3 = 6 + 4\,x_4 - 7\,x_5 \\ z = 7 - x_4 - x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 1, \quad \tilde{x}_3 = 6, \qquad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad 7$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \qquad \tilde{\pi}_4 = 1, \quad \tilde{\pi}_5 = 1 \quad \text{of value} \quad 7$$

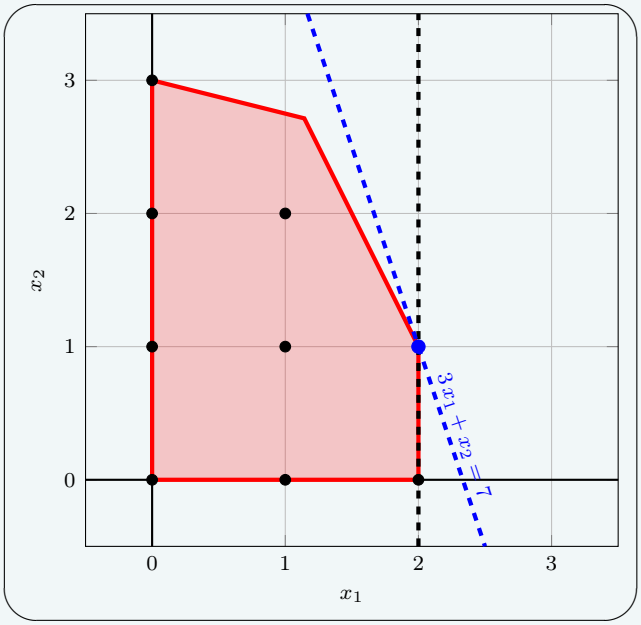


Figure 7.33: Node 2: feasible region of LP_2 (red), with the branching constraint $x_1 \leq 2$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_2)$ is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 1$$

the optimal value is:

$$z(\text{LP}_2) = 7$$

The LP_2 problem is **feasible** and we have

$$\lfloor z(\text{LP}_2) \rfloor > \text{LB} \qquad \text{since} \qquad 7 > -\infty$$

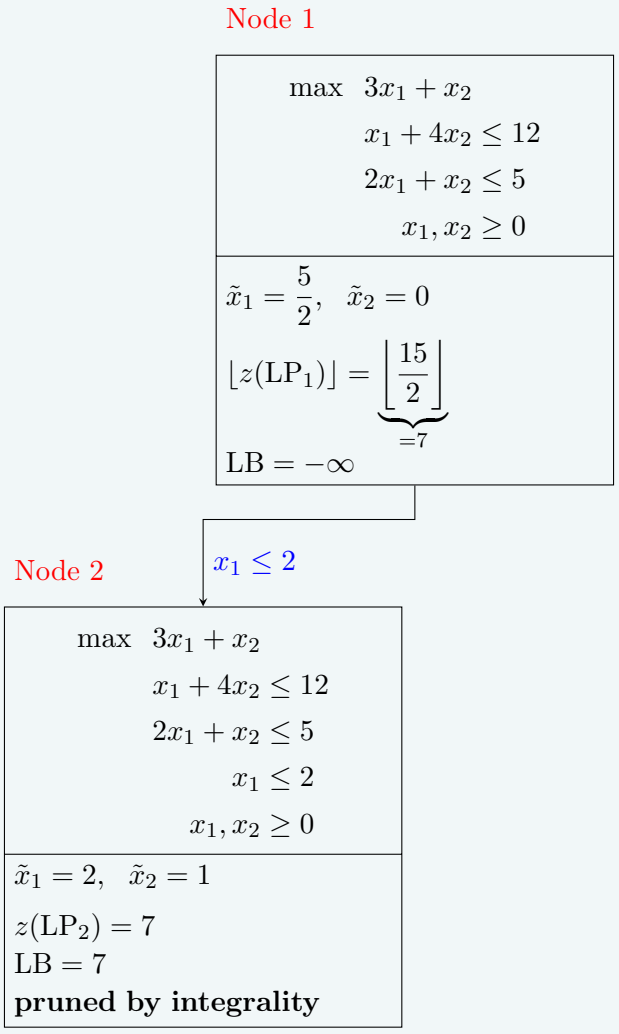
The optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_2)$ is **integer**, accordingly, the lower bound LB becomes 7 and the incumbent solution $\bar{\boldsymbol{x}}(\text{ILP})$ becomes $\bar{x}_1 = 2, \bar{x}_2 = 1$.

Accordingly, the branching node is pruned by integrality.

At the end of the first iteration, branching on x_1 from Node 1 has generated the down-branching node, Node 2. This node is pruned by integrality. The up-branching node is not solved, since after Node 2 the lower bound satisfies

$$\lfloor z(\mathrm{LP}_1) \rfloor \leq \mathrm{LB}.$$

This is the portion of the branch-and-bound tree showing the LP problems of the branching nodes that are solved:



Due to the current lower bound LB and upper bound UB, we have:

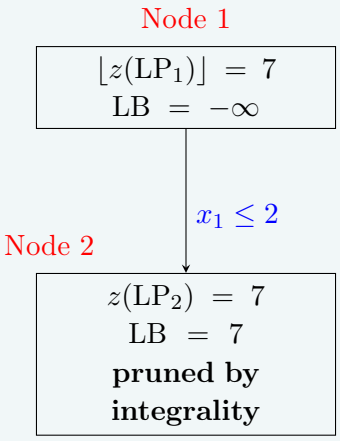
$$7 \quad \leq \quad z(\mathrm{ILP}) \quad \leq \quad 7$$

- **Termination**

Since the list L is empty, the lower bound LB is equal to the upper bound UB and the ILP problem is solved to optimality.

The optimal value is $z(\mathrm{ILP}) = 7$ (the value of the lower bound LB) and an optimal solution $\tilde{\boldsymbol{x}}(\mathrm{ILP})$ is $\tilde{x}_1 = 2, \tilde{x}_2 = 1$ (the incumbent solution $\bar{\boldsymbol{x}}(\mathrm{ILP})$).

2. The final branch-and-bound tree is:



The reasons why the branching nodes are pruned are the following.

- Node 1 has a fractional optimal solution with $\lfloor z(\text{LP}_1) \rfloor = 7$: it is branched on x_1.
- The up-branching node of Node 1 is not solved, and it is not drawn, since $\lfloor z(\text{LP}_1) \rfloor = 7 \leq \text{LB} = 7$ after its first child.
- Node 2 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_2) = (2, 1)$ is integer, with value 7, which becomes the incumbent solution.

The optimal value is $z(\text{ILP}) = 7$ and the branch-and-bound algorithm returns the optimal solution $\tilde{\boldsymbol{x}}(\text{ILP}) = (2, 1)$.

Exercise 7.8 — Branch-and-bound algorithm ($P9$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{array}{lrcrcl} \max & 2\,x_1 & + & x_2 & & \\ & -x_1 & + & x_2 & \leq & 1 \\ & 6\,x_1 & + & 4\,x_2 & \leq & 39 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

and an optimal simplex dictionary of its LP relaxation in standard form:

$$\begin{cases} x_1 = \frac{13}{2} - \frac{2}{3}\,x_2 - \frac{1}{6}\,x_4 \\ x_3 = \frac{15}{2} - \frac{5}{3}\,x_2 - \frac{1}{6}\,x_4 \\ z = 13 - \frac{1}{3}\,x_2 - \frac{1}{3}\,x_4 \end{cases}$$

Questions

1. Execute the branch-and-bound algorithm using:
   (a) the smallest-index rule as branching-variable selection strategy;
   (b) the LIFO rule as branching-node selection strategy;
   (c) solve first the down-branching node and then the up-branching node;
   (d) set the lower bound LB to $-\infty$ (no heuristic algorithm is executed).

   Solve the LP problems of the branching nodes using the dual simplex algorithm. In the dual simplex algorithm, the leaving variable is the basic variable with negative value and smallest index, and ties in the ratio test are broken by the smallest index. Show graphically the optimal solution of the LP problems associated with the branching nodes.
2. Determine and comment the branch-and-bound tree, showing the reason why each branching node is pruned.

Solution

1. Execution of the branch-and-bound algorithm to solve the ILP problem.
   - **Initialization**

     Let us consider the LP_1 problem in standard form:

$$\begin{array}{lrcrcrcrcl} \max & 2\,x_1 & + & x_2 & & & & & & \\ & -x_1 & + & x_2 & + & x_3 & & & = & 1 \\ & 6\,x_1 & + & 4\,x_2 & & & + & x_4 & = & 39 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

and its optimal simplex dictionary:

$$\begin{cases} x_1 = \dfrac{13}{2} - \dfrac{2}{3}\,x_2 - \dfrac{1}{6}\,x_4 \\ x_3 = \dfrac{15}{2} - \dfrac{5}{3}\,x_2 - \dfrac{1}{6}\,x_4 \\ z = 13 - \dfrac{1}{3}\,x_2 - \dfrac{1}{3}\,x_4 \end{cases}$$

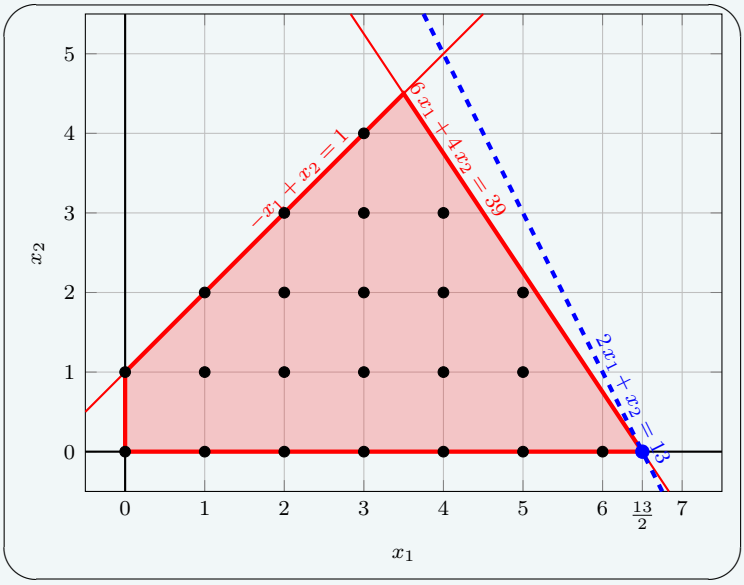


Figure 7.34: Node 1: feasible region of LP_1 (red), feasible points of the ILP problem (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is:

$$\tilde{x}_1 = \frac{13}{2}, \quad \tilde{x}_2 = 0$$

the optimal value is:

$$z(\mathrm{LP}_1) = 13$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is **fractional** and the LP_1 problem is **feasible**, so the LP_1 problem is inserted in the problem list L and the lower bound LB is set to $-\infty$.

Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \leq z(\mathrm{ILP}) \leq 13$$

- **Iteration 1**

The LP_1 problem is extracted and deleted from the problem list L.

The branching variable is x_1. The branching constraints are:

$$x_1 \leq \underbrace{\left\lfloor \frac{13}{2} \right\rfloor}_{=6} \quad \text{and} \quad x_1 \geq \underbrace{\left\lfloor \frac{13}{2} \right\rfloor + 1}_{=7}$$

Two LP_2 and LP_3 problems are created with these branching constraints.

(a) Solution of the LP_2 problem

Let us consider the LP_2 problem in standard form:

$$\begin{array}{lrcrcrcrcrcr}
\max & 2\,x_1 & + & x_2 & & & & & & & & \\
& -x_1 & + & x_2 & + & x_3 & & & & & = & 1 \\
& 6\,x_1 & + & 4\,x_2 & & & + & x_4 & & & = & 39 \\
& x_1 & & & & & & & + & x_5 & = & 6 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_1 + x_5 = 6 \quad \Rightarrow \quad x_5 = 6 - x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_5 = 6 - \left(\frac{13}{2} - \frac{2}{3}\,x_2 - \frac{1}{6}\,x_4 \right) \quad \Rightarrow \quad x_5 = -\frac{1}{2} + \frac{2}{3}\,x_2 + \frac{1}{6}\,x_4$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases}
x_1 = \frac{13}{2} - \frac{2}{3}\,x_2 - \frac{1}{6}\,x_4 \\
x_3 = \frac{15}{2} - \frac{5}{3}\,x_2 - \frac{1}{6}\,x_4 \\
x_5 = -\frac{1}{2} + \frac{2}{3}\,x_2 + \frac{1}{6}\,x_4 \\
z = 13 - \frac{1}{3}\,x_2 - \frac{1}{3}\,x_4
\end{cases}$$

where $\mathcal{B} = \{1, 3, 5\}$ and $\mathcal{N} = \{2, 4\}$ and

– the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{13}{2}, \quad \bar{x}_3 = \frac{15}{2}, \quad \bar{x}_5 = -\frac{1}{2}, \qquad \bar{x}_2 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad 13$$

– the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_5 = 0, \qquad \bar{\pi}_2 = \frac{1}{3}, \quad \bar{\pi}_4 = \frac{1}{3} \quad \text{of value} \quad 13$$

If we increase the value of π_5, we can get a better dual solution. We have:

$$\begin{cases} \pi_2 &= \frac{1}{3} - \frac{2}{3}\pi_5 \\ \pi_4 &= \frac{1}{3} - \frac{1}{6}\pi_5 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{3} - \frac{2}{3}\pi_5 \geq 0}_{\pi_2 \geq 0} \\ \underbrace{\frac{1}{3} - \frac{1}{6}\pi_5 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_5 &\leq \frac{1}{2} \\ \pi_5 &\leq 2 \end{cases}$$

By increasing the value of π_5 to $\min\left\{\frac{1}{2}, 2\right\} = \frac{1}{2}$, the value of π_2 decreases to 0. Then we have that:

π_5 becomes a dual basic variable and accordingly x_5 becomes a primal non-basic variable.

π_2 becomes a dual non-basic variable and accordingly x_2 becomes a primal basic variable.

We have:

$$x_5 = -\frac{1}{2} + \frac{2}{3}x_2 + \frac{1}{6}x_4 \quad \Rightarrow \quad x_2 = \frac{3}{4} - \frac{1}{4}x_4 + \frac{3}{2}x_5$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{13}{2} - \frac{2}{3}\left(\frac{3}{4} - \frac{1}{4}x_4 + \frac{3}{2}x_5\right) - \frac{1}{6}x_4 = 6 - x_5$$

$$x_3 = \frac{15}{2} - \frac{5}{3}\left(\frac{3}{4} - \frac{1}{4}x_4 + \frac{3}{2}x_5\right) - \frac{1}{6}x_4 = \frac{25}{4} + \frac{1}{4}x_4 - \frac{5}{2}x_5$$

$$z = 13 - \frac{1}{3}\left(\frac{3}{4} - \frac{1}{4}x_4 + \frac{3}{2}x_5\right) - \frac{1}{3}x_4 = \frac{51}{4} - \frac{1}{4}x_4 - \frac{1}{2}x_5$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 &= 6 &&&& - && x_5 \\ x_2 &= \frac{3}{4} &- \frac{1}{4}x_4 &+ \frac{3}{2}x_5 \\ x_3 &= \frac{25}{4} &+ \frac{1}{4}x_4 &- \frac{5}{2}x_5 \\ z &= \frac{51}{4} &- \frac{1}{4}x_4 &- \frac{1}{2}x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$ and

– the basic primal optimal solution is:

$$\tilde{x}_1 = 6, \quad \tilde{x}_2 = \frac{3}{4}, \quad \tilde{x}_3 = \frac{25}{4}, \qquad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad \frac{51}{4}$$

– the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \qquad \tilde{\pi}_4 = \frac{1}{4}, \quad \tilde{\pi}_5 = \frac{1}{2} \quad \text{of value} \quad \frac{51}{4}$$

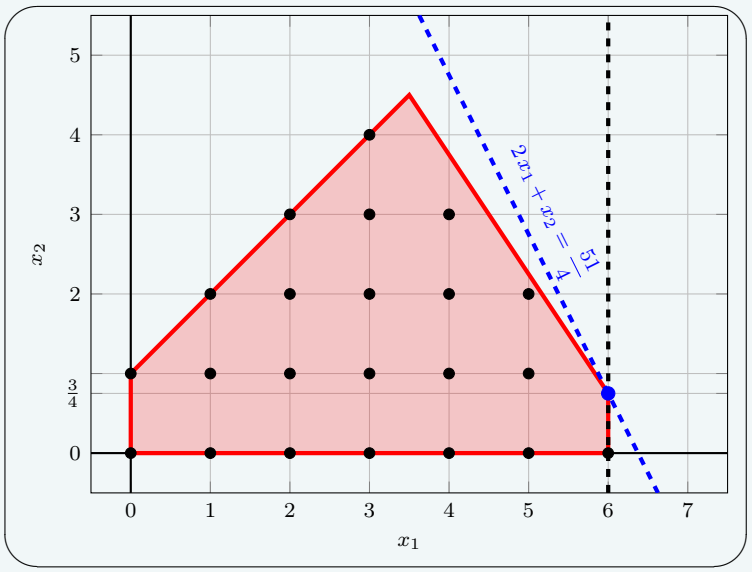


Figure 7.35: Node 2: feasible region of LP_2 (red), with the branching constraint $x_1 \leq 6$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is:

$$\tilde{x}_1 = 6, \quad \tilde{x}_2 = \frac{3}{4}$$

the optimal value is:

$$z(\mathrm{LP}_2) = \frac{51}{4}$$

The LP_2 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_2) \rfloor > \mathrm{LB} \qquad \text{since} \qquad \underbrace{\left\lfloor \frac{51}{4} \right\rfloor}_{=12} > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is **fractional**, accordingly, the LP_2 problem is **inserted in the problem list** L.

(b) Solution of the LP_3 problem

Let us consider the LP_3 problem in standard form:

$$\begin{array}{lrcrcrcrcrcl} \max & 2\,x_1 & + & x_2 & & & & & & & & \\ & -x_1 & + & x_2 & + & x_3 & & & & & = & 1 \\ & 6\,x_1 & + & 4\,x_2 & & & + & x_4 & & & = & 39 \\ & x_1 & & & & & & & - & x_5 & = & 7 \\ & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0 \end{array}$$

For the branching constraint we have:

$$x_1 - x_5 = 7 \quad \Rightarrow \quad x_5 = -7 + x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_5 = -7 + \left(\frac{13}{2} - \frac{2}{3}\,x_2 - \frac{1}{6}\,x_4\right) \quad \Rightarrow \quad x_5 = -\frac{1}{2} - \frac{2}{3}\,x_2 - \frac{1}{6}\,x_4$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = \frac{13}{2} - \frac{2}{3}\,x_2 - \frac{1}{6}\,x_4 \\ x_3 = \frac{15}{2} - \frac{5}{3}\,x_2 - \frac{1}{6}\,x_4 \\ x_5 = -\frac{1}{2} - \frac{2}{3}\,x_2 - \frac{1}{6}\,x_4 \\ z = 13 - \frac{1}{3}\,x_2 - \frac{1}{3}\,x_4 \end{cases}$$

where $\mathcal{B} = \{1, 3, 5\}$ and $\mathcal{N} = \{2, 4\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{13}{2}, \quad \bar{x}_3 = \frac{15}{2}, \quad \bar{x}_5 = -\frac{1}{2}, \qquad \bar{x}_2 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad 13$$

- the basic dual feasible solution is:

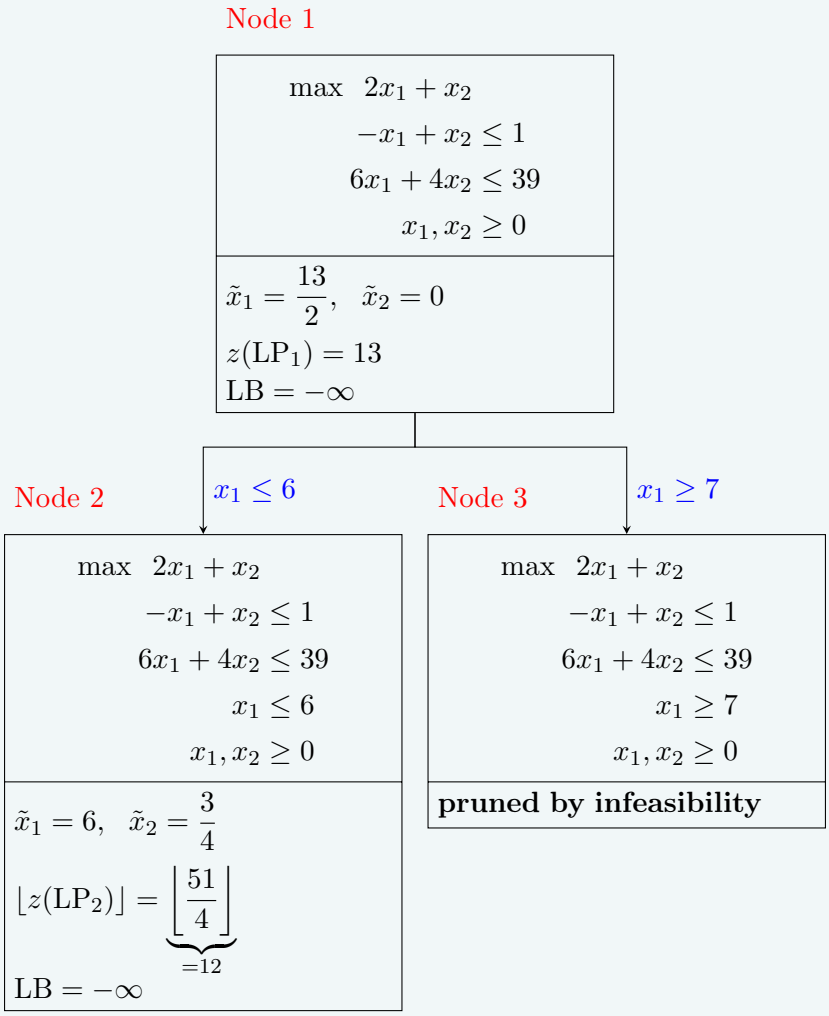
$$\bar{\pi}_1 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_5 = 0, \qquad \bar{\pi}_2 = \frac{1}{3}, \quad \bar{\pi}_4 = \frac{1}{3} \quad \text{of value} \quad 13$$

If we increase the value of π_5, we can get a better dual solution. We have:

$$\begin{cases} \pi_2 = \frac{1}{3} + \frac{2}{3}\,\pi_5 \\ \pi_4 = \frac{1}{3} + \frac{1}{6}\,\pi_5 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{3} + \frac{2}{3}\,\pi_5 \geq 0}_{\pi_2 \geq 0} \\ \underbrace{\frac{1}{3} + \frac{1}{6}\,\pi_5 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_5 \geq -\frac{1}{2} \\ \pi_5 \geq -2 \end{cases}$$

Accordingly, we can increase the value of π_5 without violating dual feasibility, which implies that the dual problem is **unbounded**. As a result, the primal problem LP_3 is **infeasible**, and the corresponding branching node is pruned by infeasibility.

At the end of the first iteration, branching on x_1 from Node 1 has generated the down-branching node, Node 2, and the up-branching node, Node 3. Node 2 is inserted in the list L, while Node 3 is pruned by infeasibility. The corresponding branch-and-bound tree is:



Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \quad \leq \quad z(\text{ILP}) \quad \leq \quad \underbrace{\left\lfloor \frac{51}{4} \right\rfloor}_{=12}$$

- **Iteration 2**

The LP_2 problem is extracted and deleted from the problem list L.

The branching variable is x_2. The branching constraints are:

$$x_2 \leq \underbrace{\left\lfloor \frac{3}{4} \right\rfloor}_{=0} \quad \text{and} \quad x_2 \geq \underbrace{\left\lfloor \frac{3}{4} \right\rfloor + 1}_{=1}$$

The LP_4 problem is created with the first branching constraint (the second child is solved only if needed).

(a) Solution of the LP_4 problem

Let us consider the LP_4 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcr}
\max & 2\,x_1 & + & x_2 & & & & & & & & & \\
 & -x_1 & + & x_2 & + & x_3 & & & & & & = & 1 \\
 & 6\,x_1 & + & 4\,x_2 & & & + & x_4 & & & & = & 39 \\
 & x_1 & & & & & & & + & x_5 & & = & 6 \\
 & & & x_2 & & & & & & & + & x_6 = & 0 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_2 + x_6 = 0 \qquad \Rightarrow \qquad x_6 = 0 - x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_6 = 0 - \left(\frac{3}{4} - \frac{1}{4}\,x_4 + \frac{3}{2}\,x_5\right) \qquad \Rightarrow \qquad x_6 = -\frac{3}{4} + \frac{1}{4}\,x_4 - \frac{3}{2}\,x_5$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases}
x_1 = 6 - x_5 \\
x_2 = \frac{3}{4} - \frac{1}{4}\,x_4 + \frac{3}{2}\,x_5 \\
x_3 = \frac{25}{4} + \frac{1}{4}\,x_4 - \frac{5}{2}\,x_5 \\
x_6 = -\frac{3}{4} + \frac{1}{4}\,x_4 - \frac{3}{2}\,x_5 \\
z = \frac{51}{4} - \frac{1}{4}\,x_4 - \frac{1}{2}\,x_5
\end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 6\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = 6, \quad \bar{x}_2 = \frac{3}{4}, \quad \bar{x}_3 = \frac{25}{4}, \quad \bar{x}_6 = -\frac{3}{4},$$
$$\bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{51}{4}$$

– the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_6 = 0,$$
$$\bar{\pi}_4 = \frac{1}{4}, \quad \bar{\pi}_5 = \frac{1}{2} \quad \text{of value} \quad \frac{51}{4}$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_4 &= \frac{1}{4} - \frac{1}{4}\pi_6 \\ \pi_5 &= \frac{1}{2} + \frac{3}{2}\pi_6 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{4} - \frac{1}{4}\pi_6 \geq 0}_{\pi_4 \geq 0} \\ \underbrace{\frac{1}{2} + \frac{3}{2}\pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 &\leq 1 \\ \pi_6 &\geq -\frac{1}{3} \end{cases}$$

By increasing the value of π_6 to 1, the value of π_4 decreases to 0. Then we have that:

π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non-basic variable.

π_4 becomes a dual non-basic variable and accordingly x_4 becomes a primal basic variable.

We have:

$$x_6 = -\frac{3}{4} + \frac{1}{4}x_4 - \frac{3}{2}x_5 \quad \Rightarrow \quad x_4 = 3 + 6\,x_5 + 4\,x_6$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_2 = \frac{3}{4} - \frac{1}{4}(3 + 6\,x_5 + 4\,x_6) + \frac{3}{2}x_5 = 0 - x_6$$
$$x_3 = \frac{25}{4} + \frac{1}{4}(3 + 6\,x_5 + 4\,x_6) - \frac{5}{2}x_5 = 7 - x_5 + x_6$$
$$z = \frac{51}{4} - \frac{1}{4}(3 + 6\,x_5 + 4\,x_6) - \frac{1}{2}x_5 = 12 - 2\,x_5 - x_6$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 &= 6 &- & x_5 & & \\ x_2 &= 0 & & & - & x_6 \\ x_3 &= 7 &- & x_5 & + & x_6 \\ x_4 &= 3 &+ & 6\,x_5 & + & 4\,x_6 \\ z &= 12 &- & 2\,x_5 & - & x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4\}$ and $\mathcal{N} = \{5, 6\}$ and

– the basic primal optimal solution is:

$$\tilde{x}_1 = 6, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = 7, \quad \tilde{x}_4 = 3,$$
$$\tilde{x}_5 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad 12$$

– the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 0,$$
$$\tilde{\pi}_5 = 2, \quad \tilde{\pi}_6 = 1 \quad \text{of value} \quad 12$$

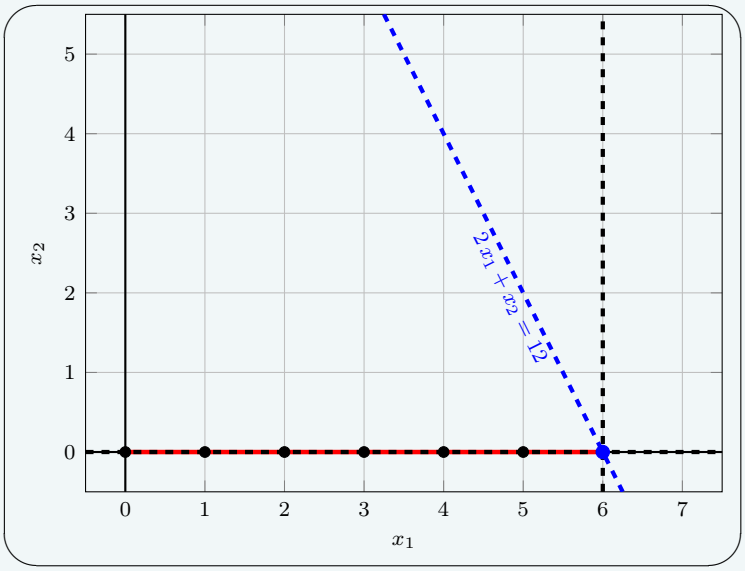


Figure 7.36: Node 4: feasible region of LP_4 (red), with the branching constraint $x_2 \leq 0$ (dashed), and its optimal solution (blue).
An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_4)$ is:

$$\tilde{x}_1 = 6, \quad \tilde{x}_2 = 0$$

the optimal value is:

$$z(\mathrm{LP}_4) = 12$$

The LP_4 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_4) \rfloor > \mathrm{LB} \qquad \text{since} \qquad 12 > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_4)$ is **integer**, accordingly, the lower bound LB becomes 12 and the incumbent solution $\bar{\boldsymbol{x}}(\mathrm{ILP})$ becomes $\bar{x}_1 = 6, \bar{x}_2 = 0$.
Accordingly, the branching node is pruned by integrality.

At the end of the second iteration, branching on x_2 from Node 2 has generated the down-branching node, Node 4. This node is pruned by integrality. The up-branching node is not solved, since after Node 4 the lower bound satisfies

$$\lfloor z(\mathrm{LP}_2) \rfloor \leq \mathrm{LB}.$$

This is the portion of the branch-and-bound tree showing the LP problems of the branching nodes that are solved:

Node 2

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ & -x_1 + x_2 \leq 1 \\ & 6x_1 + 4x_2 \leq 39 \\ & x_1 \leq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\tilde{x}_1 = 6, \quad \tilde{x}_2 = \dfrac{3}{4}$

$\lfloor z(\mathrm{LP}_2) \rfloor = \underbrace{\left\lfloor \dfrac{51}{4} \right\rfloor}_{=12}$

$\mathrm{LB} = -\infty$

$x_2 \leq 0$

Node 4

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ & -x_1 + x_2 \leq 1 \\ & 6x_1 + 4x_2 \leq 39 \\ & x_1 \leq 6 \\ & x_2 \leq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\tilde{x}_1 = 6, \quad \tilde{x}_2 = 0$

$z(\mathrm{LP}_4) = 12$

$\mathrm{LB} = 12$

pruned by integrality

Due to the current lower bound LB and upper bound UB, we have:

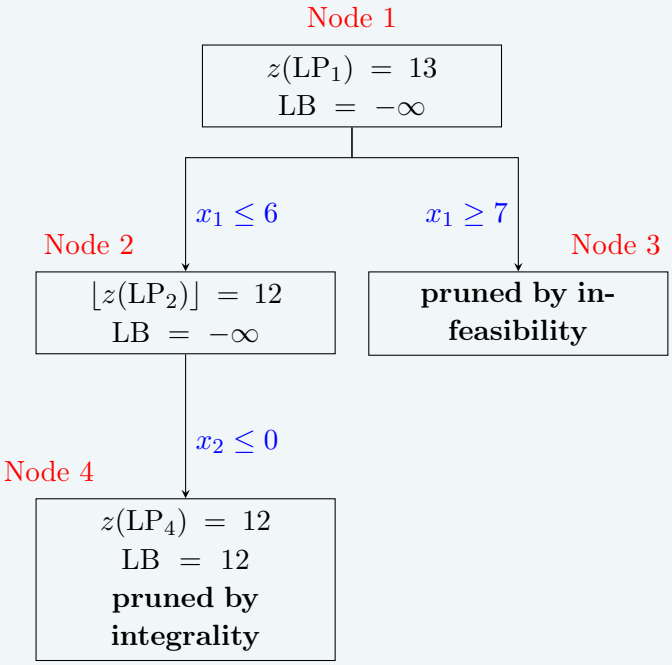
$$12 \quad \leq \quad z(\mathrm{ILP}) \quad \leq \quad 12$$

- **Termination**

Since the list L is empty, the lower bound LB is equal to the upper bound UB and the ILP problem is solved to optimality.

The optimal value is $z(\mathrm{ILP}) = 12$ (the value of the lower bound LB) and an optimal solution $\tilde{\boldsymbol{x}}(\mathrm{ILP})$ is $\tilde{x}_1 = 6, \tilde{x}_2 = 0$ (the incumbent solution $\bar{\boldsymbol{x}}(\mathrm{ILP})$).

2. The final branch-and-bound tree is:



The reasons why the branching nodes are pruned are the following.

- Node 1 has a fractional optimal solution with $z(\text{LP}_1) = 13$: it is branched on x_1.
- Node 2 has a fractional optimal solution with $\lfloor z(\text{LP}_2) \rfloor = 12$: it is branched on x_2.
- The up-branching node of Node 2 is not solved, and it is not drawn, since $\lfloor z(\text{LP}_2) \rfloor = 12 \leq \text{LB} = 12$ after its first child.
- Node 3 is pruned by infeasibility: the LP_3 problem has no feasible solution.
- Node 4 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_4) = (6, 0)$ is integer, with value 12, which becomes the incumbent solution.

The optimal value is $z(\text{ILP}) = 12$ and the branch-and-bound algorithm returns the optimal solution $\tilde{\boldsymbol{x}}(\text{ILP}) = (6, 0)$. The optimal solution is **not unique**: also $(5, 2)$ has value 12. With the rules stated in the question, the algorithm finds $(6, 0)$.

Exercise 7.9 — Branch-and-bound algorithm ($P10$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 3$ constraints:

$$\begin{array}{rrcrcrl} \max & 5\,x_1 & + & 2\,x_2 & & & \\ & -x_1 & + & 3\,x_2 & \leq & 12 & \\ & x_1 & + & 2\,x_2 & \leq & 12 & \\ & 3\,x_1 & - & 2\,x_2 & \leq & 8 & \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} & \end{array}$$

and an optimal simplex dictionary of its LP relaxation in standard form:

$$\begin{cases} x_1 = 5 - \frac{1}{4}\,x_4 - \frac{1}{4}\,x_5 \\ x_2 = \frac{7}{2} - \frac{3}{8}\,x_4 + \frac{1}{8}\,x_5 \\ x_3 = \frac{13}{2} + \frac{7}{8}\,x_4 - \frac{5}{8}\,x_5 \\ z = 32 - 2\,x_4 - x_5 \end{cases}$$

Questions

1. Execute the branch-and-bound algorithm using:
   (a) the largest-index rule as branching-variable selection strategy;
   (b) the FIFO rule as branching-node selection strategy;
   (c) solve first the down-branching node and then the up-branching node;
   (d) set the lower bound LB to $-\infty$ (no heuristic algorithm is executed).

   Solve the LP problems of the branching nodes using the dual simplex algorithm. In the dual simplex algorithm, the leaving variable is the basic variable with negative value and smallest index, and ties in the ratio test are broken by the smallest index. Show graphically the optimal solution of the LP problems associated with the branching nodes.
2. Determine and comment the branch-and-bound tree, showing the reason why each branching node is pruned.

Solution

1. Execution of the branch-and-bound algorithm to solve the ILP problem.

   - **Initialization**

     Let us consider the LP_1 problem in standard form:

$$
\begin{array}{lrcrcrcrcl}
\max & 5\,x_1 & + & 2\,x_2 & & & & & & \\
 & -x_1 & + & 3\,x_2 & + & x_3 & & & = & 12 \\
 & x_1 & + & 2\,x_2 & & & + & x_4 & = & 12 \\
 & 3\,x_1 & - & 2\,x_2 & & & & & + \quad x_5 \; = & 8 \\
 & x_1, & & x_2, & & x_3, & & x_4, & \quad x_5 \; \geq & 0
\end{array}
$$

     and its optimal simplex dictionary:

$$
\begin{cases}
x_1 = 5 - \frac{1}{4}\,x_4 - \frac{1}{4}\,x_5 \\
x_2 = \frac{7}{2} - \frac{3}{8}\,x_4 + \frac{1}{8}\,x_5 \\
x_3 = \frac{13}{2} + \frac{7}{8}\,x_4 - \frac{5}{8}\,x_5 \\
z = 32 - 2\,x_4 - x_5
\end{cases}
$$

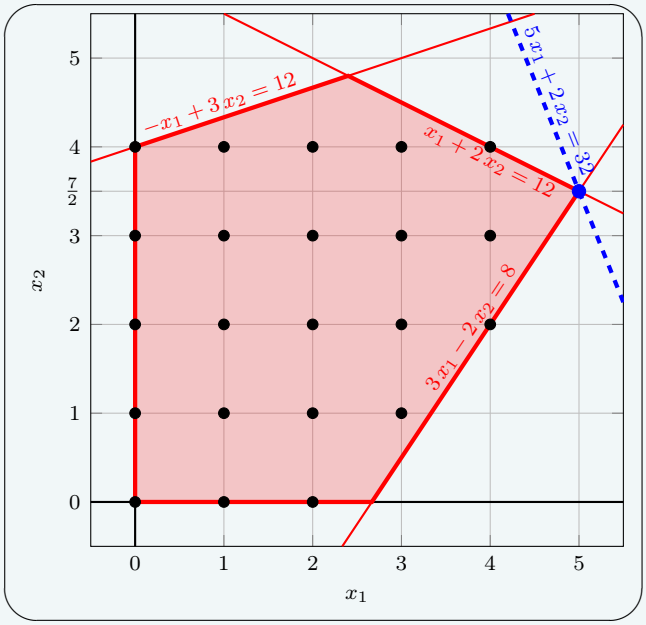


Figure 7.37: Node 1: feasible region of LP_1 (red), feasible points of the ILP problem (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is:

$$\tilde{x}_1 = 5, \quad \tilde{x}_2 = \frac{7}{2}$$

the optimal value is:

$$z(\mathrm{LP}_1) = 32$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is **fractional** and the LP_1 problem is **feasible**, so the LP_1 problem is inserted in the problem list L and the lower bound LB is set to $-\infty$.
Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \quad \leq \quad z(\mathrm{ILP}) \quad \leq \quad 32$$

- **Iteration 1**

The LP_1 problem is extracted and deleted from the problem list L.

The branching variable is x_2. The branching constraints are:

$$x_2 \leq \underbrace{\left\lfloor \frac{7}{2} \right\rfloor}_{=3} \quad \text{and} \quad x_2 \geq \underbrace{\left\lfloor \frac{7}{2} \right\rfloor + 1}_{=4}$$

Two LP_2 and LP_3 problems are created with these branching constraints.

(a) Solution of the LP_2 problem

Let us consider the LP_2 problem in standard form:

$$\begin{array}{lrcrcrcrcrcr}
\max & 5\,x_1 & + & 2\,x_2 & & & & & & & & \\
 & -x_1 & + & 3\,x_2 & + & x_3 & & & & & = & 12 \\
 & x_1 & + & 2\,x_2 & & & + & x_4 & & & = & 12 \\
 & 3\,x_1 & - & 2\,x_2 & & & & & + & x_5 & = & 8 \\
 & & & x_2 & & & & & & + \; x_6 & = & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, \quad x_6 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_2 + x_6 = 3 \qquad \Rightarrow \qquad x_6 = 3 - x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_6 = 3 - \left(\frac{7}{2} - \frac{3}{8}\, x_4 + \frac{1}{8}\, x_5 \right) \qquad \Rightarrow \qquad x_6 = -\frac{1}{2} + \frac{3}{8}\, x_4 - \frac{1}{8}\, x_5$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = 5 - \frac{1}{4}x_4 - \frac{1}{4}x_5 \\ x_2 = \frac{7}{2} - \frac{3}{8}x_4 + \frac{1}{8}x_5 \\ x_3 = \frac{13}{2} + \frac{7}{8}x_4 - \frac{5}{8}x_5 \\ x_6 = -\frac{1}{2} + \frac{3}{8}x_4 - \frac{1}{8}x_5 \\ z = 32 - 2x_4 - x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 6\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = 5, \quad \bar{x}_2 = \frac{7}{2}, \quad \bar{x}_3 = \frac{13}{2}, \quad \bar{x}_6 = -\frac{1}{2},$$
$$\bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad 32$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_6 = 0,$$
$$\bar{\pi}_4 = 2, \quad \bar{\pi}_5 = 1 \quad \text{of value} \quad 32$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_4 = 2 - \frac{3}{8}\pi_6 \\ \pi_5 = 1 + \frac{1}{8}\pi_6 \end{cases} \qquad \begin{cases} \underbrace{2 - \frac{3}{8}\pi_6 \geq 0}_{\pi_4 \geq 0} \\ \underbrace{1 + \frac{1}{8}\pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 \leq \frac{16}{3} \\ \pi_6 \geq -8 \end{cases}$$

By increasing the value of π_6 to $\frac{16}{3}$, the value of π_4 decreases to 0. Then we have that:

π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non-basic variable.

π_4 becomes a dual non-basic variable and accordingly x_4 becomes a primal basic variable.

We have:

$$x_6 = -\frac{1}{2} + \frac{3}{8}x_4 - \frac{1}{8}x_5 \quad \Rightarrow \quad x_4 = \frac{4}{3} + \frac{1}{3}x_5 + \frac{8}{3}x_6$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = 5 - \frac{1}{4}\left(\frac{4}{3} + \frac{1}{3}x_5 + \frac{8}{3}x_6\right) - \frac{1}{4}x_5 = \frac{14}{3} - \frac{1}{3}x_5 - \frac{2}{3}x_6$$

$$x_2 = \frac{7}{2} - \frac{3}{8}\left(\frac{4}{3} + \frac{1}{3}x_5 + \frac{8}{3}x_6\right) + \frac{1}{8}x_5 = 3 - x_6$$

$$x_3 = \frac{13}{2} + \frac{7}{8}\left(\frac{4}{3} + \frac{1}{3}x_5 + \frac{8}{3}x_6\right) - \frac{5}{8}x_5 = \frac{23}{3} - \frac{1}{3}x_5 + \frac{7}{3}x_6$$

$$z = 32 - 2\left(\frac{4}{3} + \frac{1}{3}x_5 + \frac{8}{3}x_6\right) - x_5 = \frac{88}{3} - \frac{5}{3}x_5 - \frac{16}{3}x_6$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = \frac{14}{3} - \frac{1}{3}x_5 - \frac{2}{3}x_6 \\ x_2 = 3 - x_6 \\ x_3 = \frac{23}{3} - \frac{1}{3}x_5 + \frac{7}{3}x_6 \\ x_4 = \frac{4}{3} + \frac{1}{3}x_5 + \frac{8}{3}x_6 \\ z = \frac{88}{3} - \frac{5}{3}x_5 - \frac{16}{3}x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4\}$ and $\mathcal{N} = \{5, 6\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = \frac{14}{3}, \quad \tilde{x}_2 = 3, \quad \tilde{x}_3 = \frac{23}{3}, \quad \tilde{x}_4 = \frac{4}{3},$$
$$\tilde{x}_5 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad \frac{88}{3}$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 0,$$
$$\tilde{\pi}_5 = \frac{5}{3}, \quad \tilde{\pi}_6 = \frac{16}{3} \quad \text{of value} \quad \frac{88}{3}$$

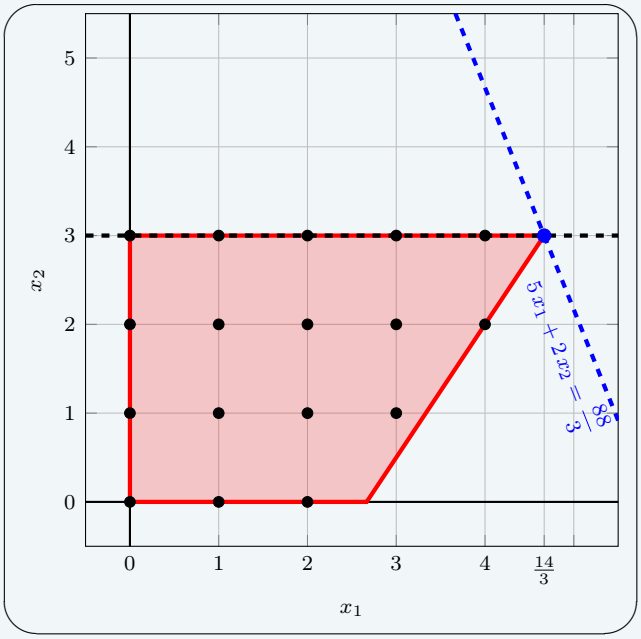


Figure 7.38: Node 2: feasible region of LP_2 (red), with the branching constraint $x_2 \leq 3$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is:

$$\tilde{x}_1 = \frac{14}{3}, \quad \tilde{x}_2 = 3$$

the optimal value is:

$$z(\mathrm{LP}_2) = \frac{88}{3}$$

The LP_2 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_2) \rfloor > \mathrm{LB} \qquad \text{since} \qquad \underbrace{\left\lfloor \frac{88}{3} \right\rfloor}_{=29} > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is **fractional**, accordingly, the LP_2 problem is **inserted in the problem list** L.

