



SAPIENZA
UNIVERSITÀ DI ROMA

Integer Linear Programming

Exercises 1 — Models, bounds and formulations

Worked exercises on chapters 1–5, with complete solutions. Each exercise cites the catalogue problem it uses.

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Online version and downloadable material:
fabiofurini.github.io/integer-linear-programming/
github.com/fabiofurini/integer-linear-programming

Academic year 2026–2027

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Part I

Models and bounds



Integer linear programming

Exercise 1.1 — Integer linear programming (P7)

Consider the following input data of an ILP problem with $n = 2$ variables and $m = 2$ constraints:

$$\mathbf{c} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$$

Questions

1. Write the ILP problem in canonical maximization form.
2. Draw the feasible region of the ILP problem and list all its feasible solutions.
3. Find an optimal solution graphically, and check it by complete enumeration.

Solution

1. The ILP problem in canonical maximization form with $n = 2$ non-negative integer variables and $m = 2$ less-than-or-equal-to linear constraints reads as follows:

$$\begin{aligned} \max \quad & 3x_1 + x_2 \\ & x_1 + 4x_2 \leq 12 \\ & 2x_1 + x_2 \leq 5 \\ & x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

2. Both constraints have non-negative coefficients, so every feasible solution satisfies $x_1 \leq \lfloor \frac{5}{2} \rfloor = 2$ (constraint 2). For each value of x_1 the constraints give the integer values of x_2 :

x_1	bounds on x_2	feasible points
0	$x_2 \leq 3, \quad x_2 \leq 5$	$(0,0), (0,1), (0,2), (0,3)$
1	$x_2 \leq \frac{11}{4}, \quad x_2 \leq 3$	$(1,0), (1,1), (1,2)$
2	$x_2 \leq \frac{5}{2}, \quad x_2 \leq 1$	$(2,0), (2,1)$

The feasible region F is made of the 9 black points of the figure, the integer points of the polyhedron P (red):

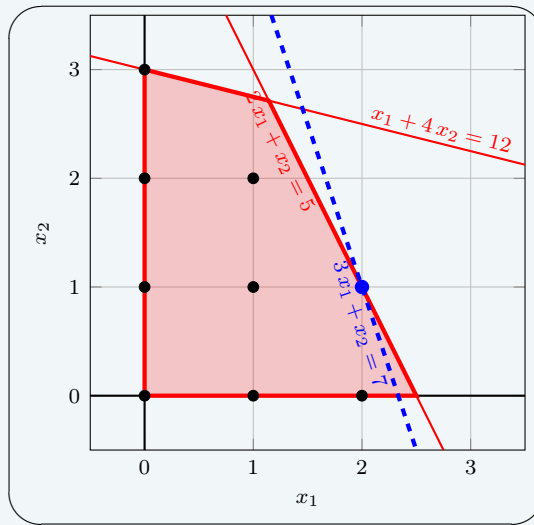


Figure 1.1: Feasible region of the ILP problem: the integer points (black) inside the polyhedron of the constraints (red). The optimal solution $(2,1)$ is blue, with the objective line $3x_1 + x_2 = 7$ (dashed).

- The gradient of the objective function is $\nabla f = (3, 1)$. Moving the level line $3x_1 + x_2 = z$ in the direction of ∇f , the last point of F reached is $(2,1)$, on the dashed line $3x_1 + x_2 = 7$ of the figure; no other point of F lies on this line.

Complete enumeration of the 9 feasible solutions confirms it (the optimum in bold):

x_1	x_2	z	x_1	x_2	z	x_1	x_2	z
0	0	0	0	3	3	1	2	5
0	1	1	1	0	3	2	0	6
0	2	2	1	1	4	2	1	7

The ILP problem has the **unique** optimal solution $\tilde{x}(\text{ILP}) = (2,1)$, with optimal value $z(\text{ILP}) = 7$.

Exercise 1.2 — Canonical maximization form (P13)

Consider the following ILP problem with $n = 2$ integer variables:

$$\begin{aligned} \min \quad & x_1 + 3x_2 \\ & x_1 + 2x_2 \geq 6 \\ & -x_1 + 4x_2 \geq 6 \\ & 2x_1 - 2x_2 \leq 3 \\ & x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

Questions

1. Write the ILP problem in canonical maximization form, and give its data $\mathbf{c}$, $\mathbf{A}$ and $\mathbf{b}$.
2. Draw the feasible region. Is it a finite set?
3. Explain why the ILP problem has an optimal solution, and find it graphically.

Solution

1. The minimization of $x_1 + 3x_2$ is the maximization of $-x_1 - 3x_2$, with $\min\{x_1 + 3x_2\} = -\max\{-x_1 - 3x_2\}$. Each constraint $\geq$ is multiplied by -1 :

$$x_1 + 2x_2 \geq 6 \iff -x_1 - 2x_2 \leq -6, \quad -x_1 + 4x_2 \geq 6 \iff x_1 - 4x_2 \leq -6.$$

The ILP problem in canonical maximization form (problem P13 of the catalogue) is

$$\begin{aligned} \max \quad & -x_1 - 3x_2 \\ & -x_1 - 2x_2 \leq -6 \\ & x_1 - 4x_2 \leq -6 \\ & 2x_1 - 2x_2 \leq 3 \\ & x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

with data

$$\mathbf{c} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} -1 & -2 \\ 1 & -4 \\ 2 & -2 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -6 \\ -6 \\ 3 \end{pmatrix}$$

2. The polyhedron of the constraints is drawn in red inside the box $0 \leq x_1, x_2 \leq 5$; it continues upwards without limit.