(b) Solution of the LP_3 problem

Let us consider the LP_3 problem in standard form:

$$\begin{array}{lrcrcrcrcrcl}
\max & 5\,x_1 & + & 2\,x_2 & & & & & & & & \\
 & -x_1 & + & 3\,x_2 & + & x_3 & & & & & = & 12 \\
 & x_1 & + & 2\,x_2 & & & + & x_4 & & & = & 12 \\
 & 3\,x_1 & - & 2\,x_2 & & & & & + & x_5 & = & 8 \\
 & & & x_2 & & & & & & - & x_6 & = & 4 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & x_6 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_2 - x_6 = 4 \qquad \Rightarrow \qquad x_6 = -4 + x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_6 = -4 + \left(\frac{7}{2} - \frac{3}{8}\,x_4 + \frac{1}{8}\,x_5\right) \qquad \Rightarrow \qquad x_6 = -\frac{1}{2} - \frac{3}{8}\,x_4 + \frac{1}{8}\,x_5$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = 5 - \frac{1}{4}\,x_4 - \frac{1}{4}\,x_5 \\ x_2 = \frac{7}{2} - \frac{3}{8}\,x_4 + \frac{1}{8}\,x_5 \\ x_3 = \frac{13}{2} + \frac{7}{8}\,x_4 - \frac{5}{8}\,x_5 \\ x_6 = -\frac{1}{2} - \frac{3}{8}\,x_4 + \frac{1}{8}\,x_5 \\ z = 32 - 2\,x_4 - x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 6\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = 5, \quad \bar{x}_2 = \frac{7}{2}, \quad \bar{x}_3 = \frac{13}{2}, \quad \bar{x}_6 = -\frac{1}{2},$$
$$\bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad 32$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_6 = 0,$$
$$\bar{\pi}_4 = 2, \quad \bar{\pi}_5 = 1 \quad \text{of value} \quad 32$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_4 = 2 + \frac{3}{8}\,\pi_6 \\ \pi_5 = 1 - \frac{1}{8}\,\pi_6 \end{cases} \qquad \begin{cases} \underbrace{2 + \frac{3}{8}\,\pi_6 \geq 0}_{\pi_4 \geq 0} \\ \underbrace{1 - \frac{1}{8}\,\pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 \geq -\frac{16}{3} \\ \pi_6 \leq 8 \end{cases}$$

By increasing the value of π_6 to 8, the value of π_5 decreases to 0. Then we have that:

π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non-basic variable.

π_5 becomes a dual non-basic variable and accordingly x_5 becomes a primal basic variable.

We have:

$$x_6 = -\frac{1}{2} - \frac{3}{8}\,x_4 + \frac{1}{8}\,x_5 \quad \Rightarrow \quad x_5 = 4 + 3\,x_4 + 8\,x_6$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = 5 - \frac{1}{4}\,x_4 - \frac{1}{4}\,(4 + 3\,x_4 + 8\,x_6) = 4 - x_4 - 2\,x_6$$

$$x_2 = \frac{7}{2} - \frac{3}{8}\,x_4 + \frac{1}{8}\,(4 + 3\,x_4 + 8\,x_6) = 4 + x_6$$

$$x_3 = \frac{13}{2} + \frac{7}{8}\,x_4 - \frac{5}{8}\,(4 + 3\,x_4 + 8\,x_6) = 4 - x_4 - 5\,x_6$$

$$z = 32 - 2\,x_4 - (4 + 3\,x_4 + 8\,x_6) = 28 - 5\,x_4 - 8\,x_6$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 4 - x_4 - 2\,x_6 \\ x_2 = 4 + x_6 \\ x_3 = 4 - x_4 - 5\,x_6 \\ x_5 = 4 + 3\,x_4 + 8\,x_6 \\ z = 28 - 5\,x_4 - 8\,x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 5\}$ and $\mathcal{N} = \{4, 6\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = 4, \quad \tilde{x}_3 = 4, \quad \tilde{x}_5 = 4,$$
$$\tilde{x}_4 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad 28$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_5 = 0,$$
$$\tilde{\pi}_4 = 5, \quad \tilde{\pi}_6 = 8 \quad \text{of value} \quad 28$$

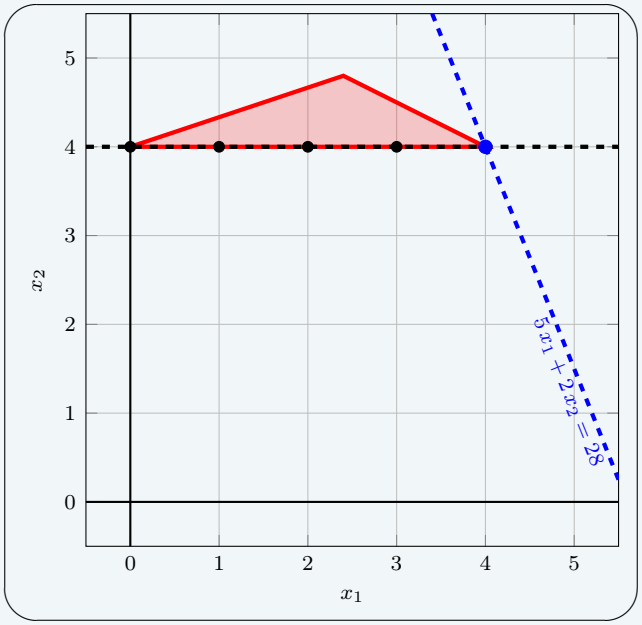


Figure 7.39: Node 3: feasible region of LP_3 (red), with the branching constraint $x_2 \geq 4$ (dashed), and its optimal solution (blue).
An optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_3)$ is:

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = 4$$

the optimal value is:

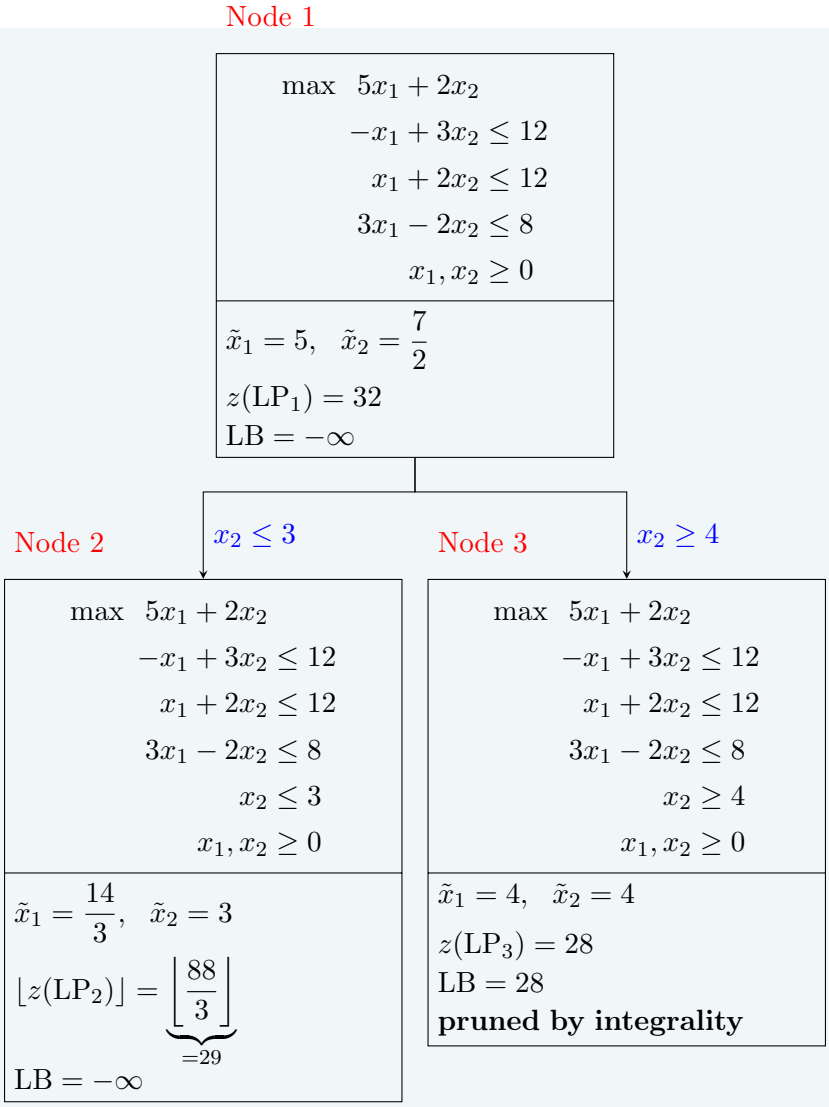
$$z(\text{LP}_3) = 28$$

The LP_3 problem is **feasible** and we have

$$\lfloor z(\text{LP}_3) \rfloor > \text{LB} \qquad \text{since} \qquad 28 > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_3)$ is **integer**, accordingly, the lower bound LB becomes 28 and the incumbent solution $\bar{\boldsymbol{x}}(\text{ILP})$ becomes $\bar{x}_1 = 4, \bar{x}_2 = 4$.
Accordingly, the branching node is pruned by integrality.

At the end of the first iteration, branching on x_2 from Node 1 has generated the down-branching node, Node 2, and the up-branching node, Node 3. Node 2 is inserted in the list L, while Node 3 is pruned by integrality. The corresponding branch-and-bound tree is:



Due to the current lower bound LB and upper bound UB, we have:

$$28 \quad \leq \quad z(\text{ILP}) \quad \leq \quad \underbrace{\left\lfloor \frac{88}{3} \right\rfloor}_{=29}$$

- **Iteration 2**

The LP_2 problem is extracted and deleted from the problem list L.

The branching variable is x_1. The branching constraints are:

$$x_1 \leq \underbrace{\left\lfloor \frac{14}{3} \right\rfloor}_{=4} \quad \text{and} \quad x_1 \geq \underbrace{\left\lfloor \frac{14}{3} \right\rfloor + 1}_{=5}$$

Two LP_4 and LP_5 problems are created with these branching constraints.

(a) Solution of the LP_4 problem

Let us consider the LP_4 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcrcl}
\max & 5\,x_1 & + & 2\,x_2 & & & & & & & & & & & \\
& -x_1 & + & 3\,x_2 & + & x_3 & & & & & & & & = & 12 \\
& x_1 & + & 2\,x_2 & & & + & x_4 & & & & & & = & 12 \\
& 3\,x_1 & - & 2\,x_2 & & & & & + & x_5 & & & & = & 8 \\
& & & x_2 & & & & & & & + & x_6 & & = & 3 \\
& x_1 & & & & & & & & & & & + \; x_7 & = & 4 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & & x_7 \; \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_1 + x_7 = 4 \qquad \Rightarrow \quad x_7 = 4 - x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_7 = 4 - \left(\frac{14}{3} - \frac{1}{3}\,x_5 - \frac{2}{3}\,x_6\right) \qquad \Rightarrow \quad x_7 = -\frac{2}{3} + \frac{1}{3}\,x_5 + \frac{2}{3}\,x_6$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases}
x_1 = \frac{14}{3} - \frac{1}{3}\,x_5 - \frac{2}{3}\,x_6 \\
x_2 = 3 - x_6 \\
x_3 = \frac{23}{3} - \frac{1}{3}\,x_5 + \frac{7}{3}\,x_6 \\
x_4 = \frac{4}{3} + \frac{1}{3}\,x_5 + \frac{8}{3}\,x_6 \\
x_7 = -\frac{2}{3} + \frac{1}{3}\,x_5 + \frac{2}{3}\,x_6 \\
z = \frac{88}{3} - \frac{5}{3}\,x_5 - \frac{16}{3}\,x_6
\end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4, 7\}$ and $\mathcal{N} = \{5, 6\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{14}{3}, \quad \bar{x}_2 = 3, \quad \bar{x}_3 = \frac{23}{3}, \quad \bar{x}_4 = \frac{4}{3}, \quad \bar{x}_7 = -\frac{2}{3},$$
$$\bar{x}_5 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad \frac{88}{3}$$

– the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_7 = 0,$$
$$\bar{\pi}_5 = \frac{5}{3}, \quad \bar{\pi}_6 = \frac{16}{3} \quad \text{of value} \quad \frac{88}{3}$$

If we increase the value of π_7, we can get a better dual solution. We have:

$$\begin{cases} \pi_5 = \frac{5}{3} - \frac{1}{3}\pi_7 \\ \pi_6 = \frac{16}{3} - \frac{2}{3}\pi_7 \end{cases} \qquad \begin{cases} \underbrace{\frac{5}{3} - \frac{1}{3}\pi_7 \geq 0}_{\pi_5 \geq 0} \\ \underbrace{\frac{16}{3} - \frac{2}{3}\pi_7 \geq 0}_{\pi_6 \geq 0} \end{cases} \qquad \begin{cases} \pi_7 \leq 5 \\ \pi_7 \leq 8 \end{cases}$$

By increasing the value of π_7 to $\min\{5, 8\} = 5$, the value of π_5 decreases to 0. Then we have that:

π_7 becomes a dual basic variable and accordingly x_7 becomes a primal non-basic variable.

π_5 becomes a dual non-basic variable and accordingly x_5 becomes a primal basic variable.

We have:

$$x_7 = -\frac{2}{3} + \frac{1}{3}x_5 + \frac{2}{3}x_6 \quad \Rightarrow \quad x_5 = 2 - 2x_6 + 3x_7$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{14}{3} - \frac{1}{3}(2 - 2x_6 + 3x_7) - \frac{2}{3}x_6 = 4 - x_7$$
$$x_3 = \frac{23}{3} - \frac{1}{3}(2 - 2x_6 + 3x_7) + \frac{7}{3}x_6 = 7 + 3x_6 - x_7$$
$$x_4 = \frac{4}{3} + \frac{1}{3}(2 - 2x_6 + 3x_7) + \frac{8}{3}x_6 = 2 + 2x_6 + x_7$$
$$z = \frac{88}{3} - \frac{5}{3}(2 - 2x_6 + 3x_7) - \frac{16}{3}x_6 = 26 - 2x_6 - 5x_7$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 4 - x_7 \\ x_2 = 3 - x_6 \\ x_3 = 7 + 3x_6 - x_7 \\ x_4 = 2 + 2x_6 + x_7 \\ x_5 = 2 - 2x_6 + 3x_7 \\ z = 26 - 2x_6 - 5x_7 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4, 5\}$ and $\mathcal{N} = \{6, 7\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = 3, \quad \tilde{x}_3 = 7, \quad \tilde{x}_4 = 2, \quad \tilde{x}_5 = 2,$$
$$\tilde{x}_6 = 0, \quad \tilde{x}_7 = 0 \quad \text{of value} \quad 26$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 0, \quad \tilde{\pi}_5 = 0,$$
$$\tilde{\pi}_6 = 2, \quad \tilde{\pi}_7 = 5 \quad \text{of value} \quad 26$$

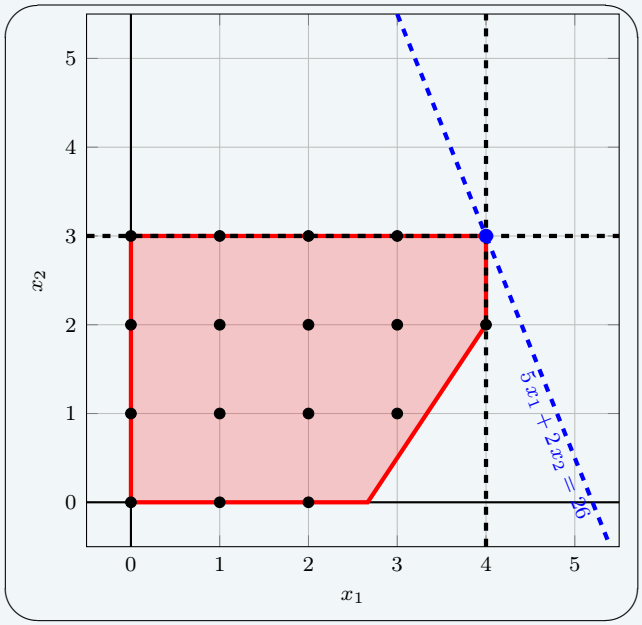


Figure 7.40: Node 4: feasible region of LP_4 (red), with the branching constraint $x_1 \leq 4$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_4)$ is:

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = 3$$

the optimal value is:

$$z(\text{LP}_4) = 26$$

The LP_4 problem is **feasible** and we have

$$\lfloor z(\text{LP}_4) \rfloor \leq \text{LB} \qquad \text{since} \qquad 26 \leq 28$$

The optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_4)$ is **integer**, but its value does not improve the lower bound LB, which remains 28.

Accordingly, the branching node is pruned by integrality.

(b) Solution of the LP_5 problem

Let us consider the LP_5 problem in standard form:

$$
\begin{array}{lrcrcrcrcrcrcrcl}
\max & 5\,x_1 & + & 2\,x_2 & & & & & & & & & & & \\
 & -x_1 & + & 3\,x_2 & + & x_3 & & & & & & & & = & 12 \\
 & x_1 & + & 2\,x_2 & & & + & x_4 & & & & & & = & 12 \\
 & 3\,x_1 & - & 2\,x_2 & & & & & + & x_5 & & & & = & 8 \\
 & & & x_2 & & & & & & & + & x_6 & & = & 3 \\
 & x_1 & & & & & & & & & & & - & x_7 = & 5 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & & x_7 \geq & 0
\end{array}
$$

For the branching constraint we have:

$$x_1 - x_7 = 5 \qquad \Rightarrow \qquad x_7 = -5 + x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_7 = -5 + \left(\frac{14}{3} - \frac{1}{3}\,x_5 - \frac{2}{3}\,x_6\right) \qquad \Rightarrow \qquad x_7 = -\frac{1}{3} - \frac{1}{3}\,x_5 - \frac{2}{3}\,x_6$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$
\begin{cases}
x_1 = \frac{14}{3} - \frac{1}{3}\,x_5 - \frac{2}{3}\,x_6 \\
x_2 = 3 - x_6 \\
x_3 = \frac{23}{3} - \frac{1}{3}\,x_5 + \frac{7}{3}\,x_6 \\
x_4 = \frac{4}{3} + \frac{1}{3}\,x_5 + \frac{8}{3}\,x_6 \\
x_7 = -\frac{1}{3} - \frac{1}{3}\,x_5 - \frac{2}{3}\,x_6 \\
z = \frac{88}{3} - \frac{5}{3}\,x_5 - \frac{16}{3}\,x_6
\end{cases}
$$

where $\mathcal{B} = \{1, 2, 3, 4, 7\}$ and $\mathcal{N} = \{5, 6\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{14}{3}, \quad \bar{x}_2 = 3, \quad \bar{x}_3 = \frac{23}{3}, \quad \bar{x}_4 = \frac{4}{3}, \quad \bar{x}_7 = -\frac{1}{3},$$
$$\bar{x}_5 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad \frac{88}{3}$$

- the basic dual feasible solution is:

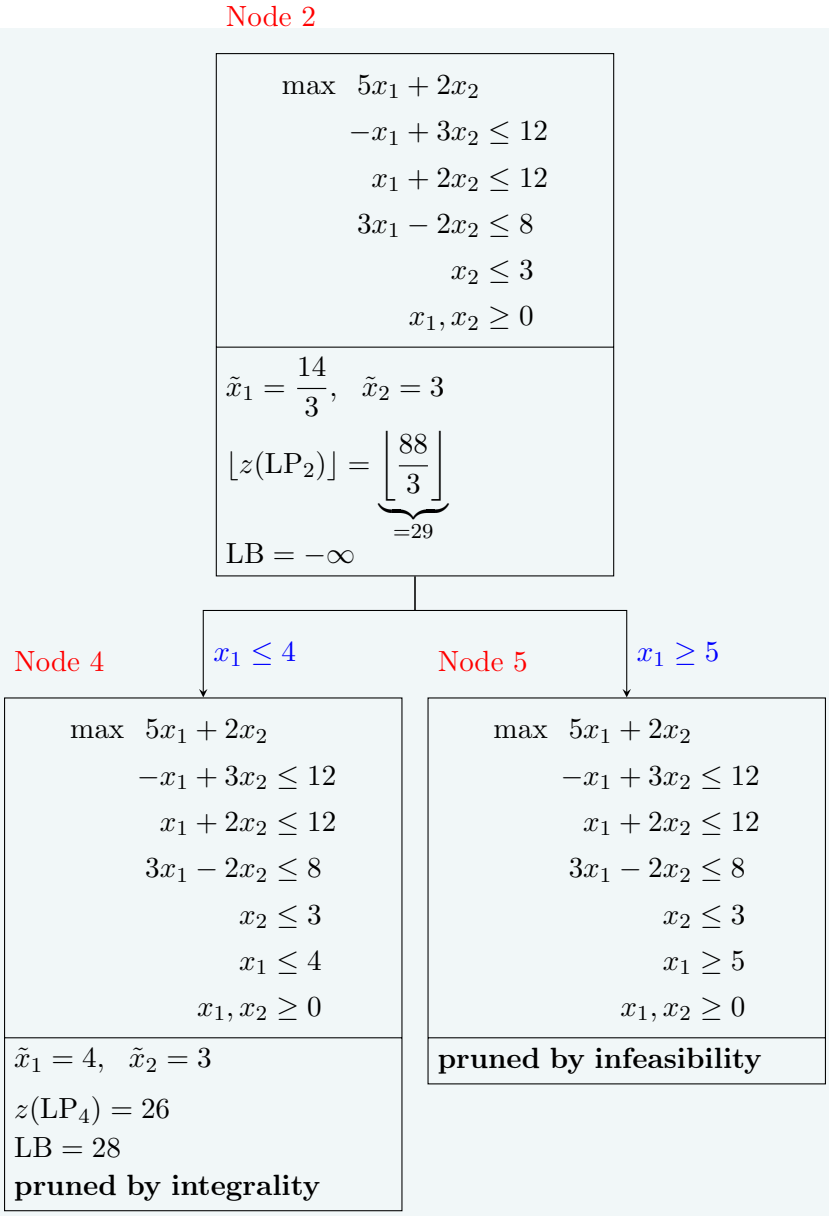
$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_7 = 0,$$
$$\bar{\pi}_5 = \frac{5}{3}, \quad \bar{\pi}_6 = \frac{16}{3} \quad \text{of value} \quad \frac{88}{3}$$

If we increase the value of π_7, we can get a better dual solution. We have:

$$\begin{cases} \pi_5 &= \frac{5}{3} + \frac{1}{3}\pi_7 \\ \pi_6 &= \frac{16}{3} + \frac{2}{3}\pi_7 \end{cases} \qquad \begin{cases} \underbrace{\frac{5}{3} + \frac{1}{3}\pi_7 \geq 0}_{\pi_5 \geq 0} \\ \underbrace{\frac{16}{3} + \frac{2}{3}\pi_7 \geq 0}_{\pi_6 \geq 0} \end{cases} \qquad \begin{cases} \pi_7 &\geq -5 \\ \pi_7 &\geq -8 \end{cases}$$

Accordingly, we can increase the value of π_7 without violating dual feasibility, which implies that the dual problem is **unbounded**. As a result, the primal problem LP_5 is **infeasible**, and the corresponding branching node is pruned by infeasibility.

At the end of the second iteration, branching on x_1 from Node 2 has generated the down-branching node, Node 4, and the up-branching node, Node 5. Node 4 is pruned by integrality, while Node 5 is pruned by infeasibility. The corresponding branch-and-bound tree is:



Due to the current lower bound LB and upper bound UB, we have:

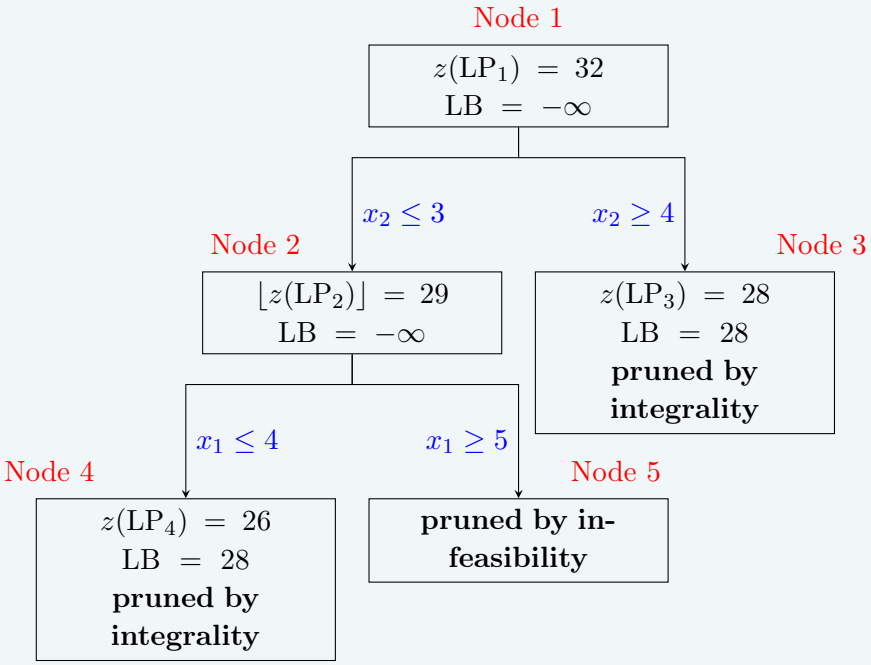
$$28 \quad \leq \quad z(\text{ILP}) \quad \leq \quad 28$$

- **Termination**

Since the list L is empty, the lower bound LB is equal to the upper bound UB and the ILP problem is solved to optimality.

The optimal value is $z(\text{ILP}) = 28$ (the value of the lower bound LB) and an optimal solution $\tilde{\boldsymbol{x}}(\text{ILP})$ is $\tilde{x}_1 = 4, \tilde{x}_2 = 4$ (the incumbent solution $\bar{\boldsymbol{x}}(\text{ILP})$).

2. The final branch-and-bound tree is:



The reasons why the branching nodes are pruned are the following.

- Node 1 has a fractional optimal solution with $z(\text{LP}_1) = 32$: it is branched on x_2.
- Node 2 has a fractional optimal solution with $\lfloor z(\text{LP}_2) \rfloor = 29$: it is branched on x_1.
- Node 3 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_3) = (4, 4)$ is integer, with value 28, which becomes the incumbent solution.
- Node 4 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_4) = (4, 3)$ is integer, with value 26, which does not improve the lower bound $\text{LB} = 28$.
- Node 5 is pruned by infeasibility: the LP_5 problem has no feasible solution.

The optimal value is $z(\text{ILP}) = 28$ and the branch-and-bound algorithm returns the optimal solution $\tilde{\boldsymbol{x}}(\text{ILP}) = (4, 4)$.

Exercise 7.10 — Branch-and-bound algorithm ($P21$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 3$ constraints:

$$\begin{array}{lrcrcl} \max & x_1 & + & 2\,x_2 & & \\ & -x_1 & - & x_2 & \leq & -2 \\ & 4\,x_1 & - & 2\,x_2 & \leq & 7 \\ & x_1 & + & 5\,x_2 & \leq & 21 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

and an optimal simplex dictionary of its LP relaxation in standard form:

$$\begin{cases} x_1 = \frac{7}{2} - \frac{5}{22}\,x_4 - \frac{1}{11}\,x_5 \\ x_2 = \frac{7}{2} + \frac{1}{22}\,x_4 - \frac{2}{11}\,x_5 \\ x_3 = 5 - \frac{2}{11}\,x_4 - \frac{3}{11}\,x_5 \\ z = \frac{21}{2} - \frac{3}{22}\,x_4 - \frac{5}{11}\,x_5 \end{cases}$$

Questions

1. Execute the branch-and-bound algorithm using:
   (a) the largest-index rule as branching-variable selection strategy;
   (b) the LIFO rule as branching-node selection strategy;
   (c) solve first the down-branching node and then the up-branching node;
   (d) set the lower bound LB to $-\infty$ (no heuristic algorithm is executed).

   Solve the LP problems of the branching nodes using the dual simplex algorithm. In the dual simplex algorithm, the leaving variable is the basic variable with negative value and smallest index, and ties in the ratio test are broken by the smallest index. Show graphically the optimal solution of the LP problems associated with the branching nodes.
2. Determine and comment the branch-and-bound tree, showing the reason why each branching node is pruned.
3. Re-execute the branch-and-bound algorithm, changing only the branching-variable selection strategy: use the smallest-index rule. Determine and comment the corresponding branch-and-bound tree.

Solution

1. Execution of the branch-and-bound algorithm to solve the ILP problem.

   - **Initialization**

     Let us consider the LP_1 problem in standard form:

$$
\begin{array}{lrcrcrcrcl}
\max & x_1 & + & 2\,x_2 & & & & & & \\
 & -x_1 & - & x_2 & + & x_3 & & & = & -2 \\
 & 4\,x_1 & - & 2\,x_2 & & & + & x_4 & = & 7 \\
 & x_1 & + & 5\,x_2 & & & & & + \quad x_5 & = \quad 21 \\
 & x_1, & & x_2, & & x_3, & & x_4, & x_5 & \geq \quad 0
\end{array}
$$

     and its optimal simplex dictionary:

$$
\begin{cases}
x_1 = \dfrac{7}{2} - \dfrac{5}{22}\,x_4 - \dfrac{1}{11}\,x_5 \\
x_2 = \dfrac{7}{2} + \dfrac{1}{22}\,x_4 - \dfrac{2}{11}\,x_5 \\
x_3 = 5 - \dfrac{2}{11}\,x_4 - \dfrac{3}{11}\,x_5 \\
z = \dfrac{21}{2} - \dfrac{3}{22}\,x_4 - \dfrac{5}{11}\,x_5
\end{cases}
$$

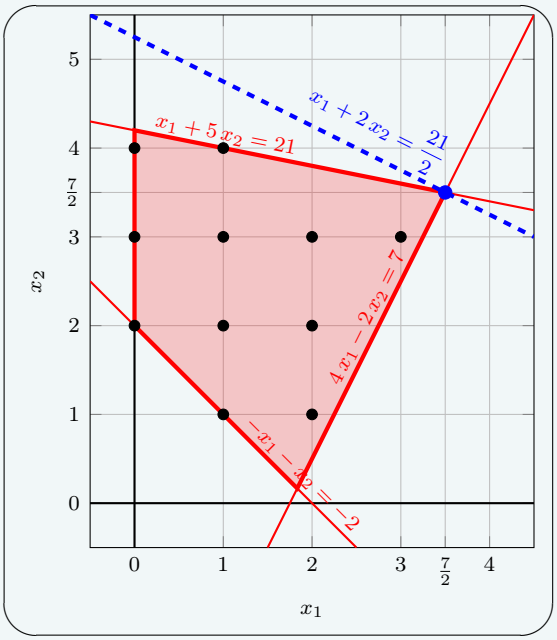


Figure 7.41: Node 1: feasible region of LP_1 (red), feasible points of the ILP problem (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is:

$$\tilde{x}_1 = \frac{7}{2}, \quad \tilde{x}_2 = \frac{7}{2}$$

the optimal value is:

$$z(\mathrm{LP}_1) = \frac{21}{2}$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is **fractional** and the LP_1 problem is **feasible**, so the LP_1 problem is inserted in the problem list L and the lower bound LB is set to $-\infty$.
Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \quad \leq \quad z(\mathrm{ILP}) \quad \leq \quad \underbrace{\left\lfloor \frac{21}{2} \right\rfloor}_{=10}$$

- **Iteration 1**

The LP_1 problem is extracted and deleted from the problem list L.

The branching variable is x_2. The branching constraints are:

$$x_2 \leq \underbrace{\left\lfloor \frac{7}{2} \right\rfloor}_{=3} \quad \text{and} \quad x_2 \geq \underbrace{\left\lfloor \frac{7}{2} \right\rfloor + 1}_{=4}$$

Two LP_2 and LP_3 problems are created with these branching constraints.

(a) Solution of the LP_2 problem

Let us consider the LP_2 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcr}
\max & x_1 & + & 2\,x_2 & & & & & & & & & \\
 & -x_1 & - & x_2 & + & x_3 & & & & & & = & -2 \\
 & 4\,x_1 & - & 2\,x_2 & & & + & x_4 & & & & = & 7 \\
 & x_1 & + & 5\,x_2 & & & & & + & x_5 & & = & 21 \\
 & & & x_2 & & & & & & & + & x_6 & = & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_2 + x_6 = 3 \qquad \Rightarrow \qquad x_6 = 3 - x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_6 = 3 - \left(\frac{7}{2} + \frac{1}{22}\, x_4 - \frac{2}{11}\, x_5 \right) \qquad \Rightarrow \qquad x_6 = -\frac{1}{2} - \frac{1}{22}\, x_4 + \frac{2}{11}\, x_5$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = \frac{7}{2} - \frac{5}{22}x_4 - \frac{1}{11}x_5 \\ x_2 = \frac{7}{2} + \frac{1}{22}x_4 - \frac{2}{11}x_5 \\ x_3 = 5 - \frac{2}{11}x_4 - \frac{3}{11}x_5 \\ x_6 = -\frac{1}{2} - \frac{1}{22}x_4 + \frac{2}{11}x_5 \\ z = \frac{21}{2} - \frac{3}{22}x_4 - \frac{5}{11}x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 6\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{7}{2}, \quad \bar{x}_2 = \frac{7}{2}, \quad \bar{x}_3 = 5, \quad \bar{x}_6 = -\frac{1}{2},$$
$$\bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{21}{2}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_6 = 0,$$
$$\bar{\pi}_4 = \frac{3}{22}, \quad \bar{\pi}_5 = \frac{5}{11} \quad \text{of value} \quad \frac{21}{2}$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_4 = \frac{3}{22} + \frac{1}{22}\pi_6 \\ \pi_5 = \frac{5}{11} - \frac{2}{11}\pi_6 \end{cases} \qquad \begin{cases} \underbrace{\frac{3}{22} + \frac{1}{22}\pi_6 \geq 0}_{\pi_4 \geq 0} \\ \underbrace{\frac{5}{11} - \frac{2}{11}\pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 \geq -3 \\ \pi_6 \leq \frac{5}{2} \end{cases}$$

By increasing the value of π_6 to $\frac{5}{2}$, the value of π_5 decreases to 0. Then we have that:

π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non-basic variable.

π_5 becomes a dual non-basic variable and accordingly x_5 becomes a primal basic variable.

We have:

$$x_6 = -\frac{1}{2} - \frac{1}{22}x_4 + \frac{2}{11}x_5 \quad \Rightarrow \quad x_5 = \frac{11}{4} + \frac{1}{4}x_4 + \frac{11}{2}x_6$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{7}{2} - \frac{5}{22}\,x_4 - \frac{1}{11}\left(\frac{11}{4} + \frac{1}{4}\,x_4 + \frac{11}{2}\,x_6\right) = \frac{13}{4} - \frac{1}{4}\,x_4 - \frac{1}{2}\,x_6$$

$$x_2 = \frac{7}{2} + \frac{1}{22}\,x_4 - \frac{2}{11}\left(\frac{11}{4} + \frac{1}{4}\,x_4 + \frac{11}{2}\,x_6\right) = 3 - x_6$$

$$x_3 = 5 - \frac{2}{11}\,x_4 - \frac{3}{11}\left(\frac{11}{4} + \frac{1}{4}\,x_4 + \frac{11}{2}\,x_6\right) = \frac{17}{4} - \frac{1}{4}\,x_4 - \frac{3}{2}\,x_6$$

$$z = \frac{21}{2} - \frac{3}{22}\,x_4 - \frac{5}{11}\left(\frac{11}{4} + \frac{1}{4}\,x_4 + \frac{11}{2}\,x_6\right) = \frac{37}{4} - \frac{1}{4}\,x_4 - \frac{5}{2}\,x_6$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = \frac{13}{4} - \frac{1}{4}\,x_4 - \frac{1}{2}\,x_6 \\ x_2 = 3 - x_6 \\ x_3 = \frac{17}{4} - \frac{1}{4}\,x_4 - \frac{3}{2}\,x_6 \\ x_5 = \frac{11}{4} + \frac{1}{4}\,x_4 + \frac{11}{2}\,x_6 \\ z = \frac{37}{4} - \frac{1}{4}\,x_4 - \frac{5}{2}\,x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 5\}$ and $\mathcal{N} = \{4, 6\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = \frac{13}{4}, \quad \tilde{x}_2 = 3, \quad \tilde{x}_3 = \frac{17}{4}, \quad \tilde{x}_5 = \frac{11}{4},$$
$$\tilde{x}_4 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad \frac{37}{4}$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_5 = 0,$$
$$\tilde{\pi}_4 = \frac{1}{4}, \quad \tilde{\pi}_6 = \frac{5}{2} \quad \text{of value} \quad \frac{37}{4}$$

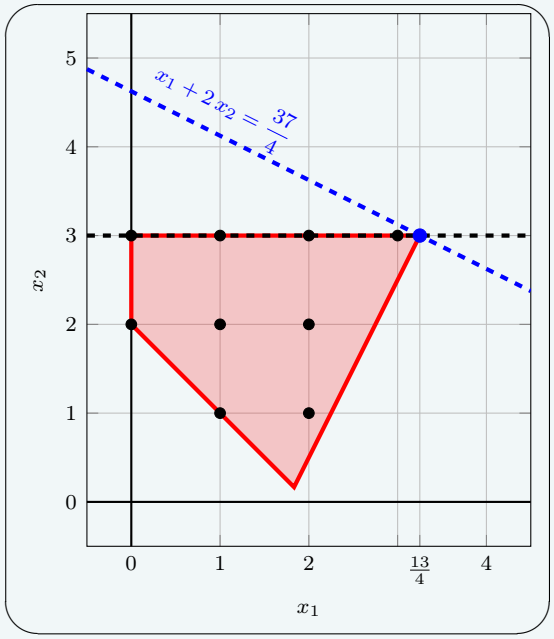


Figure 7.42: Node 2: feasible region of LP_2 (red), with the branching constraint $x_2 \leq 3$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is:

$$\tilde{x}_1 = \frac{13}{4}, \quad \tilde{x}_2 = 3$$

the optimal value is:

$$z(\mathrm{LP}_2) = \frac{37}{4}$$

The LP_2 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_2) \rfloor > \mathrm{LB} \qquad \text{since} \qquad \underbrace{\left\lfloor \frac{37}{4} \right\rfloor}_{=9} > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is **fractional**, accordingly, the LP_2 problem is **inserted in the problem list** L.

(b) Solution of the LP_3 problem

Let us consider the LP_3 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcr}
\max & x_1 & + & 2\,x_2 & & & & & & & & & \\
 & -x_1 & - & x_2 & + & x_3 & & & & & & = & -2 \\
 & 4\,x_1 & - & 2\,x_2 & & & + & x_4 & & & & = & 7 \\
 & x_1 & + & 5\,x_2 & & & & & + & x_5 & & = & 21 \\
 & & & x_2 & & & & & & & - & x_6 & = & 4 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_2 - x_6 = 4 \qquad \Rightarrow \qquad x_6 = -4 + x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_6 = -4 + \left(\frac{7}{2} + \frac{1}{22}\, x_4 - \frac{2}{11}\, x_5\right) \qquad \Rightarrow \qquad x_6 = -\frac{1}{2} + \frac{1}{22}\, x_4 - \frac{2}{11}\, x_5$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = \frac{7}{2} - \frac{5}{22}\, x_4 - \frac{1}{11}\, x_5 \\ x_2 = \frac{7}{2} + \frac{1}{22}\, x_4 - \frac{2}{11}\, x_5 \\ x_3 = 5 - \frac{2}{11}\, x_4 - \frac{3}{11}\, x_5 \\ x_6 = -\frac{1}{2} + \frac{1}{22}\, x_4 - \frac{2}{11}\, x_5 \\ z = \frac{21}{2} - \frac{3}{22}\, x_4 - \frac{5}{11}\, x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 6\}$ and $\mathcal{N} = \{4, 5\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{7}{2}, \quad \bar{x}_2 = \frac{7}{2}, \quad \bar{x}_3 = 5, \quad \bar{x}_6 = -\frac{1}{2}, \quad \bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{21}{2}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_6 = 0, \quad \bar{\pi}_4 = \frac{3}{22}, \quad \bar{\pi}_5 = \frac{5}{11} \quad \text{of value} \quad \frac{21}{2}$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_4 = \frac{3}{22} - \frac{1}{22}\, \pi_6 \\ \pi_5 = \frac{5}{11} + \frac{2}{11}\, \pi_6 \end{cases} \qquad \begin{cases} \underbrace{\frac{3}{22} - \frac{1}{22}\, \pi_6 \geq 0}_{\pi_4 \geq 0} \\ \underbrace{\frac{5}{11} + \frac{2}{11}\, \pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 \leq 3 \\ \pi_6 \geq -\frac{5}{2} \end{cases}$$

By increasing the value of π_6 to 3, the value of π_4 decreases to 0. Then we have that:

π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non-basic variable.

π_4 becomes a dual non-basic variable and accordingly x_4 becomes a primal basic variable.

We have:

$$x_6 = -\frac{1}{2} + \frac{1}{22}\,x_4 - \frac{2}{11}\,x_5 \quad \Rightarrow \quad x_4 = 11 + 4\,x_5 + 22\,x_6$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{7}{2} - \frac{5}{22}\,(11 + 4\,x_5 + 22\,x_6) - \frac{1}{11}\,x_5 = 1 - x_5 - 5\,x_6$$

$$x_2 = \frac{7}{2} + \frac{1}{22}\,(11 + 4\,x_5 + 22\,x_6) - \frac{2}{11}\,x_5 = 4 + x_6$$

$$x_3 = 5 - \frac{2}{11}\,(11 + 4\,x_5 + 22\,x_6) - \frac{3}{11}\,x_5 = 3 - x_5 - 4\,x_6$$

$$z = \frac{21}{2} - \frac{3}{22}\,(11 + 4\,x_5 + 22\,x_6) - \frac{5}{11}\,x_5 = 9 - x_5 - 3\,x_6$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 1 - x_5 - 5\,x_6 \\ x_2 = 4 + x_6 \\ x_3 = 3 - x_5 - 4\,x_6 \\ x_4 = 11 + 4\,x_5 + 22\,x_6 \\ z = 9 - x_5 - 3\,x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4\}$ and $\mathcal{N} = \{5, 6\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 4, \quad \tilde{x}_3 = 3, \quad \tilde{x}_4 = 11,$$
$$\tilde{x}_5 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad 9$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 0,$$
$$\tilde{\pi}_5 = 1, \quad \tilde{\pi}_6 = 3 \quad \text{of value} \quad 9$$

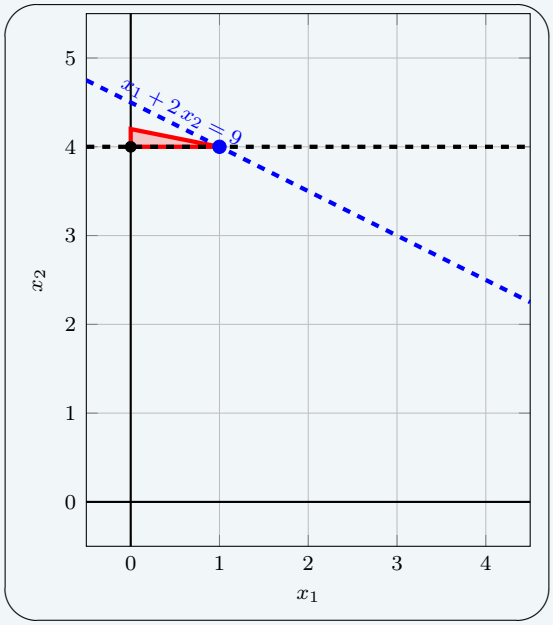


Figure 7.43: Node 3: feasible region of LP_3 (red), with the branching constraint $x_2 \geq 4$ (dashed), and its optimal solution (blue).
An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_3)$ is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 4$$

the optimal value is:

$$z(\mathrm{LP}_3) = 9$$

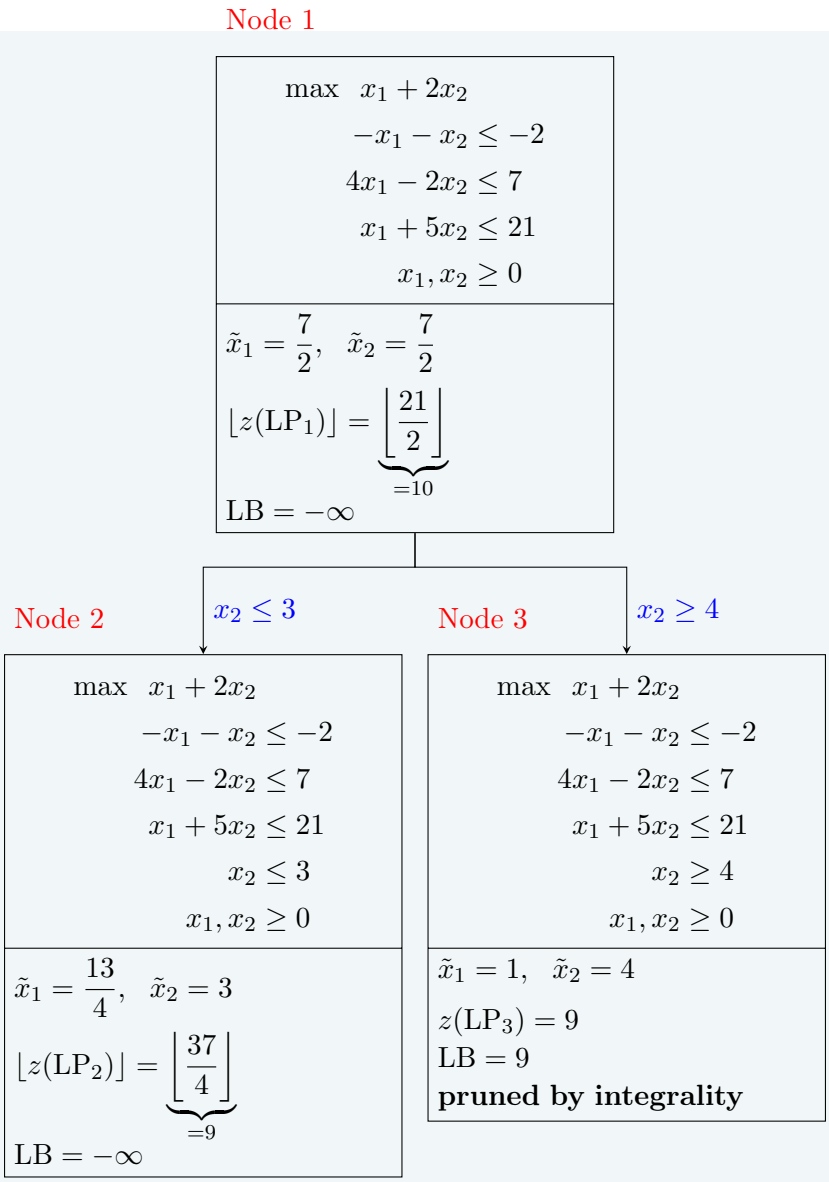
The LP_3 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_3) \rfloor > \mathrm{LB} \qquad \text{since} \qquad 9 > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_3)$ is **integer**, accordingly, the lower bound LB becomes 9 and the incumbent solution $\bar{\boldsymbol{x}}(\mathrm{ILP})$ becomes $\bar{x}_1 = 1, \bar{x}_2 = 4$.
Accordingly, the branching node is pruned by integrality.
Node 2 is removed from the problem list L (pruned by bound), since now

$$\lfloor z(\mathrm{LP}_2) \rfloor = 9 \leq \mathrm{LB} = 9$$

At the end of the first iteration, branching on x_2 from Node 1 has generated the down-branching node, Node 2, and the up-branching node, Node 3. Node 2 is inserted in the list L, while Node 3 is pruned by integrality. Node 2 is then removed from the problem list L (pruned by bound). The corresponding branch-and-bound tree is:



Due to the current lower bound LB and upper bound UB, we have:

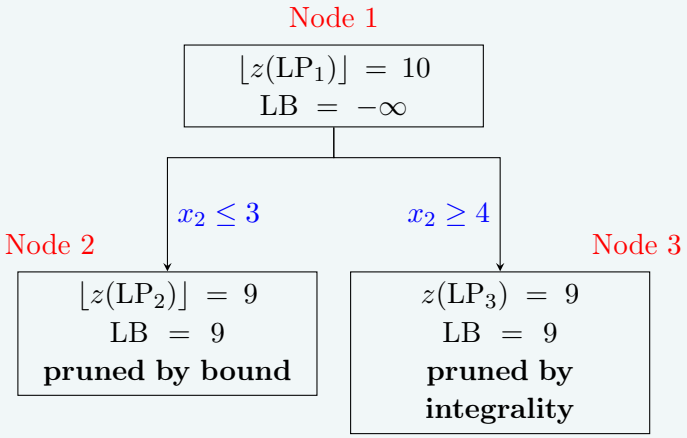
$$9 \quad \leq \quad z(\text{ILP}) \quad \leq \quad 9$$

- **Termination**

Since the list L is empty, the lower bound LB is equal to the upper bound UB and the ILP problem is solved to optimality.

The optimal value is $z(\text{ILP}) = 9$ (the value of the lower bound LB) and an optimal solution $\tilde{\boldsymbol{x}}(\text{ILP})$ is $\tilde{x}_1 = 1, \tilde{x}_2 = 4$ (the incumbent solution $\bar{\boldsymbol{x}}(\text{ILP})$).

2. The final branch-and-bound tree is:



The reasons why the branching nodes are pruned are the following.

- Node 1 has a fractional optimal solution with $\lfloor z(\text{LP}_1) \rfloor = 10$: it is branched on x_2.
- Node 2 is pruned by bound: after an improvement of the lower bound it is removed from the list, since $\lfloor z(\text{LP}_2) \rfloor = 9 \leq \text{LB} = 9$.
- Node 3 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_3) = (1, 4)$ is integer, with value 9, which becomes the incumbent solution.

The optimal value is $z(\text{ILP}) = 9$ and the branch-and-bound algorithm returns the optimal solution $\tilde{\boldsymbol{x}}(\text{ILP}) = (1, 4)$. The optimal solution is **not unique**: also $(3, 3)$ has value 9. With the rules stated in the question, the algorithm finds $(1, 4)$.

3. Execution of the branch-and-bound algorithm with the smallest-index rule as branching-variable selection strategy.

   - **Initialization**

     Let us consider the LP_1 problem in standard form:

$$
\begin{array}{lrcrcrcrcrcr}
\max & x_1 & + & 2\,x_2 & & & & & & & & \\
 & -x_1 & - & x_2 & + & x_3 & & & & & = & -2 \\
 & 4\,x_1 & - & 2\,x_2 & & & + & x_4 & & & = & 7 \\
 & x_1 & + & 5\,x_2 & & & & & + & x_5 & = & 21 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
\end{array}
$$

     and its optimal simplex dictionary:

$$\begin{cases} x_1 = \frac{7}{2} - \frac{5}{22} x_4 - \frac{1}{11} x_5 \\ x_2 = \frac{7}{2} + \frac{1}{22} x_4 - \frac{2}{11} x_5 \\ x_3 = 5 - \frac{2}{11} x_4 - \frac{3}{11} x_5 \\ z = \frac{21}{2} - \frac{3}{22} x_4 - \frac{5}{11} x_5 \end{cases}$$

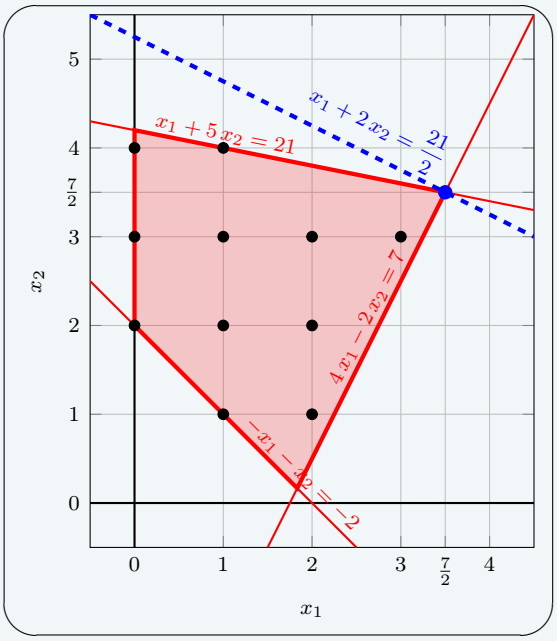


Figure 7.44: Node 1: feasible region of LP_1 (red), feasible points of the ILP problem (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is:

$$\tilde{x}_1 = \frac{7}{2}, \quad \tilde{x}_2 = \frac{7}{2}$$

the optimal value is:

$$z(\mathrm{LP}_1) = \frac{21}{2}$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is **fractional** and the LP_1 problem is **feasible**, so the LP_1 problem is inserted in the problem list L and the lower bound LB is set to $-\infty$.

Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \leq z(\mathrm{ILP}) \leq \underbrace{\left\lfloor \frac{21}{2} \right\rfloor}_{=10}$$

- **Iteration 1**

The LP_1 problem is extracted and deleted from the problem list L.

The branching variable is x_1. The branching constraints are:

$$x_1 \leq \underbrace{\left\lfloor \frac{7}{2} \right\rfloor}_{=3} \quad \text{and} \quad x_1 \geq \underbrace{\left\lfloor \frac{7}{2} \right\rfloor + 1}_{=4}$$

Two LP_2 and LP_3 problems are created with these branching constraints.

(a) Solution of the LP_2 problem

Let us consider the LP_2 problem in standard form:

$$\begin{array}{lrcrcrcrcrcr}
\max & x_1 & + & 2\,x_2 & & & & & & & & \\
 & -x_1 & - & x_2 & + & x_3 & & & & & = & -2 \\
 & 4\,x_1 & - & 2\,x_2 & & & + & x_4 & & & = & 7 \\
 & x_1 & + & 5\,x_2 & & & & & + & x_5 & = & 21 \\
 & x_1 & & & & & & & & + \; x_6 & = & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, \quad x_6 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_1 + x_6 = 3 \qquad \Rightarrow \qquad x_6 = 3 - x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_6 = 3 - \left(\frac{7}{2} - \frac{5}{22}\, x_4 - \frac{1}{11}\, x_5 \right) \qquad \Rightarrow \qquad x_6 = -\frac{1}{2} + \frac{5}{22}\, x_4 + \frac{1}{11}\, x_5$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases}
x_1 = \frac{7}{2} - \frac{5}{22}\, x_4 - \frac{1}{11}\, x_5 \\
x_2 = \frac{7}{2} + \frac{1}{22}\, x_4 - \frac{2}{11}\, x_5 \\
x_3 = 5 - \frac{2}{11}\, x_4 - \frac{3}{11}\, x_5 \\
x_6 = -\frac{1}{2} + \frac{5}{22}\, x_4 + \frac{1}{11}\, x_5 \\
z = \frac{21}{2} - \frac{3}{22}\, x_4 - \frac{5}{11}\, x_5
\end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 6\}$ and $\mathcal{N} = \{4, 5\}$ and

– the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{7}{2}, \quad \bar{x}_2 = \frac{7}{2}, \quad \bar{x}_3 = 5, \quad \bar{x}_6 = -\frac{1}{2},$$
$$\bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{21}{2}$$

– the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_6 = 0,$$
$$\bar{\pi}_4 = \frac{3}{22}, \quad \bar{\pi}_5 = \frac{5}{11} \quad \text{of value} \quad \frac{21}{2}$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_4 = \frac{3}{22} - \frac{5}{22}\pi_6 \\ \pi_5 = \frac{5}{11} - \frac{1}{11}\pi_6 \end{cases} \qquad \begin{cases} \underbrace{\frac{3}{22} - \frac{5}{22}\pi_6 \geq 0}_{\pi_4 \geq 0} \\ \underbrace{\frac{5}{11} - \frac{1}{11}\pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 \leq \frac{3}{5} \\ \pi_6 \leq 5 \end{cases}$$

By increasing the value of π_6 to $\min\left\{\frac{3}{5}, 5\right\} = \frac{3}{5}$, the value of π_4 decreases to 0. Then we have that:

π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non-basic variable.

π_4 becomes a dual non-basic variable and accordingly x_4 becomes a primal basic variable.

We have:

$$x_6 = -\frac{1}{2} + \frac{5}{22}x_4 + \frac{1}{11}x_5 \quad \Rightarrow \quad x_4 = \frac{11}{5} - \frac{2}{5}x_5 + \frac{22}{5}x_6$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{7}{2} - \frac{5}{22}\left(\frac{11}{5} - \frac{2}{5}x_5 + \frac{22}{5}x_6\right) - \frac{1}{11}x_5 = 3 - x_6$$
$$x_2 = \frac{7}{2} + \frac{1}{22}\left(\frac{11}{5} - \frac{2}{5}x_5 + \frac{22}{5}x_6\right) - \frac{2}{11}x_5 = \frac{18}{5} - \frac{1}{5}x_5 + \frac{1}{5}x_6$$
$$x_3 = 5 - \frac{2}{11}\left(\frac{11}{5} - \frac{2}{5}x_5 + \frac{22}{5}x_6\right) - \frac{3}{11}x_5 = \frac{23}{5} - \frac{1}{5}x_5 - \frac{4}{5}x_6$$
$$z = \frac{21}{2} - \frac{3}{22}\left(\frac{11}{5} - \frac{2}{5}x_5 + \frac{22}{5}x_6\right) - \frac{5}{11}x_5 = \frac{51}{5} - \frac{2}{5}x_5 - \frac{3}{5}x_6$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 3 - x_6 \\ x_2 = \frac{18}{5} - \frac{1}{5}x_5 + \frac{1}{5}x_6 \\ x_3 = \frac{23}{5} - \frac{1}{5}x_5 - \frac{4}{5}x_6 \\ x_4 = \frac{11}{5} - \frac{2}{5}x_5 + \frac{22}{5}x_6 \\ z = \frac{51}{5} - \frac{2}{5}x_5 - \frac{3}{5}x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4\}$ and $\mathcal{N} = \{5, 6\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 3, \quad \tilde{x}_2 = \frac{18}{5}, \quad \tilde{x}_3 = \frac{23}{5}, \quad \tilde{x}_4 = \frac{11}{5},$$
$$\tilde{x}_5 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad \frac{51}{5}$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 0,$$
$$\tilde{\pi}_5 = \frac{2}{5}, \quad \tilde{\pi}_6 = \frac{3}{5} \quad \text{of value} \quad \frac{51}{5}$$

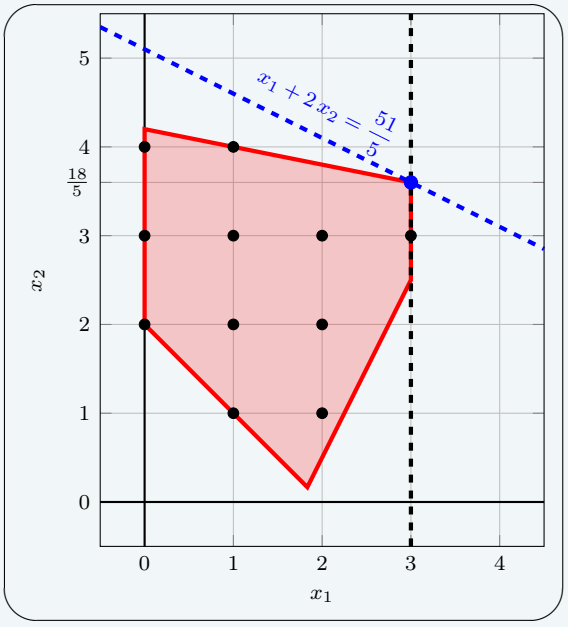


Figure 7.45: Node 2: feasible region of LP_2 (red), with the branching constraint $x_1 \leq 3$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_2)$ is:

$$\tilde{x}_1 = 3, \quad \tilde{x}_2 = \frac{18}{5}$$

the optimal value is:

$$z(\mathrm{LP}_2) = \frac{51}{5}$$

The LP_2 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_2) \rfloor > \mathrm{LB} \qquad \text{since} \qquad \underbrace{\left\lfloor \frac{51}{5} \right\rfloor}_{=10} > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is **fractional**, accordingly, the LP_2 problem is **inserted in the problem list** L.

(b) Solution of the LP_3 problem

Let us consider the LP_3 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcr}
\max & x_1 & + & 2\,x_2 & & & & & & & & & \\
& -x_1 & - & x_2 & + & x_3 & & & & & & = & -2 \\
& 4\,x_1 & - & 2\,x_2 & & & + & x_4 & & & & = & 7 \\
& x_1 & + & 5\,x_2 & & & & & + & x_5 & & = & 21 \\
& x_1 & & & & & & & & & - & x_6 = & 4 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_1 - x_6 = 4 \qquad \Rightarrow \qquad x_6 = -4 + x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_6 = -4 + \left(\frac{7}{2} - \frac{5}{22}\, x_4 - \frac{1}{11}\, x_5 \right) \qquad \Rightarrow \qquad x_6 = -\frac{1}{2} - \frac{5}{22}\, x_4 - \frac{1}{11}\, x_5$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases}
x_1 = \frac{7}{2} - \frac{5}{22}\, x_4 - \frac{1}{11}\, x_5 \\
x_2 = \frac{7}{2} + \frac{1}{22}\, x_4 - \frac{2}{11}\, x_5 \\
x_3 = 5 - \frac{2}{11}\, x_4 - \frac{3}{11}\, x_5 \\
x_6 = -\frac{1}{2} - \frac{5}{22}\, x_4 - \frac{1}{11}\, x_5 \\
z = \frac{21}{2} - \frac{3}{22}\, x_4 - \frac{5}{11}\, x_5
\end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 6\}$ and $\mathcal{N} = \{4, 5\}$ and

– the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{7}{2}, \quad \bar{x}_2 = \frac{7}{2}, \quad \bar{x}_3 = 5, \quad \bar{x}_6 = -\frac{1}{2},$$
$$\bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{21}{2}$$

– the basic dual feasible solution is:

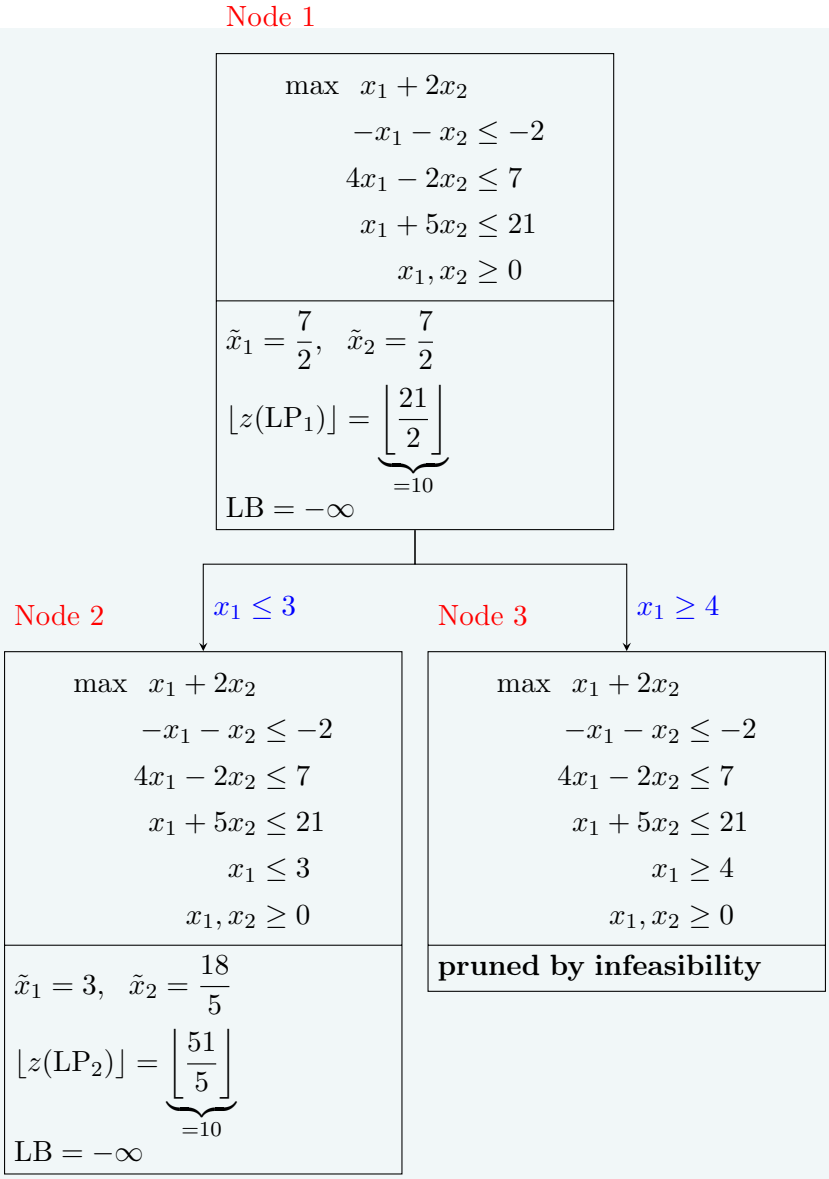
$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_6 = 0,$$
$$\bar{\pi}_4 = \frac{3}{22}, \quad \bar{\pi}_5 = \frac{5}{11} \quad \text{of value} \quad \frac{21}{2}$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_4 = \frac{3}{22} + \frac{5}{22}\pi_6 \\ \pi_5 = \frac{5}{11} + \frac{1}{11}\pi_6 \end{cases} \qquad \begin{cases} \underbrace{\frac{3}{22} + \frac{5}{22}\pi_6 \geq 0}_{\pi_4 \geq 0} \\ \underbrace{\frac{5}{11} + \frac{1}{11}\pi_6 \geq 0}_{\pi_5 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 \geq -\frac{3}{5} \\ \pi_6 \geq -5 \end{cases}$$

Accordingly, we can increase the value of π_6 without violating dual feasibility, which implies that the dual problem is **unbounded**. As a result, the primal problem LP_3 is **infeasible**, and the corresponding branching node is pruned by infeasibility.

At the end of the first iteration, branching on x_1 from Node 1 has generated the down-branching node, Node 2, and the up-branching node, Node 3. Node 2 is inserted in the list L, while Node 3 is pruned by infeasibility. The corresponding branch-and-bound tree is:



Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \quad \leq \quad z(\text{ILP}) \quad \leq \quad \underbrace{\left\lfloor \frac{51}{5} \right\rfloor}_{=10}$$

- **Iteration 2**

The LP_2 problem is extracted and deleted from the problem list L.

The branching variable is x_2. The branching constraints are:

$$x_2 \leq \underbrace{\left\lfloor \frac{18}{5} \right\rfloor}_{=3} \quad \text{and} \quad x_2 \geq \underbrace{\left\lfloor \frac{18}{5} \right\rfloor + 1}_{=4}$$

Two LP_4 and LP_5 problems are created with these branching constraints.

(a) Solution of the LP_4 problem

Let us consider the LP_4 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcr}
\max & x_1 & + & 2\,x_2 & & & & & & & & & & \\
 & -x_1 & - & x_2 & + & x_3 & & & & & & & = & -2 \\
 & 4\,x_1 & - & 2\,x_2 & & & + & x_4 & & & & & = & 7 \\
 & x_1 & + & 5\,x_2 & & & & & + & x_5 & & & = & 21 \\
 & x_1 & & & & & & & & & + & x_6 & = & 3 \\
 & & & x_2 & & & & & & & & & = & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_2 + x_7 = 3 \qquad \Rightarrow \quad x_7 = 3 - x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_7 = 3 - \left(\frac{18}{5} - \frac{1}{5}\,x_5 + \frac{1}{5}\,x_6\right) \qquad \Rightarrow \quad x_7 = -\frac{3}{5} + \frac{1}{5}\,x_5 - \frac{1}{5}\,x_6$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases}
x_1 = 3 - x_6 \\
x_2 = \frac{18}{5} - \frac{1}{5}\,x_5 + \frac{1}{5}\,x_6 \\
x_3 = \frac{23}{5} - \frac{1}{5}\,x_5 - \frac{4}{5}\,x_6 \\
x_4 = \frac{11}{5} - \frac{2}{5}\,x_5 + \frac{22}{5}\,x_6 \\
x_7 = -\frac{3}{5} + \frac{1}{5}\,x_5 - \frac{1}{5}\,x_6 \\
z = \frac{51}{5} - \frac{2}{5}\,x_5 - \frac{3}{5}\,x_6
\end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4, 7\}$ and $\mathcal{N} = \{5, 6\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = 3, \quad \bar{x}_2 = \frac{18}{5}, \quad \bar{x}_3 = \frac{23}{5}, \quad \bar{x}_4 = \frac{11}{5}, \quad \bar{x}_7 = -\frac{3}{5},$$
$$\bar{x}_5 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad \frac{51}{5}$$

– the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_7 = 0,$$
$$\bar{\pi}_5 = \frac{2}{5}, \quad \bar{\pi}_6 = \frac{3}{5} \quad \text{of value} \quad \frac{51}{5}$$

If we increase the value of π_7, we can get a better dual solution. We have:

$$\begin{cases} \pi_5 = \frac{2}{5} - \frac{1}{5}\pi_7 \\ \pi_6 = \frac{3}{5} + \frac{1}{5}\pi_7 \end{cases} \qquad \begin{cases} \underbrace{\frac{2}{5} - \frac{1}{5}\pi_7 \geq 0}_{\pi_5 \geq 0} \\ \underbrace{\frac{3}{5} + \frac{1}{5}\pi_7 \geq 0}_{\pi_6 \geq 0} \end{cases} \qquad \begin{cases} \pi_7 \leq 2 \\ \pi_7 \geq -3 \end{cases}$$

By increasing the value of π_7 to 2, the value of π_5 decreases to 0. Then we have that:

π_7 becomes a dual basic variable and accordingly x_7 becomes a primal non-basic variable.

π_5 becomes a dual non-basic variable and accordingly x_5 becomes a primal basic variable.

We have:

$$x_7 = -\frac{3}{5} + \frac{1}{5}x_5 - \frac{1}{5}x_6 \quad \Rightarrow \quad x_5 = 3 + x_6 + 5\,x_7$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_2 = \frac{18}{5} - \frac{1}{5}(3 + x_6 + 5\,x_7) + \frac{1}{5}x_6 = 3 - x_7$$
$$x_3 = \frac{23}{5} - \frac{1}{5}(3 + x_6 + 5\,x_7) - \frac{4}{5}x_6 = 4 - x_6 - x_7$$
$$x_4 = \frac{11}{5} - \frac{2}{5}(3 + x_6 + 5\,x_7) + \frac{22}{5}x_6 = 1 + 4\,x_6 - 2\,x_7$$
$$z = \frac{51}{5} - \frac{2}{5}(3 + x_6 + 5\,x_7) - \frac{3}{5}x_6 = 9 - x_6 - 2\,x_7$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 3 - x_6 \\ x_2 = 3 - x_7 \\ x_3 = 4 - x_6 - x_7 \\ x_4 = 1 + 4\,x_6 - 2\,x_7 \\ x_5 = 3 + x_6 + 5\,x_7 \\ z = 9 - x_6 - 2\,x_7 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4, 5\}$ and $\mathcal{N} = \{6, 7\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 3, \quad \tilde{x}_2 = 3, \quad \tilde{x}_3 = 4, \quad \tilde{x}_4 = 1, \quad \tilde{x}_5 = 3,$$
$$\tilde{x}_6 = 0, \quad \tilde{x}_7 = 0 \quad \text{of value} \quad 9$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 0, \quad \tilde{\pi}_5 = 0,$$
$$\tilde{\pi}_6 = 1, \quad \tilde{\pi}_7 = 2 \quad \text{of value} \quad 9$$

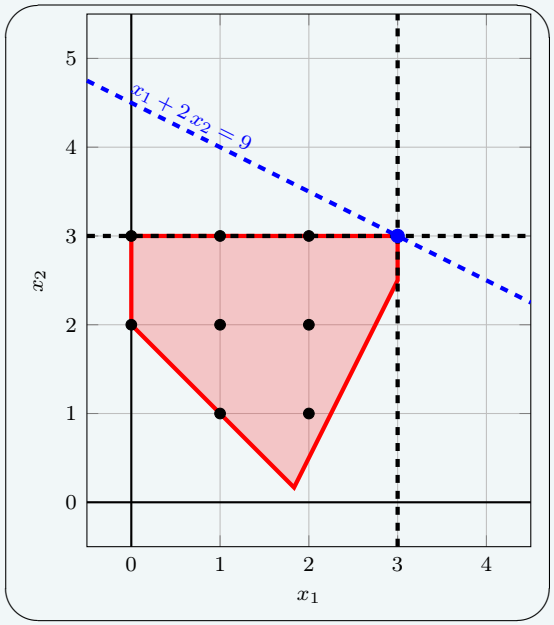


Figure 7.46: Node 4: feasible region of LP_4 (red), with the branching constraint $x_2 \leq 3$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_4)$ is:

$$\tilde{x}_1 = 3, \quad \tilde{x}_2 = 3$$

the optimal value is:

$$z(\text{LP}_4) = 9$$

The LP_4 problem is **feasible** and we have

$$\lfloor z(\text{LP}_4) \rfloor > \text{LB} \qquad \text{since} \qquad 9 > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_4)$ is **integer**, accordingly, the lower bound LB becomes 9 and the incumbent solution $\bar{\boldsymbol{x}}(\text{ILP})$ becomes $\bar{x}_1 = 3, \bar{x}_2 = 3$.

Accordingly, the branching node is pruned by integrality.

(b) Solution of the LP_5 problem

Let us consider the LP_5 problem in standard form:

$$
\begin{array}{lrcrcrcrcrcrcr}
\max & x_1 & + & 2\,x_2 & & & & & & & & & & \\
 & -x_1 & - & x_2 & + & x_3 & & & & & & & = & -2 \\
 & 4\,x_1 & - & 2\,x_2 & & & + & x_4 & & & & & = & 7 \\
 & x_1 & + & 5\,x_2 & & & & & + & x_5 & & & = & 21 \\
 & x_1 & & & & & & & & & + & x_6 & = & 3 \\
 & & & x_2 & & & & & & & & & = & 4 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & \geq & 0
\end{array}
$$

For the branching constraint we have:

$$x_2 - x_7 = 4 \quad \Rightarrow \quad x_7 = -4 + x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_7 = -4 + \left(\frac{18}{5} - \frac{1}{5}\,x_5 + \frac{1}{5}\,x_6\right) \quad \Rightarrow \quad x_7 = -\frac{2}{5} - \frac{1}{5}\,x_5 + \frac{1}{5}\,x_6$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$
\begin{cases}
x_1 = 3 - x_6 \\
x_2 = \frac{18}{5} - \frac{1}{5}\,x_5 + \frac{1}{5}\,x_6 \\
x_3 = \frac{23}{5} - \frac{1}{5}\,x_5 - \frac{4}{5}\,x_6 \\
x_4 = \frac{11}{5} - \frac{2}{5}\,x_5 + \frac{22}{5}\,x_6 \\
x_7 = -\frac{2}{5} - \frac{1}{5}\,x_5 + \frac{1}{5}\,x_6 \\
z = \frac{51}{5} - \frac{2}{5}\,x_5 - \frac{3}{5}\,x_6
\end{cases}
$$

where $\mathcal{B} = \{1, 2, 3, 4, 7\}$ and $\mathcal{N} = \{5, 6\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = 3, \quad \bar{x}_2 = \frac{18}{5}, \quad \bar{x}_3 = \frac{23}{5}, \quad \bar{x}_4 = \frac{11}{5}, \quad \bar{x}_7 = -\frac{2}{5},$$
$$\bar{x}_5 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad \frac{51}{5}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_7 = 0,$$
$$\bar{\pi}_5 = \frac{2}{5}, \quad \bar{\pi}_6 = \frac{3}{5} \quad \text{of value} \quad \frac{51}{5}$$

If we increase the value of π_7, we can get a better dual solution. We have:

$$\begin{cases} \pi_5 = \frac{2}{5} + \frac{1}{5}\pi_7 \\ \pi_6 = \frac{3}{5} - \frac{1}{5}\pi_7 \end{cases} \qquad \begin{cases} \underbrace{\frac{2}{5} + \frac{1}{5}\pi_7 \geq 0}_{\pi_5 \geq 0} \\ \underbrace{\frac{3}{5} - \frac{1}{5}\pi_7 \geq 0}_{\pi_6 \geq 0} \end{cases} \qquad \begin{cases} \pi_7 \geq -2 \\ \pi_7 \leq 3 \end{cases}$$

By increasing the value of π_7 to 3, the value of π_6 decreases to 0. Then we have that:

π_7 becomes a dual basic variable and accordingly x_7 becomes a primal non-basic variable.

π_6 becomes a dual non-basic variable and accordingly x_6 becomes a primal basic variable.

We have:

$$x_7 = -\frac{2}{5} - \frac{1}{5}x_5 + \frac{1}{5}x_6 \quad \Rightarrow \quad x_6 = 2 + x_5 + 5\,x_7$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = 3 - (2 + x_5 + 5\,x_7) = 1 - x_5 - 5\,x_7$$

$$x_2 = \frac{18}{5} - \frac{1}{5}x_5 + \frac{1}{5}(2 + x_5 + 5\,x_7) = 4 + x_7$$

$$x_3 = \frac{23}{5} - \frac{1}{5}x_5 - \frac{4}{5}(2 + x_5 + 5\,x_7) = 3 - x_5 - 4\,x_7$$

$$x_4 = \frac{11}{5} - \frac{2}{5}x_5 + \frac{22}{5}(2 + x_5 + 5\,x_7) = 11 + 4\,x_5 + 22\,x_7$$

$$z = \frac{51}{5} - \frac{2}{5}x_5 - \frac{3}{5}(2 + x_5 + 5\,x_7) = 9 - x_5 - 3\,x_7$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 1 - x_5 - 5\,x_7 \\ x_2 = 4 + x_7 \\ x_3 = 3 - x_5 - 4\,x_7 \\ x_4 = 11 + 4\,x_5 + 22\,x_7 \\ x_6 = 2 + x_5 + 5\,x_7 \\ z = 9 - x_5 - 3\,x_7 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4, 6\}$ and $\mathcal{N} = \{5, 7\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 4, \quad \tilde{x}_3 = 3, \quad \tilde{x}_4 = 11, \quad \tilde{x}_6 = 2,$$
$$\tilde{x}_5 = 0, \quad \tilde{x}_7 = 0 \quad \text{of value} \quad 9$$

– the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 0, \quad \tilde{\pi}_6 = 0,$$
$$\tilde{\pi}_5 = 1, \quad \tilde{\pi}_7 = 3 \quad \text{of value} \quad 9$$

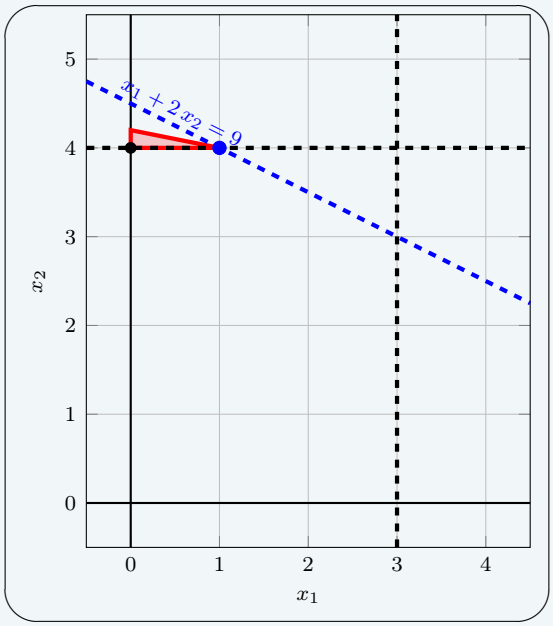


Figure 7.47: Node 5: feasible region of LP_5 (red), with the branching constraint $x_2 \geq 4$ (dashed), and its optimal solution (blue).
An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_5)$ is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 4$$

the optimal value is:

$$z(\mathrm{LP}_5) = 9$$

The LP_5 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_5) \rfloor \leq \mathrm{LB} \qquad \text{since} \qquad 9 \leq 9$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_5)$ is **integer**, but its value does not improve the lower bound LB, which remains 9.
Accordingly, the branching node is pruned by integrality.

At the end of the second iteration, branching on x_2 from Node 2 has generated the down-branching node, Node 4, and the up-branching node, Node 5. Both nodes are pruned by integrality. The corresponding branch-and-bound tree is:

Node 2

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ & -x_1 - x_2 \leq -2 \\ & 4x_1 - 2x_2 \leq 7 \\ & x_1 + 5x_2 \leq 21 \\ & x_1 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\tilde{x}_1 = 3, \quad \tilde{x}_2 = \frac{18}{5}$

$\lfloor z(\mathrm{LP}_2) \rfloor = \underbrace{\left\lfloor \frac{51}{5} \right\rfloor}_{=10}$

$\mathrm{LB} = -\infty$

$x_2 \leq 3$ | $x_2 \geq 4$

Node 4

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ & -x_1 - x_2 \leq -2 \\ & 4x_1 - 2x_2 \leq 7 \\ & x_1 + 5x_2 \leq 21 \\ & x_1 \leq 3 \\ & x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\tilde{x}_1 = 3, \quad \tilde{x}_2 = 3$

$z(\mathrm{LP}_4) = 9$

$\mathrm{LB} = 9$

pruned by integrality

Node 5

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ & -x_1 - x_2 \leq -2 \\ & 4x_1 - 2x_2 \leq 7 \\ & x_1 + 5x_2 \leq 21 \\ & x_1 \leq 3 \\ & x_2 \geq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 4$

$z(\mathrm{LP}_5) = 9$

$\mathrm{LB} = 9$

pruned by integrality

Due to the current lower bound LB and upper bound UB, we have:

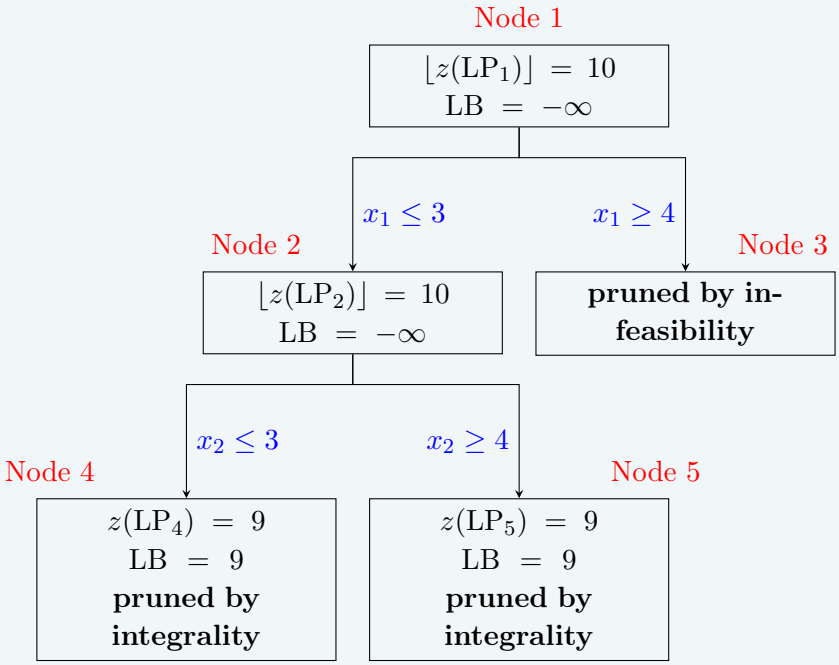
$$9 \quad \leq \quad z(\mathrm{ILP}) \quad \leq \quad 9$$

- **Termination**

Since the list L is empty, the lower bound LB is equal to the upper bound UB and the ILP problem is solved to optimality.

The optimal value is $z(\mathrm{ILP}) = 9$ (the value of the lower bound LB) and an optimal solution $\tilde{\boldsymbol{x}}(\mathrm{ILP})$ is $\tilde{x}_1 = 3, \tilde{x}_2 = 3$ (the incumbent solution $\bar{\boldsymbol{x}}(\mathrm{ILP})$).

The final branch-and-bound tree of this second execution is:



The reasons why the branching nodes are pruned are the following.

- Node 1 has a fractional optimal solution with $\lfloor z(\text{LP}_1) \rfloor = 10$: it is branched on x_1.
- Node 2 has a fractional optimal solution with $\lfloor z(\text{LP}_2) \rfloor = 10$: it is branched on x_2.
- Node 3 is pruned by infeasibility: the LP_3 problem has no feasible solution.
- Node 4 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_4) = (3, 3)$ is integer, with value 9, which becomes the incumbent solution.
- Node 5 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_5) = (1, 4)$ is integer, with value 9, which does not improve the lower bound LB $= 9$.