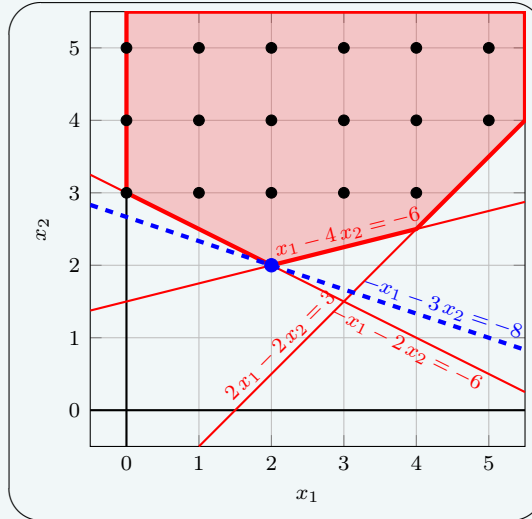


Figure 1.2: Feasible region of $P13$ inside the box (red), its integer points (black) and the optimal solution $(2, 2)$ (blue), with the objective line $-x_1 - 3x_2 = -8$ (dashed).

The feasible region is **not finite**: for every integer $k \geq 3$ the point $(0, k)$ satisfies $-2k \leq -6$, $-4k \leq -6$ and $-2k \leq 3$, so it belongs to F .

3. All the costs of the maximization problem are negative and $x_1, x_2 \geq 0$, so $-x_1 - 3x_2 \leq 0$ on F : the objective function is bounded from above. The feasible point $(0, 3)$ has value -9 , and only the finitely many points with $x_1 + 3x_2 < 9$ can do better: an optimal solution exists.

The gradient is $\nabla f = (-1, -3)$: the level line moves towards the origin. The last point of F reached is $(2, 2)$, on the dashed line $-x_1 - 3x_2 = -8$. The row $x_2 = 1$ contains no feasible point, because $x_1 + 2 \geq 6$ requires $x_1 \geq 4$ and $-x_1 + 4 \geq 6$ requires $x_1 \leq -2$.

The ILP problem has the **unique** optimal solution $\tilde{x}(\text{ILP}) = (2, 2)$, with optimal value $z(\text{ILP}) = -8$ in canonical maximization form, that is, a minimum value $x_1 + 3x_2 = 8$ of the original problem.

Exercise 1.3 — Complete enumeration ($P43$)

Consider the following input data of an ILP problem with $n = 3$ variables and $m = 2$ constraints:

$$c = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Questions

1. Write the ILP problem in canonical maximization form.

2. Derive an upper bound on each variable from the constraints, and list all the feasible solutions.
3. Find an optimal solution by complete enumeration. Is it unique?

Solution

1. The ILP problem in canonical maximization form with $n = 3$ non-negative integer variables and $m = 2$ less-than-or-equal-to linear constraints reads as follows:

$$\begin{array}{rclcl}
 \max & x_1 & + & 2x_2 & + & 7x_3 \\
 & x_1 & & & + & 2x_3 & \leq & 2 \\
 & & & x_2 & + & 2x_3 & \leq & 2 \\
 & x_1, & & x_2, & & x_3 & \in & \mathbb{Z}_{\geq 0}
 \end{array}$$

2. All the coefficients are non-negative, so each variable is at most the right-hand side divided by its coefficient, rounded down:

$$x_1 \leq \lfloor 2/1 \rfloor = 2, \quad x_2 \leq \lfloor 2/1 \rfloor = 2, \quad x_3 \leq \min \{ \lfloor 2/2 \rfloor, \lfloor 2/2 \rfloor \} = 1.$$

- If $x_3 = 1$, the constraints become $x_1 \leq 0$ and $x_2 \leq 0$: the only point is $(0, 0, 1)$.
- If $x_3 = 0$, the constraints become $x_1 \leq 2$ and $x_2 \leq 2$: the 9 points $(x_1, x_2, 0)$ with $x_1, x_2 \in \{0, 1, 2\}$.

The feasible region has 10 points.

3. The values of the objective function $x_1 + 2x_2 + 7x_3$ are (the optimum in bold):

x_1	x_2	x_3	z	x_1	x_2	x_3	z
0	0	0	0	1	2	0	5
0	1	0	2	2	0	0	2
0	2	0	4	2	1	0	4
1	0	0	1	2	2	0	6
1	1	0	3	0	0	1	7

The best point with $x_3 = 0$ is $(2, 2, 0)$, of value 6: it is worse than $(0, 0, 1)$, of value 7.

Complete enumeration gives the **unique** optimal solution $\tilde{\mathbf{x}}(\text{ILP}) = (0, 0, 1)$, with optimal value $z(\text{ILP}) = 7$.

Exercise 1.4 — Possible outcomes: an infeasible ILP problem

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 4$ constraints:

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ & 2x_1 - 2x_2 \leq 1 \\ & -2x_1 + 2x_2 \leq 1 \\ & -2x_1 - 2x_2 \leq -1 \\ & 2x_1 + 2x_2 \leq 3 \\ & x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

Questions

1. Draw the polyhedron P of the constraints, and decide whether the LP problem with the same data (with $x_1, x_2 \geq 0$ instead of $x_1, x_2 \in \mathbb{Z}_{\geq 0}$) is feasible.
2. Decide the outcome of the ILP problem.
3. Find an optimal solution of the LP problem with the same data.

Solution

1. The four constraints are $-\frac{1}{2} \leq x_1 - x_2 \leq \frac{1}{2}$ and $\frac{1}{2} \leq x_1 + x_2 \leq \frac{3}{2}$: the polyhedron P is the square with vertices $(0, \frac{1}{2})$, $(\frac{1}{2}, 0)$, $(\frac{1}{2}, 1)$, $(1, \frac{1}{2})$.