The optimal value is $z(\text{ILP}) = 9$ and the branch-and-bound algorithm returns the optimal solution $\tilde{\boldsymbol{x}}(\text{ILP}) = (3, 3)$. The optimal solution is **not unique**: also $(1, 4)$ has value 9. With the rules stated in the question, the algorithm finds $(3, 3)$.

The two executions return two different optimal solutions, $(1, 4)$ and $(3, 3)$, with the same optimal value 9.

Exercise 7.11 — Branch-and-bound algorithm ($P23$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 3$ constraints:

$$\begin{array}{lrcrcl} \max & x_1 & + & 4\ x_2 & & \\ & 3\ x_1 & + & 8\ x_2 & \leq & 24 \\ & -x_1 & & & \leq & -1 \\ & & & -x_2 & \leq & -1 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

The constraints $x_1 \geq 1$ and $x_2 \geq 1$ are written in canonical form as $-x_1 \leq -1$ and $-x_2 \leq -1$.

and an optimal simplex dictionary of its LP relaxation in standard form:

$$\begin{cases} x_1 = 1 + x_4 \\ x_2 = \frac{21}{8} - \frac{1}{8}\ x_3 - \frac{3}{8}\ x_4 \\ x_5 = \frac{13}{8} - \frac{1}{8}\ x_3 - \frac{3}{8}\ x_4 \\ z = \frac{23}{2} - \frac{1}{2}\ x_3 - \frac{1}{2}\ x_4 \end{cases}$$

Questions

1. Execute the branch-and-bound algorithm using:
   (a) the smallest-index rule as branching-variable selection strategy;
   (b) the LIFO rule as branching-node selection strategy;
   (c) solve first the down-branching node and then the up-branching node;
   (d) set the lower bound LB to $-\infty$ (no heuristic algorithm is executed).

   Solve the LP problems of the branching nodes using the dual simplex algorithm. In the dual simplex algorithm, the leaving variable is the basic variable with negative value and smallest index, and ties in the ratio test are broken by the smallest index. Show graphically the optimal solution of the LP problems associated with the branching nodes.
2. Determine and comment the branch-and-bound tree, showing the reason why each branching node is pruned.

Solution

1. Execution of the branch-and-bound algorithm to solve the ILP problem.

   - **Initialization**

     Let us consider the LP_1 problem in standard form:

$$
\begin{array}{lrcrcrcrcrcl}
\max & x_1 & + & 4\,x_2 & & & & & & & & \\
 & 3\,x_1 & + & 8\,x_2 & + & x_3 & & & & & = & 24 \\
 & -x_1 & & & & & + & x_4 & & & = & -1 \\
 & & & -x_2 & & & & & + & x_5 & = & -1 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
\end{array}
$$

     and its optimal simplex dictionary:

$$
\begin{cases}
x_1 = 1 + x_4 \\
x_2 = \dfrac{21}{8} - \dfrac{1}{8}\,x_3 - \dfrac{3}{8}\,x_4 \\
x_5 = \dfrac{13}{8} - \dfrac{1}{8}\,x_3 - \dfrac{3}{8}\,x_4 \\
z = \dfrac{23}{2} - \dfrac{1}{2}\,x_3 - \dfrac{1}{2}\,x_4
\end{cases}
$$

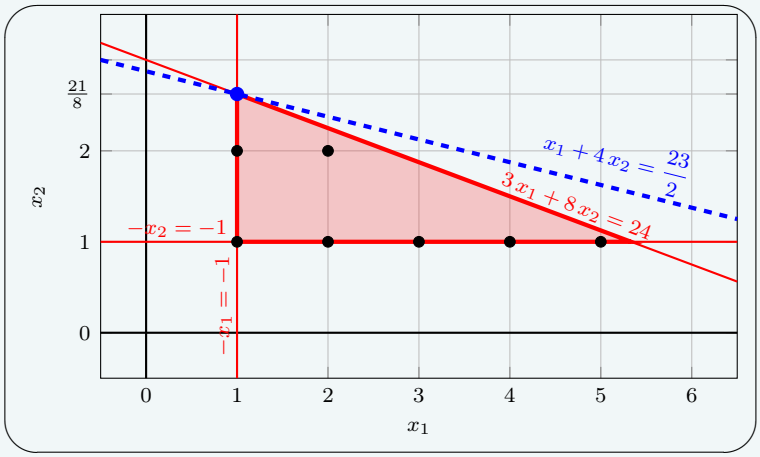


Figure 7.48: Node 1: feasible region of LP_1 (red), feasible points of the ILP problem (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_1)$ is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = \frac{21}{8}$$

the optimal value is:

$$z(\mathrm{LP}_1) = \frac{23}{2}$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is **fractional** and the LP_1 problem is **feasible**, so the LP_1 problem is inserted in the problem list L and the lower bound LB is set to $-\infty$.
Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \quad \leq \quad z(\mathrm{ILP}) \quad \leq \quad \underbrace{\left\lfloor \frac{23}{2} \right\rfloor}_{=11}$$

- **Iteration 1**

The LP_1 problem is extracted and deleted from the problem list L.

The branching variable is x_2. The branching constraints are:

$$x_2 \leq \underbrace{\left\lfloor \frac{21}{8} \right\rfloor}_{=2} \quad \text{and} \quad x_2 \geq \underbrace{\left\lfloor \frac{21}{8} \right\rfloor + 1}_{=3}$$

Two LP_2 and LP_3 problems are created with these branching constraints.

(a) Solution of the LP_2 problem

Let us consider the LP_2 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcr}
\max & x_1 & + & 4\,x_2 & & & & & & & & & \\
 & 3\,x_1 & + & 8\,x_2 & + & x_3 & & & & & & = & 24 \\
 & -x_1 & & & & & + & x_4 & & & & = & -1 \\
 & & & -x_2 & & & & & + & x_5 & & = & -1 \\
 & & & x_2 & & & & & & & + & x_6 & = & 2 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_2 + x_6 = 2 \qquad \Rightarrow \qquad x_6 = 2 - x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_6 = 2 - \left(\frac{21}{8} - \frac{1}{8}\,x_3 - \frac{3}{8}\,x_4 \right) \qquad \Rightarrow \qquad x_6 = -\frac{5}{8} + \frac{1}{8}\,x_3 + \frac{3}{8}\,x_4$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 &= 1 &&&+ x_4 \\ x_2 &= \frac{21}{8} &- \frac{1}{8} x_3 &- \frac{3}{8} x_4 \\ x_5 &= \frac{13}{8} &- \frac{1}{8} x_3 &- \frac{3}{8} x_4 \\ x_6 &= -\frac{5}{8} &+ \frac{1}{8} x_3 &+ \frac{3}{8} x_4 \\ z &= \frac{23}{2} &- \frac{1}{2} x_3 &- \frac{1}{2} x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2, 5, 6\}$ and $\mathcal{N} = \{3, 4\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = 1, \quad \bar{x}_2 = \frac{21}{8}, \quad \bar{x}_5 = \frac{13}{8}, \quad \bar{x}_6 = -\frac{5}{8},$$
$$\bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad \frac{23}{2}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_5 = 0, \quad \bar{\pi}_6 = 0,$$
$$\bar{\pi}_3 = \frac{1}{2}, \quad \bar{\pi}_4 = \frac{1}{2} \quad \text{of value} \quad \frac{23}{2}$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 &= \frac{1}{2} - \frac{1}{8} \pi_6 \\ \pi_4 &= \frac{1}{2} - \frac{3}{8} \pi_6 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{2} - \frac{1}{8} \pi_6 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{1}{2} - \frac{3}{8} \pi_6 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 &\leq 4 \\ \pi_6 &\leq \frac{4}{3} \end{cases}$$

By increasing the value of π_6 to $\min\left\{4, \frac{4}{3}\right\} = \frac{4}{3}$, the value of π_4 decreases to 0. Then we have that:

π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non-basic variable.

π_4 becomes a dual non-basic variable and accordingly x_4 becomes a primal basic variable.

We have:

$$x_6 = -\frac{5}{8} + \frac{1}{8} x_3 + \frac{3}{8} x_4 \quad \Rightarrow \quad x_4 = \frac{5}{3} - \frac{1}{3} x_3 + \frac{8}{3} x_6$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = 1 + \left(\frac{5}{3} - \frac{1}{3} x_3 + \frac{8}{3} x_6\right) = \frac{8}{3} - \frac{1}{3} x_3 + \frac{8}{3} x_6$$

$$x_2 = \frac{21}{8} - \frac{1}{8}\,x_3 - \frac{3}{8}\left(\frac{5}{3} - \frac{1}{3}\,x_3 + \frac{8}{3}\,x_6\right) = 2 - x_6$$

$$x_5 = \frac{13}{8} - \frac{1}{8}\,x_3 - \frac{3}{8}\left(\frac{5}{3} - \frac{1}{3}\,x_3 + \frac{8}{3}\,x_6\right) = 1 - x_6$$

$$z = \frac{23}{2} - \frac{1}{2}\,x_3 - \frac{1}{2}\left(\frac{5}{3} - \frac{1}{3}\,x_3 + \frac{8}{3}\,x_6\right) = \frac{32}{3} - \frac{1}{3}\,x_3 - \frac{4}{3}\,x_6$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = \frac{8}{3} - \frac{1}{3}\,x_3 + \frac{8}{3}\,x_6 \\ x_2 = 2 - x_6 \\ x_4 = \frac{5}{3} - \frac{1}{3}\,x_3 + \frac{8}{3}\,x_6 \\ x_5 = 1 - x_6 \\ z = \frac{32}{3} - \frac{1}{3}\,x_3 - \frac{4}{3}\,x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4, 5\}$ and $\mathcal{N} = \{3, 6\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = \frac{8}{3}, \quad \tilde{x}_2 = 2, \quad \tilde{x}_4 = \frac{5}{3}, \quad \tilde{x}_5 = 1,$$
$$\tilde{x}_3 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad \frac{32}{3}$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_4 = 0, \quad \tilde{\pi}_5 = 0,$$
$$\tilde{\pi}_3 = \frac{1}{3}, \quad \tilde{\pi}_6 = \frac{4}{3} \quad \text{of value} \quad \frac{32}{3}$$

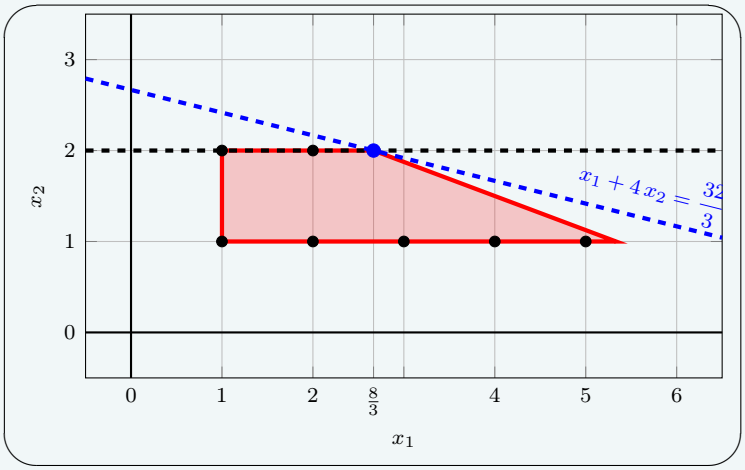


Figure 7.49: Node 2: feasible region of LP_2 (red), with the branching constraint $x_2 \leq 2$ (dashed), and its optimal solution (blue).
An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is:

$$\tilde{x}_1 = \frac{8}{3}, \quad \tilde{x}_2 = 2$$

the optimal value is:

$$z(\mathrm{LP}_2) = \frac{32}{3}$$

The LP_2 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_2) \rfloor > \mathrm{LB} \qquad \text{since} \qquad \underbrace{\left\lfloor \frac{32}{3} \right\rfloor}_{=10} > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is **fractional**, accordingly, the LP_2 problem is **inserted in the problem list** L.

(b) Solution of the LP_3 problem

Let us consider the LP_3 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcr}
\max & x_1 & + & 4\,x_2 & & & & & & & & & \\
& 3\,x_1 & + & 8\,x_2 & + & x_3 & & & & & & = & 24 \\
& -x_1 & & & & & + & x_4 & & & & = & -1 \\
& & & -x_2 & & & & & + & x_5 & & = & -1 \\
& & & x_2 & & & & & & & - & x_6 & = & 3 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_2 - x_6 = 3 \qquad \Rightarrow \qquad x_6 = -3 + x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_6 = -3 + \left(\frac{21}{8} - \frac{1}{8}\, x_3 - \frac{3}{8}\, x_4 \right) \qquad \Rightarrow \qquad x_6 = -\frac{3}{8} - \frac{1}{8}\, x_3 - \frac{3}{8}\, x_4$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = 1 + x_4 \\ x_2 = \frac{21}{8} - \frac{1}{8}x_3 - \frac{3}{8}x_4 \\ x_5 = \frac{13}{8} - \frac{1}{8}x_3 - \frac{3}{8}x_4 \\ x_6 = -\frac{3}{8} - \frac{1}{8}x_3 - \frac{3}{8}x_4 \\ z = \frac{23}{2} - \frac{1}{2}x_3 - \frac{1}{2}x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2, 5, 6\}$ and $\mathcal{N} = \{3, 4\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = 1, \quad \bar{x}_2 = \frac{21}{8}, \quad \bar{x}_5 = \frac{13}{8}, \quad \bar{x}_6 = -\frac{3}{8},$$
$$\bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad \frac{23}{2}$$

- the basic dual feasible solution is:

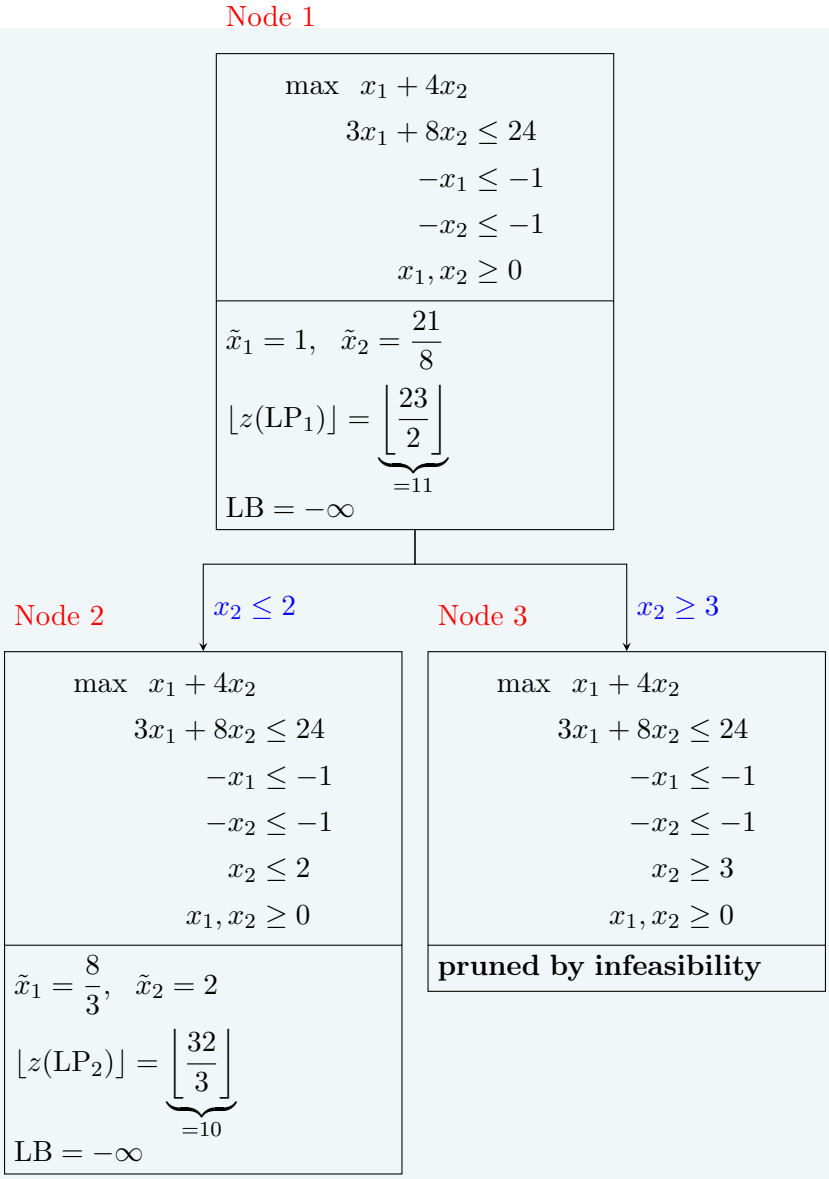
$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_5 = 0, \quad \bar{\pi}_6 = 0,$$
$$\bar{\pi}_3 = \frac{1}{2}, \quad \bar{\pi}_4 = \frac{1}{2} \quad \text{of value} \quad \frac{23}{2}$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 = \frac{1}{2} + \frac{1}{8}\pi_6 \\ \pi_4 = \frac{1}{2} + \frac{3}{8}\pi_6 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{2} + \frac{1}{8}\pi_6 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{1}{2} + \frac{3}{8}\pi_6 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 \geq -4 \\ \pi_6 \geq -\frac{4}{3} \end{cases}$$

Accordingly, we can increase the value of π_6 without violating dual feasibility, which implies that the dual problem is **unbounded**. As a result, the primal problem LP_3 is **infeasible**, and the corresponding branching node is pruned by infeasibility.

At the end of the first iteration, branching on x_2 from Node 1 has generated the down-branching node, Node 2, and the up-branching node, Node 3. Node 2 is inserted in the list L, while Node 3 is pruned by infeasibility. The corresponding branch-and-bound tree is:



Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \quad \leq \quad z(\mathrm{ILP}) \quad \leq \quad \underbrace{\left\lfloor \frac{32}{3} \right\rfloor}_{=10}$$

- **Iteration 2**

The LP_2 problem is extracted and deleted from the problem list L.

The branching variable is x_1. The branching constraints are:

$$x_1 \leq \underbrace{\left\lfloor \frac{8}{3} \right\rfloor}_{=2} \quad \text{and} \quad x_1 \geq \underbrace{\left\lfloor \frac{8}{3} \right\rfloor + 1}_{=3}$$

The LP_4 problem is created with the first branching constraint (the second child is solved only if needed).

(a) Solution of the LP_4 problem

Let us consider the LP_4 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcr}
\max & x_1 & + & 4\,x_2 & & & & & & & & & & \\
& 3\,x_1 & + & 8\,x_2 & + & x_3 & & & & & & & = & 24 \\
& -x_1 & & & & & + & x_4 & & & & & = & -1 \\
& & & -x_2 & & & & & + & x_5 & & & = & -1 \\
& & & x_2 & & & & & & & + & x_6 & = & 2 \\
& x_1 & & & & & & & & & & & + \; x_7 \; = & 2 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & \; x_7 \; \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_1 + x_7 = 2 \qquad \Rightarrow \qquad x_7 = 2 - x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_7 = 2 - \left(\frac{8}{3} - \frac{1}{3}\,x_3 + \frac{8}{3}\,x_6\right) \qquad \Rightarrow \qquad x_7 = -\frac{2}{3} + \frac{1}{3}\,x_3 - \frac{8}{3}\,x_6$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases}
x_1 = \frac{8}{3} - \frac{1}{3}\,x_3 + \frac{8}{3}\,x_6 \\
x_2 = 2 - x_6 \\
x_4 = \frac{5}{3} - \frac{1}{3}\,x_3 + \frac{8}{3}\,x_6 \\
x_5 = 1 - x_6 \\
x_7 = -\frac{2}{3} + \frac{1}{3}\,x_3 - \frac{8}{3}\,x_6 \\
z = \frac{32}{3} - \frac{1}{3}\,x_3 - \frac{4}{3}\,x_6
\end{cases}$$

where $\mathcal{B} = \{1, 2, 4, 5, 7\}$ and $\mathcal{N} = \{3, 6\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{8}{3}, \quad \bar{x}_2 = 2, \quad \bar{x}_4 = \frac{5}{3}, \quad \bar{x}_5 = 1, \quad \bar{x}_7 = -\frac{2}{3},$$
$$\bar{x}_3 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad \frac{32}{3}$$

– the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_5 = 0, \quad \bar{\pi}_7 = 0,$$
$$\bar{\pi}_3 = \frac{1}{3}, \quad \bar{\pi}_6 = \frac{4}{3} \quad \text{of value} \quad \frac{32}{3}$$

If we increase the value of π_7, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 = \frac{1}{3} - \frac{1}{3}\pi_7 \\ \pi_6 = \frac{4}{3} + \frac{8}{3}\pi_7 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{3} - \frac{1}{3}\pi_7 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{4}{3} + \frac{8}{3}\pi_7 \geq 0}_{\pi_6 \geq 0} \end{cases} \qquad \begin{cases} \pi_7 \leq 1 \\ \pi_7 \geq -\frac{1}{2} \end{cases}$$

By increasing the value of π_7 to 1, the value of π_3 decreases to 0. Then we have that:

π_7 becomes a dual basic variable and accordingly x_7 becomes a primal non-basic variable.

π_3 becomes a dual non-basic variable and accordingly x_3 becomes a primal basic variable.

We have:

$$x_7 = -\frac{2}{3} + \frac{1}{3}x_3 - \frac{8}{3}x_6 \quad \Rightarrow \quad x_3 = 2 + 8\,x_6 + 3\,x_7$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{8}{3} - \frac{1}{3}(2 + 8\,x_6 + 3\,x_7) + \frac{8}{3}x_6 = 2 - x_7$$
$$x_4 = \frac{5}{3} - \frac{1}{3}(2 + 8\,x_6 + 3\,x_7) + \frac{8}{3}x_6 = 1 - x_7$$
$$z = \frac{32}{3} - \frac{1}{3}(2 + 8\,x_6 + 3\,x_7) - \frac{4}{3}x_6 = 10 - 4\,x_6 - x_7$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 2 - x_7 \\ x_2 = 2 - x_6 \\ x_3 = 2 + 8\,x_6 + 3\,x_7 \\ x_4 = 1 - x_7 \\ x_5 = 1 - x_6 \\ z = 10 - 4\,x_6 - x_7 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4, 5\}$ and $\mathcal{N} = \{6, 7\}$ and

– the basic primal optimal solution is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 2, \quad \tilde{x}_3 = 2, \quad \tilde{x}_4 = 1, \quad \tilde{x}_5 = 1,$$
$$\tilde{x}_6 = 0, \quad \tilde{x}_7 = 0 \quad \text{of value} \quad 10$$

– the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 0, \quad \tilde{\pi}_5 = 0,$$
$$\tilde{\pi}_6 = 4, \quad \tilde{\pi}_7 = 1 \quad \text{of value} \quad 10$$

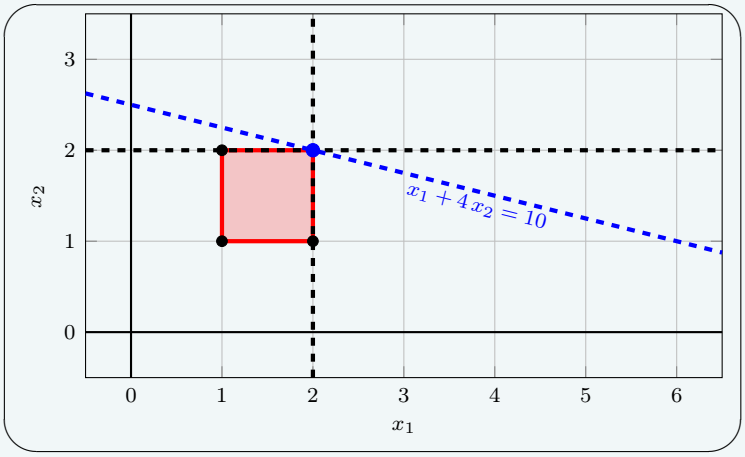


Figure 7.50: Node 4: feasible region of LP_4 (red), with the branching constraint $x_1 \leq 2$ (dashed), and its optimal solution (blue).
An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_4)$ is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 2$$

the optimal value is:

$$z(\mathrm{LP}_4) = 10$$

The LP_4 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_4) \rfloor > \mathrm{LB} \qquad \text{since} \qquad 10 > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_4)$ is **integer**, accordingly, the lower bound LB becomes 10 and the incumbent solution $\bar{\boldsymbol{x}}(\mathrm{ILP})$ becomes $\bar{x}_1 = 2, \bar{x}_2 = 2$.
Accordingly, the branching node is pruned by integrality.

At the end of the second iteration, branching on x_1 from Node 2 has generated the down-branching node, Node 4. This node is pruned by integrality. The up-branching node is not solved, since after Node 4 the lower bound satisfies

$$\lfloor z(\mathrm{LP}_2) \rfloor \leq \mathrm{LB}.$$

This is the portion of the branch-and-bound tree showing the LP problems of the branching nodes that are solved:

Node 2

$$\begin{aligned} \max \quad & x_1 + 4x_2 \\ & 3x_1 + 8x_2 \leq 24 \\ & -x_1 \leq -1 \\ & -x_2 \leq -1 \\ & x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\tilde{x}_1 = \frac{8}{3}, \quad \tilde{x}_2 = 2$

$\lfloor z(\text{LP}_2) \rfloor = \underbrace{\left\lfloor \frac{32}{3} \right\rfloor}_{=10}$

$\text{LB} = -\infty$

$x_1 \leq 2$

Node 4

$$\begin{aligned} \max \quad & x_1 + 4x_2 \\ & 3x_1 + 8x_2 \leq 24 \\ & -x_1 \leq -1 \\ & -x_2 \leq -1 \\ & x_2 \leq 2 \\ & x_1 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 2$

$z(\text{LP}_4) = 10$

$\text{LB} = 10$

pruned by integrality

Due to the current lower bound LB and upper bound UB, we have:

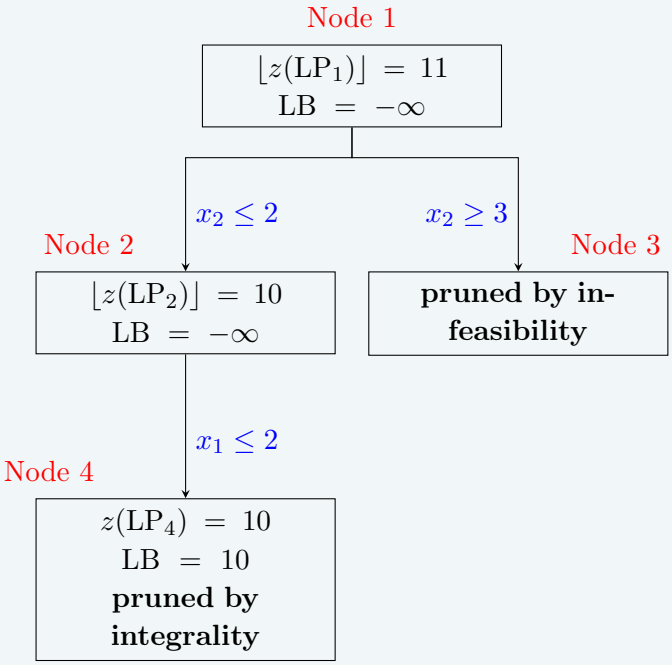
$$10 \quad \leq \quad z(\text{ILP}) \quad \leq \quad 10$$

- **Termination**

Since the list L is empty, the lower bound LB is equal to the upper bound UB and the ILP problem is solved to optimality.

The optimal value is $z(\text{ILP}) = 10$ (the value of the lower bound LB) and an optimal solution $\tilde{\boldsymbol{x}}(\text{ILP})$ is $\tilde{x}_1 = 2, \tilde{x}_2 = 2$ (the incumbent solution $\bar{\boldsymbol{x}}(\text{ILP})$).

2. The final branch-and-bound tree is:



The reasons why the branching nodes are pruned are the following.

- Node 1 has a fractional optimal solution with $\lfloor z(\text{LP}_1) \rfloor = 11$: it is branched on x_2.
- Node 2 has a fractional optimal solution with $\lfloor z(\text{LP}_2) \rfloor = 10$: it is branched on x_1.
- The up-branching node of Node 2 is not solved, and it is not drawn, since $\lfloor z(\text{LP}_2) \rfloor = 10 \leq \text{LB} = 10$ after its first child.
- Node 3 is pruned by infeasibility: the LP_3 problem has no feasible solution.
- Node 4 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_4) = (2, 2)$ is integer, with value 10, which becomes the incumbent solution.

The optimal value is $z(\text{ILP}) = 10$ and the branch-and-bound algorithm returns the optimal solution $\tilde{\boldsymbol{x}}(\text{ILP}) = (2, 2)$.

Exercise 7.12 — Branch-and-bound algorithm ($P24$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 4$ constraints:

$$\begin{array}{rrcrcrcr} \max & 4\,x_1 & + & 3\,x_2 & & & \\ & 7\,x_1 & + & 3\,x_2 & \leq & 24 \\ & -x_1 & + & x_2 & \leq & 3 \\ & -x_1 & & & \leq & -1 \\ & & & -x_2 & \leq & -2 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

The constraints $x_1 \geq 1$ and $x_2 \geq 2$ are written in canonical form as $-x_1 \leq -1$ and $-x_2 \leq -2$.

and an optimal simplex dictionary of its LP relaxation in standard form:

$$\begin{cases} x_1 = \frac{3}{2} - \frac{1}{10}\,x_3 + \frac{3}{10}\,x_4 \\ x_2 = \frac{9}{2} - \frac{1}{10}\,x_3 - \frac{7}{10}\,x_4 \\ x_5 = \frac{1}{2} - \frac{1}{10}\,x_3 + \frac{3}{10}\,x_4 \\ x_6 = \frac{5}{2} - \frac{1}{10}\,x_3 - \frac{7}{10}\,x_4 \\ z = \frac{39}{2} - \frac{7}{10}\,x_3 - \frac{9}{10}\,x_4 \end{cases}$$

Questions

1. Execute the branch-and-bound algorithm using:
   (a) the largest-index rule as branching-variable selection strategy;
   (b) the LIFO rule as branching-node selection strategy;
   (c) solve first the down-branching node and then the up-branching node;
   (d) set the lower bound LB to $-\infty$ (no heuristic algorithm is executed).

   Solve the LP problems of the branching nodes using the dual simplex algorithm. In the dual simplex algorithm, the leaving variable is the basic variable with negative value and smallest index, and ties in the ratio test are broken by the smallest index. Show graphically the optimal solution of the LP problems associated with the branching nodes.

2. Determine and comment the branch-and-bound tree, showing the reason why each branching node is pruned.

Solution

1. Execution of the branch-and-bound algorithm to solve the ILP problem.

   - **Initialization**

     Let us consider the LP_1 problem in standard form:

$$
\begin{array}{lrcrcrcrcrcrcr}
\max & 4\,x_1 & + & 3\,x_2 & & & & & & & & & & \\
 & 7\,x_1 & + & 3\,x_2 & + & x_3 & & & & & & & = & 24 \\
 & -x_1 & + & x_2 & & & + & x_4 & & & & & = & 3 \\
 & -x_1 & & & & & & & + & x_5 & & & = & -1 \\
 & & & -x_2 & & & & & & & + & x_6 & = & -2 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0
\end{array}
$$

     and its optimal simplex dictionary:

$$
\begin{cases}
x_1 = \frac{3}{2} - \frac{1}{10}\,x_3 + \frac{3}{10}\,x_4 \\
x_2 = \frac{9}{2} - \frac{1}{10}\,x_3 - \frac{7}{10}\,x_4 \\
x_5 = \frac{1}{2} - \frac{1}{10}\,x_3 + \frac{3}{10}\,x_4 \\
x_6 = \frac{5}{2} - \frac{1}{10}\,x_3 - \frac{7}{10}\,x_4 \\
z = \frac{39}{2} - \frac{7}{10}\,x_3 - \frac{9}{10}\,x_4
\end{cases}
$$

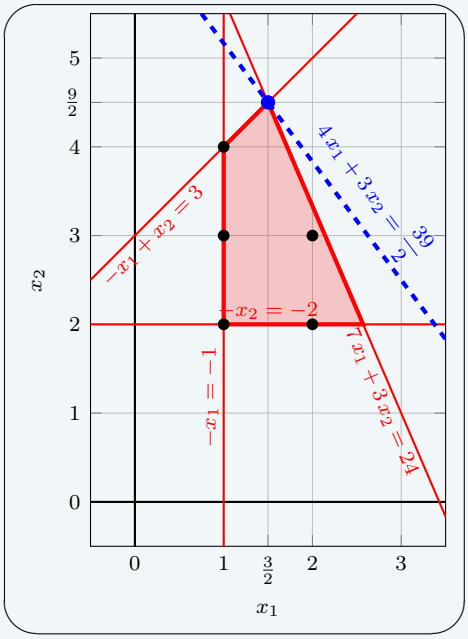


Figure 7.51: Node 1: feasible region of LP_1 (red), feasible points of the ILP problem (black) and optimal solution of LP_1 (blue) with the objective function line through it.
An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is:

$$\tilde{x}_1 = \frac{3}{2}, \quad \tilde{x}_2 = \frac{9}{2}$$

the optimal value is:

$$z(\mathrm{LP}_1) = \frac{39}{2}$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is **fractional** and the LP_1 problem is **feasible**, so the LP_1 problem is inserted in the problem list L and the lower bound LB is set to $-\infty$.
Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \leq z(\mathrm{ILP}) \leq \underbrace{\left\lfloor \frac{39}{2} \right\rfloor}_{=19}$$

- **Iteration 1**

The LP_1 problem is extracted and deleted from the problem list L.

The branching variable is x_2. The branching constraints are:

$$x_2 \leq \underbrace{\left\lfloor \frac{9}{2} \right\rfloor}_{=4} \quad \text{and} \quad x_2 \geq \underbrace{\left\lfloor \frac{9}{2} \right\rfloor + 1}_{=5}$$

Two LP_2 and LP_3 problems are created with these branching constraints.

(a) Solution of the LP_2 problem

Let us consider the LP_2 problem in standard form:

$$
\begin{array}{lrcrcrcrcrcrcr}
\max & 4\,x_1 & + & 3\,x_2 & & & & & & & & & & \\
 & 7\,x_1 & + & 3\,x_2 & + & x_3 & & & & & & & = & 24 \\
 & -x_1 & + & x_2 & & & + & x_4 & & & & & = & 3 \\
 & -x_1 & & & & & & & + & x_5 & & & = & -1 \\
 & & & -x_2 & & & & & & & + & x_6 & = & -2 \\
 & & & x_2 & & & & & & & & & + \; x_7 \; = & 4 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & x_7 \; \geq & 0
\end{array}
$$

For the branching constraint we have:

$$x_2 + x_7 = 4 \qquad \Rightarrow \qquad x_7 = 4 - x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_7 = 4 - \left(\frac{9}{2} - \frac{1}{10}\,x_3 - \frac{7}{10}\,x_4\right) \qquad \Rightarrow \qquad x_7 = -\frac{1}{2} + \frac{1}{10}\,x_3 + \frac{7}{10}\,x_4$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$
\begin{cases}
x_1 = \frac{3}{2} - \frac{1}{10}\,x_3 + \frac{3}{10}\,x_4 \\
x_2 = \frac{9}{2} - \frac{1}{10}\,x_3 - \frac{7}{10}\,x_4 \\
x_5 = \frac{1}{2} - \frac{1}{10}\,x_3 + \frac{3}{10}\,x_4 \\
x_6 = \frac{5}{2} - \frac{1}{10}\,x_3 - \frac{7}{10}\,x_4 \\
x_7 = -\frac{1}{2} + \frac{1}{10}\,x_3 + \frac{7}{10}\,x_4 \\
z = \frac{39}{2} - \frac{7}{10}\,x_3 - \frac{9}{10}\,x_4
\end{cases}
$$

where $\mathcal{B} = \{1, 2, 5, 6, 7\}$ and $\mathcal{N} = \{3, 4\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{3}{2}, \quad \bar{x}_2 = \frac{9}{2}, \quad \bar{x}_5 = \frac{1}{2}, \quad \bar{x}_6 = \frac{5}{2}, \quad \bar{x}_7 = -\frac{1}{2},$$
$$\bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad \frac{39}{2}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_5 = 0, \quad \bar{\pi}_6 = 0, \quad \bar{\pi}_7 = 0,$$
$$\bar{\pi}_3 = \frac{7}{10}, \quad \bar{\pi}_4 = \frac{9}{10} \quad \text{of value} \quad \frac{39}{2}$$

If we increase the value of π_7, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 = \frac{7}{10} - \frac{1}{10}\pi_7 \\ \pi_4 = \frac{9}{10} - \frac{7}{10}\pi_7 \end{cases} \qquad \begin{cases} \underbrace{\frac{7}{10} - \frac{1}{10}\pi_7 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{9}{10} - \frac{7}{10}\pi_7 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_7 \leq 7 \\ \pi_7 \leq \frac{9}{7} \end{cases}$$

By increasing the value of π_7 to $\min\left\{7, \frac{9}{7}\right\} = \frac{9}{7}$, the value of π_4 decreases to 0. Then we have that:

π_7 becomes a dual basic variable and accordingly x_7 becomes a primal non-basic variable.

π_4 becomes a dual non-basic variable and accordingly x_4 becomes a primal basic variable.

We have:

$$x_7 = -\frac{1}{2} + \frac{1}{10}x_3 + \frac{7}{10}x_4 \quad \Rightarrow \quad x_4 = \frac{5}{7} - \frac{1}{7}x_3 + \frac{10}{7}x_7$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{3}{2} - \frac{1}{10}x_3 + \frac{3}{10}\left(\frac{5}{7} - \frac{1}{7}x_3 + \frac{10}{7}x_7\right) = \frac{12}{7} - \frac{1}{7}x_3 + \frac{3}{7}x_7$$

$$x_2 = \frac{9}{2} - \frac{1}{10}x_3 - \frac{7}{10}\left(\frac{5}{7} - \frac{1}{7}x_3 + \frac{10}{7}x_7\right) = 4 - x_7$$

$$x_5 = \frac{1}{2} - \frac{1}{10}x_3 + \frac{3}{10}\left(\frac{5}{7} - \frac{1}{7}x_3 + \frac{10}{7}x_7\right) = \frac{5}{7} - \frac{1}{7}x_3 + \frac{3}{7}x_7$$

$$x_6 = \frac{5}{2} - \frac{1}{10}x_3 - \frac{7}{10}\left(\frac{5}{7} - \frac{1}{7}x_3 + \frac{10}{7}x_7\right) = 2 - x_7$$

$$z = \frac{39}{2} - \frac{7}{10}x_3 - \frac{9}{10}\left(\frac{5}{7} - \frac{1}{7}x_3 + \frac{10}{7}x_7\right) = \frac{132}{7} - \frac{4}{7}x_3 - \frac{9}{7}x_7$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = \frac{12}{7} - \frac{1}{7}x_3 + \frac{3}{7}x_7 \\ x_2 = 4 - x_7 \\ x_4 = \frac{5}{7} - \frac{1}{7}x_3 + \frac{10}{7}x_7 \\ x_5 = \frac{5}{7} - \frac{1}{7}x_3 + \frac{3}{7}x_7 \\ x_6 = 2 - x_7 \\ z = \frac{132}{7} - \frac{4}{7}x_3 - \frac{9}{7}x_7 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4, 5, 6\}$ and $\mathcal{N} = \{3, 7\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = \frac{12}{7}, \quad \tilde{x}_2 = 4, \quad \tilde{x}_4 = \frac{5}{7}, \quad \tilde{x}_5 = \frac{5}{7}, \quad \tilde{x}_6 = 2,$$
$$\tilde{x}_3 = 0, \quad \tilde{x}_7 = 0 \quad \text{of value} \quad \frac{132}{7}$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_4 = 0, \quad \tilde{\pi}_5 = 0, \quad \tilde{\pi}_6 = 0,$$
$$\tilde{\pi}_3 = \frac{4}{7}, \quad \tilde{\pi}_7 = \frac{9}{7} \quad \text{of value} \quad \frac{132}{7}$$

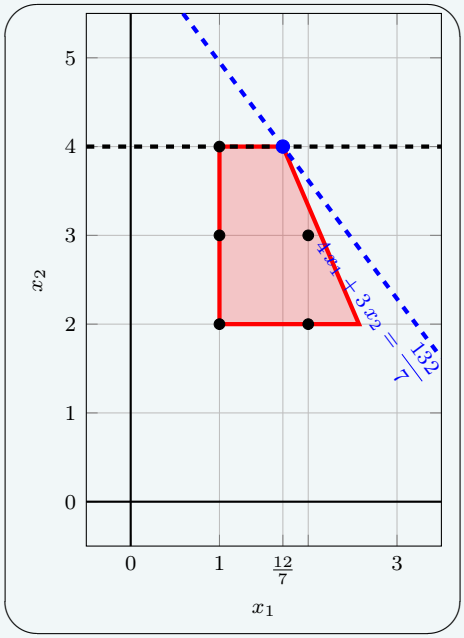


Figure 7.52: Node 2: feasible region of LP_2 (red), with the branching constraint $x_2 \leq 4$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is:

$$\tilde{x}_1 = \frac{12}{7}, \quad \tilde{x}_2 = 4$$

the optimal value is:

$$z(\mathrm{LP}_2) = \frac{132}{7}$$

The LP_2 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_2) \rfloor > \mathrm{LB} \qquad \text{since} \qquad \underbrace{\left\lfloor \frac{132}{7} \right\rfloor}_{=18} > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_2)$ is **fractional**, accordingly, the LP_2 problem is **inserted in the problem list** L.

(b) Solution of the LP_3 problem

Let us consider the LP_3 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcr}
\max & 4\,x_1 & + & 3\,x_2 & & & & & & & & & & \\
& 7\,x_1 & + & 3\,x_2 & + & x_3 & & & & & & & = & 24 \\
& -x_1 & + & x_2 & & & + & x_4 & & & & & = & 3 \\
& -x_1 & & & & & & & + & x_5 & & & = & -1 \\
& & & -x_2 & & & & & & & + & x_6 & = & -2 \\
& & & x_2 & & & & & & & & - \; x_7 & = & 5 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, \quad x_7 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_2 - x_7 = 5 \qquad \Rightarrow \qquad x_7 = -5 + x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_7 = -5 + \left(\frac{9}{2} - \frac{1}{10}\,x_3 - \frac{7}{10}\,x_4\right) \qquad \Rightarrow \qquad x_7 = -\frac{1}{2} - \frac{1}{10}\,x_3 - \frac{7}{10}\,x_4$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases}
x_1 = \frac{3}{2} - \frac{1}{10}\,x_3 + \frac{3}{10}\,x_4 \\
x_2 = \frac{9}{2} - \frac{1}{10}\,x_3 - \frac{7}{10}\,x_4 \\
x_5 = \frac{1}{2} - \frac{1}{10}\,x_3 + \frac{3}{10}\,x_4 \\
x_6 = \frac{5}{2} - \frac{1}{10}\,x_3 - \frac{7}{10}\,x_4 \\
x_7 = -\frac{1}{2} - \frac{1}{10}\,x_3 - \frac{7}{10}\,x_4 \\
z = \frac{39}{2} - \frac{7}{10}\,x_3 - \frac{9}{10}\,x_4
\end{cases}$$

where $\mathcal{B} = \{1, 2, 5, 6, 7\}$ and $\mathcal{N} = \{3, 4\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{3}{2}, \quad \bar{x}_2 = \frac{9}{2}, \quad \bar{x}_5 = \frac{1}{2}, \quad \bar{x}_6 = \frac{5}{2}, \quad \bar{x}_7 = -\frac{1}{2},$$
$$\bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad \frac{39}{2}$$

– the basic dual feasible solution is:

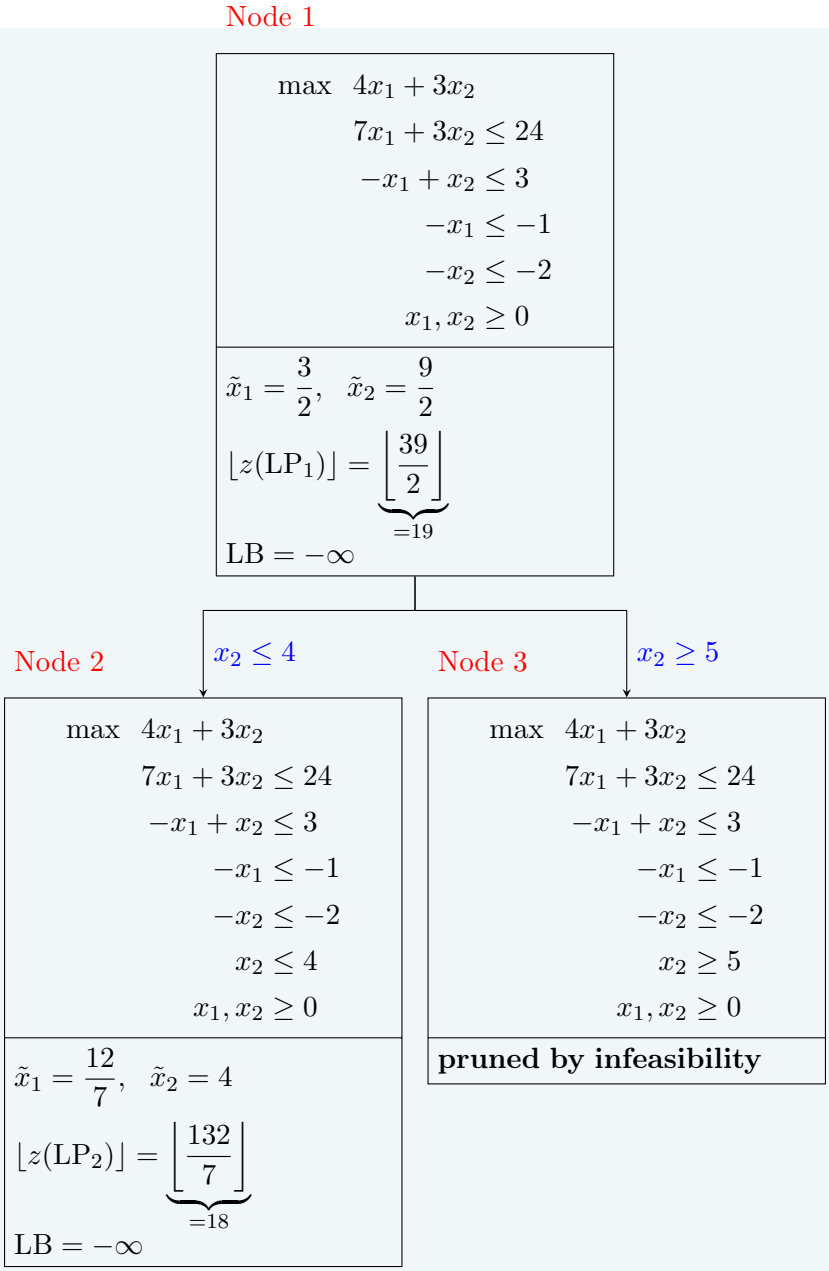
$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_5 = 0, \quad \bar{\pi}_6 = 0, \quad \bar{\pi}_7 = 0,$$
$$\bar{\pi}_3 = \frac{7}{10}, \quad \bar{\pi}_4 = \frac{9}{10} \quad \text{of value} \quad \frac{39}{2}$$

If we increase the value of π_7, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 = \frac{7}{10} + \frac{1}{10}\pi_7 \\ \pi_4 = \frac{9}{10} + \frac{7}{10}\pi_7 \end{cases} \qquad \begin{cases} \underbrace{\frac{7}{10} + \frac{1}{10}\pi_7 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{9}{10} + \frac{7}{10}\pi_7 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_7 \geq -7 \\ \pi_7 \geq -\frac{9}{7} \end{cases}$$

Accordingly, we can increase the value of π_7 without violating dual feasibility, which implies that the dual problem is **unbounded**. As a result, the primal problem LP_3 is **infeasible**, and the corresponding branching node is pruned by infeasibility.

At the end of the first iteration, branching on x_2 from Node 1 has generated the down-branching node, Node 2, and the up-branching node, Node 3. Node 2 is inserted in the list L, while Node 3 is pruned by infeasibility. The corresponding branch-and-bound tree is:



Due to the current lower bound LB and upper bound UB, we have:

$$-\infty \quad \leq \quad z(\text{ILP}) \quad \leq \quad \underbrace{\left\lfloor \frac{132}{7} \right\rfloor}_{=18}$$

- **Iteration 2**

The LP_2 problem is extracted and deleted from the problem list L.

The branching variable is x_1. The branching constraints are:

$$x_1 \leq \underbrace{\left\lfloor \frac{12}{7} \right\rfloor}_{=1} \quad \text{and} \quad x_1 \geq \underbrace{\left\lfloor \frac{12}{7} \right\rfloor + 1}_{=2}$$

Two LP_4 and LP_5 problems are created with these branching constraints.

(a) Solution of the LP_4 problem

Let us consider the LP_4 problem in standard form:

$$
\begin{array}{lrcrcrcrcrcrcrcl}
\max & 4\,x_1 & + & 3\,x_2 & & & & & & & & & & & \\
 & 7\,x_1 & + & 3\,x_2 & + & x_3 & & & & & & & & = & 24 \\
 & -x_1 & + & x_2 & & & + & x_4 & & & & & & = & 3 \\
 & -x_1 & & & & & & & + & x_5 & & & & = & -1 \\
 & & & -x_2 & & & & & & & + & x_6 & & = & -2 \\
 & & & x_2 & & & & & & & & & + \; x_7 & = & 4 \\
 & x_1 & & & & & & & & & & & & + \; x_8 & = & 1 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & & x_7, & & x_8 & \geq & 0
\end{array}
$$

For the branching constraint we have:

$$x_1 + x_8 = 1 \qquad \Rightarrow \quad x_8 = 1 - x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_8 = 1 - \left(\frac{12}{7} - \frac{1}{7}\, x_3 + \frac{3}{7}\, x_7 \right) \qquad \Rightarrow \quad x_8 = -\frac{5}{7} + \frac{1}{7}\, x_3 - \frac{3}{7}\, x_7$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$
\begin{cases}
x_1 = \frac{12}{7} - \frac{1}{7}\, x_3 + \frac{3}{7}\, x_7 \\
x_2 = 4 - x_7 \\
x_4 = \frac{5}{7} - \frac{1}{7}\, x_3 + \frac{10}{7}\, x_7 \\
x_5 = \frac{5}{7} - \frac{1}{7}\, x_3 + \frac{3}{7}\, x_7 \\
x_6 = 2 - x_7 \\
x_8 = -\frac{5}{7} + \frac{1}{7}\, x_3 - \frac{3}{7}\, x_7 \\
z = \frac{132}{7} - \frac{4}{7}\, x_3 - \frac{9}{7}\, x_7
\end{cases}
$$

where $\mathcal{B} = \{1, 2, 4, 5, 6, 8\}$ and $\mathcal{N} = \{3, 7\}$ and

– the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{12}{7}, \quad \bar{x}_2 = 4, \quad \bar{x}_4 = \frac{5}{7},$$
$$\bar{x}_5 = \frac{5}{7}, \quad \bar{x}_6 = 2, \quad \bar{x}_8 = -\frac{5}{7},$$
$$\bar{x}_3 = 0, \quad \bar{x}_7 = 0 \quad \text{of value} \quad \frac{132}{7}$$

– the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_4 = 0,$$
$$\bar{\pi}_5 = 0, \quad \bar{\pi}_6 = 0, \quad \bar{\pi}_8 = 0,$$
$$\bar{\pi}_3 = \frac{4}{7}, \quad \bar{\pi}_7 = \frac{9}{7} \quad \text{of value} \quad \frac{132}{7}$$

If we increase the value of π_8, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 = \frac{4}{7} - \frac{1}{7}\pi_8 \\ \pi_7 = \frac{9}{7} + \frac{3}{7}\pi_8 \end{cases} \qquad \begin{cases} \underbrace{\frac{4}{7} - \frac{1}{7}\pi_8 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{9}{7} + \frac{3}{7}\pi_8 \geq 0}_{\pi_7 \geq 0} \end{cases} \qquad \begin{cases} \pi_8 \leq 4 \\ \pi_8 \geq -3 \end{cases}$$

By increasing the value of π_8 to 4, the value of π_3 decreases to 0. Then we have that:

π_8 becomes a dual basic variable and accordingly x_8 becomes a primal non-basic variable.

π_3 becomes a dual non-basic variable and accordingly x_3 becomes a primal basic variable.

We have:

$$x_8 = -\frac{5}{7} + \frac{1}{7}x_3 - \frac{3}{7}x_7 \quad \Rightarrow \quad x_3 = 5 + 3x_7 + 7x_8$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{12}{7} - \frac{1}{7}(5 + 3x_7 + 7x_8) + \frac{3}{7}x_7 = 1 - x_8$$
$$x_4 = \frac{5}{7} - \frac{1}{7}(5 + 3x_7 + 7x_8) + \frac{10}{7}x_7 = 0 + x_7 - x_8$$
$$x_5 = \frac{5}{7} - \frac{1}{7}(5 + 3x_7 + 7x_8) + \frac{3}{7}x_7 = 0 - x_8$$
$$z = \frac{132}{7} - \frac{4}{7}(5 + 3x_7 + 7x_8) - \frac{9}{7}x_7 = 16 - 3x_7 - 4x_8$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 1 - x_8 \\ x_2 = 4 - x_7 \\ x_3 = 5 + 3\,x_7 + 7\,x_8 \\ x_4 = 0 + x_7 - x_8 \\ x_5 = 0 - x_8 \\ x_6 = 2 - x_7 \\ z = 16 - 3\,x_7 - 4\,x_8 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{N} = \{7, 8\}$ and

- the basic primal optimal solution is:

$$\begin{gathered} \tilde{x}_1 = 1, \quad \tilde{x}_2 = 4, \quad \tilde{x}_3 = 5, \\ \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0, \quad \tilde{x}_6 = 2, \\ \tilde{x}_7 = 0, \quad \tilde{x}_8 = 0 \quad \text{of value} \quad 16 \end{gathered}$$

- the basic dual optimal solution is:

$$\begin{gathered} \tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \\ \tilde{\pi}_4 = 0, \quad \tilde{\pi}_5 = 0, \quad \tilde{\pi}_6 = 0, \\ \tilde{\pi}_7 = 3, \quad \tilde{\pi}_8 = 4 \quad \text{of value} \quad 16 \end{gathered}$$

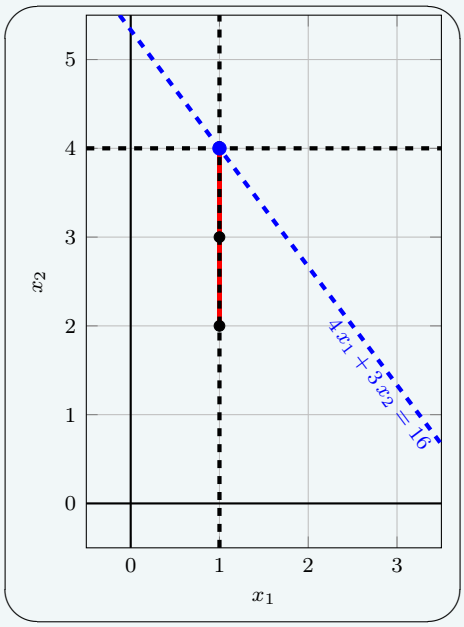


Figure 7.53: Node 4: feasible region of LP_4 (red), with the branching constraint $x_1 \leq 1$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_4)$ is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 4$$

the optimal value is:

$$z(\mathrm{LP}_4) = 16$$

The LP_4 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_4) \rfloor > \mathrm{LB} \qquad \text{since} \qquad 16 > -\infty$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_4)$ is **integer**, accordingly, the lower bound LB becomes 16 and the incumbent solution $\bar{\boldsymbol{x}}(\mathrm{ILP})$ becomes $\bar{x}_1 = 1, \bar{x}_2 = 4$.
Accordingly, the branching node is pruned by integrality.

(b) Solution of the LP_5 problem

Let us consider the LP_5 problem in standard form:

$$\begin{array}{rrcrcrcrcrcrcrcrcl}
\max & 4\,x_1 & + & 3\,x_2 & & & & & & & & & & & & & \\
 & 7\,x_1 & + & 3\,x_2 & + & x_3 & & & & & & & & & & = & 24 \\
 & -x_1 & + & x_2 & & & + & x_4 & & & & & & & & = & 3 \\
 & -x_1 & & & & & & & + & x_5 & & & & & & = & -1 \\
 & & & -x_2 & & & & & & & + & x_6 & & & & = & -2 \\
 & & & x_2 & & & & & & & & & + & x_7 & & = & 4 \\
 & x_1 & & & & & & & & & & & & & - & x_8 & = & 2 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & & x_7, & & x_8 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_1 - x_8 = 2 \qquad \Rightarrow \qquad x_8 = -2 + x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_8 = -2 + \left(\frac{12}{7} - \frac{1}{7}\,x_3 + \frac{3}{7}\,x_7\right) \qquad \Rightarrow \qquad x_8 = -\frac{2}{7} - \frac{1}{7}\,x_3 + \frac{3}{7}\,x_7$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = \frac{12}{7} - \frac{1}{7}x_3 + \frac{3}{7}x_7 \\ x_2 = 4 - x_7 \\ x_4 = \frac{5}{7} - \frac{1}{7}x_3 + \frac{10}{7}x_7 \\ x_5 = \frac{5}{7} - \frac{1}{7}x_3 + \frac{3}{7}x_7 \\ x_6 = 2 - x_7 \\ x_8 = -\frac{2}{7} - \frac{1}{7}x_3 + \frac{3}{7}x_7 \\ z = \frac{132}{7} - \frac{4}{7}x_3 - \frac{9}{7}x_7 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4, 5, 6, 8\}$ and $\mathcal{N} = \{3, 7\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{12}{7}, \quad \bar{x}_2 = 4, \quad \bar{x}_4 = \frac{5}{7},$$
$$\bar{x}_5 = \frac{5}{7}, \quad \bar{x}_6 = 2, \quad \bar{x}_8 = -\frac{2}{7},$$
$$\bar{x}_3 = 0, \quad \bar{x}_7 = 0 \quad \text{of value} \quad \frac{132}{7}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_4 = 0,$$
$$\bar{\pi}_5 = 0, \quad \bar{\pi}_6 = 0, \quad \bar{\pi}_8 = 0,$$
$$\bar{\pi}_3 = \frac{4}{7}, \quad \bar{\pi}_7 = \frac{9}{7} \quad \text{of value} \quad \frac{132}{7}$$

If we increase the value of π_8, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 = \frac{4}{7} + \frac{1}{7}\pi_8 \\ \pi_7 = \frac{9}{7} - \frac{3}{7}\pi_8 \end{cases} \qquad \begin{cases} \underbrace{\frac{4}{7} + \frac{1}{7}\pi_8 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{9}{7} - \frac{3}{7}\pi_8 \geq 0}_{\pi_7 \geq 0} \end{cases} \qquad \begin{cases} \pi_8 \geq -4 \\ \pi_8 \leq 3 \end{cases}$$

By increasing the value of π_8 to 3, the value of π_7 decreases to 0. Then we have that:

π_8 becomes a dual basic variable and accordingly x_8 becomes a primal non-basic variable.

π_7 becomes a dual non-basic variable and accordingly x_7 becomes a primal basic variable.

We have:

$$x_8 = -\frac{2}{7} - \frac{1}{7}x_3 + \frac{3}{7}x_7 \quad \Rightarrow \quad x_7 = \frac{2}{3} + \frac{1}{3}x_3 + \frac{7}{3}x_8$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{12}{7} - \frac{1}{7}\,x_3 + \frac{3}{7}\left(\frac{2}{3} + \frac{1}{3}\,x_3 + \frac{7}{3}\,x_8\right) = 2 + x_8$$

$$x_2 = 4 - \left(\frac{2}{3} + \frac{1}{3}\,x_3 + \frac{7}{3}\,x_8\right) = \frac{10}{3} - \frac{1}{3}\,x_3 - \frac{7}{3}\,x_8$$

$$x_4 = \frac{5}{7} - \frac{1}{7}\,x_3 + \frac{10}{7}\left(\frac{2}{3} + \frac{1}{3}\,x_3 + \frac{7}{3}\,x_8\right) = \frac{5}{3} + \frac{1}{3}\,x_3 + \frac{10}{3}\,x_8$$

$$x_5 = \frac{5}{7} - \frac{1}{7}\,x_3 + \frac{3}{7}\left(\frac{2}{3} + \frac{1}{3}\,x_3 + \frac{7}{3}\,x_8\right) = 1 + x_8$$

$$x_6 = 2 - \left(\frac{2}{3} + \frac{1}{3}\,x_3 + \frac{7}{3}\,x_8\right) = \frac{4}{3} - \frac{1}{3}\,x_3 - \frac{7}{3}\,x_8$$

$$z = \frac{132}{7} - \frac{4}{7}\,x_3 - \frac{9}{7}\left(\frac{2}{3} + \frac{1}{3}\,x_3 + \frac{7}{3}\,x_8\right) = 18 - x_3 - 3\,x_8$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = 2 + x_8 \\ x_2 = \frac{10}{3} - \frac{1}{3}\,x_3 - \frac{7}{3}\,x_8 \\ x_4 = \frac{5}{3} + \frac{1}{3}\,x_3 + \frac{10}{3}\,x_8 \\ x_5 = 1 + x_8 \\ x_6 = \frac{4}{3} - \frac{1}{3}\,x_3 - \frac{7}{3}\,x_8 \\ x_7 = \frac{2}{3} + \frac{1}{3}\,x_3 + \frac{7}{3}\,x_8 \\ z = 18 - x_3 - 3\,x_8 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4, 5, 6, 7\}$ and $\mathcal{N} = \{3, 8\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = \frac{10}{3}, \quad \tilde{x}_4 = \frac{5}{3},$$
$$\tilde{x}_5 = 1, \quad \tilde{x}_6 = \frac{4}{3}, \quad \tilde{x}_7 = \frac{2}{3},$$
$$\tilde{x}_3 = 0, \quad \tilde{x}_8 = 0 \quad \text{of value} \quad 18$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_4 = 0,$$
$$\tilde{\pi}_5 = 0, \quad \tilde{\pi}_6 = 0, \quad \tilde{\pi}_7 = 0,$$
$$\tilde{\pi}_3 = 1, \quad \tilde{\pi}_8 = 3 \quad \text{of value} \quad 18$$

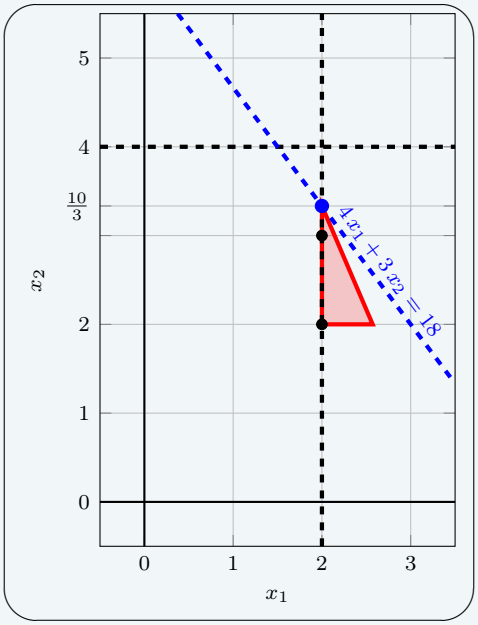


Figure 7.54: Node 5: feasible region of LP_5 (red), with the branching constraint $x_1 \geq 2$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_5)$ is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = \frac{10}{3}$$

the optimal value is:

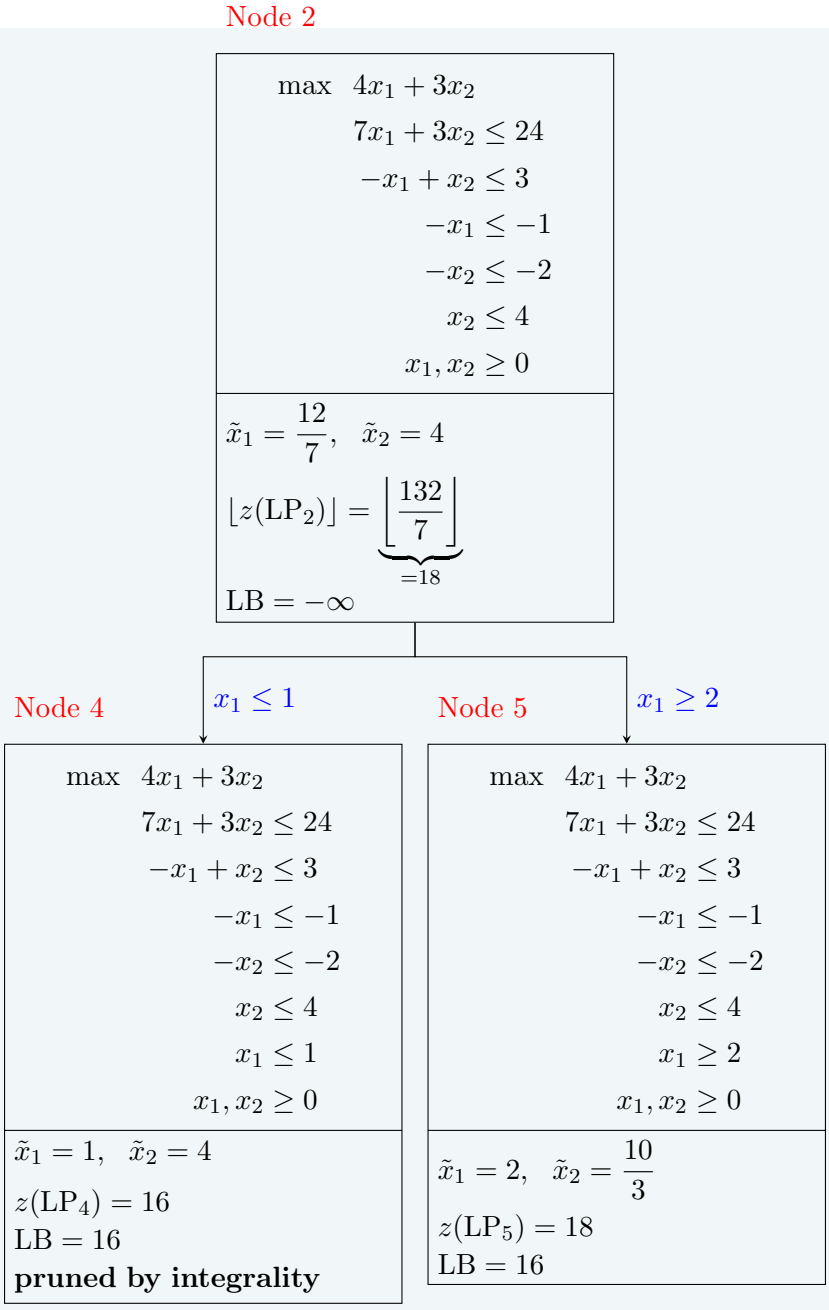
$$z(\mathrm{LP}_5) = 18$$

The LP_5 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_5) \rfloor > \mathrm{LB} \qquad \text{since} \qquad 18 > 16$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_5)$ is **fractional**, accordingly, the LP_5 problem is **inserted in the problem list** L.

At the end of the second iteration, branching on x_1 from Node 2 has generated the down-branching node, Node 4, and the up-branching node, Node 5. Node 4 is pruned by integrality, while Node 5 is inserted in the list L. The corresponding branch-and-bound tree is:



Due to the current lower bound LB and upper bound UB, we have:

$$16 \quad \leq \quad z(\text{ILP}) \quad \leq \quad 18$$

- **Iteration 3**

The LP_5 problem is extracted and deleted from the problem list L.

The branching variable is x_2. The branching constraints are:

$$x_2 \leq \underbrace{\left\lfloor \frac{10}{3} \right\rfloor}_{=3} \quad \text{and} \quad x_2 \geq \underbrace{\left\lfloor \frac{10}{3} \right\rfloor + 1}_{=4}$$

Two LP_6 and LP_7 problems are created with these branching constraints.

(a) Solution of the LP_6 problem

Let us consider the LP_6 problem in standard form:

$$
\begin{array}{lrcrcrcrcrcrcrcrcl}
\max & 4\,x_1 & + & 3\,x_2 & & & & & & & & & & & & & \\
 & 7\,x_1 & + & 3\,x_2 & + & x_3 & & & & & & & & & & = & 24 \\
 & -x_1 & + & x_2 & & & + & x_4 & & & & & & & & = & 3 \\
 & -x_1 & & & & & & & + & x_5 & & & & & & = & -1 \\
 & & & -x_2 & & & & & & & + & x_6 & & & & = & -2 \\
 & & & x_2 & & & & & & & & & + & x_7 & & = & 4 \\
 & x_1 & & & & & & & & & & & & & - & x_8 & = & 2 \\
 & & & x_2 & & & & & & & & & & & + & x_9 & = & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & & x_7, & & x_8, & & x_9 & \geq & 0
\end{array}
$$

For the branching constraint we have:

$$x_2 + x_9 = 3 \qquad \Rightarrow \qquad x_9 = 3 - x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_9 = 3 - \left(\frac{10}{3} - \frac{1}{3}\,x_3 - \frac{7}{3}\,x_8\right) \qquad \Rightarrow \qquad x_9 = -\frac{1}{3} + \frac{1}{3}\,x_3 + \frac{7}{3}\,x_8$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$
\begin{cases}
x_1 = 2 + x_8 \\
x_2 = \frac{10}{3} - \frac{1}{3}\,x_3 - \frac{7}{3}\,x_8 \\
x_4 = \frac{5}{3} + \frac{1}{3}\,x_3 + \frac{10}{3}\,x_8 \\
x_5 = 1 + x_8 \\
x_6 = \frac{4}{3} - \frac{1}{3}\,x_3 - \frac{7}{3}\,x_8 \\
x_7 = \frac{2}{3} + \frac{1}{3}\,x_3 + \frac{7}{3}\,x_8 \\
x_9 = -\frac{1}{3} + \frac{1}{3}\,x_3 + \frac{7}{3}\,x_8 \\
z = 18 - x_3 - 3\,x_8
\end{cases}
$$

where $\mathcal{B} = \{1, 2, 4, 5, 6, 7, 9\}$ and $\mathcal{N} = \{3, 8\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = 2, \quad \bar{x}_2 = \frac{10}{3}, \quad \bar{x}_4 = \frac{5}{3}, \quad \bar{x}_5 = 1,$$
$$\bar{x}_6 = \frac{4}{3}, \quad \bar{x}_7 = \frac{2}{3}, \quad \bar{x}_9 = -\frac{1}{3},$$
$$\bar{x}_3 = 0, \quad \bar{x}_8 = 0 \quad \text{of value} \quad 18$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_5 = 0,$$
$$\bar{\pi}_6 = 0, \quad \bar{\pi}_7 = 0, \quad \bar{\pi}_9 = 0,$$
$$\bar{\pi}_3 = 1, \quad \bar{\pi}_8 = 3 \quad \text{of value} \quad 18$$

If we increase the value of π_9, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 &= 1 - \frac{1}{3}\pi_9 \\ \pi_8 &= 3 - \frac{7}{3}\pi_9 \end{cases} \qquad \begin{cases} \underbrace{1 - \frac{1}{3}\pi_9 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{3 - \frac{7}{3}\pi_9 \geq 0}_{\pi_8 \geq 0} \end{cases} \qquad \begin{cases} \pi_9 &\leq 3 \\ \pi_9 &\leq \frac{9}{7} \end{cases}$$

By increasing the value of π_9 to $\min\left\{3, \frac{9}{7}\right\} = \frac{9}{7}$, the value of π_8 decreases to 0. Then we have that:

π_9 becomes a dual basic variable and accordingly x_9 becomes a primal non-basic variable.

π_8 becomes a dual non-basic variable and accordingly x_8 becomes a primal basic variable.

We have:

$$x_9 = -\frac{1}{3} + \frac{1}{3}x_3 + \frac{7}{3}x_8 \quad \Rightarrow \quad x_8 = \frac{1}{7} - \frac{1}{7}x_3 + \frac{3}{7}x_9$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = 2 + \left(\frac{1}{7} - \frac{1}{7}x_3 + \frac{3}{7}x_9\right) = \frac{15}{7} - \frac{1}{7}x_3 + \frac{3}{7}x_9$$

$$x_2 = \frac{10}{3} - \frac{1}{3}x_3 - \frac{7}{3}\left(\frac{1}{7} - \frac{1}{7}x_3 + \frac{3}{7}x_9\right) = 3 - x_9$$

$$x_4 = \frac{5}{3} + \frac{1}{3}x_3 + \frac{10}{3}\left(\frac{1}{7} - \frac{1}{7}x_3 + \frac{3}{7}x_9\right) = \frac{15}{7} - \frac{1}{7}x_3 + \frac{10}{7}x_9$$

$$x_5 = 1 + \left(\frac{1}{7} - \frac{1}{7}x_3 + \frac{3}{7}x_9\right) = \frac{8}{7} - \frac{1}{7}x_3 + \frac{3}{7}x_9$$

$$x_6 = \frac{4}{3} - \frac{1}{3}\,x_3 - \frac{7}{3}\left(\frac{1}{7} - \frac{1}{7}\,x_3 + \frac{3}{7}\,x_9\right) = 1 - x_9$$

$$x_7 = \frac{2}{3} + \frac{1}{3}\,x_3 + \frac{7}{3}\left(\frac{1}{7} - \frac{1}{7}\,x_3 + \frac{3}{7}\,x_9\right) = 1 + x_9$$

$$z = 18 - x_3 - 3\left(\frac{1}{7} - \frac{1}{7}\,x_3 + \frac{3}{7}\,x_9\right) = \frac{123}{7} - \frac{4}{7}\,x_3 - \frac{9}{7}\,x_9$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases}
x_1 = \frac{15}{7} - \frac{1}{7}\,x_3 + \frac{3}{7}\,x_9 \\
x_2 = 3 - x_9 \\
x_4 = \frac{15}{7} - \frac{1}{7}\,x_3 + \frac{10}{7}\,x_9 \\
x_5 = \frac{8}{7} - \frac{1}{7}\,x_3 + \frac{3}{7}\,x_9 \\
x_6 = 1 - x_9 \\
x_7 = 1 + x_9 \\
x_8 = \frac{1}{7} - \frac{1}{7}\,x_3 + \frac{3}{7}\,x_9 \\
z = \frac{123}{7} - \frac{4}{7}\,x_3 - \frac{9}{7}\,x_9
\end{cases}$$

where $\mathcal{B} = \{1, 2, 4, 5, 6, 7, 8\}$ and $\mathcal{N} = \{3, 9\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = \frac{15}{7}, \quad \tilde{x}_2 = 3, \quad \tilde{x}_4 = \frac{15}{7}, \quad \tilde{x}_5 = \frac{8}{7},$$
$$\tilde{x}_6 = 1, \quad \tilde{x}_7 = 1, \quad \tilde{x}_8 = \frac{1}{7},$$
$$\tilde{x}_3 = 0, \quad \tilde{x}_9 = 0 \quad \text{of value} \quad \frac{123}{7}$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_4 = 0, \quad \tilde{\pi}_5 = 0,$$
$$\tilde{\pi}_6 = 0, \quad \tilde{\pi}_7 = 0, \quad \tilde{\pi}_8 = 0,$$
$$\tilde{\pi}_3 = \frac{4}{7}, \quad \tilde{\pi}_9 = \frac{9}{7} \quad \text{of value} \quad \frac{123}{7}$$

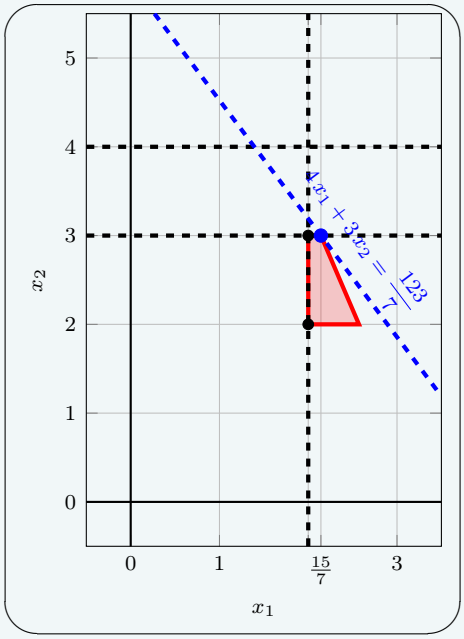


Figure 7.55: Node 6: feasible region of LP_6 (red), with the branching constraint $x_2 \leq 3$ (dashed), and its optimal solution (blue).

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_6)$ is:

$$\tilde{x}_1 = \frac{15}{7}, \quad \tilde{x}_2 = 3$$

the optimal value is:

$$z(\mathrm{LP}_6) = \frac{123}{7}$$

The LP_6 problem is **feasible** and we have

$$\lfloor z(\mathrm{LP}_6) \rfloor > \mathrm{LB} \qquad \text{since} \qquad \underbrace{\left\lfloor \frac{123}{7} \right\rfloor}_{=17} > 16$$

The optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_6)$ is **fractional**, accordingly, the LP_6 problem is **inserted in the problem list** L.

(b) Solution of the LP_7 problem

Let us consider the LP_7 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcrcrcl}
\max & 4\,x_1 & + & 3\,x_2 & & & & & & & & & & & & & \\
 & 7\,x_1 & + & 3\,x_2 & + & x_3 & & & & & & & & & & = & 24 \\
 & -x_1 & + & x_2 & & & + & x_4 & & & & & & & & = & 3 \\
 & -x_1 & & & & & & & + & x_5 & & & & & & = & -1 \\
 & & & -x_2 & & & & & & & + & x_6 & & & & = & -2 \\
 & & & x_2 & & & & & & & & & + & x_7 & & = & 4 \\
 & x_1 & & & & & & & & & & & & & - & x_8 & = & 2 \\
 & & & x_2 & & & & & & & & & & & & x_9 & = & 4 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & & x_7, & & x_8, & x_9 & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_2 - x_9 = 4 \qquad \Rightarrow \qquad x_9 = -4 + x_2$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_9 = -4 + \left(\frac{10}{3} - \frac{1}{3}\,x_3 - \frac{7}{3}\,x_8\right) \qquad \Rightarrow \qquad x_9 = -\frac{2}{3} - \frac{1}{3}\,x_3 - \frac{7}{3}\,x_8$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 = 2 + x_8 \\ x_2 = \frac{10}{3} - \frac{1}{3}\,x_3 - \frac{7}{3}\,x_8 \\ x_4 = \frac{5}{3} + \frac{1}{3}\,x_3 + \frac{10}{3}\,x_8 \\ x_5 = 1 + x_8 \\ x_6 = \frac{4}{3} - \frac{1}{3}\,x_3 - \frac{7}{3}\,x_8 \\ x_7 = \frac{2}{3} + \frac{1}{3}\,x_3 + \frac{7}{3}\,x_8 \\ x_9 = -\frac{2}{3} - \frac{1}{3}\,x_3 - \frac{7}{3}\,x_8 \\ z = 18 - x_3 - 3\,x_8 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4, 5, 6, 7, 9\}$ and $\mathcal{N} = \{3, 8\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = 2, \quad \bar{x}_2 = \frac{10}{3}, \quad \bar{x}_4 = \frac{5}{3}, \quad \bar{x}_5 = 1,$$
$$\bar{x}_6 = \frac{4}{3}, \quad \bar{x}_7 = \frac{2}{3}, \quad \bar{x}_9 = -\frac{2}{3},$$
$$\bar{x}_3 = 0, \quad \bar{x}_8 = 0 \quad \text{of value} \quad 18$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_5 = 0,$$
$$\bar{\pi}_6 = 0, \quad \bar{\pi}_7 = 0, \quad \bar{\pi}_9 = 0,$$
$$\bar{\pi}_3 = 1, \quad \bar{\pi}_8 = 3 \quad \text{of value} \quad 18$$

If we increase the value of π_9, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 &= 1 + \frac{1}{3}\pi_9 \\ \pi_8 &= 3 + \frac{7}{3}\pi_9 \end{cases} \qquad \begin{cases} \underbrace{1 + \frac{1}{3}\pi_9 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{3 + \frac{7}{3}\pi_9 \geq 0}_{\pi_8 \geq 0} \end{cases} \qquad \begin{cases} \pi_9 &\geq -3 \\ \pi_9 &\geq -\frac{9}{7} \end{cases}$$

Accordingly, we can increase the value of π_9 without violating dual feasibility, which implies that the dual problem is **unbounded**. As a result, the primal problem LP_7 is **infeasible**, and the corresponding branching node is pruned by infeasibility.

At the end of the third iteration, branching on x_2 from Node 5 has generated the down-branching node, Node 6, and the up-branching node, Node 7. Node 6 is inserted in the list L, while Node 7 is pruned by infeasibility. The corresponding branch-and-bound tree is:

Node 5

$$\begin{aligned} \max \quad & 4x_1 + 3x_2 \\ & 7x_1 + 3x_2 \leq 24 \\ & -x_1 + x_2 \leq 3 \\ & -x_1 \leq -1 \\ & -x_2 \leq -2 \\ & x_2 \leq 4 \\ & x_1 \geq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\tilde{x}_1 = 2, \quad \tilde{x}_2 = \dfrac{10}{3}$

$z(\text{LP}_5) = 18$

$\text{LB} = 16$

$x_2 \leq 3$ | $x_2 \geq 4$

Node 6

$$\begin{aligned} \max \quad & 4x_1 + 3x_2 \\ & 7x_1 + 3x_2 \leq 24 \\ & -x_1 + x_2 \leq 3 \\ & -x_1 \leq -1 \\ & -x_2 \leq -2 \\ & x_2 \leq 4 \\ & x_1 \geq 2 \\ & x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\tilde{x}_1 = \dfrac{15}{7}, \quad \tilde{x}_2 = 3$

$\lfloor z(\text{LP}_6) \rfloor = \underbrace{\left\lfloor \dfrac{123}{7} \right\rfloor}_{=17}$

$\text{LB} = 16$

Node 7

$$\begin{aligned} \max \quad & 4x_1 + 3x_2 \\ & 7x_1 + 3x_2 \leq 24 \\ & -x_1 + x_2 \leq 3 \\ & -x_1 \leq -1 \\ & -x_2 \leq -2 \\ & x_2 \leq 4 \\ & x_1 \geq 2 \\ & x_2 \geq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

pruned by infeasibility

Due to the current lower bound LB and upper bound UB, we have:

$$16 \quad \leq \quad z(\text{ILP}) \quad \leq \quad \underbrace{\left\lfloor \frac{123}{7} \right\rfloor}_{=17}$$

- **Iteration 4**

The LP_6 problem is extracted and deleted from the problem list L.

The branching variable is x_1. The branching constraints are:

$$x_1 \leq \underbrace{\left\lfloor \frac{15}{7} \right\rfloor}_{=2} \quad \text{and} \quad x_1 \geq \underbrace{\left\lfloor \frac{15}{7} \right\rfloor + 1}_{=3}$$

The LP_8 problem is created with the first branching constraint (the second child is solved only if needed).

(a) Solution of the LP_8 problem

Let us consider the LP_8 problem in standard form:

$$\begin{array}{lrcrcrcrcrcrcrcrcrcr}
\max & 4\,x_1 & + & 3\,x_2 & & & & & & & & & & & & & & \\
& 7\,x_1 & + & 3\,x_2 & + & x_3 & & & & & & & & & & & = & 24 \\
& -x_1 & + & x_2 & & & + & x_4 & & & & & & & & & = & 3 \\
& -x_1 & & & & & & & + & x_5 & & & & & & & = & -1 \\
& & & -x_2 & & & & & & & + & x_6 & & & & & = & -2 \\
& & & x_2 & & & & & & & & & + & x_7 & & & = & 4 \\
& x_1 & & & & & & & & & & & & & - & x_8 & = & 2 \\
& & & x_2 & & & & & & & & & & & & & = & 3 \\
& x_1 & & & & & & & & & & & & & & & = & 2 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & & x_7, & & x_8, & \geq & 0
\end{array}$$

For the branching constraint we have:

$$x_1 + x_{10} = 2 \qquad \Rightarrow \qquad x_{10} = 2 - x_1$$

Using the equations of the optimal simplex dictionary we can express this branching constraint as a function of the non-basic variables. Substituting, we get

$$x_{10} = 2 - \left(\frac{15}{7} - \frac{1}{7}\, x_3 + \frac{3}{7}\, x_9 \right) \qquad \Rightarrow \qquad x_{10} = -\frac{1}{7} + \frac{1}{7}\, x_3 - \frac{3}{7}\, x_9$$

We then add this equation to the previous optimal simplex dictionary and we obtain the following simplex dictionary:

$$\begin{cases} x_1 &= \frac{15}{7} - \frac{1}{7}\, x_3 + \frac{3}{7}\, x_9 \\ x_2 &= 3 - x_9 \\ x_4 &= \frac{15}{7} - \frac{1}{7}\, x_3 + \frac{10}{7}\, x_9 \\ x_5 &= \frac{8}{7} - \frac{1}{7}\, x_3 + \frac{3}{7}\, x_9 \\ x_6 &= 1 - x_9 \\ x_7 &= 1 + x_9 \\ x_8 &= \frac{1}{7} - \frac{1}{7}\, x_3 + \frac{3}{7}\, x_9 \\ x_{10} &= -\frac{1}{7} + \frac{1}{7}\, x_3 - \frac{3}{7}\, x_9 \\ z &= \frac{123}{7} - \frac{4}{7}\, x_3 - \frac{9}{7}\, x_9 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4, 5, 6, 7, 8, 10\}$ and $\mathcal{N} = \{3, 9\}$ and

- the basic primal non-feasible solution is:

$$\bar{x}_1 = \frac{15}{7}, \quad \bar{x}_2 = 3, \quad \bar{x}_4 = \frac{15}{7}, \quad \bar{x}_5 = \frac{8}{7},$$
$$\bar{x}_6 = 1, \quad \bar{x}_7 = 1, \quad \bar{x}_8 = \frac{1}{7}, \quad \bar{x}_{10} = -\frac{1}{7},$$
$$\bar{x}_3 = 0, \quad \bar{x}_9 = 0 \quad \text{of value} \quad \frac{123}{7}$$

- the basic dual feasible solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_5 = 0,$$
$$\bar{\pi}_6 = 0, \quad \bar{\pi}_7 = 0, \quad \bar{\pi}_8 = 0, \quad \bar{\pi}_{10} = 0,$$
$$\bar{\pi}_3 = \frac{4}{7}, \quad \bar{\pi}_9 = \frac{9}{7} \quad \text{of value} \quad \frac{123}{7}$$

If we increase the value of π_{10}, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 &= \frac{4}{7} - \frac{1}{7}\,\pi_{10} \\ \pi_9 &= \frac{9}{7} + \frac{3}{7}\,\pi_{10} \end{cases} \qquad \begin{cases} \underbrace{\frac{4}{7} - \frac{1}{7}\,\pi_{10} \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{9}{7} + \frac{3}{7}\,\pi_{10} \geq 0}_{\pi_9 \geq 0} \end{cases} \qquad \begin{cases} \pi_{10} &\leq 4 \\ \pi_{10} &\geq -3 \end{cases}$$

By increasing the value of π_{10} to 4, the value of π_3 decreases to 0. Then we have that:

π_{10} becomes a dual basic variable and accordingly x_{10} becomes a primal non-basic variable.