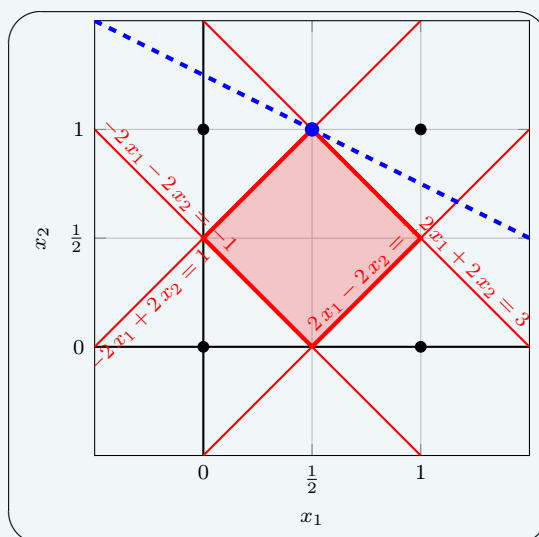


Figure 1.3: The polyhedron P (red) contains no integer point: the four integer points nearest to it (black) are outside. The optimal solution $(\frac{1}{2}, 1)$ of the LP problem is blue, with the objective line $x_1 + 2x_2 = \frac{5}{2}$ (dashed).

The point $(\frac{1}{2}, \frac{1}{2})$ satisfies all the constraints, so the LP problem is **feasible**.

2.
 - From the first two constraints, an integer point has $x_1 - x_2 \in \mathbb{Z}$ and $|x_1 - x_2| \leq \frac{1}{2}$, hence $x_1 = x_2$.
 - With $x_1 = x_2$ the last two constraints read $1 \leq 4x_1 \leq 3$, that is $\frac{1}{4} \leq x_1 \leq \frac{3}{4}$: no integer value of x_1 satisfies them.

The feasible region $F = P \cap \mathbb{Z}^2$ is empty: the ILP problem is **infeasible**, although the LP problem with the same data is feasible.

3. The polyhedron P is bounded and non-empty, so the LP problem has an optimal vertex. The values of $x_1 + 2x_2$ at the four vertices are

$$(0, \frac{1}{2}) \mapsto 1, \quad (\frac{1}{2}, 0) \mapsto \frac{1}{2}, \quad (\frac{1}{2}, 1) \mapsto \frac{5}{2}, \quad (1, \frac{1}{2}) \mapsto 2$$

so the optimal solution is $\tilde{\mathbf{x}}(\text{LP}) = (\frac{1}{2}, 1)$, with $z(\text{LP}) = \frac{5}{2}$ (see the figure).

Exercise 1.5 — Possible outcomes: an unbounded feasible region

Consider the constraints

$$-x_1 + x_2 \leq 2$$

$$x_1 - 2x_2 \leq 2$$

$$x_1, x_2 \in \mathbb{Z}_{\geq 0}$$

and the three ILP problems in canonical maximization form with these constraints and the objective functions

$$(a) \max x_1 + x_2, \quad (b) \max -2x_1 + x_2, \quad (c) \max x_1 - 3x_2.$$

Questions

1. Draw the feasible region. Is it a finite set?
2. Decide the outcome of each of the three ILP problems (a), (b) and (c), and give an optimal solution when it exists.

Solution

1. The feasible region is **not finite**: for every $k \in \mathbb{Z}_{\geq 0}$ the point $(2k, k)$ satisfies $-2k + k = -k \leq 2$ and $2k - 2k = 0 \leq 2$.

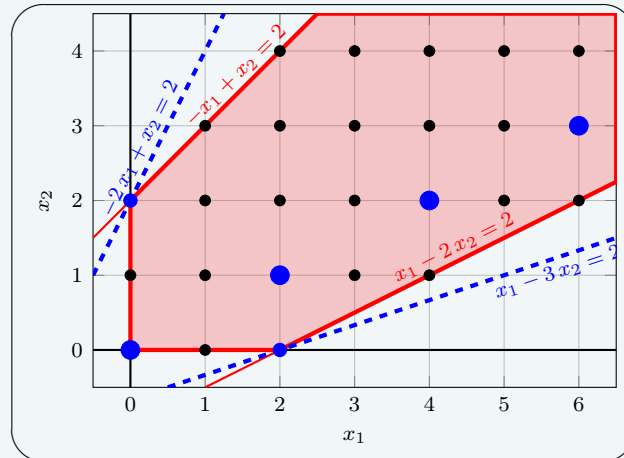


Figure 1.4: Feasible region inside the box $0 \leq x_1 \leq 6$, $0 \leq x_2 \leq 4$ (red), its integer points (black), the points $(2k, k)$ (blue circles), and the optimal solutions $(0, 2)$ of (b) and $(2, 0)$ of (c) (blue) with their objective lines (dashed).

2.
 - (a) The points $(2k, k)$ are feasible with value $3k$, which grows without limit: the ILP problem (a) is **unbounded**.
 - (b) For every feasible point, $-2x_1 + x_2 = (-x_1 + x_2) - x_1 \leq 2 - x_1 \leq 2$. The value 2 requires $x_1 = 0$ and $-x_1 + x_2 = 2$, that is, the point $(0, 2)$, which is feasible: the **unique** optimal solution is $(0, 2)$, with optimal value 2.
 - (c) For every feasible point, $x_1 - 3x_2 = (x_1 - 2x_2) - x_2 \leq 2 - x_2 \leq 2$. The value 2 requires $x_2 = 0$ and $x_1 - 2x_2 = 2$, that is, the point $(2, 0)$: the **unique** optimal solution is $(2, 0)$, with optimal value 2.