π_3 becomes a dual non-basic variable and accordingly x_3 becomes a primal basic variable.

We have:

$$x_{10} = -\frac{1}{7} + \frac{1}{7}\,x_3 - \frac{3}{7}\,x_9 \quad \Rightarrow \quad x_3 = 1 + 3\,x_9 + 7\,x_{10}$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{15}{7} - \frac{1}{7}\,(1 + 3\,x_9 + 7\,x_{10}) + \frac{3}{7}\,x_9 = 2 - x_{10}$$

$$x_4 = \frac{15}{7} - \frac{1}{7}\,(1 + 3\,x_9 + 7\,x_{10}) + \frac{10}{7}\,x_9 = 2 + x_9 - x_{10}$$

$$x_5 = \frac{8}{7} - \frac{1}{7}\,(1 + 3\,x_9 + 7\,x_{10}) + \frac{3}{7}\,x_9 = 1 - x_{10}$$

$$x_8 = \frac{1}{7} - \frac{1}{7}\,(1 + 3\,x_9 + 7\,x_{10}) + \frac{3}{7}\,x_9 = 0 - x_{10}$$

$$z = \frac{123}{7} - \frac{4}{7}\,(1 + 3\,x_9 + 7\,x_{10}) - \frac{9}{7}\,x_9 = 17 - 3\,x_9 - 4\,x_{10}$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases}
x_1 = 2 - x_{10} \\
x_2 = 3 - x_9 \\
x_3 = 1 + 3\,x_9 + 7\,x_{10} \\
x_4 = 2 + x_9 - x_{10} \\
x_5 = 1 - x_{10} \\
x_6 = 1 - x_9 \\
x_7 = 1 + x_9 \\
x_8 = 0 - x_{10} \\
z = 17 - 3\,x_9 - 4\,x_{10}
\end{cases}$$

where $\mathcal{B} = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $\mathcal{N} = \{9, 10\}$ and

– the basic primal optimal solution is:

$$\begin{aligned}
&\tilde{x}_1 = 2, \quad \tilde{x}_2 = 3, \quad \tilde{x}_3 = 1, \quad \tilde{x}_4 = 2, \\
&\tilde{x}_5 = 1, \quad \tilde{x}_6 = 1, \quad \tilde{x}_7 = 1, \quad \tilde{x}_8 = 0, \\
&\tilde{x}_9 = 0, \quad \tilde{x}_{10} = 0 \quad \text{of value} \quad 17
\end{aligned}$$

– the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad \tilde{\pi}_4 = 0,$$
$$\tilde{\pi}_5 = 0, \quad \tilde{\pi}_6 = 0, \quad \tilde{\pi}_7 = 0, \quad \tilde{\pi}_8 = 0,$$
$$\tilde{\pi}_9 = 3, \quad \tilde{\pi}_{10} = 4 \quad \text{of value} \quad 17$$

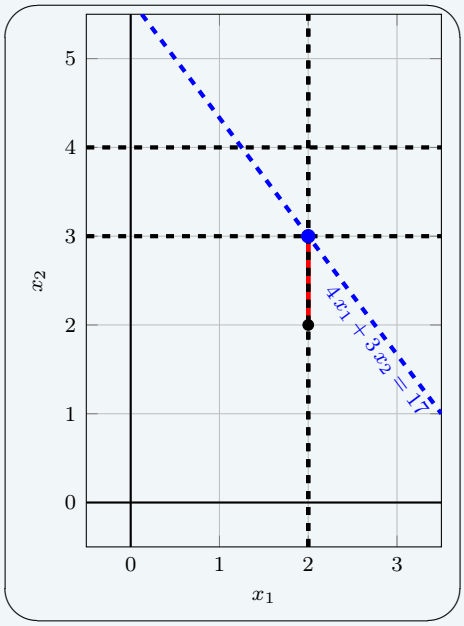


Figure 7.56: Node 8: feasible region of LP_8 (red), with the branching constraint $x_1 \leq 2$ (dashed), and its optimal solution (blue).
An optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_8)$ is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 3$$

the optimal value is:

$$z(\text{LP}_8) = 17$$

The LP_8 problem is **feasible** and we have

$$\lfloor z(\text{LP}_8) \rfloor > \text{LB} \qquad \text{since} \qquad 17 > 16$$

The optimal solution $\tilde{\boldsymbol{x}}(\text{LP}_8)$ is **integer**, accordingly, the lower bound LB becomes 17 and the incumbent solution $\bar{\boldsymbol{x}}(\text{ILP})$ becomes $\bar{x}_1 = 2, \bar{x}_2 = 3$.
Accordingly, the branching node is pruned by integrality.

At the end of the fourth iteration, branching on x_1 from Node 6 has generated the down-branching node, Node 8. This node is pruned by integrality. The up-branching node is not solved, since after Node 8 the lower bound satisfies

$$\lfloor z(\text{LP}_6) \rfloor \leq \text{LB}.$$

This is the portion of the branch-and-bound tree showing the LP problems of the branching nodes that are solved:

Node 6

$$
\begin{aligned}
\max \quad & 4x_1 + 3x_2 \\
& 7x_1 + 3x_2 \leq 24 \\
& -x_1 + x_2 \leq 3 \\
& -x_1 \leq -1 \\
& -x_2 \leq -2 \\
& x_2 \leq 4 \\
& x_1 \geq 2 \\
& x_2 \leq 3 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

$\tilde{x}_1 = \frac{15}{7}, \quad \tilde{x}_2 = 3$

$\lfloor z(\text{LP}_6) \rfloor = \underbrace{\left\lfloor \frac{123}{7} \right\rfloor}_{=17}$

$\text{LB} = 16$

$x_1 \leq 2$

Node 8

$$
\begin{aligned}
\max \quad & 4x_1 + 3x_2 \\
& 7x_1 + 3x_2 \leq 24 \\
& -x_1 + x_2 \leq 3 \\
& -x_1 \leq -1 \\
& -x_2 \leq -2 \\
& x_2 \leq 4 \\
& x_1 \geq 2 \\
& x_2 \leq 3 \\
& x_1 \leq 2 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 3$

$z(\text{LP}_8) = 17$

$\text{LB} = 17$

pruned by integrality

Due to the current lower bound LB and upper bound UB, we have:

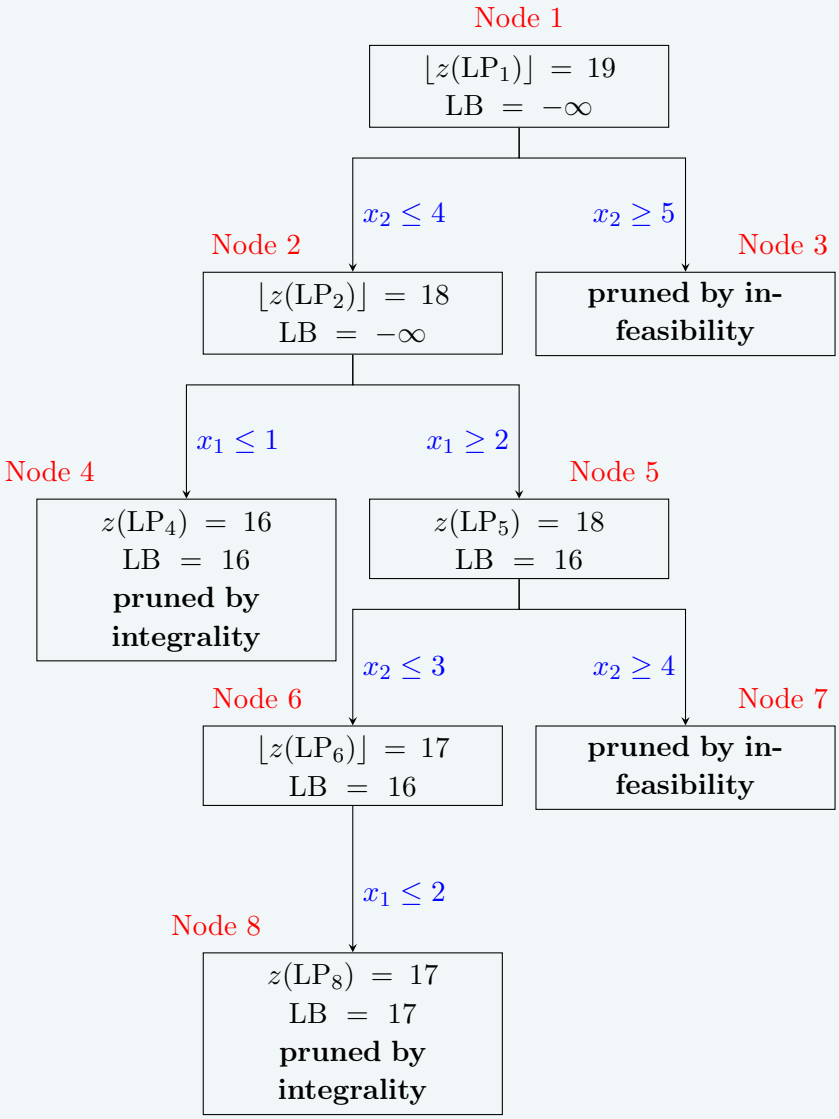
$$17 \quad \leq \quad z(\text{ILP}) \quad \leq \quad 17$$

- **Termination**

Since the list L is empty, the lower bound LB is equal to the upper bound UB and the ILP problem is solved to optimality.

The optimal value is $z(\text{ILP}) = 17$ (the value of the lower bound LB) and an optimal solution $\tilde{\boldsymbol{x}}(\text{ILP})$ is $\tilde{x}_1 = 2, \tilde{x}_2 = 3$ (the incumbent solution $\bar{\boldsymbol{x}}(\text{ILP})$).

2. The final branch-and-bound tree is:



The reasons why the branching nodes are pruned are the following.

- Node 1 has a fractional optimal solution with $\lfloor z(\text{LP}_1) \rfloor = 19$: it is branched on x_2.
- Node 2 has a fractional optimal solution with $\lfloor z(\text{LP}_2) \rfloor = 18$: it is branched on x_1.
- Node 3 is pruned by infeasibility: the LP_3 problem has no feasible solution.
- Node 4 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_4) = (1, 4)$ is integer, with value 16, which becomes the incumbent solution.

- Node 5 has a fractional optimal solution with $z(\text{LP}_5) = 18$: it is branched on x_2.
- Node 6 has a fractional optimal solution with $\lfloor z(\text{LP}_6) \rfloor = 17$: it is branched on x_1.
- The up-branching node of Node 6 is not solved, and it is not drawn, since $\lfloor z(\text{LP}_6) \rfloor = 17 \leq \text{LB} = 17$ after its first child.
- Node 7 is pruned by infeasibility: the LP_7 problem has no feasible solution.
- Node 8 is pruned by integrality: $\tilde{\boldsymbol{x}}(\text{LP}_8) = (2, 3)$ is integer, with value 17, which becomes the incumbent solution.

The optimal value is $z(\text{ILP}) = 17$ and the branch-and-bound algorithm returns the optimal solution $\tilde{\boldsymbol{x}}(\text{ILP}) = (2, 3)$.

7.2 Strong branching

Exercise 7.13 — Strong branching ($P1$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 1$ constraint:

$$\begin{array}{lrcrcrl} \max & x_1 & + & 8\ x_2 & & & \\ & 3\ x_1 & + & 6\ x_2 & \leq & 16 & \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} & \end{array}$$

Questions

1. Solve graphically the LP relaxation LP_1 of the ILP problem.
2. Determine the branching variable with the strong branching branching-variable selection strategy (in case of ties, choose the variable with the smallest index). Solve graphically the LP problems of the branching nodes.

Solution

For a fractional variable x_j of $\tilde{\boldsymbol{x}}(\text{LP}_1)$, the two branching nodes are LP_2 (with $x_j \leq \lfloor \tilde{x}_j \rfloor$) and LP_3 (with $x_j \geq \lfloor \tilde{x}_j \rfloor + 1$), and

$$\delta_1^d(x_j) = z(\text{LP}_1) - z(\text{LP}_2), \qquad \delta_1^u(x_j) = z(\text{LP}_1) - z(\text{LP}_3),$$

with $\delta = +\infty$ for an infeasible node. Strong branching selects the variable with the largest score $\min\{\delta_1^d(x_j), \delta_1^u(x_j)\}$.

1. For the LP_1 problem, graphically we have:

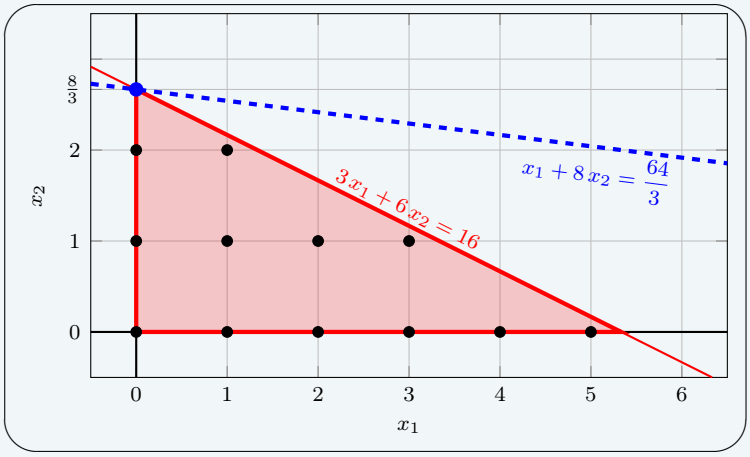


Figure 7.57: Feasible region of LP_1 (red), integer points (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = \frac{8}{3}$$

the optimal value is:

$$z(\mathrm{LP}_1) = \frac{64}{3}$$

Only x_2 is fractional: it is the only candidate branching variable, and strong branching computes its score anyway.

2. Strong branching. Graphically, we have:

   (a) Branching on x_2:

   - With $x_2 \leq 2$, an optimal solution of LP_2 is

     $$\tilde{x}_1 = \frac{4}{3}, \quad \tilde{x}_2 = 2 \quad \text{and} \quad z(\mathrm{LP}_2) = \frac{52}{3},$$

     accordingly

     $$\delta_1^d(x_2) = \frac{64}{3} - \frac{52}{3} = 4.$$

   - With $x_2 \geq 3$, the LP_3 problem is **infeasible**, accordingly

     $$\delta_1^u(x_2) = +\infty$$

   For the variable x_2 we have:

   $$\min\left\{\delta_1^d(x_2),\, \delta_1^u(x_2)\right\} = \min\{4,\, +\infty\} = 4.$$

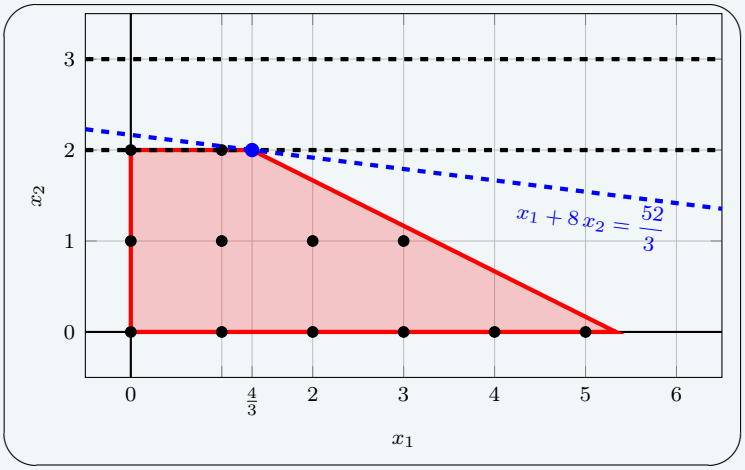


Figure 7.58: Branching on x_2: feasible regions of LP_2 ($x_2 \leq 2$) and LP_3 ($x_2 \geq 3$) in red, integer points (black) and optimal solutions (blue).

Therefore, strong branching selects x_2, the only fractional variable, with score 4.

Exercise 7.14 — Strong branching ($P2$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{array}{rrcrcr} \max & x_1 & + & 3\ x_2 & & \\ & -4\ x_1 & + & 10\ x_2 & \leq & 35 \\ & 3\ x_1 & + & x_2 & \leq & 12 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

Questions

1. Solve graphically the LP relaxation LP_1 of the ILP problem.
2. Determine the branching variable with the strong branching branching-variable selection strategy (in case of ties, choose the variable with the smallest index). Solve graphically the LP problems of the branching nodes.

Solution

For a fractional variable x_j of $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$, the two branching nodes are LP_2 (with $x_j \leq \lfloor \tilde{x}_j \rfloor$) and LP_3 (with $x_j \geq \lfloor \tilde{x}_j \rfloor + 1$), and

$$\delta_1^d(x_j) = z(\mathrm{LP}_1) - z(\mathrm{LP}_2), \qquad \delta_1^u(x_j) = z(\mathrm{LP}_1) - z(\mathrm{LP}_3),$$

with $\delta = +\infty$ for an infeasible node. Strong branching selects the variable with the largest score $\min\{\delta_1^d(x_j), \delta_1^u(x_j)\}$.

1. For the LP_1 problem, graphically we have:

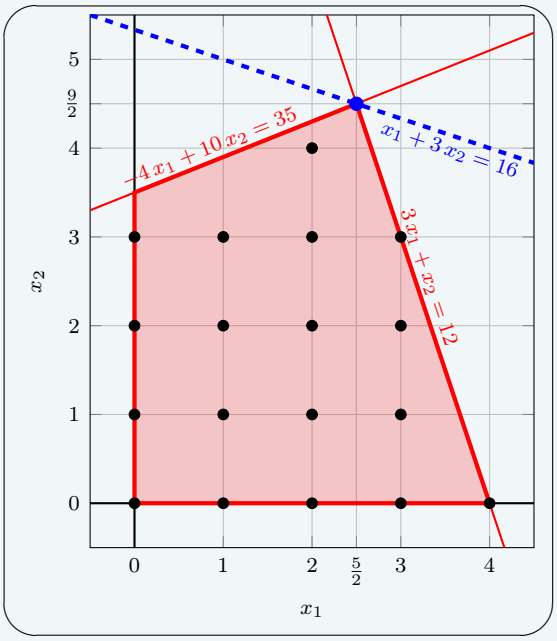


Figure 7.59: Feasible region of LP_1 (red), integer points (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is:

$$\tilde{x}_1 = \frac{5}{2}, \quad \tilde{x}_2 = \frac{9}{2}$$

the optimal value is:

$$z(\mathrm{LP}_1) = 16$$

2. Strong branching. Graphically, we have:

   (a) Branching on x_1:

   - With $x_1 \leq 2$, an optimal solution of LP_2 is

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = \frac{43}{10} \quad \text{and} \quad z(\mathrm{LP}_2) = \frac{149}{10},$$

     accordingly

$$\delta_1^d(x_1) = 16 - \frac{149}{10} = \frac{11}{10}.$$

- With $x_1 \geq 3$, an optimal solution of LP_3 is

$$\tilde{x}_1 = 3, \quad \tilde{x}_2 = 3 \quad \text{and} \quad z(\text{LP}_3) = 12,$$

accordingly

$$\delta_1^u(x_1) = 16 - 12 = 4.$$

For the variable x_1 we have:

$$\min\left\{\delta_1^d(x_1),\, \delta_1^u(x_1)\right\} = \min\left\{\frac{11}{10},\, 4\right\} = \frac{11}{10}.$$

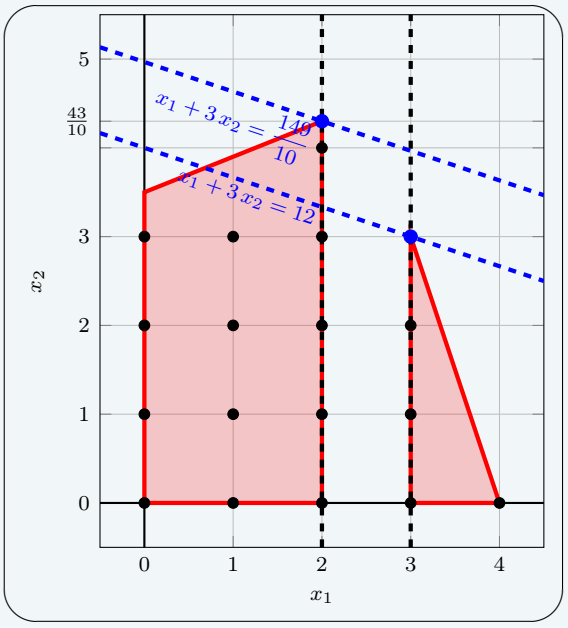


Figure 7.60: Branching on x_1: feasible regions of LP_2 ($x_1 \leq 2$) and LP_3 ($x_1 \geq 3$) in red, integer points (black) and optimal solutions (blue).

(b) Branching on x_2:

- With $x_2 \leq 4$, an optimal solution of LP_2 is

$$\tilde{x}_1 = \frac{8}{3}, \quad \tilde{x}_2 = 4 \quad \text{and} \quad z(\text{LP}_2) = \frac{44}{3},$$

accordingly

$$\delta_1^d(x_2) = 16 - \frac{44}{3} = \frac{4}{3}.$$

- With $x_2 \geq 5$, the LP_3 problem is **infeasible**, accordingly

$$\delta_1^u(x_2) = +\infty$$

For the variable x_2 we have:

$$\min\left\{\delta_1^d(x_2),\, \delta_1^u(x_2)\right\} = \min\left\{\frac{4}{3},\, +\infty\right\} = \frac{4}{3}.$$

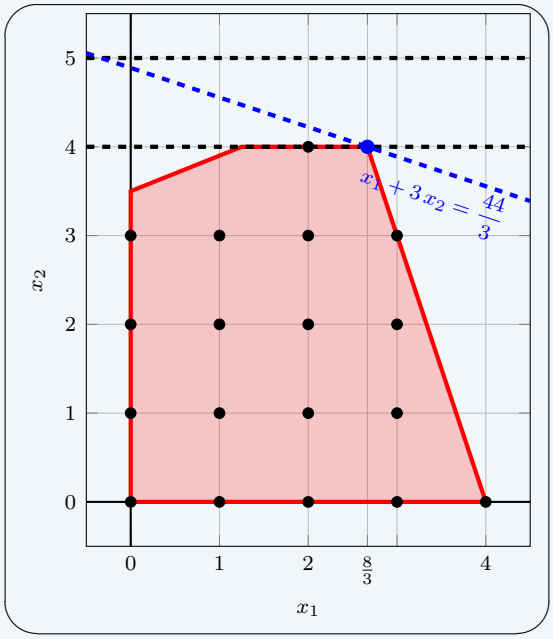


Figure 7.61: Branching on x_2: feasible regions of LP_2 ($x_2 \leq 4$) and LP_3 ($x_2 \geq 5$) in red, integer points (black) and optimal solutions (blue).

Therefore, strong branching selects x_2, since its score is the largest:

$$\frac{4}{3} > \frac{11}{10}.$$

Exercise 7.15 — Strong branching ($P3$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{array}{rrcrcr} \max & 2\ x_1 & + & 4\ x_2 & & \\ & -x_1 & + & x_2 & \leq & 2 \\ & 8\ x_1 & + & 2\ x_2 & \leq & 19 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

Questions

1. Solve graphically the LP relaxation LP_1 of the ILP problem.
2. Determine the branching variable with the strong branching branching-variable selection strategy (in case of ties, choose the variable with the smallest index). Solve graphically the LP problems of the branching nodes.

Solution

For a fractional variable x_j of $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$, the two branching nodes are LP_2 (with $x_j \leq \lfloor \tilde{x}_j \rfloor$) and LP_3 (with $x_j \geq \lfloor \tilde{x}_j \rfloor + 1$), and

$$\delta_1^d(x_j) = z(\mathrm{LP}_1) - z(\mathrm{LP}_2), \qquad \delta_1^u(x_j) = z(\mathrm{LP}_1) - z(\mathrm{LP}_3),$$

with $\delta = +\infty$ for an infeasible node. Strong branching selects the variable with the largest score $\min\{\delta_1^d(x_j), \delta_1^u(x_j)\}$.

1. For the LP_1 problem, graphically we have:

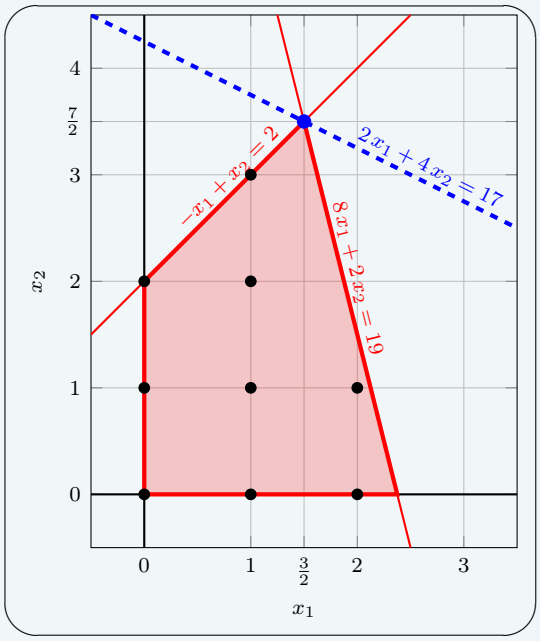


Figure 7.62: Feasible region of LP_1 (red), integer points (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is:

$$\tilde{x}_1 = \frac{3}{2}, \quad \tilde{x}_2 = \frac{7}{2}$$

the optimal value is:

$$z(\mathrm{LP}_1) = 17$$

2. Strong branching. Graphically, we have:

   (a) Branching on x_1:

   - With $x_1 \leq 1$, an optimal solution of LP_2 is

     $$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 3 \quad \text{and} \quad z(\mathrm{LP}_2) = 14,$$

     accordingly

     $$\delta_1^d(x_1) = 17 - 14 = 3.$$

- With $x_1 \geq 2$, an optimal solution of LP_3 is

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = \frac{3}{2} \quad \text{and} \quad z(\text{LP}_3) = 10,$$

accordingly

$$\delta_1^u(x_1) = 17 - 10 = 7.$$

For the variable x_1 we have:

$$\min\left\{\delta_1^d(x_1),\, \delta_1^u(x_1)\right\} = \min\{3,\, 7\} = 3.$$

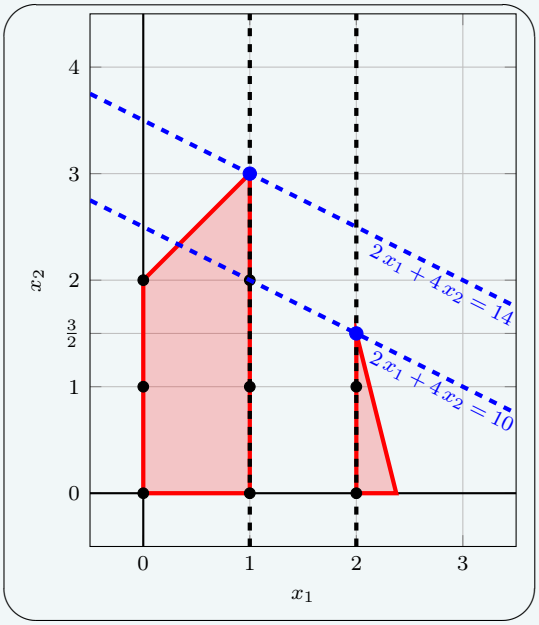


Figure 7.63: Branching on x_1: feasible regions of LP_2 ($x_1 \leq 1$) and LP_3 ($x_1 \geq 2$) in red, integer points (black) and optimal solutions (blue).

(b) Branching on x_2:

- With $x_2 \leq 3$, an optimal solution of LP_2 is

$$\tilde{x}_1 = \frac{13}{8}, \quad \tilde{x}_2 = 3 \quad \text{and} \quad z(\text{LP}_2) = \frac{61}{4},$$

accordingly

$$\delta_1^d(x_2) = 17 - \frac{61}{4} = \frac{7}{4}.$$

- With $x_2 \geq 4$, the LP_3 problem is **infeasible**, accordingly

$$\delta_1^u(x_2) = +\infty$$

For the variable x_2 we have:

$$\min\left\{\delta_1^d(x_2),\, \delta_1^u(x_2)\right\} = \min\left\{\frac{7}{4},\, +\infty\right\} = \frac{7}{4}.$$

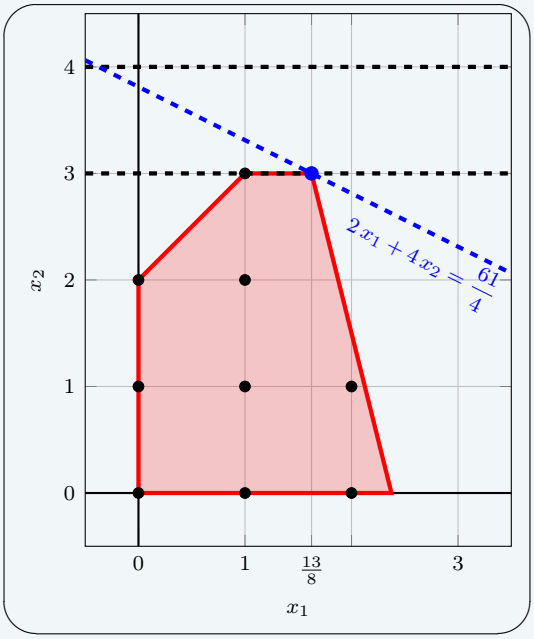


Figure 7.64: Branching on x_2: feasible regions of LP_2 ($x_2 \leq 3$) and LP_3 ($x_2 \geq 4$) in red, integer points (black) and optimal solutions (blue).

Therefore, strong branching selects x_1, since its score is the largest:

$$3 > \frac{7}{4}.$$

Exercise 7.16 — Strong branching ($P4$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{array}{lrcrcl} \max & 5\,x_1 & + & 3\,x_2 & & \\ & 2\,x_1 & + & x_2 & \leq & 14 \\ & x_1 & + & 8\,x_2 & \leq & 32 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

Questions

1. Solve graphically the LP relaxation LP_1 of the ILP problem.
2. Determine the branching variable with the strong branching branching-variable selection strategy (in case of ties, choose the variable with the smallest index). Solve graphically the LP problems of the branching nodes.

Solution

For a fractional variable x_j of $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$, the two branching nodes are LP_2 (with $x_j \leq \lfloor \tilde{x}_j \rfloor$) and LP_3 (with $x_j \geq \lfloor \tilde{x}_j \rfloor + 1$), and

$$\delta_1^d(x_j) = z(\mathrm{LP}_1) - z(\mathrm{LP}_2), \qquad \delta_1^u(x_j) = z(\mathrm{LP}_1) - z(\mathrm{LP}_3),$$

with $\delta = +\infty$ for an infeasible node. Strong branching selects the variable with the largest score $\min\{\delta_1^d(x_j), \delta_1^u(x_j)\}$.

1. For the LP_1 problem, graphically we have:

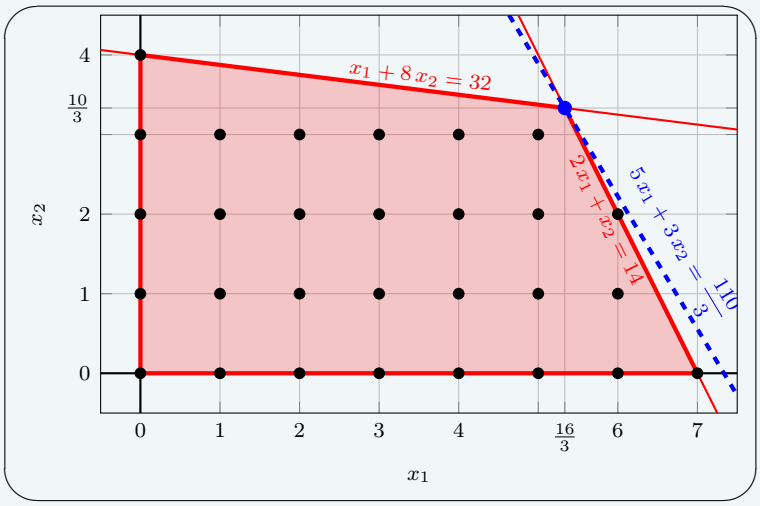


Figure 7.65: Feasible region of LP_1 (red), integer points (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is:

$$\tilde{x}_1 = \frac{16}{3}, \quad \tilde{x}_2 = \frac{10}{3}$$

the optimal value is:

$$z(\mathrm{LP}_1) = \frac{110}{3}$$

2. Strong branching. Graphically, we have:

   (a) Branching on x_1:

   - With $x_1 \leq 5$, an optimal solution of LP_2 is

     $$\tilde{x}_1 = 5, \quad \tilde{x}_2 = \frac{27}{8} \quad \text{and} \quad z(\mathrm{LP}_2) = \frac{281}{8},$$

     accordingly

     $$\delta_1^d(x_1) = \frac{110}{3} - \frac{281}{8} = \frac{37}{24}.$$

   - With $x_1 \geq 6$, an optimal solution of LP_3 is

     $$\tilde{x}_1 = 6, \quad \tilde{x}_2 = 2 \quad \text{and} \quad z(\mathrm{LP}_3) = 36,$$

accordingly

$$\delta_1^u(x_1) = \frac{110}{3} - 36 = \frac{2}{3}.$$

For the variable x_1 we have:

$$\min\left\{\delta_1^d(x_1),\, \delta_1^u(x_1)\right\} = \min\left\{\frac{37}{24},\, \frac{2}{3}\right\} = \frac{2}{3}.$$

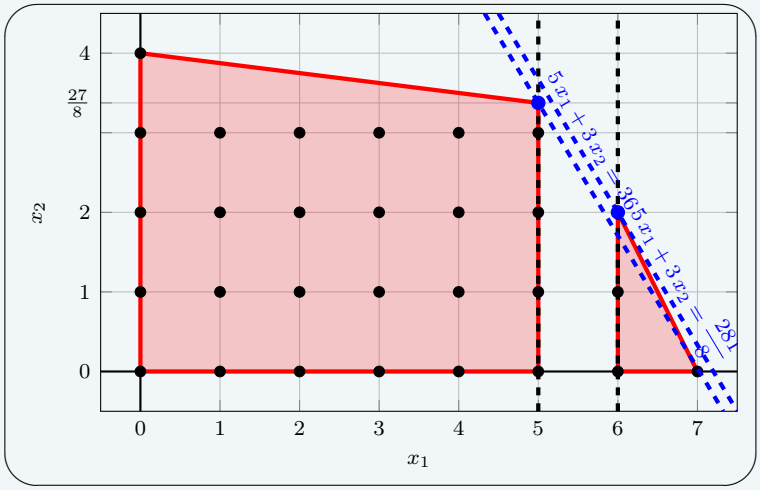


Figure 7.66: Branching on x_1: feasible regions of LP_2 ($x_1 \leq 5$) and LP_3 ($x_1 \geq 6$) in red, integer points (black) and optimal solutions (blue).

(b) Branching on x_2:

- With $x_2 \leq 3$, an optimal solution of LP_2 is

$$\tilde{x}_1 = \frac{11}{2}, \quad \tilde{x}_2 = 3 \quad \text{and} \quad z(\mathrm{LP}_2) = \frac{73}{2},$$

accordingly

$$\delta_1^d(x_2) = \frac{110}{3} - \frac{73}{2} = \frac{1}{6}.$$

- With $x_2 \geq 4$, an optimal solution of LP_3 is

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = 4 \quad \text{and} \quad z(\mathrm{LP}_3) = 12,$$

accordingly

$$\delta_1^u(x_2) = \frac{110}{3} - 12 = \frac{74}{3}.$$

For the variable x_2 we have:

$$\min\left\{\delta_1^d(x_2),\, \delta_1^u(x_2)\right\} = \min\left\{\frac{1}{6},\, \frac{74}{3}\right\} = \frac{1}{6}.$$

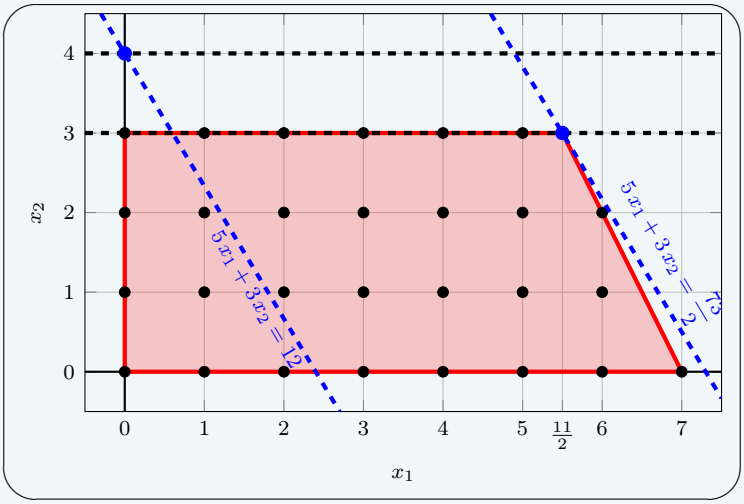


Figure 7.67: Branching on x_2: feasible regions of LP_2 ($x_2 \leq 3$) and LP_3 ($x_2 \geq 4$) in red, integer points (black) and optimal solutions (blue).

Therefore, strong branching selects x_1, since its score is the largest:

$$\frac{2}{3} > \frac{1}{6}.$$

Exercise 7.17 — Strong branching ($P5$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{array}{lrcrcl} \max & 2\ x_1 & + & 3\ x_2 & & \\ & 6\ x_1 & + & 21\ x_2 & \leq & 77 \\ & 8\ x_1 & + & 3\ x_2 & \leq & 36 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

Questions

1. Solve graphically the LP relaxation LP_1 of the ILP problem.
2. Determine the branching variable with the strong branching branching-variable selection strategy (in case of ties, choose the variable with the smallest index). Solve graphically the LP problems of the branching nodes.

Solution

For a fractional variable x_j of $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$, the two branching nodes are LP_2 (with $x_j \leq \lfloor \tilde{x}_j \rfloor$) and LP_3 (with $x_j \geq \lfloor \tilde{x}_j \rfloor + 1$), and

$$\delta_1^d(x_j) = z(\mathrm{LP}_1) - z(\mathrm{LP}_2), \qquad \delta_1^u(x_j) = z(\mathrm{LP}_1) - z(\mathrm{LP}_3),$$

with $\delta = +\infty$ for an infeasible node. Strong branching selects the variable with the largest score $\min\{\delta_1^d(x_j), \delta_1^u(x_j)\}$.

1. For the LP_1 problem, graphically we have:

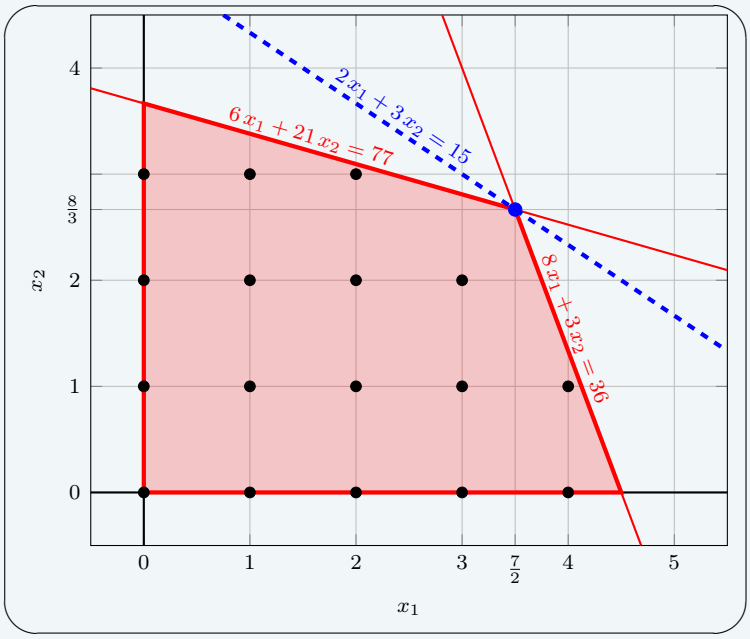


Figure 7.68: Feasible region of LP_1 (red), integer points (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is:

$$\tilde{x}_1 = \frac{7}{2}, \quad \tilde{x}_2 = \frac{8}{3}$$

the optimal value is:

$$z(\mathrm{LP}_1) = 15$$

2. Strong branching. Graphically, we have:

   (a) Branching on x_1:

   - With $x_1 \leq 3$, an optimal solution of LP_2 is

$$\tilde{x}_1 = 3, \quad \tilde{x}_2 = \frac{59}{21} \quad \text{and} \quad z(\mathrm{LP}_2) = \frac{101}{7},$$

   accordingly

$$\delta_1^d(x_1) = 15 - \frac{101}{7} = \frac{4}{7}.$$

- With $x_1 \geq 4$, an optimal solution of LP_3 is

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = \frac{4}{3} \quad \text{and} \quad z(\text{LP}_3) = 12,$$

accordingly

$$\delta_1^u(x_1) = 15 - 12 = 3.$$

For the variable x_1 we have:

$$\min\left\{\delta_1^d(x_1),\, \delta_1^u(x_1)\right\} = \min\left\{\frac{4}{7},\, 3\right\} = \frac{4}{7}.$$

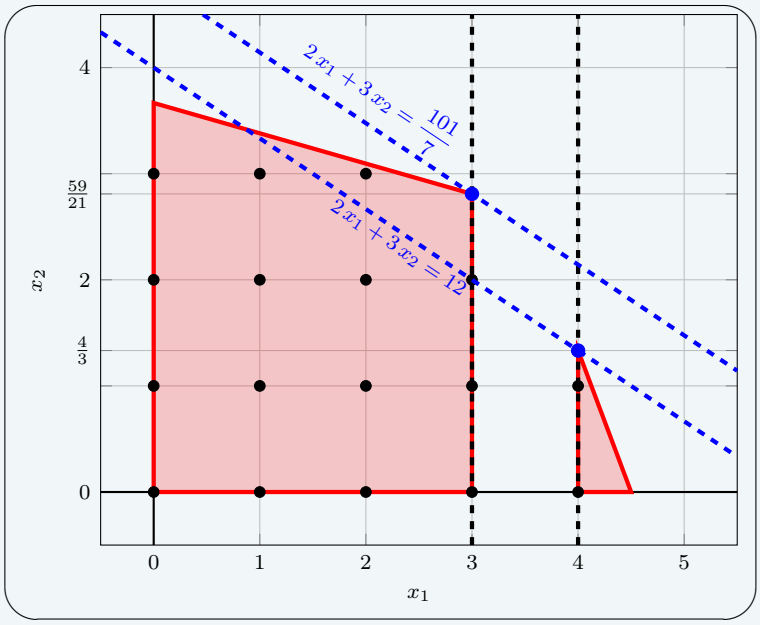


Figure 7.69: Branching on x_1: feasible regions of LP_2 ($x_1 \leq 3$) and LP_3 ($x_1 \geq 4$) in red, integer points (black) and optimal solutions (blue).