With the same unbounded feasible region, problem (a) is **unbounded**, while problems (b) and (c) have an **optimal solution**: $(0, 2)$ and $(2, 0)$, both of value 2.

Primal and dual bounds

Exercise 2.1 — Primal and dual bounds (P5)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ & 6x_1 + 21x_2 \leq 77 \\ & 8x_1 + 3x_2 \leq 36 \\ & x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

Questions

1. Apply the one-variable heuristic and the two-step heuristic, in both orders of the variables, and give the best primal bound LB.
2. Solve the LP relaxation graphically and give the dual bound UB it provides.
3. Compute the optimality gap. Is the best heuristic solution certified optimal?

Solution

1. The **one-variable heuristic** sets a single variable to its maximum value, with the other variables equal to 0. The **two-step heuristic** sets first one variable to its maximum value, and then the other one to its maximum value with the **residual right-hand sides**.

- Only x_1 ($x_2 = 0$): $\bar{x}_1 = \min\{\lfloor \frac{77}{6} \rfloor, \lfloor \frac{9}{2} \rfloor\} = 4$, so $\bar{\mathbf{x}} = (4, 0)$ and $\text{LB} = 2 \cdot 4 + 3 \cdot 0 = 8$.
- Only x_2 ($x_1 = 0$): $\bar{x}_2 = \min\{\lfloor \frac{11}{3} \rfloor, \lfloor 12 \rfloor\} = 3$, so $\bar{\mathbf{x}} = (0, 3)$ and $\text{LB} = 2 \cdot 0 + 3 \cdot 3 = 9$.
- x_1 first: $\bar{x}_1 = \min\{\lfloor \frac{77}{6} \rfloor, \lfloor \frac{9}{2} \rfloor\} = 4$, leaving the residual right-hand sides 53 and 4. Then $\bar{x}_2 = \min\{\lfloor \frac{53}{21} \rfloor, \lfloor \frac{4}{3} \rfloor\} = 1$, so $\bar{\mathbf{x}} = (4, 1)$ and $\text{LB} = 11$.
- x_2 first: $\bar{x}_2 = \min\{\lfloor \frac{11}{3} \rfloor, \lfloor 12 \rfloor\} = 3$, leaving the residual right-hand sides 14 and 27. Then $\bar{x}_1 = \min\{\lfloor \frac{7}{3} \rfloor, \lfloor \frac{27}{8} \rfloor\} = 2$, so $\bar{\mathbf{x}} = (2, 3)$ and $\text{LB} = 13$.

Heuristic	Order	$\bar{x}_1$	$\bar{x}_2$	LB
one-variable	x_1	4	0	8
one-variable	x_2	0	3	9
two-step	x_1 first	4	1	11
two-step	x_2 first	2	3	13

The best primal bound is $LB = 13$, given by $\bar{x} = (2, 3)$.

2. Graphically, we have:

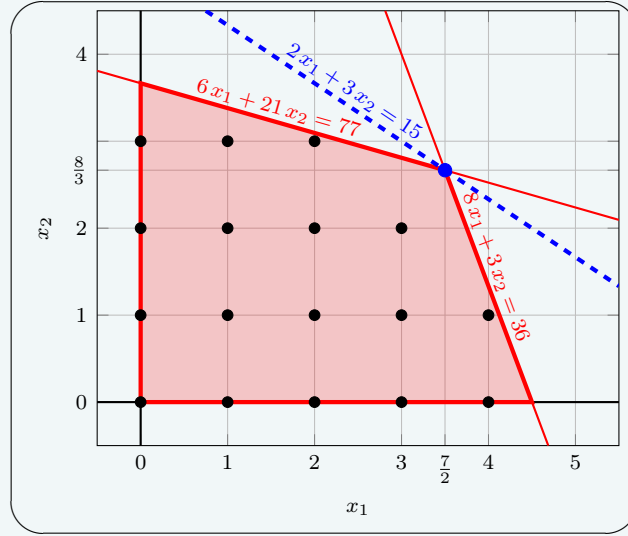


Figure 2.1: Feasible region of the LP relaxation (red), integer points (black) and optimal solution of the LP relaxation (blue), with the objective line through it (dashed).

The optimal vertex is the intersection of the lines of constraints 1 and 2:

$$\begin{cases} 6x_1 + 21x_2 = 77 \\ 8x_1 + 3x_2 = 36 \end{cases} \implies x_1 = \frac{7}{2}, \quad x_2 = \frac{8}{3}$$

The optimal value is $z(\text{LP}) = 2 \cdot \frac{7}{2} + 3 \cdot \frac{8}{3} = 15$. The data are integer, so the dual bound is

$$z(\text{ILP}) \leq z(\text{LP}) = 15$$

3. With $LB = 13$ and $UB = 15$ the optimality gap is

$$\frac{UB - LB}{|LB|} = \frac{15 - 13}{13} = \frac{2}{13} \approx 15.4\%$$

The gap is not zero: the bounds give only $13 \leq z(\text{ILP}) \leq 15$, and $\bar{x} = (2, 3)$ is **not certified** optimal.

$LB = 13$ from the two-step heuristic, $UB = 15$ from the LP relaxation, optimality gap $\frac{2}{13}$. Complete enumeration shows that $z(\text{ILP}) = 13$: the heuristic solution $(2, 3)$ is optimal, but the two bounds do not prove it.

Exercise 2.2 — Rounding and certificates (P4)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 5x_1 + 3x_2 \\ & 2x_1 + x_2 \leq 14 \\ & x_1 + 8x_2 \leq 32 \\ & x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

Questions

1. Solve the LP relaxation graphically and give the dual bound UB.
2. Round the optimal solution of the LP relaxation in the four possible ways (each component down or up). Which rounded points are feasible? Give the best primal bound LB obtained by rounding.
3. Compute the optimality gap.
4. Show that the point $(6, 2)$ is feasible and certify that it is an optimal solution.

Solution

1. Graphically, we have:

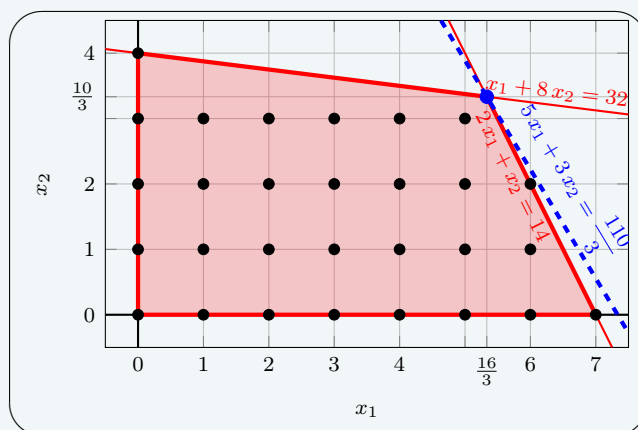


Figure 2.2: Feasible region of the LP relaxation (red), integer points (black) and optimal solution of the LP relaxation (blue), with the objective line through it (dashed).

The optimal vertex is the intersection of the lines of constraints 1 and 2:

$$\begin{cases} 2x_1 + x_2 = 14 \\ x_1 + 8x_2 = 32 \end{cases} \implies x_1 = \frac{16}{3}, \quad x_2 = \frac{10}{3}$$

The optimal value is $z(\text{LP}) = 5 \cdot \frac{16}{3} + 3 \cdot \frac{10}{3} = \frac{110}{3}$. The data are integer, so

$$z(\text{ILP}) \leq \lfloor z(\text{LP}) \rfloor = \left\lfloor \frac{110}{3} \right\rfloor = 36$$

2. The optimal solution of the LP relaxation is $(\frac{16}{3}, \frac{10}{3})$, and its rounding to the nearest integer point is $(5, 3)$. The four rounded points are:

Rounded point	feasible	violated constraint	value
$(5, 3)$	yes	–	34
$(5, 4)$	no	$1 \cdot 5 + 8 \cdot 4 = 37 > 32$	–
$(6, 3)$	no	$2 \cdot 6 + 1 \cdot 3 = 15 > 14$	–
$(6, 4)$	no	$2 \cdot 6 + 1 \cdot 4 = 16 > 14$	–

Only $(5, 3)$ is feasible, so rounding gives $\text{LB} = 5 \cdot 5 + 3 \cdot 3 = 34$.

3. With $\text{LB} = 34$ and $\text{UB} = 36$ the optimality gap is

$$\frac{\text{UB} - \text{LB}}{|\text{LB}|} = \frac{36 - 34}{34} = \frac{1}{17} \approx 5.9\%$$

4. The point $(6, 2)$ satisfies the constraints, $2 \cdot 6 + 1 \cdot 2 = 14 \leq 14$, $1 \cdot 6 + 8 \cdot 2 = 22 \leq 32$, and its value is $5 \cdot 6 + 3 \cdot 2 = 36$. Hence $\text{LB} = 36 = \text{UB}$: the optimality gap is zero.

The feasible solution $(6, 2)$ has value $36 = \lfloor z(\text{LP}) \rfloor$: it is **certified optimal**, $z(\text{ILP}) = 36$. Rounding the LP solution gives only $(5, 3)$, of value 34.

Exercise 2.3 — Rounding fails

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 4x_1 + 7x_2 \\ & 6x_1 + 10x_2 \leq 30 \\ & -6x_1 + 3x_2 \leq 4 \\ & x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

Questions

1. Solve the LP relaxation graphically and give the dual bound UB.
2. Round the optimal solution of the LP relaxation in the four possible ways. Is the

rounding down feasible? Is the rounding to the nearest integer point feasible? Give the best primal bound obtained by rounding.

- Find an optimal solution of the ILP problem graphically, and compare it with the solution of the LP relaxation.

Solution

- Graphically, we have:

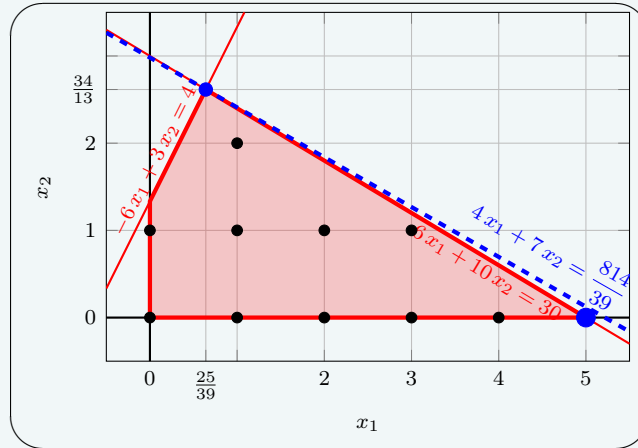


Figure 2.3: Feasible region of the LP relaxation (red), its integer points (black), the optimal solution of the LP relaxation (blue) with the objective line through it (dashed), and the optimal ILP solution (5,0) (blue circle).