(b) Branching on x_2:

- With $x_2 \leq 2$, an optimal solution of LP_2 is

$$\tilde{x}_1 = \frac{15}{4}, \quad \tilde{x}_2 = 2 \quad \text{and} \quad z(\text{LP}_2) = \frac{27}{2},$$

accordingly

$$\delta_1^d(x_2) = 15 - \frac{27}{2} = \frac{3}{2}.$$

- With $x_2 \geq 3$, an optimal solution of LP_3 is

$$\tilde{x}_1 = \frac{7}{3}, \quad \tilde{x}_2 = 3 \quad \text{and} \quad z(\text{LP}_3) = \frac{41}{3},$$

accordingly

$$\delta_1^u(x_2) = 15 - \frac{41}{3} = \frac{4}{3}.$$

For the variable x_2 we have:

$$\min\left\{\delta_1^d(x_2),\, \delta_1^u(x_2)\right\} = \min\left\{\frac{3}{2},\, \frac{4}{3}\right\} = \frac{4}{3}.$$

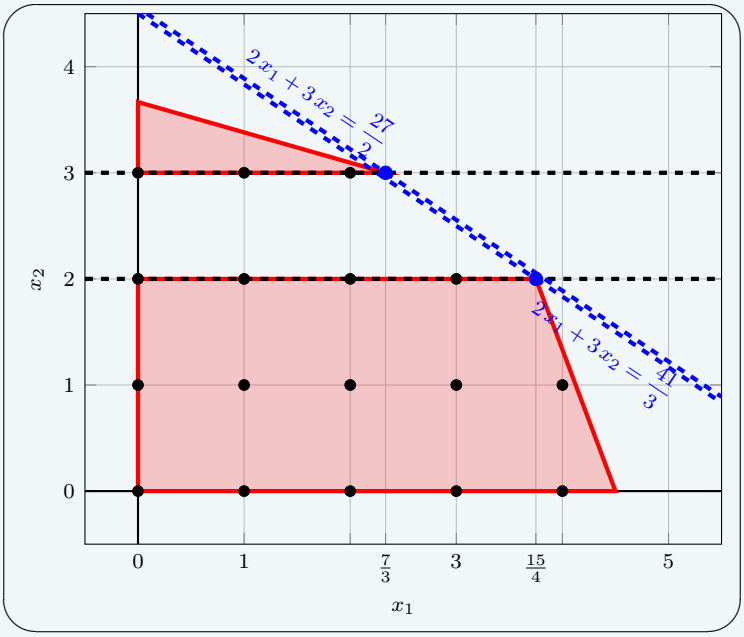


Figure 7.70: Branching on x_2: feasible regions of LP_2 ($x_2 \leq 2$) and LP_3 ($x_2 \geq 3$) in red, integer points (black) and optimal solutions (blue).

Therefore, strong branching selects x_2, since its score is the largest:

$$\frac{4}{3} > \frac{4}{7}.$$

Exercise 7.18 — Strong branching ($P6$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{array}{rrcrcl} \max & x_1 & + & x_2 & & \\ & 4\,x_1 & + & 15\,x_2 & \leq & 70 \\ & 12\,x_1 & + & 3\,x_2 & \leq & 52 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

Questions

1. Solve graphically the LP relaxation LP_1 of the ILP problem.
2. Determine the branching variable with the strong branching branching-variable selection strategy (in case of ties, choose the variable with the smallest index). Solve graphically the LP problems of the branching nodes.

Solution

For a fractional variable x_j of $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$, the two branching nodes are LP_2 (with $x_j \leq \lfloor \tilde{x}_j \rfloor$) and LP_3 (with $x_j \geq \lfloor \tilde{x}_j \rfloor + 1$), and

$$\delta_1^d(x_j) = z(\mathrm{LP}_1) - z(\mathrm{LP}_2), \qquad \delta_1^u(x_j) = z(\mathrm{LP}_1) - z(\mathrm{LP}_3),$$

with $\delta = +\infty$ for an infeasible node. Strong branching selects the variable with the largest score $\min\{\delta_1^d(x_j), \delta_1^u(x_j)\}$.

1. For the LP_1 problem, graphically we have:

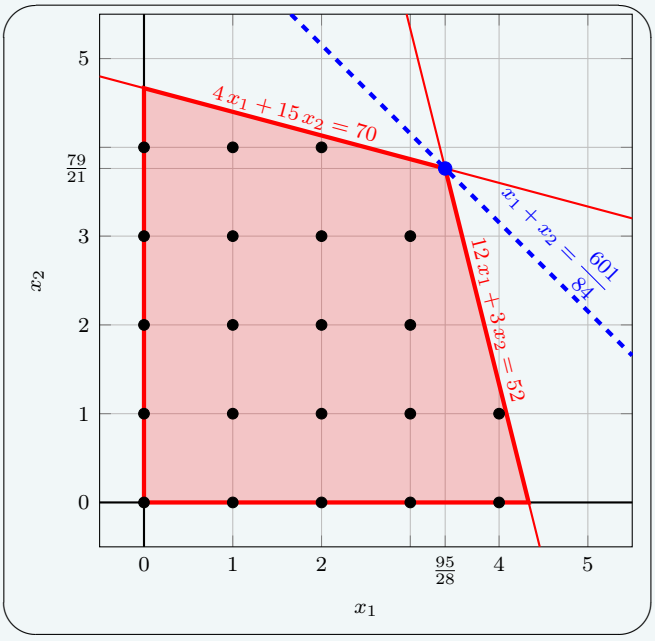


Figure 7.71: Feasible region of LP_1 (red), integer points (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is:

$$\tilde{x}_1 = \frac{95}{28}, \quad \tilde{x}_2 = \frac{79}{21}$$

the optimal value is:

$$z(\mathrm{LP}_1) = \frac{601}{84}$$

2. Strong branching. Graphically, we have:

   (a) Branching on x_1:

   - With $x_1 \leq 3$, an optimal solution of LP_2 is

   $$\tilde{x}_1 = 3, \quad \tilde{x}_2 = \frac{58}{15} \quad \text{and} \quad z(\mathrm{LP}_2) = \frac{103}{15},$$

   accordingly

   $$\delta_1^d(x_1) = \frac{601}{84} - \frac{103}{15} = \frac{121}{420}.$$

- With $x_1 \geq 4$, an optimal solution of LP_3 is

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = \frac{4}{3} \quad \text{and} \quad z(\text{LP}_3) = \frac{16}{3},$$

accordingly

$$\delta_1^u(x_1) = \frac{601}{84} - \frac{16}{3} = \frac{51}{28}.$$

For the variable x_1 we have:

$$\min\left\{\delta_1^d(x_1),\, \delta_1^u(x_1)\right\} = \min\left\{\frac{121}{420},\, \frac{51}{28}\right\} = \frac{121}{420}.$$

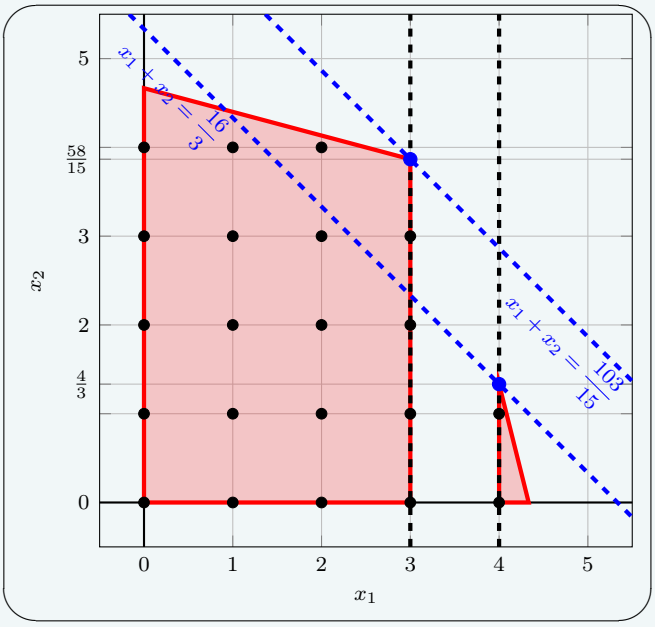


Figure 7.72: Branching on x_1: feasible regions of LP_2 ($x_1 \leq 3$) and LP_3 ($x_1 \geq 4$) in red, integer points (black) and optimal solutions (blue).

(b) Branching on x_2:

- With $x_2 \leq 3$, an optimal solution of LP_2 is

$$\tilde{x}_1 = \frac{43}{12}, \quad \tilde{x}_2 = 3 \quad \text{and} \quad z(\text{LP}_2) = \frac{79}{12},$$

accordingly

$$\delta_1^d(x_2) = \frac{601}{84} - \frac{79}{12} = \frac{4}{7}.$$

- With $x_2 \geq 4$, an optimal solution of LP_3 is

$$\tilde{x}_1 = \frac{5}{2}, \quad \tilde{x}_2 = 4 \quad \text{and} \quad z(\text{LP}_3) = \frac{13}{2},$$

accordingly

$$\delta_1^u(x_2) = \frac{601}{84} - \frac{13}{2} = \frac{55}{84}.$$

For the variable x_2 we have:

$$\min\left\{\delta_1^d(x_2),\, \delta_1^u(x_2)\right\} = \min\left\{\frac{4}{7},\, \frac{55}{84}\right\} = \frac{4}{7}.$$

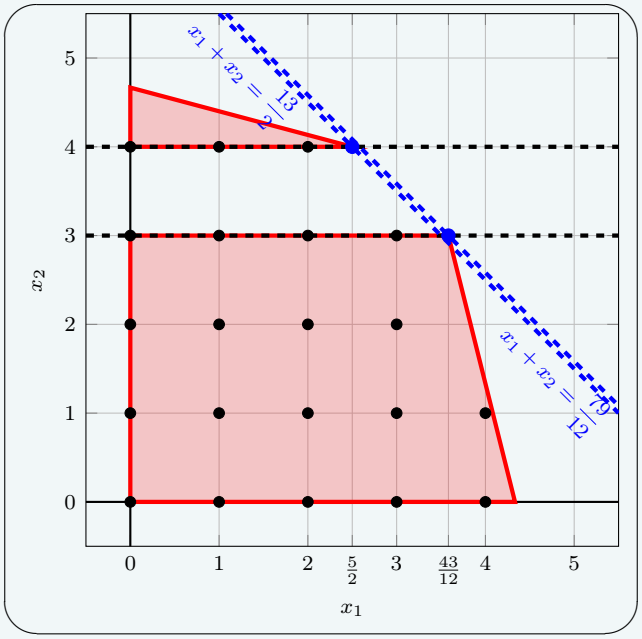


Figure 7.73: Branching on x_2: feasible regions of LP_2 ($x_2 \leq 3$) and LP_3 ($x_2 \geq 4$) in red, integer points (black) and optimal solutions (blue).

Therefore, strong branching selects x_2, since its score is the largest:

$$\frac{4}{7} > \frac{121}{420}.$$

Exercise 7.19 — Strong branching ($P7$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{array}{rrcrcr} \max & 3\,x_1 & + & x_2 & & \\ & x_1 & + & 4\,x_2 & \leq & 12 \\ & 2\,x_1 & + & x_2 & \leq & 5 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

Questions

1. Solve graphically the LP relaxation LP_1 of the ILP problem.
2. Determine the branching variable with the strong branching branching-variable selection strategy (in case of ties, choose the variable with the smallest index). Solve graphically the LP problems of the branching nodes.

Solution

For a fractional variable x_j of $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$, the two branching nodes are LP_2 (with $x_j \leq \lfloor \tilde{x}_j \rfloor$) and LP_3 (with $x_j \geq \lfloor \tilde{x}_j \rfloor + 1$), and

$$\delta_1^d(x_j) = z(\mathrm{LP}_1) - z(\mathrm{LP}_2), \qquad \delta_1^u(x_j) = z(\mathrm{LP}_1) - z(\mathrm{LP}_3),$$

with $\delta = +\infty$ for an infeasible node. Strong branching selects the variable with the largest score $\min\{\delta_1^d(x_j), \delta_1^u(x_j)\}$.

1. For the LP_1 problem, graphically we have:

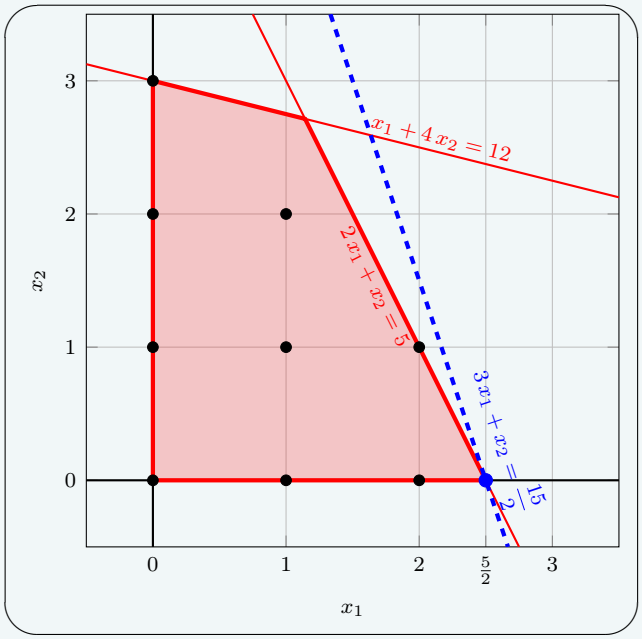


Figure 7.74: Feasible region of LP_1 (red), integer points (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is:

$$\tilde{x}_1 = \frac{5}{2}, \quad \tilde{x}_2 = 0$$

the optimal value is:

$$z(\mathrm{LP}_1) = \frac{15}{2}$$

Only x_1 is fractional: it is the only candidate branching variable, and strong branching computes its score anyway.

2. Strong branching. Graphically, we have:

   (a) Branching on x_1:

   - With $x_1 \leq 2$, an optimal solution of LP_2 is

     $$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 1 \quad \text{and} \quad z(\mathrm{LP}_2) = 7,$$

     accordingly

     $$\delta_1^d(x_1) = \frac{15}{2} - 7 = \frac{1}{2}.$$

- With $x_1 \geq 3$, the LP_3 problem is **infeasible**, accordingly

$$\delta_1^u(x_1) = +\infty$$

For the variable x_1 we have:

$$\min\left\{\delta_1^d(x_1),\, \delta_1^u(x_1)\right\} = \min\left\{\frac{1}{2},\, +\infty\right\} = \frac{1}{2}.$$

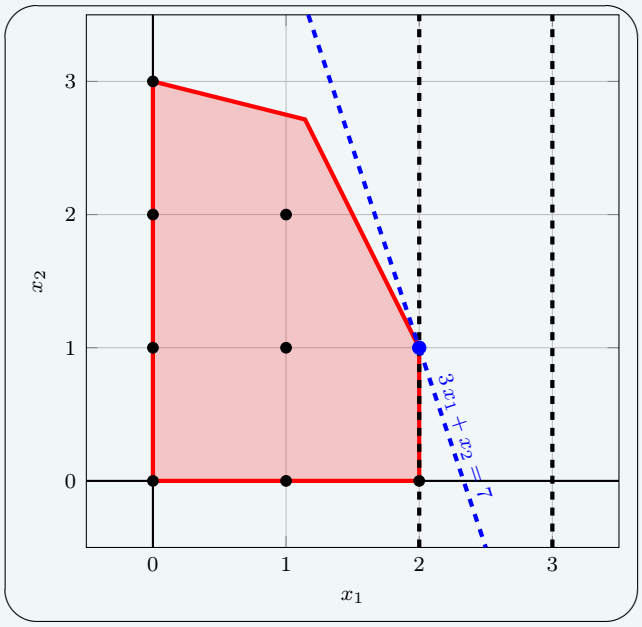


Figure 7.75: Branching on x_1: feasible regions of LP_2 ($x_1 \leq 2$) and LP_3 ($x_1 \geq 3$) in red, integer points (black) and optimal solutions (blue).

Therefore, strong branching selects x_1, the only fractional variable, with score $\frac{1}{2}$.

Exercise 7.20 — Strong branching ($P8$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{array}{lrcrcl} \max & x_1 & + & 2\,x_2 & & \\ & -x_1 & + & 2\,x_2 & \leq & 2 \\ & x_1 & + & x_2 & \leq & 3 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

Questions

1. Solve graphically the LP relaxation LP_1 of the ILP problem.

2. Determine the branching variable with the strong branching branching-variable selection strategy (in case of ties, choose the variable with the smallest index). Solve graphically the LP problems of the branching nodes.

Solution

For a fractional variable x_j of $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$, the two branching nodes are LP_2 (with $x_j \leq \lfloor \tilde{x}_j \rfloor$) and LP_3 (with $x_j \geq \lfloor \tilde{x}_j \rfloor + 1$), and

$$\delta_1^d(x_j) = z(\mathrm{LP}_1) - z(\mathrm{LP}_2), \qquad \delta_1^u(x_j) = z(\mathrm{LP}_1) - z(\mathrm{LP}_3),$$

with $\delta = +\infty$ for an infeasible node. Strong branching selects the variable with the largest score $\min\{\delta_1^d(x_j), \delta_1^u(x_j)\}$.

1. For the LP_1 problem, graphically we have:

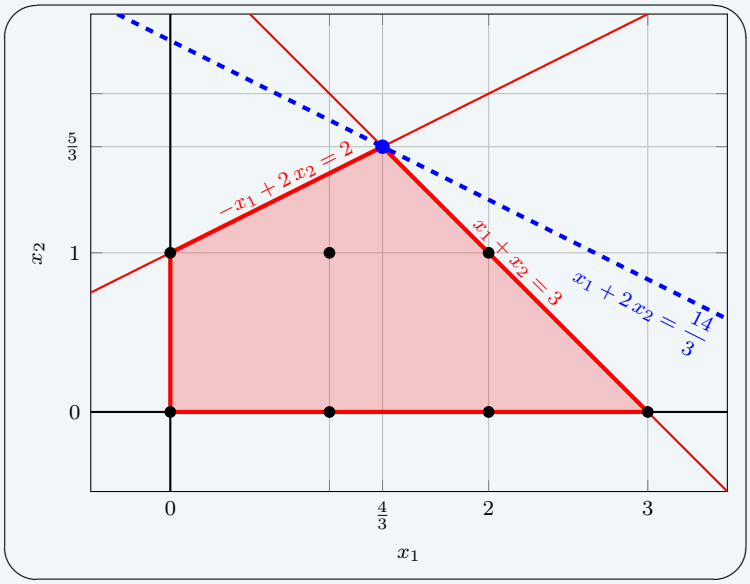


Figure 7.76: Feasible region of LP_1 (red), integer points (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is:

$$\tilde{x}_1 = \frac{4}{3}, \quad \tilde{x}_2 = \frac{5}{3}$$

the optimal value is:

$$z(\mathrm{LP}_1) = \frac{14}{3}$$

2. Strong branching. Graphically, we have:

   (a) Branching on x_1:

   - With $x_1 \leq 1$, an optimal solution of LP_2 is

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = \frac{3}{2} \quad \text{and} \quad z(\mathrm{LP}_2) = 4,$$

accordingly

$$\delta_1^d(x_1) = \frac{14}{3} - 4 = \frac{2}{3}.$$

- With $x_1 \geq 2$, an optimal solution of LP_3 is

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 1 \quad \text{and} \quad z(\text{LP}_3) = 4,$$

accordingly

$$\delta_1^u(x_1) = \frac{14}{3} - 4 = \frac{2}{3}.$$

For the variable x_1 we have:

$$\min\left\{\delta_1^d(x_1),\, \delta_1^u(x_1)\right\} = \min\left\{\frac{2}{3},\, \frac{2}{3}\right\} = \frac{2}{3}.$$

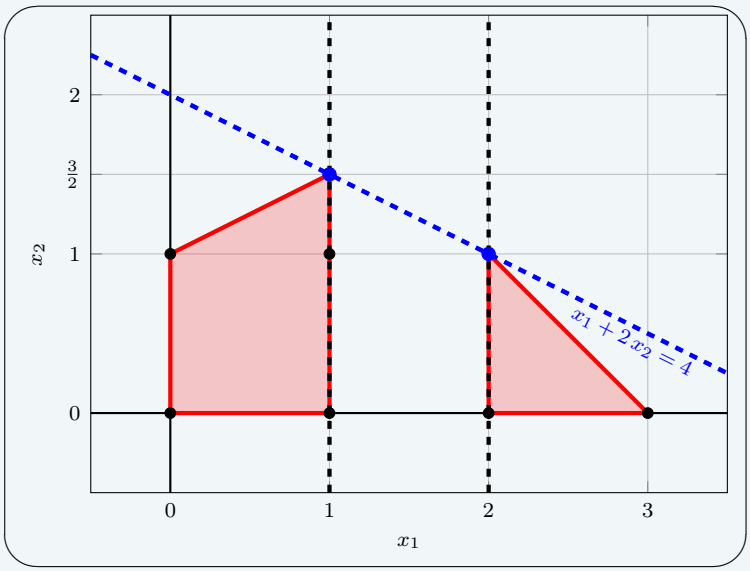


Figure 7.77: Branching on x_1: feasible regions of LP_2 ($x_1 \leq 1$) and LP_3 ($x_1 \geq 2$) in red, integer points (black) and optimal solutions (blue).

(b) Branching on x_2:

- With $x_2 \leq 1$, an optimal solution of LP_2 is

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 1 \quad \text{and} \quad z(\text{LP}_2) = 4,$$

accordingly

$$\delta_1^d(x_2) = \frac{14}{3} - 4 = \frac{2}{3}.$$

- With $x_2 \geq 2$, the LP_3 problem is **infeasible**, accordingly

$$\delta_1^u(x_2) = +\infty$$

For the variable x_2 we have:

$$\min\left\{\delta_1^d(x_2),\, \delta_1^u(x_2)\right\} = \min\left\{\frac{2}{3},\, +\infty\right\} = \frac{2}{3}.$$

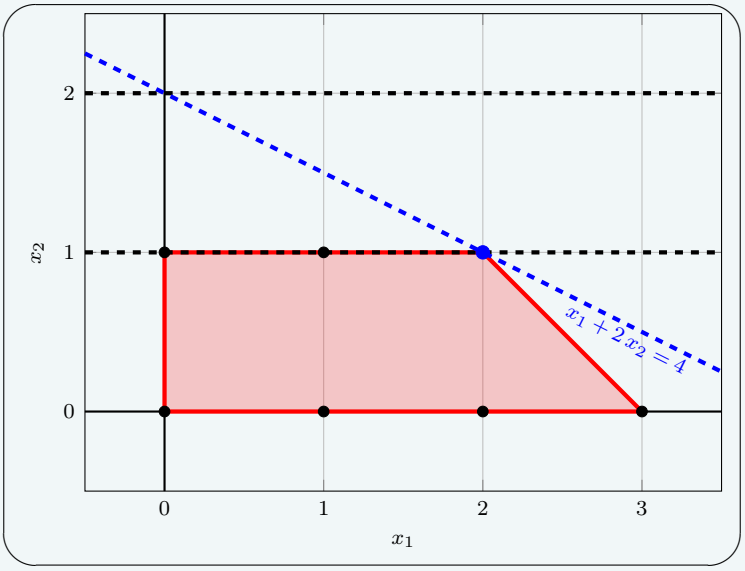


Figure 7.78: Branching on x_2: feasible regions of LP_2 ($x_2 \leq 1$) and LP_3 ($x_2 \geq 2$) in red, integer points (black) and optimal solutions (blue).

The two variables have the same score $\frac{2}{3}$: with the tie-breaking rule (smallest index), strong branching selects x_1.

Exercise 7.21 — Strong branching ($P9$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{array}{lrcrcl} \max & 2\ x_1 & + & x_2 & & \\ & -x_1 & + & x_2 & \leq & 1 \\ & 6\ x_1 & + & 4\ x_2 & \leq & 39 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

Questions

1. Solve graphically the LP relaxation LP_1 of the ILP problem.
2. Determine the branching variable with the strong branching branching-variable selection strategy (in case of ties, choose the variable with the smallest index). Solve graphically the LP problems of the branching nodes.

Solution

For a fractional variable x_j of $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$, the two branching nodes are LP_2 (with $x_j \leq \lfloor \tilde{x}_j \rfloor$) and LP_3 (with $x_j \geq \lfloor \tilde{x}_j \rfloor + 1$), and

$$\delta_1^d(x_j) = z(\mathrm{LP}_1) - z(\mathrm{LP}_2), \qquad \delta_1^u(x_j) = z(\mathrm{LP}_1) - z(\mathrm{LP}_3),$$

with $\delta = +\infty$ for an infeasible node. Strong branching selects the variable with the largest score $\min\{\delta_1^d(x_j), \delta_1^u(x_j)\}$.

1. For the LP_1 problem, graphically we have:

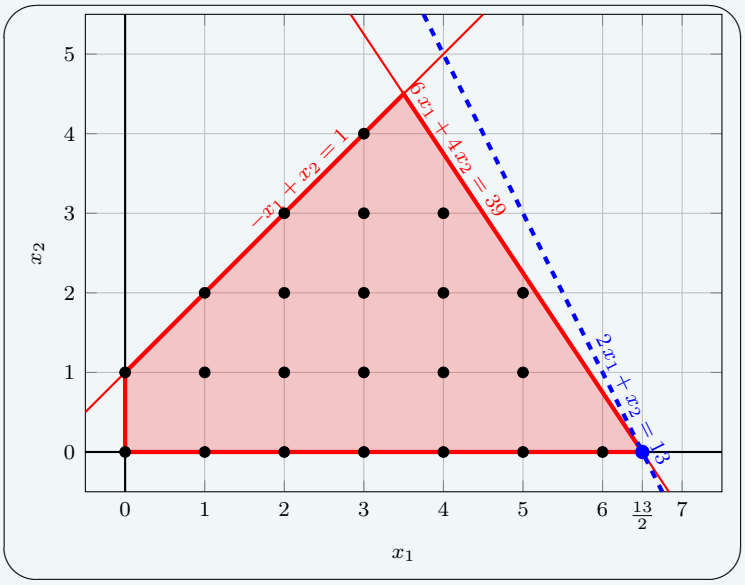


Figure 7.79: Feasible region of LP_1 (red), integer points (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is:

$$\tilde{x}_1 = \frac{13}{2}, \quad \tilde{x}_2 = 0$$

the optimal value is:

$$z(\mathrm{LP}_1) = 13$$

Only x_1 is fractional: it is the only candidate branching variable, and strong branching computes its score anyway.

2. Strong branching. Graphically, we have:

   (a) Branching on x_1:

   - With $x_1 \leq 6$, an optimal solution of LP_2 is

$$\tilde{x}_1 = 6, \quad \tilde{x}_2 = \frac{3}{4} \quad \text{and} \quad z(\mathrm{LP}_2) = \frac{51}{4},$$

   accordingly

$$\delta_1^d(x_1) = 13 - \frac{51}{4} = \frac{1}{4}.$$

- With $x_1 \geq 7$, the LP_3 problem is **infeasible**, accordingly

$$\delta_1^u(x_1) = +\infty$$

For the variable x_1 we have:

$$\min\left\{\delta_1^d(x_1),\, \delta_1^u(x_1)\right\} = \min\left\{\frac{1}{4},\, +\infty\right\} = \frac{1}{4}.$$

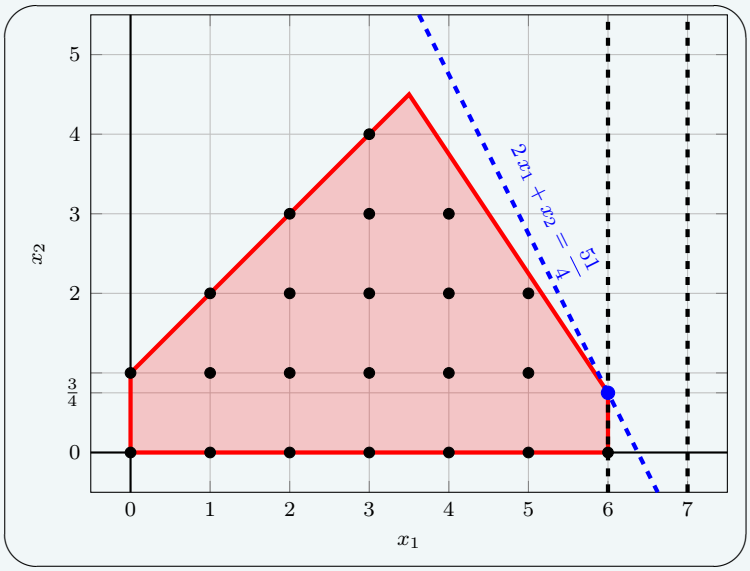


Figure 7.80: Branching on x_1: feasible regions of LP_2 ($x_1 \leq 6$) and LP_3 ($x_1 \geq 7$) in red, integer points (black) and optimal solutions (blue).

Therefore, strong branching selects x_1, the only fractional variable, with score $\frac{1}{4}$.

Exercise 7.22 — Strong branching ($P10$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 3$ constraints:

$$\begin{array}{rrcrcr} \max & 5\,x_1 & + & 2\,x_2 & & \\ & -x_1 & + & 3\,x_2 & \leq & 12 \\ & x_1 & + & 2\,x_2 & \leq & 12 \\ & 3\,x_1 & - & 2\,x_2 & \leq & 8 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

Questions

1. Solve graphically the LP relaxation LP_1 of the ILP problem.

2. Determine the branching variable with the strong branching branching-variable selection strategy (in case of ties, choose the variable with the smallest index). Solve graphically the LP problems of the branching nodes.

Solution

For a fractional variable x_j of $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$, the two branching nodes are LP_2 (with $x_j \leq \lfloor \tilde{x}_j \rfloor$) and LP_3 (with $x_j \geq \lfloor \tilde{x}_j \rfloor + 1$), and

$$\delta_1^d(x_j) = z(\mathrm{LP}_1) - z(\mathrm{LP}_2), \qquad \delta_1^u(x_j) = z(\mathrm{LP}_1) - z(\mathrm{LP}_3),$$

with $\delta = +\infty$ for an infeasible node. Strong branching selects the variable with the largest score $\min\{\delta_1^d(x_j), \delta_1^u(x_j)\}$.

1. For the LP_1 problem, graphically we have:

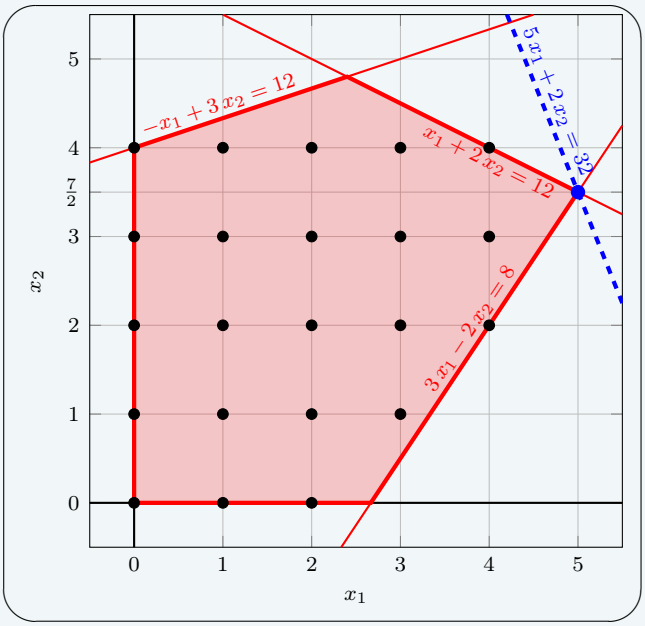


Figure 7.81: Feasible region of LP_1 (red), integer points (black) and optimal solution of LP_1 (blue) with the objective function line through it.

An optimal solution $\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$ is:

$$\tilde{x}_1 = 5, \quad \tilde{x}_2 = \frac{7}{2}$$

the optimal value is:

$$z(\mathrm{LP}_1) = 32$$

Only x_2 is fractional: it is the only candidate branching variable, and strong branching computes its score anyway.

2. Strong branching. Graphically, we have:

   (a) Branching on x_2:

- With $x_2 \leq 3$, an optimal solution of LP_2 is

$$\tilde{x}_1 = \frac{14}{3}, \quad \tilde{x}_2 = 3 \quad \text{and} \quad z(\text{LP}_2) = \frac{88}{3},$$

accordingly

$$\delta_1^d(x_2) = 32 - \frac{88}{3} = \frac{8}{3}.$$

- With $x_2 \geq 4$, an optimal solution of LP_3 is

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = 4 \quad \text{and} \quad z(\text{LP}_3) = 28,$$

accordingly

$$\delta_1^u(x_2) = 32 - 28 = 4.$$

For the variable x_2 we have:

$$\min\left\{\delta_1^d(x_2),\, \delta_1^u(x_2)\right\} = \min\left\{\frac{8}{3},\, 4\right\} = \frac{8}{3}.$$

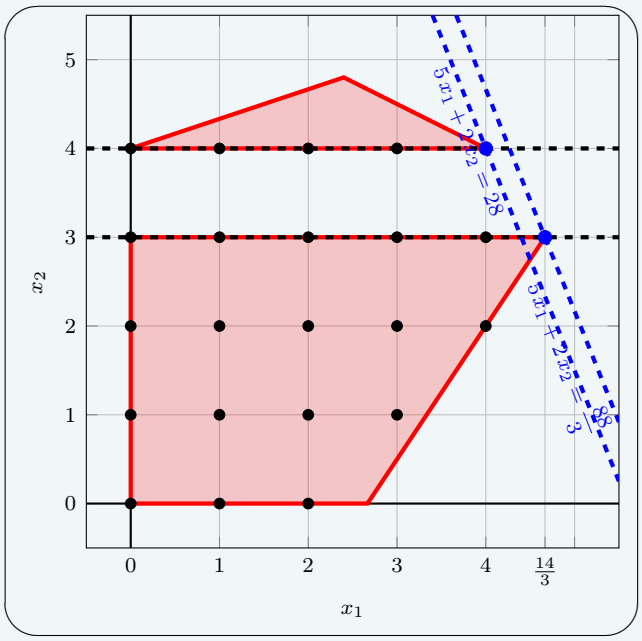


Figure 7.82: Branching on x_2: feasible regions of LP_2 ($x_2 \leq 3$) and LP_3 ($x_2 \geq 4$) in red, integer points (black) and optimal solutions (blue).

Therefore, strong branching selects x_2, the only fractional variable, with score $\frac{8}{3}$.

Branch-and-cut

Exercise 8.1 — Cut-and-branch ($P1$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 1$ constraint:

$$\begin{array}{lrcrcl} \max & x_1 & + & 8\,x_2 & & \\ & 3\,x_1 & + & 6\,x_2 & \leq & 16 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

and the following optimal simplex dictionary of its LP relaxation in standard form:

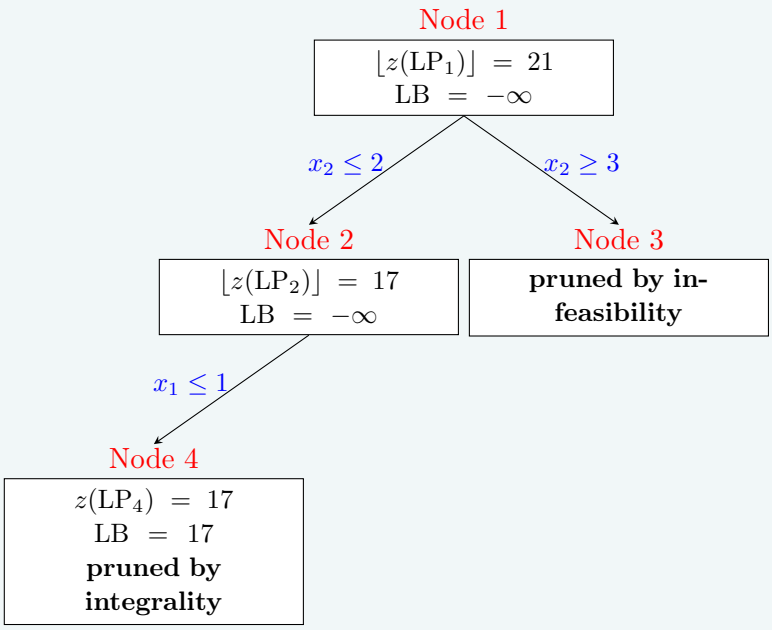
$$\begin{cases} x_2 = \dfrac{8}{3} - \dfrac{1}{2}\,x_1 - \dfrac{1}{6}\,x_3 \\ z = \dfrac{64}{3} - 3\,x_1 - \dfrac{4}{3}\,x_3 \end{cases}$$

Questions

1. Execute the branch-and-bound algorithm using the smallest-index rule as branching-variable selection strategy, the LIFO rule as branching-node selection strategy, the down-branching node solved first, and LB $= -\infty$ at the start (no heuristic algorithm). Give the branch-and-bound tree and the number of LP problems solved.
2. Execute one round of a cut-and-branch algorithm at the root node: generate the Gomory fractional cut of the row with the largest fractional right-hand side (smallest index of the basic variable in case of a tie, the objective-function row last), written in the original variables and made primitive before its slack variable is added.
3. Re-optimize the LP problem of the root node with the cut, using the dual simplex algorithm (leaving variable: the basic variable with negative value and smallest index; ties in the ratio test broken by the smallest index). Show the effect of the cut graphically.
4. Execute the branch-and-bound algorithm on the strengthened problem, with the same rules, and compare the number of LP problems solved with the first question.

Solution

1. The LP_1 problem has the optimal solution $(0, \frac{8}{3})$ with $z(\mathrm{LP}_1) = \dfrac{64}{3}$. The branch-and-bound algorithm solves 4 LP problems:



Node	Parent	Branching	$\tilde{\boldsymbol{x}}(\mathrm{LP}_k)$	$z(\mathrm{LP}_k)$	LB	Outcome
1	–	–	$(0, \frac{8}{3})$	$\frac{64}{3}$	$-\infty$	branching on x_2
2	1	$x_2 \leq 2$	$(\frac{4}{3}, 2)$	$\frac{52}{3}$	$-\infty$	branching on x_1
3	1	$x_2 \geq 3$	–	–	$-\infty$	pruned by infeasibility
4	2	$x_1 \leq 1$	$(1, 2)$	17	17	pruned by integrality

The optimal solution is $\tilde{\boldsymbol{x}}(\mathrm{ILP}) = (1, 2)$, with $z(\mathrm{ILP}) = 17$.

2. The rows of the optimal dictionary with a fractional right-hand side are the candidates:

Row	fractional part of the right-hand side
x_2	$\frac{2}{3}$
z	$\frac{1}{3}$

The largest fractional part is $\frac{2}{3}$, in the row of x_2.

In equation form, the row of x_2 reads

$$x_2 + \tfrac{1}{2}\, x_1 + \tfrac{1}{6}\, x_3 = \frac{8}{3},$$

and the Gomory fractional cut generated by this row is

$$\underbrace{\left(\tfrac{1}{2} - \left\lfloor \tfrac{1}{2} \right\rfloor\right)}_{=\frac{1}{2}} x_1 + \underbrace{\left(\tfrac{1}{6} - \left\lfloor \tfrac{1}{6} \right\rfloor\right)}_{=\frac{1}{6}} x_3 \;\geq\; \underbrace{\tfrac{8}{3} - \left\lfloor \tfrac{8}{3} \right\rfloor}_{=\frac{2}{3}},$$

that is,

$$\frac{1}{2}\, x_1 + \frac{1}{6}\, x_3 \geq \frac{2}{3}.$$

Substituting the slack variables with their expressions in the original variables, the cut becomes

$$x_2 \leq 2.$$

The coefficients have greatest common divisor 1: the cut $x_2 \leq 2$ is primitive.

At the root node no branching constraint exists, so the cut is **global**. It cuts off $(0, \frac{8}{3})$: $0 \cdot 0 + 1 \cdot \frac{8}{3} = \frac{8}{3} > 2$.

3. The cut $x_2 \leq 2$ is added with its slack variable $x_4 = 2 - x_2 \geq 0$. Expressing it with the non-basic variables and adding it to the optimal dictionary we get a dictionary that is dual feasible but not primal feasible:

$$\begin{cases} x_2 &= \dfrac{8}{3} - \dfrac{1}{2}\, x_1 - \dfrac{1}{6}\, x_3 \\ x_4 &= -\dfrac{2}{3} + \dfrac{1}{2}\, x_1 + \dfrac{1}{6}\, x_3 \\ z &= \dfrac{64}{3} - 3\, x_1 - \dfrac{4}{3}\, x_3 \end{cases}$$

If we increase the value of π_4, we can get a better dual solution. We have:

$$\begin{cases} \pi_1 &= 3 - \dfrac{1}{2}\, \pi_4 \\ \pi_3 &= \dfrac{4}{3} - \dfrac{1}{6}\, \pi_4 \end{cases} \qquad \begin{cases} \underbrace{3 - \dfrac{1}{2}\, \pi_4 \geq 0}_{\pi_1 \geq 0} \\ \underbrace{\dfrac{4}{3} - \dfrac{1}{6}\, \pi_4 \geq 0}_{\pi_3 \geq 0} \end{cases} \qquad \begin{cases} \pi_4 &\leq 6 \\ \pi_4 &\leq 8 \end{cases}$$

By increasing the value of π_4 to $\min\{6, 8\} = 6$, the value of π_1 decreases to 0. Then we have that:

> π_4 becomes a dual basic variable and accordingly x_4 becomes a primal non-basic variable.
>
> π_1 becomes a dual non-basic variable and accordingly x_1 becomes a primal basic variable.

We have:

$$x_4 = -\frac{2}{3} + \frac{1}{2}\, x_1 + \frac{1}{6}\, x_3 \quad \Rightarrow \quad x_1 = \frac{4}{3} - \frac{1}{3}\, x_3 + 2\, x_4$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_2 = \frac{8}{3} - \frac{1}{2}\left(\frac{4}{3} - \frac{1}{3}x_3 + 2\,x_4\right) - \frac{1}{6}\,x_3 = 2 - x_4$$

$$z = \frac{64}{3} - 3\left(\frac{4}{3} - \frac{1}{3}x_3 + 2\,x_4\right) - \frac{4}{3}\,x_3 = \frac{52}{3} - \frac{1}{3}x_3 - 6\,x_4$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = \frac{4}{3} - \frac{1}{3}\,x_3 + 2\,x_4 \\ x_2 = 2 - x_4 \\ z = \frac{52}{3} - \frac{1}{3}\,x_3 - 6\,x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = \frac{4}{3}, \quad \tilde{x}_2 = 2, \qquad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{of value} \quad \frac{52}{3}$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \qquad \tilde{\pi}_3 = \frac{1}{3}, \quad \tilde{\pi}_4 = 6 \quad \text{of value} \quad \frac{52}{3}$$

The new optimal solution of LP_1 is $(\frac{4}{3}, 2)$, with $z(\mathrm{LP}_1) = \dfrac{52}{3}$ and the upper bound 17. Graphically, we have:

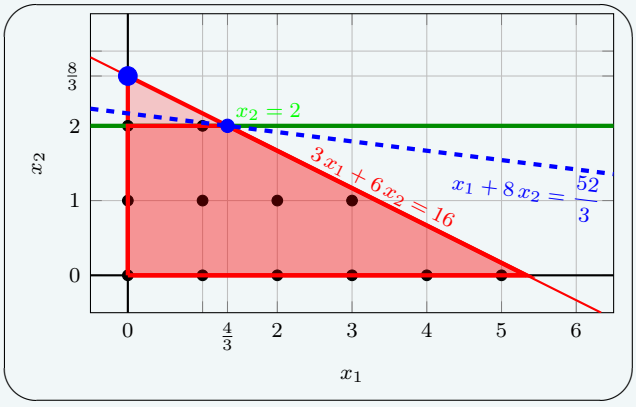
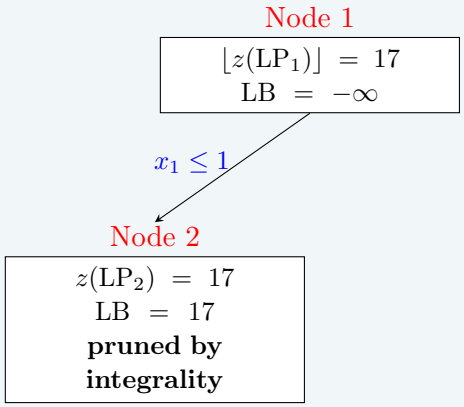


Figure 8.1: The root LP problem with the Gomory cut (green): the old optimal solution (blue circle) is cut off, and the new optimal solution is blue, with its objective line (dashed).