The optimal vertex is the intersection of the lines of constraints 1 and 2:

$$\begin{cases} 6x_1 + 10x_2 = 30 \\ -6x_1 + 3x_2 = 4 \end{cases} \implies x_1 = \frac{25}{39}, \quad x_2 = \frac{34}{13}$$

with $z(\text{LP}) = 4 \cdot \frac{25}{39} + 7 \cdot \frac{34}{13} = \frac{814}{39} \approx 20.87$. The data are integer, so

$$z(\text{ILP}) \leq \lfloor z(\text{LP}) \rfloor = \left\lfloor \frac{814}{39} \right\rfloor = 20$$

- The four roundings of $(\frac{25}{39}, \frac{34}{13}) \approx (0.64, 2.62)$ are:

Rounded point	feasible	violated constraint	value
(0, 2)	no	$-6 \cdot 0 + 3 \cdot 2 = 6 > 4$	–
(0, 3)	no	$-6 \cdot 0 + 3 \cdot 3 = 9 > 4$	–
(1, 2)	yes	–	18
(1, 3)	no	$6 \cdot 1 + 10 \cdot 3 = 36 > 30$	–

The rounding down $(0, 2)$ and the rounding to the nearest point $(1, 3)$ are both **infeasible**. The only feasible rounding is $(1, 2)$, with $LB = 18$.

3. The gradient of the objective function is $\nabla f = (4, 7)$. The integer points near the LP optimum lie under the steep line of constraint 2, and the last point of F reached by the level line is $(5, 0)$, on the line $4x_1 + 7x_2 = 20$. Its value 20 equals $UB = 20$, so it is certified optimal.

The optimal ILP solution $\tilde{\mathbf{x}}(\text{ILP}) = (5, 0)$, with $z(\text{ILP}) = 20$, is far from $\tilde{\mathbf{x}}(\text{LP}) \approx (0.64, 2.62)$: **no rounding** finds it, while the bound $\lfloor z(\text{LP}) \rfloor = 20$ is exact.

Exercise 2.4 — Dual bounds ($P2$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & x_1 + 3x_2 \\ & -4x_1 + 10x_2 \leq 35 \\ & 3x_1 + x_2 \leq 12 \\ & x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

The optimal solution of its LP relaxation is $\tilde{\mathbf{x}}(\text{LP}) = (\frac{5}{2}, \frac{9}{2})$, with $z(\text{LP}) = 16$.

Questions

1. Multiply the constraints by the multipliers (π_1, π_2) of the table below. Which multipliers are valid, and which upper bound on $z(\text{ILP})$ does each valid choice give?

$$(0, 3), \quad (\frac{1}{2}, 1), \quad (\frac{3}{10}, 0), \quad (\frac{1}{4}, \frac{2}{3})$$

2. Write the dual problem $D(\text{LP})$ of the LP relaxation, and verify that $\tilde{\boldsymbol{\pi}} = (\frac{4}{17}, \frac{11}{17})$ is an optimal dual solution.
3. Derive upper bounds on the variables from the constraints. Do they improve the bound of the LP relaxation?

Solution

1. With multipliers $\pi_1, \pi_2 \geq 0$ the sum of the constraints is

$$(-4\pi_1 + 3\pi_2)x_1 + (10\pi_1 + \pi_2)x_2 \leq 35\pi_1 + 12\pi_2,$$

and it gives an upper bound when the coefficients are at least the costs: $-4\pi_1 + 3\pi_2 \geq 1$ and $10\pi_1 + \pi_2 \geq 3$.

(π_1, π_2)	coefficients of x_1, x_2	valid multipliers	upper bound $35\pi_1 + 12\pi_2$
$(0, 3)$	9, 3	yes	36
$(\frac{1}{2}, 1)$	1, 6	yes	$\frac{59}{2}$
$(\frac{3}{10}, 0)$	$-\frac{6}{5}, 3$	no: $-\frac{6}{5} < 1$	—
$(\frac{1}{4}, \frac{2}{3})$	$1, \frac{19}{6}$	yes	$\frac{67}{4}$

The best of these bounds is $\frac{67}{4}$, given by $(\frac{1}{4}, \frac{2}{3})$. The multipliers $(\frac{3}{10}, 0)$ are not valid: the first constraint has a negative coefficient of x_1 .

2. The dual problem D(LP) is

$$\begin{aligned}
 \min \quad & 35\pi_1 + 12\pi_2 \\
 & -4\pi_1 + 3\pi_2 \geq 1 \\
 & 10\pi_1 + \pi_2 \geq 3 \\
 & \pi_1, \pi_2 \in \mathbb{R}_{\geq 0}
 \end{aligned}$$

The point $(\frac{4}{17}, \frac{11}{17})$ is feasible, since $-4 \cdot \frac{4}{17} + 3 \cdot \frac{11}{17} = 1 \geq 1$ and $10 \cdot \frac{4}{17} + 1 \cdot \frac{11}{17} = 3 \geq 3$, and its value is $35 \cdot \frac{4}{17} + 12 \cdot \frac{11}{17} = 16 = z(\text{LP})$. By weak duality no dual feasible solution has a smaller value, so it is optimal.

3. Only constraint 2, $3x_1 + x_2 \leq 12$, has non-negative coefficients; constraint 1 has a negative coefficient and gives no bound. From constraint 2:

$$x_1 \leq \lfloor 12/3 \rfloor = 4, \quad x_2 \leq \lfloor 12/1 \rfloor = 12.$$

The optimal LP solution $(\frac{5}{2}, \frac{9}{2})$ satisfies both bounds, so it is not cut off and $z(\text{LP}) = 16$ does not change.

The best multipliers $\tilde{\pi} = (\frac{4}{17}, \frac{11}{17})$ give the bound $16 = z(\text{LP})$, which the variable upper bounds do not improve: $z(\text{ILP}) \leq 16$ (in fact $z(\text{ILP}) = 14$).

Exercise 2.5 — Bounds with three variables (P46)

Consider the following ILP problem in canonical maximization form with $n = 3$ integer variables and $m = 3$ constraints:

$$\begin{aligned} \max \quad & 5x_1 + 6x_2 + 10x_3 \\ & 9x_1 + 2x_2 + 2x_3 \leq 11 \\ & 2x_1 + 6x_2 + 6x_3 \leq 16 \\ & 4x_1 + 2x_2 + 7x_3 \leq 10 \\ & x_1, x_2, x_3 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

The optimal solution of its LP relaxation is $\tilde{\mathbf{x}}(\text{LP}) = (\frac{17}{25}, \frac{49}{25}, \frac{12}{25})$, with $z(\text{LP}) = \frac{499}{25}$.

Questions

1. Give the dual bound UB provided by the LP relaxation.
2. Round down the optimal solution of the LP relaxation. Is the rounded point feasible, and which primal bound does it give?
3. Apply the two-step heuristic (extended to three steps) with the variables in order of decreasing cost, x_3, x_2, x_1 .
4. Compute the optimality gap with the best primal bound.

Solution

1. The data are integer, so

$$z(\text{ILP}) \leq \lfloor z(\text{LP}) \rfloor = \left\lfloor \frac{499}{25} \right\rfloor = 19$$

2. Rounding down gives $(0, 1, 0)$, which is feasible: $9 \cdot 0 + 2 \cdot 1 + 2 \cdot 0 = 2 \leq 11$, $2 \cdot 0 + 6 \cdot 1 + 6 \cdot 0 = 6 \leq 16$, $4 \cdot 0 + 2 \cdot 1 + 7 \cdot 0 = 2 \leq 10$. Its value gives LB = $5 \cdot 0 + 6 \cdot 1 + 10 \cdot 0 = 6$, far from the upper bound.
3. Each variable, in the order x_3, x_2, x_1 , takes its maximum value with the residual right-hand sides:

(a) $\bar{x}_3 = \min\{\lfloor \frac{11}{2} \rfloor, \lfloor \frac{8}{3} \rfloor, \lfloor \frac{10}{7} \rfloor\} = 1$; the residual right-hand sides become 9, 10, 3.

(b) $\bar{x}_2 = \min\{\lfloor \frac{9}{2} \rfloor, \lfloor \frac{5}{3} \rfloor, \lfloor \frac{3}{2} \rfloor\} = 1$; the residual right-hand sides become 7, 4, 1.

(c) $\bar{x}_1 = \min\{\lfloor \frac{7}{9} \rfloor, \lfloor 2 \rfloor, \lfloor \frac{1}{4} \rfloor\} = 0$; the residual right-hand sides become 7, 4, 1.

The heuristic solution is $\bar{\mathbf{x}} = (0, 1, 1)$, with LB = $5 \cdot 0 + 6 \cdot 1 + 10 \cdot 1 = 16$.

4. With LB = 16 and UB = 19 the optimality gap is

$$\frac{\text{UB} - \text{LB}}{|\text{LB}|} = \frac{19 - 16}{16} = \frac{3}{16} \approx 18.8\%$$

$16 \leq z(\text{ILP}) \leq 19$. Complete enumeration shows that the heuristic solution $(0, 1, 1)$ is optimal, $z(\text{ILP}) = 16$, but the gap $\frac{3}{16}$ does not certify it.

Part II

Strengthening the formulation



Ideal formulations

Exercise 3.1 — Convex hull and ideal formulation (P8)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ & -x_1 + 2x_2 \leq 2 \\ & x_1 + x_2 \leq 3 \\ & x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

Questions

1. List the feasible solutions and compute the convex hull $\text{conv}(F)$ of the feasible region. Classify the feasible solutions as vertices, boundary points or interior points of the hull.
2. Describe $\text{conv}(F)$ with linear inequalities and write the corresponding ideal formulation.
3. Solve the LP relaxation of the given formulation and the LP problem over the ideal formulation, and compare the two upper bounds.

Solution

1. For $x_2 = 0$ the constraints give $x_1 \leq 3$, for $x_2 = 1$ they give $x_1 \leq 2$, and for $x_2 = 2$ they require $x_1 \geq 2$ and $x_1 \leq 1$. The feasible region has the 7 points

$$(0, 0), \quad (1, 0), \quad (2, 0), \quad (3, 0), \quad (0, 1), \quad (1, 1), \quad (2, 1).$$

Graphically, we have:

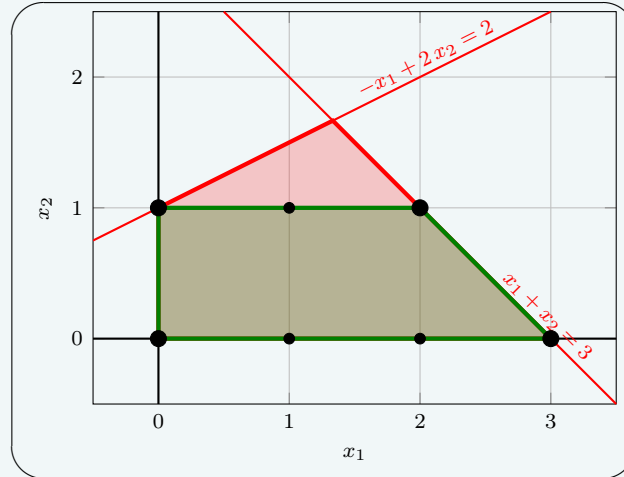


Figure 3.1: The polyhedron of the formulation (red) and the convex hull of the integer points (green): vertices of the hull as black squares, the other integer points as black dots.