4. Every node inherits the cut. The branch-and-bound algorithm now solves 2 LP problems:



Node	Parent	Branching	$\tilde{\boldsymbol{x}}(\mathrm{LP}_k)$	$z(\mathrm{LP}_k)$	LB	Outcome
1	–	–	$(\frac{4}{3}, 2)$	$\frac{52}{3}$	$-\infty$	branching on x_1
2	1	$x_1 \leq 1$	$(1, 2)$	17	17	pruned by integrality

Branch-and-bound: 4 LP problems; cut-and-branch with one Gomory cut: 2 LP problems. The cut lowers the root bound and **reduces** the tree. Both find $\tilde{\boldsymbol{x}}(\mathrm{ILP}) = (1, 2)$, $z(\mathrm{ILP}) = 17$.

Exercise 8.2 — Cut-and-branch ($P23$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 3$ constraints:

$$\begin{array}{lrcrcl} \max & x_1 & + & 4\,x_2 & & \\ & 3\,x_1 & + & 8\,x_2 & \leq & 24 \\ & -x_1 & & & \leq & -1 \\ & & & -x_2 & \leq & -1 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

The constraints $x_1 \geq 1$ and $x_2 \geq 1$ are written in canonical form as $-x_1 \leq -1$ and $-x_2 \leq -1$.
and the following optimal simplex dictionary of its LP relaxation in standard form:

$$\begin{cases} x_1 = 1 + x_4 \\ x_2 = \frac{21}{8} - \frac{1}{8}\,x_3 - \frac{3}{8}\,x_4 \\ x_5 = \frac{13}{8} - \frac{1}{8}\,x_3 - \frac{3}{8}\,x_4 \\ z = \frac{23}{2} - \frac{1}{2}\,x_3 - \frac{1}{2}\,x_4 \end{cases}$$

Questions

1. Execute the branch-and-bound algorithm using the smallest-index rule as branching-variable selection strategy, the LIFO rule as branching-node selection strategy, the down-branching node solved first, and LB $= -\infty$ at the start (no heuristic algorithm). Give the branch-and-bound tree and the number of LP problems solved.

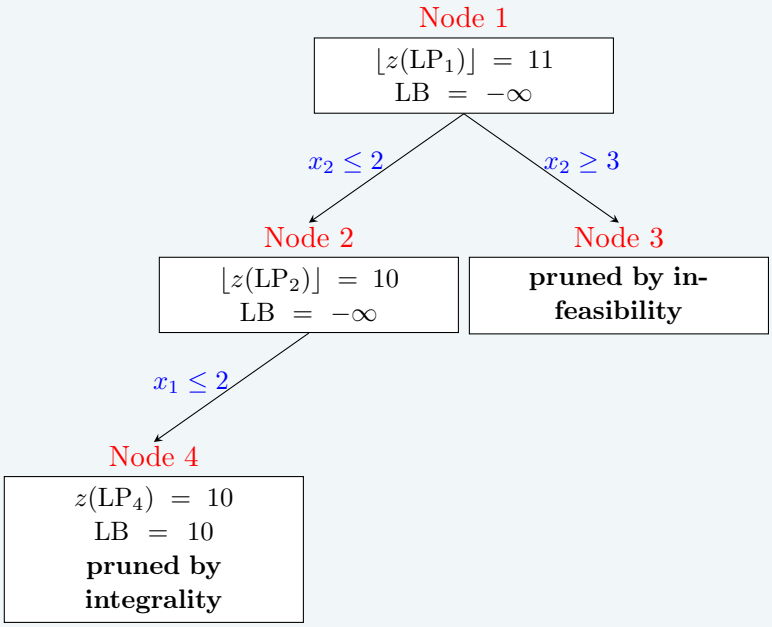
2. Execute one round of a cut-and-branch algorithm at the root node: generate the Gomory fractional cut of the row with the largest fractional right-hand side (smallest index of the basic variable in case of a tie, the objective-function row last), written in the original variables and made primitive before its slack variable is added.

3. Re-optimize the LP problem of the root node with the cut, using the dual simplex algorithm (leaving variable: the basic variable with negative value and smallest index; ties in the ratio test broken by the smallest index). Show the effect of the cut graphically.

4. Execute the branch-and-bound algorithm on the strengthened problem, with the same rules, and compare the number of LP problems solved with the first question.

Solution

1. The LP_1 problem has the optimal solution $(1, \frac{21}{8})$ with $z(\mathrm{LP}_1) = \dfrac{23}{2}$. The branch-and-bound algorithm solves 4 LP problems:



Node	Parent	Branching	$\tilde{\boldsymbol{x}}(\mathrm{LP}_k)$	$z(\mathrm{LP}_k)$	LB	Outcome
1	–	–	$(1, \frac{21}{8})$	$\frac{23}{2}$	$-\infty$	branching on x_2
2	1	$x_2 \leq 2$	$(\frac{8}{3}, 2)$	$\frac{32}{3}$	$-\infty$	branching on x_1
3	1	$x_2 \geq 3$	–	–	$-\infty$	pruned by infeasibility
4	2	$x_1 \leq 2$	$(2, 2)$	10	10	pruned by integrality

The optimal solution is $\tilde{\boldsymbol{x}}(\mathrm{ILP}) = (2, 2)$, with $z(\mathrm{ILP}) = 10$.

2. The rows of the optimal dictionary with a fractional right-hand side are the candidates:

Row	fractional part of the right-hand side
x_2	$\frac{5}{8}$
x_5	$\frac{5}{8}$
z	$\frac{1}{2}$

The largest fractional part is $\frac{5}{8}$, in the row of x_2.

In equation form, the row of x_2 reads

$$x_2 + \tfrac{1}{8}\, x_3 + \tfrac{3}{8}\, x_4 = \frac{21}{8},$$

and the Gomory fractional cut generated by this row is

$$\underbrace{\left(\tfrac{1}{8} - \left\lfloor \tfrac{1}{8} \right\rfloor\right)}_{=\frac{1}{8}} x_3 + \underbrace{\left(\tfrac{3}{8} - \left\lfloor \tfrac{3}{8} \right\rfloor\right)}_{=\frac{3}{8}} x_4 \;\geq\; \underbrace{\tfrac{21}{8} - \left\lfloor \tfrac{21}{8} \right\rfloor}_{=\frac{5}{8}},$$

that is,

$$\frac{1}{8}\, x_3 + \frac{3}{8}\, x_4 \geq \frac{5}{8}.$$

Substituting the slack variables with their expressions in the original variables, the cut becomes

$$x_2 \leq 2.$$

The coefficients have greatest common divisor 1: the cut $x_2 \leq 2$ is primitive.

At the root node no branching constraint exists, so the cut is **global**. It cuts off $(1, \frac{21}{8})$: $0 \cdot 1 + 1 \cdot \frac{21}{8} = \frac{21}{8} > 2$.

3. The cut $x_2 \leq 2$ is added with its slack variable $x_6 = 2 - x_2 \geq 0$. Expressing it with the non-basic variables and adding it to the optimal dictionary we get a dictionary that is dual feasible but not primal feasible:

$$\begin{cases} x_1 = 1 + x_4 \\ x_2 = \frac{21}{8} - \frac{1}{8} x_3 - \frac{3}{8} x_4 \\ x_5 = \frac{13}{8} - \frac{1}{8} x_3 - \frac{3}{8} x_4 \\ x_6 = -\frac{5}{8} + \frac{1}{8} x_3 + \frac{3}{8} x_4 \\ z = \frac{23}{2} - \frac{1}{2} x_3 - \frac{1}{2} x_4 \end{cases}$$

If we increase the value of π_6, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 = \frac{1}{2} - \frac{1}{8}\pi_6 \\ \pi_4 = \frac{1}{2} - \frac{3}{8}\pi_6 \end{cases} \qquad \begin{cases} \underbrace{\frac{1}{2} - \frac{1}{8}\pi_6 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{1}{2} - \frac{3}{8}\pi_6 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_6 \leq 4 \\ \pi_6 \leq \frac{4}{3} \end{cases}$$

By increasing the value of π_6 to $\min\left\{4, \frac{4}{3}\right\} = \frac{4}{3}$, the value of π_4 decreases to 0. Then we have that:

π_6 becomes a dual basic variable and accordingly x_6 becomes a primal non-basic variable.

π_4 becomes a dual non-basic variable and accordingly x_4 becomes a primal basic variable.

We have:

$$x_6 = -\frac{5}{8} + \frac{1}{8}x_3 + \frac{3}{8}x_4 \quad \Rightarrow \quad x_4 = \frac{5}{3} - \frac{1}{3}x_3 + \frac{8}{3}x_6$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = 1 + \left(\frac{5}{3} - \frac{1}{3}x_3 + \frac{8}{3}x_6\right) = \frac{8}{3} - \frac{1}{3}x_3 + \frac{8}{3}x_6$$

$$x_2 = \frac{21}{8} - \frac{1}{8}x_3 - \frac{3}{8}\left(\frac{5}{3} - \frac{1}{3}x_3 + \frac{8}{3}x_6\right) = 2 - x_6$$

$$x_5 = \frac{13}{8} - \frac{1}{8}x_3 - \frac{3}{8}\left(\frac{5}{3} - \frac{1}{3}x_3 + \frac{8}{3}x_6\right) = 1 - x_6$$

$$z = \frac{23}{2} - \frac{1}{2}x_3 - \frac{1}{2}\left(\frac{5}{3} - \frac{1}{3}x_3 + \frac{8}{3}x_6\right) = \frac{32}{3} - \frac{1}{3}x_3 - \frac{4}{3}x_6$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = \frac{8}{3} - \frac{1}{3}x_3 + \frac{8}{3}x_6 \\ x_2 = 2 - x_6 \\ x_4 = \frac{5}{3} - \frac{1}{3}x_3 + \frac{8}{3}x_6 \\ x_5 = 1 - x_6 \\ z = \frac{32}{3} - \frac{1}{3}x_3 - \frac{4}{3}x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4, 5\}$ and $\mathcal{N} = \{3, 6\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = \frac{8}{3}, \quad \tilde{x}_2 = 2, \quad \tilde{x}_4 = \frac{5}{3}, \quad \tilde{x}_5 = 1, \qquad \tilde{x}_3 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad \frac{32}{3}$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_4 = 0, \quad \tilde{\pi}_5 = 0, \qquad \tilde{\pi}_3 = \frac{1}{3}, \quad \tilde{\pi}_6 = \frac{4}{3} \quad \text{of value} \quad \frac{32}{3}$$

The new optimal solution of LP_1 is $(\frac{8}{3}, 2)$, with $z(\text{LP}_1) = \frac{32}{3}$ and the upper bound 10. Graphically, we have:

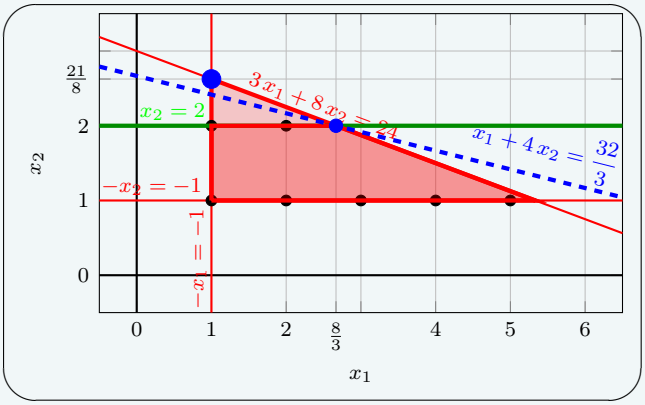
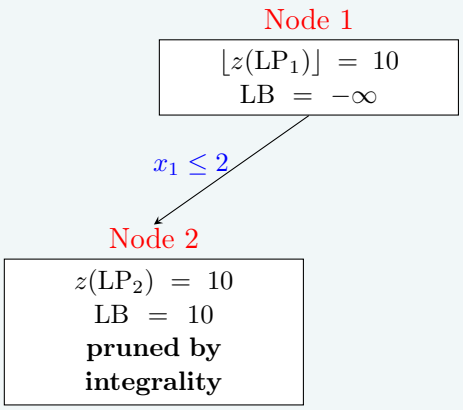


Figure 8.2: The root LP problem with the Gomory cut (green): the old optimal solution (blue circle) is cut off, and the new optimal solution is blue, with its objective line (dashed).

4. Every node inherits the cut. The branch-and-bound algorithm now solves 2 LP problems:



Node	Parent	Branching	$\tilde{\boldsymbol{x}}(\text{LP}_k)$	$z(\text{LP}_k)$	LB	Outcome
1	–	–	$(\frac{8}{3}, 2)$	$\frac{32}{3}$	$-\infty$	branching on x_1
2	1	$x_1 \leq 2$	$(2, 2)$	10	10	pruned by integrality

Branch-and-bound: 4 LP problems; cut-and-branch with one Gomory cut: 2 LP problems. The cut lowers the root bound and **reduces** the tree. Both find $\tilde{\boldsymbol{x}}(\text{ILP}) = (2, 2)$, $z(\text{ILP}) = 10$.

Exercise 8.3 — Cut-and-branch ($P24$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 4$ constraints:

$$\begin{array}{lrcrcl} \max & 4\,x_1 & + & 3\,x_2 & & \\ & 7\,x_1 & + & 3\,x_2 & \leq & 24 \\ & -x_1 & + & x_2 & \leq & 3 \\ & -x_1 & & & \leq & -1 \\ & & & -x_2 & \leq & -2 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

The constraints $x_1 \geq 1$ and $x_2 \geq 2$ are written in canonical form as $-x_1 \leq -1$ and $-x_2 \leq -2$.
and the following optimal simplex dictionary of its LP relaxation in standard form:

$$\begin{cases} x_1 = \frac{3}{2} - \frac{1}{10}\,x_3 + \frac{3}{10}\,x_4 \\ x_2 = \frac{9}{2} - \frac{1}{10}\,x_3 - \frac{7}{10}\,x_4 \\ x_5 = \frac{1}{2} - \frac{1}{10}\,x_3 + \frac{3}{10}\,x_4 \\ x_6 = \frac{5}{2} - \frac{1}{10}\,x_3 - \frac{7}{10}\,x_4 \\ z = \frac{39}{2} - \frac{7}{10}\,x_3 - \frac{9}{10}\,x_4 \end{cases}$$

Questions

1. Execute the branch-and-bound algorithm using the smallest-index rule as branching-variable selection strategy, the LIFO rule as branching-node selection strategy, the down-branching node solved first, and LB $= -\infty$ at the start (no heuristic algorithm). Give the branch-and-bound tree and the number of LP problems solved.

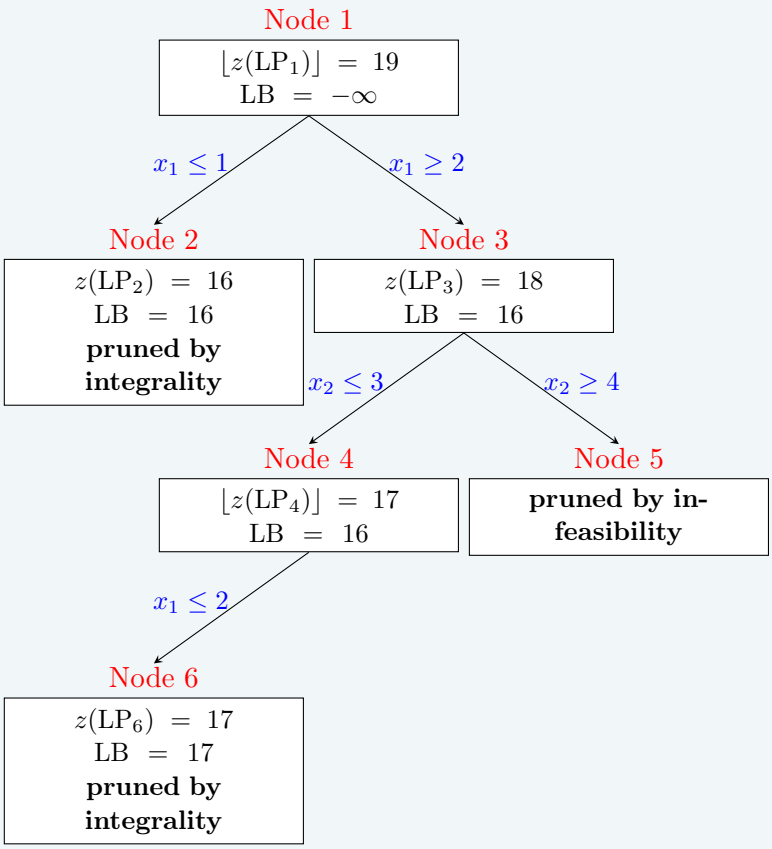
2. Execute one round of a cut-and-branch algorithm at the root node: generate the Gomory fractional cut of the row with the largest fractional right-hand side (smallest index of the basic variable in case of a tie, the objective-function row last), written in the original variables and made primitive before its slack variable is added.

3. Re-optimize the LP problem of the root node with the cut, using the dual simplex algorithm (leaving variable: the basic variable with negative value and smallest index; ties in the ratio test broken by the smallest index). Show the effect of the cut graphically.

4. Execute the branch-and-bound algorithm on the strengthened problem, with the same rules, and compare the number of LP problems solved with the first question.

Solution

1. The LP_1 problem has the optimal solution $(\frac{3}{2}, \frac{9}{2})$ with $z(\text{LP}_1) = \frac{39}{2}$. The branch-and-bound algorithm solves 6 LP problems:



Node	Parent	Branching	$\tilde{\boldsymbol{x}}(\text{LP}_k)$	$z(\text{LP}_k)$	LB	Outcome
1	–	–	$(\frac{3}{2}, \frac{9}{2})$	$\frac{39}{2}$	$-\infty$	branching on x_1
2	1	$x_1 \leq 1$	$(1, 4)$	16	16	pruned by integrality
3	1	$x_1 \geq 2$	$(2, \frac{10}{3})$	18	16	branching on x_2
4	3	$x_2 \leq 3$	$(\frac{15}{7}, 3)$	$\frac{123}{7}$	16	branching on x_1
5	3	$x_2 \geq 4$	–	–	16	pruned by infeasibility
6	4	$x_1 \leq 2$	$(2, 3)$	17	17	pruned by integrality

The optimal solution is $\tilde{\boldsymbol{x}}(\text{ILP}) = (2, 3)$, with $z(\text{ILP}) = 17$.

2. The rows of the optimal dictionary with a fractional right-hand side are the candidates:

Row	fractional part of the right-hand side
x_1	$\frac{1}{2}$
x_2	$\frac{1}{2}$
x_5	$\frac{1}{2}$
x_6	$\frac{1}{2}$
z	$\frac{1}{2}$

The largest fractional part is $\frac{1}{2}$, in the row of x_1.

In equation form, the row of x_1 reads

$$x_1 + \tfrac{1}{10}\, x_3 - \tfrac{3}{10}\, x_4 = \frac{3}{2},$$

and the Gomory fractional cut generated by this row is

$$\underbrace{\left(\tfrac{1}{10} - \left\lfloor \tfrac{1}{10} \right\rfloor\right)}_{=\frac{1}{10}} x_3 + \underbrace{\left(-\tfrac{3}{10} - \left\lfloor -\tfrac{3}{10} \right\rfloor\right)}_{=\frac{7}{10}} x_4 \;\geq\; \underbrace{\tfrac{3}{2} - \left\lfloor \tfrac{3}{2} \right\rfloor}_{=\frac{1}{2}},$$

that is,

$$\frac{1}{10}\, x_3 + \frac{7}{10}\, x_4 \geq \frac{1}{2}.$$

Substituting the slack variables with their expressions in the original variables, the cut becomes

$$x_2 \leq 4.$$

The coefficients have greatest common divisor 1: the cut $x_2 \leq 4$ is primitive.

At the root node no branching constraint exists, so the cut is **global**. It cuts off $(\frac{3}{2}, \frac{9}{2})$: $0 \cdot \frac{3}{2} + 1 \cdot \frac{9}{2} = \frac{9}{2} > 4$.

3. The cut $x_2 \leq 4$ is added with its slack variable $x_7 = 4 - x_2 \geq 0$. Expressing it with the non-basic variables and adding it to the optimal dictionary we get a dictionary that is dual feasible but not primal feasible:

$$\begin{cases} x_1 = \frac{3}{2} - \frac{1}{10}\, x_3 + \frac{3}{10}\, x_4 \\ x_2 = \frac{9}{2} - \frac{1}{10}\, x_3 - \frac{7}{10}\, x_4 \\ x_5 = \frac{1}{2} - \frac{1}{10}\, x_3 + \frac{3}{10}\, x_4 \\ x_6 = \frac{5}{2} - \frac{1}{10}\, x_3 - \frac{7}{10}\, x_4 \\ x_7 = -\frac{1}{2} + \frac{1}{10}\, x_3 + \frac{7}{10}\, x_4 \\ z = \frac{39}{2} - \frac{7}{10}\, x_3 - \frac{9}{10}\, x_4 \end{cases}$$

If we increase the value of π_7, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 &= \frac{7}{10} - \frac{1}{10}\pi_7 \\ \pi_4 &= \frac{9}{10} - \frac{7}{10}\pi_7 \end{cases} \qquad \begin{cases} \underbrace{\frac{7}{10} - \frac{1}{10}\pi_7 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{9}{10} - \frac{7}{10}\pi_7 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_7 &\leq 7 \\ \pi_7 &\leq \frac{9}{7} \end{cases}$$

By increasing the value of π_7 to $\min\left\{7, \frac{9}{7}\right\} = \frac{9}{7}$, the value of π_4 decreases to 0. Then we have that:

π_7 becomes a dual basic variable and accordingly x_7 becomes a primal non-basic variable.

π_4 becomes a dual non-basic variable and accordingly x_4 becomes a primal basic variable.

We have:

$$x_7 = -\frac{1}{2} + \frac{1}{10}x_3 + \frac{7}{10}x_4 \quad \Rightarrow \quad x_4 = \frac{5}{7} - \frac{1}{7}x_3 + \frac{10}{7}x_7$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{3}{2} - \frac{1}{10}x_3 + \frac{3}{10}\left(\frac{5}{7} - \frac{1}{7}x_3 + \frac{10}{7}x_7\right) = \frac{12}{7} - \frac{1}{7}x_3 + \frac{3}{7}x_7$$

$$x_2 = \frac{9}{2} - \frac{1}{10}x_3 - \frac{7}{10}\left(\frac{5}{7} - \frac{1}{7}x_3 + \frac{10}{7}x_7\right) = 4 - x_7$$

$$x_5 = \frac{1}{2} - \frac{1}{10}x_3 + \frac{3}{10}\left(\frac{5}{7} - \frac{1}{7}x_3 + \frac{10}{7}x_7\right) = \frac{5}{7} - \frac{1}{7}x_3 + \frac{3}{7}x_7$$

$$x_6 = \frac{5}{2} - \frac{1}{10}x_3 - \frac{7}{10}\left(\frac{5}{7} - \frac{1}{7}x_3 + \frac{10}{7}x_7\right) = 2 - x_7$$

$$z = \frac{39}{2} - \frac{7}{10}x_3 - \frac{9}{10}\left(\frac{5}{7} - \frac{1}{7}x_3 + \frac{10}{7}x_7\right) = \frac{132}{7} - \frac{4}{7}x_3 - \frac{9}{7}x_7$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 &= \frac{12}{7} - \frac{1}{7}x_3 + \frac{3}{7}x_7 \\ x_2 &= 4 - x_7 \\ x_4 &= \frac{5}{7} - \frac{1}{7}x_3 + \frac{10}{7}x_7 \\ x_5 &= \frac{5}{7} - \frac{1}{7}x_3 + \frac{3}{7}x_7 \\ x_6 &= 2 - x_7 \\ z &= \frac{132}{7} - \frac{4}{7}x_3 - \frac{9}{7}x_7 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4, 5, 6\}$ and $\mathcal{N} = \{3, 7\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = \frac{12}{7},\ \tilde{x}_2 = 4,\ \tilde{x}_4 = \frac{5}{7},\ \tilde{x}_5 = \frac{5}{7},\ \tilde{x}_6 = 2,\quad \tilde{x}_3 = 0,\ \tilde{x}_7 = 0 \quad \text{of value} \quad \frac{132}{7}$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0,\ \tilde{\pi}_2 = 0,\ \tilde{\pi}_4 = 0,\ \tilde{\pi}_5 = 0,\ \tilde{\pi}_6 = 0,\quad \tilde{\pi}_3 = \frac{4}{7},\ \tilde{\pi}_7 = \frac{9}{7} \quad \text{of value} \quad \frac{132}{7}$$

The new optimal solution of LP_1 is $(\frac{12}{7}, 4)$, with $z(\text{LP}_1) = \frac{132}{7}$ and the upper bound 18. Graphically, we have:

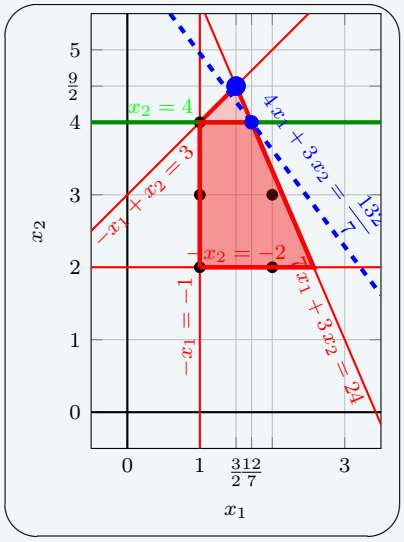
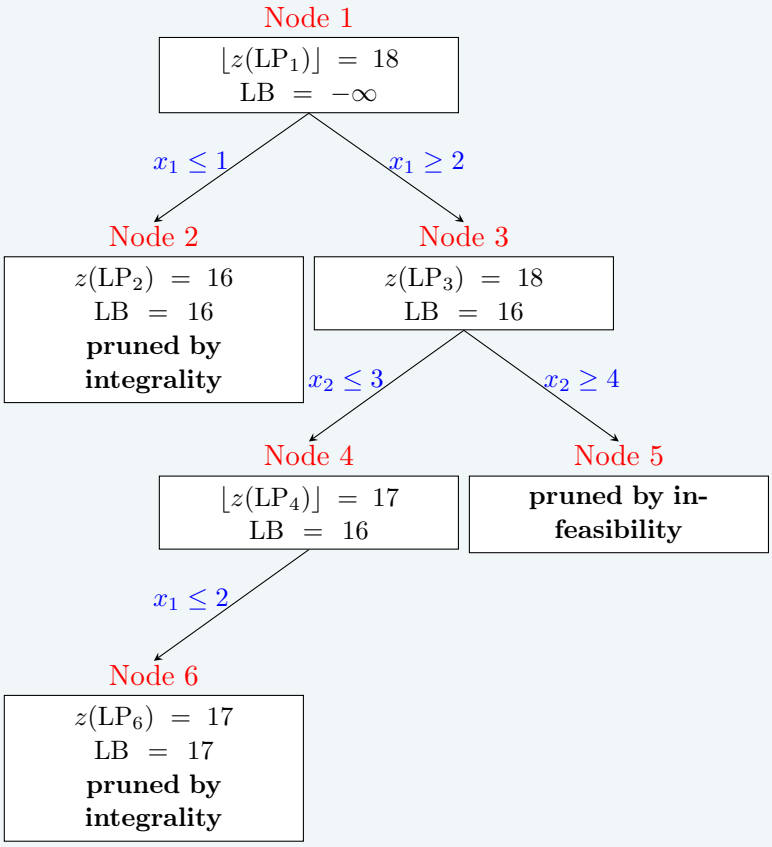


Figure 8.3: The root LP problem with the Gomory cut (green): the old optimal solution (blue circle) is cut off, and the new optimal solution is blue, with its objective line (dashed).

4. Every node inherits the cut. The branch-and-bound algorithm now solves 6 LP problems:



Node	Parent	Branching	$\tilde{\boldsymbol{x}}(\text{LP}_k)$	$z(\text{LP}_k)$	LB	Outcome
1	–	–	$(\frac{12}{7}, 4)$	$\frac{132}{7}$	$-\infty$	branching on x_1
2	1	$x_1 \leq 1$	$(1, 4)$	16	16	pruned by integrality
3	1	$x_1 \geq 2$	$(2, \frac{10}{3})$	18	16	branching on x_2
4	3	$x_2 \leq 3$	$(\frac{15}{7}, 3)$	$\frac{123}{7}$	16	branching on x_1
5	3	$x_2 \geq 4$	–	–	16	pruned by infeasibility
6	4	$x_1 \leq 2$	$(2, 3)$	17	17	pruned by integrality

Branch-and-bound: 6 LP problems; cut-and-branch with one Gomory cut: 6 LP problems. The cut lowers the root bound but does **not** reduce the tree: the quality of the cut matters. Both find $\tilde{\boldsymbol{x}}(\text{ILP}) = (2, 3)$, $z(\text{ILP}) = 17$.

Exercise 8.4 — Global and local cuts

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{array}{lrcrcl} \max & 2\,x_1 & + & 3\,x_2 & & \\ & 5\,x_1 & + & x_2 & \leq & 22 \\ & 3\,x_1 & + & 9\,x_2 & \leq & 35 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

The branch-and-bound algorithm (the smallest-index rule as branching-variable selection strategy, the LIFO rule as branching-node selection strategy, the down-branching node solved first, and $\mathrm{LB} = -\infty$ at the start (no heuristic algorithm)) branches at the root on x_1. Node 2 is the down-branching node, with the branching constraint $x_1 \leq 3$ and its slack variable $x_5 = 3 - x_1$. The optimal dictionary of LP_2 is:

$$\begin{cases} x_1 = 3 - x_5 \\ x_2 = \frac{26}{9} - \frac{1}{9}\,x_4 + \frac{1}{3}\,x_5 \\ x_3 = \frac{37}{9} + \frac{1}{9}\,x_4 + \frac{14}{3}\,x_5 \\ z = \frac{44}{3} - \frac{1}{3}\,x_4 - x_5 \end{cases}$$

Questions

1. Generate at Node 2 the Gomory fractional cut of the row with the largest fractional right-hand side (smallest index of the basic variable in case of a tie, the objective-function row last), written in the original variables and made primitive before its slack variable is added.
2. From its derivation, is the cut a global cut or a local cut of Node 2?
3. Check by enumeration whether the cut is satisfied by the integer feasible solutions of Node 2 and by all the integer feasible solutions of the ILP problem.
4. What would happen if the cut were stored in the global pool and used at every node?

Solution

1. An optimal solution of LP_2 is $(3, \frac{26}{9})$, with $z(\mathrm{LP}_2) = \frac{44}{3}$. The candidate rows are:

Row	fractional part of the right-hand side
x_2	$\frac{8}{9}$
x_3	$\frac{1}{9}$
z	$\frac{2}{3}$

The largest fractional part is $\frac{8}{9}$, in the row of x_2.

In equation form, the row of x_2 reads

$$x_2 + \tfrac{1}{9}\, x_4 - \tfrac{1}{3}\, x_5 = \frac{26}{9},$$

and the Gomory fractional cut generated by this row is

$$\underbrace{\left(\tfrac{1}{9} - \left\lfloor \tfrac{1}{9} \right\rfloor\right)}_{=\frac{1}{9}} x_4 + \underbrace{\left(-\tfrac{1}{3} - \left\lfloor -\tfrac{1}{3} \right\rfloor\right)}_{=\frac{2}{3}} x_5 \;\geq\; \underbrace{\tfrac{26}{9} - \left\lfloor \tfrac{26}{9} \right\rfloor}_{=\frac{8}{9}},$$

that is,

$$\frac{1}{9}\, x_4 + \frac{2}{3}\, x_5 \geq \frac{8}{9}.$$

Substituting the slack variables with their expressions in the original variables, the cut becomes

$$x_1 + x_2 \leq 5.$$

The coefficients have greatest common divisor 1: the cut $x_1 + x_2 \leq 5$ is primitive.

2. The row used contains the slack variable x_5 of the branching constraint $x_1 \leq 3$ with the non-zero coefficient $\frac{2}{3}$ in the cut: its derivation uses the branching constraint, so the algorithm treats it as a **local cut** of Node 2, valid only in the subtree of Node 2.

3. The feasible solutions of Node 2 are the 15 points of F with $x_1 \leq 3$: all satisfy $x_1 + x_2 \leq 5$ (the largest value of $x_1 + x_2$ among them is 5). Among all the 18 feasible solutions, instead, $(4, 2)$ violates the cut. Graphically, we have:

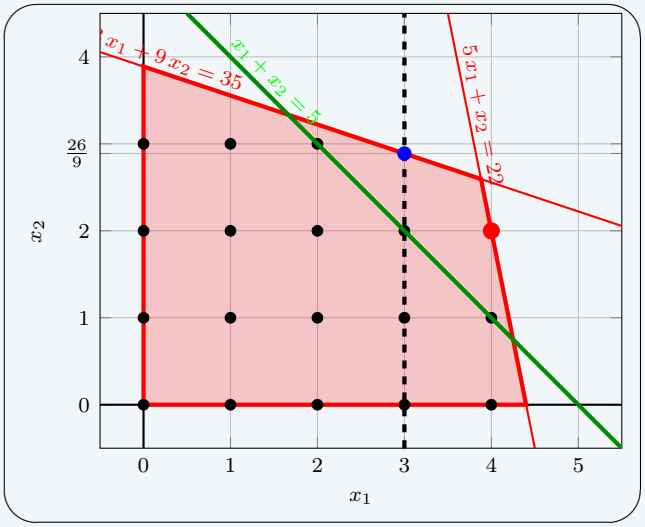


Figure 8.4: The LP relaxation (red), the branching constraint $x_1 = 3$ (dashed), the cut $x_1 + x_2 = 5$ (green), the optimal solution of LP_2 (blue) and the feasible solutions that violate the cut (red squares).

The cut is locally valid at Node 2 but **not** globally valid.

4. The point $(4, 2)$ is the unique optimal solution of the ILP problem, with value 14. With the cut in the global pool it would be removed also from Node 3 ($x_1 \geq 4$), and the algorithm would return $(2, 3)$, of value 13.

The Gomory cut $x_1 + x_2 \leq 5$ of Node 2 is **local**: it must stay in the subtree of Node 2. Used globally it would cut off the optimal solution $(4, 2)$ and the algorithm would return a solution of value 13 instead of 14.

Exercise 8.5 — Rounds of cuts ($P5$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{array}{lrcrcl} \max & 2\,x_1 & + & 3\,x_2 & & \\ & 6\,x_1 & + & 21\,x_2 & \leq & 77 \\ & 8\,x_1 & + & 3\,x_2 & \leq & 36 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

and the following optimal simplex dictionary of its LP relaxation in standard form:

$$\begin{cases} x_1 = \frac{7}{2} + \frac{1}{50}\,x_3 - \frac{7}{50}\,x_4 \\ x_2 = \frac{8}{3} - \frac{4}{75}\,x_3 + \frac{1}{25}\,x_4 \\ z = 15 - \frac{3}{25}\,x_3 - \frac{4}{25}\,x_4 \end{cases}$$

Questions

1. Generate the first Gomory cut of the root node: the Gomory fractional cut of the row with the largest fractional right-hand side (smallest index of the basic variable in case of a tie, the objective-function row last), written in the original variables and made primitive before its slack variable is added.

2. A cut-and-branch algorithm executes R rounds of Gomory cuts at the root (one cut per round, with the same rule), and then the branch-and-bound algorithm with the smallest-index rule as branching-variable selection strategy, the LIFO rule as branching-node selection strategy, the down-branching node solved first, and $LB = -\infty$ at the start (no heuristic algorithm). The table below reports, for each R, the last cut added, the root LP problem and the number of LP problems solved. Verify the line $R = 1$ and comment on the table.

Rounds	last cut added	$\tilde{\boldsymbol{x}}(\mathrm{LP}_1)$	$z(\mathrm{LP}_1)$	UB at the root	LP problems solved
0	–	$(\frac{7}{2}, \frac{8}{3})$	15	15	11
1	$2\,x_1 + x_2 \leq 9$	$(\frac{28}{9}, \frac{25}{9})$	$\frac{131}{9}$	14	8
2	$x_1 + x_2 \leq 5$	$(\frac{28}{15}, \frac{47}{15})$	$\frac{197}{15}$	13	3
3	$3\,x_1 + 10\,x_2 \leq 36$	$(2, 3)$	13	13	1

3. How many rounds are needed to solve the problem at the root node? Which cuts would you add in practice?

Solution

1. The candidate rows are:

Row	fractional part of the right-hand side
x_1	$\frac{1}{2}$
x_2	$\frac{2}{3}$

The largest fractional part is $\frac{2}{3}$, in the row of x_2.

In equation form, the row of x_2 reads

$$x_2 + \tfrac{4}{75}\,x_3 - \tfrac{1}{25}\,x_4 = \frac{8}{3},$$

and the Gomory fractional cut generated by this row is

$$\underbrace{\left(\tfrac{4}{75} - \left\lfloor \tfrac{4}{75} \right\rfloor\right)}_{=\frac{4}{75}} x_3 + \underbrace{\left(-\tfrac{1}{25} - \left\lfloor -\tfrac{1}{25} \right\rfloor\right)}_{=\frac{24}{25}} x_4 \;\geq\; \underbrace{\tfrac{8}{3} - \left\lfloor \tfrac{8}{3} \right\rfloor}_{=\frac{2}{3}},$$

that is,

$$\frac{4}{75}\,x_3 + \frac{24}{25}\,x_4 \geq \frac{2}{3}.$$

Substituting the slack variables with their expressions in the original variables, the cut becomes

$$8\,x_1 + 4\,x_2 \leq 38.$$

Dividing by the greatest common divisor $g = 4$ of the coefficients and rounding down the right-hand side we get the primitive cut

$$2\,x_1 + x_2 \leq 9.$$

2. The cut $2\,x_1 + x_2 \leq 9$ is added with its slack variable $x_5 = 9 - 2\,x_1 - x_2 \geq 0$. Expressing it with the non-basic variables and adding it to the optimal dictionary we

get a dictionary that is dual feasible but not primal feasible:

$$\begin{cases} x_1 = \frac{7}{2} + \frac{1}{50}x_3 - \frac{7}{50}x_4 \\ x_2 = \frac{8}{3} - \frac{4}{75}x_3 + \frac{1}{25}x_4 \\ x_5 = -\frac{2}{3} + \frac{1}{75}x_3 + \frac{6}{25}x_4 \\ z = 15 - \frac{3}{25}x_3 - \frac{4}{25}x_4 \end{cases}$$

If we increase the value of π_5, we can get a better dual solution. We have:

$$\begin{cases} \pi_3 = \frac{3}{25} - \frac{1}{75}\pi_5 \\ \pi_4 = \frac{4}{25} - \frac{6}{25}\pi_5 \end{cases} \qquad \begin{cases} \underbrace{\frac{3}{25} - \frac{1}{75}\pi_5 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{\frac{4}{25} - \frac{6}{25}\pi_5 \geq 0}_{\pi_4 \geq 0} \end{cases} \qquad \begin{cases} \pi_5 \leq 9 \\ \pi_5 \leq \frac{2}{3} \end{cases}$$

By increasing the value of π_5 to $\min\left\{9, \frac{2}{3}\right\} = \frac{2}{3}$, the value of π_4 decreases to 0. Then we have that:

> π_5 becomes a dual basic variable and accordingly x_5 becomes a primal non-basic variable.
>
> π_4 becomes a dual non-basic variable and accordingly x_4 becomes a primal basic variable.

We have:

$$x_5 = -\frac{2}{3} + \frac{1}{75}x_3 + \frac{6}{25}x_4 \quad \Rightarrow \quad x_4 = \frac{25}{9} - \frac{1}{18}x_3 + \frac{25}{6}x_5$$

Substituting into the other equations of the primal simplex dictionary, we get:

$$x_1 = \frac{7}{2} + \frac{1}{50}x_3 - \frac{7}{50}\left(\frac{25}{9} - \frac{1}{18}x_3 + \frac{25}{6}x_5\right) = \frac{28}{9} + \frac{1}{36}x_3 - \frac{7}{12}x_5$$

$$x_2 = \frac{8}{3} - \frac{4}{75}x_3 + \frac{1}{25}\left(\frac{25}{9} - \frac{1}{18}x_3 + \frac{25}{6}x_5\right) = \frac{25}{9} - \frac{1}{18}x_3 + \frac{1}{6}x_5$$

$$z = 15 - \frac{3}{25}x_3 - \frac{4}{25}\left(\frac{25}{9} - \frac{1}{18}x_3 + \frac{25}{6}x_5\right) = \frac{131}{9} - \frac{1}{9}x_3 - \frac{2}{3}x_5$$

Accordingly, we now have the following updated primal simplex dictionary:

$$\begin{cases} x_1 = \frac{28}{9} + \frac{1}{36}\, x_3 - \frac{7}{12}\, x_5 \\ x_2 = \frac{25}{9} - \frac{1}{18}\, x_3 + \frac{1}{6}\, x_5 \\ x_4 = \frac{25}{9} - \frac{1}{18}\, x_3 + \frac{25}{6}\, x_5 \\ z = \frac{131}{9} - \frac{1}{9}\, x_3 - \frac{2}{3}\, x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4\}$ and $\mathcal{N} = \{3, 5\}$ and

- the basic primal optimal solution is:

$$\tilde{x}_1 = \frac{28}{9}, \quad \tilde{x}_2 = \frac{25}{9}, \quad \tilde{x}_4 = \frac{25}{9}, \qquad \tilde{x}_3 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad \frac{131}{9}$$

- the basic dual optimal solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_4 = 0, \qquad \tilde{\pi}_3 = \frac{1}{9}, \quad \tilde{\pi}_5 = \frac{2}{3} \quad \text{of value} \quad \frac{131}{9}$$

The root LP problem with the cut has the optimal solution $(\frac{28}{9}, \frac{25}{9})$ and $z(\text{LP}_1) = \frac{131}{9}$, as in the line $R = 1$. The number of LP problems decreases from 11 to 8, 3 and 1: each round lowers $z(\text{LP}_1)$ a little, and the tree shrinks.

3. After 3 rounds the optimal solution of the root LP problem is integer and the tree reduces to the root node. The first round alone barely changes the bound; in practice a cut-and-branch algorithm stops cutting when the decrease of $z(\text{LP}_1)$ becomes small (tailing off), since each round makes every node LP problem larger.

With $0, 1, 2, 3$ rounds of Gomory cuts the algorithm solves $11, 8, 3, 1$ LP problems: on $P5$ the cuts pay off, and 3 rounds solve the problem at the root, $z(\text{ILP}) = 13$.