The convex hull $\text{conv}(F)$ is the quadrilateral with

- the 4 **vertices** $(0,0)$, $(3,0)$, $(0,1)$, $(2,1)$;
- 3 points on the boundary, $(1,0)$, $(2,0)$, $(1,1)$;
- no point in the interior.

2. The edges of the hull lie on the lines

$$x_1 + x_2 = 3, \quad x_2 = 1,$$

together with the axes $x_1 = 0$ and $x_2 = 0$. The ideal formulation is therefore

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ & x_1 + x_2 \leq 3 \\ & x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Its integer points are the same 7 points of F , and its polyhedron has only integer vertices.

3. For the given formulation the optimal vertex is the intersection of the lines of constraints 1 and 2:

$$\begin{cases} -x_1 + 2x_2 = 2 \\ x_1 + x_2 = 3 \end{cases} \implies x_1 = \frac{4}{3}, \quad x_2 = \frac{5}{3}$$

with $z(\text{LP}) = \frac{14}{3}$ and the bound $\lfloor z(\text{LP}) \rfloor = 4$. Over the ideal formulation the optimal vertex is the integer point $(2,1)$, with value 4.

The LP problem over the ideal formulation gives $z = 4 = z(\text{ILP})$ at the integer vertex $(2, 1)$; the given formulation gives only the bound $z(\text{LP}) = \frac{14}{3}$, rounded down to 4.

Exercise 3.2 — Ideal formulation ($P3$)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 2x_1 + 4x_2 \\ & -x_1 + x_2 \leq 2 \\ & 8x_1 + 2x_2 \leq 19 \\ & x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

Questions

1. Compute the convex hull $\text{conv}(F)$ of the feasible region and classify the feasible solutions as vertices, boundary points or interior points.
2. Write an ideal formulation of the ILP problem.
3. The optimal solution of the LP relaxation of the given formulation is $\tilde{x}(\text{LP}) = (\frac{3}{2}, \frac{7}{2})$. Show that it does not belong to $\text{conv}(F)$, and conclude that the given formulation is not ideal.
4. Compare the upper bounds of the two formulations.

Solution

1. The feasible region has 9 points. Graphically, we have:

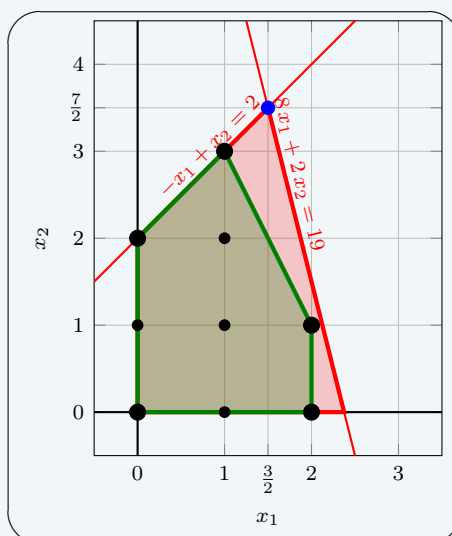


Figure 3.2: The polyhedron of the given formulation (red), the convex hull of the integer points (green) with its vertices (black squares), and the optimal solution of the LP relaxation (blue), outside the hull.

The convex hull $\text{conv}(F)$ is a polygon with

- the 5 **vertices** $(0, 0)$, $(2, 0)$, $(2, 1)$, $(0, 2)$, $(1, 3)$;
 - 2 points on the boundary, $(1, 0)$, $(0, 1)$;
 - 2 points in the interior, $(1, 1)$, $(1, 2)$.
2. The edges of the hull lie on the lines $x_1 = 2$, $2x_1 + x_2 = 5$, $-x_1 + x_2 = 2$ and on the axes. The ideal formulation is

$$\begin{array}{rcll} \max & 2x_1 & + & 4x_2 \\ & x_1 & & \leq 2 \\ & 2x_1 & + & x_2 \leq 5 \\ & -x_1 & + & x_2 \leq 2 \\ & x_1, & & x_2 \geq 0 \end{array}$$

3. The point $(\frac{3}{2}, \frac{7}{2})$ violates the inequality $2x_1 + x_2 \leq 5$ of the hull: $2 \cdot \frac{3}{2} + 1 \cdot \frac{7}{2} = \frac{13}{2} > 5$. Every point of $\text{conv}(F)$ satisfies this inequality, so $\tilde{\mathbf{x}}(\text{LP}) \notin \text{conv}(F)$ and $\text{conv}(F) \subsetneq P$: the given formulation is **not ideal**.
4. The given formulation gives $z(\text{LP}) = 17$, hence $z(\text{ILP}) \leq 17$. The ideal formulation has the optimal vertex $(1, 3)$, with value 14.

The ideal formulation gives the exact value $z(\text{ILP}) = 14$, attained at the integer vertex $(1, 3)$; the given formulation gives the weaker bound 17.

Exercise 3.3 — Comparing formulations (P23)

Consider the ILP problem in canonical maximization form

$$\begin{aligned} \max \quad & x_1 + 4x_2 \\ & 3x_1 + 8x_2 \leq 24 \\ & -x_1 \leq -1 \\ & -x_2 \leq -1 \\ & x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

and the two polyhedra (both with $x_1, x_2 \geq 0$)

$$P_A = \{ 3x_1 + 8x_2 \leq 24, x_1 \geq 1, x_2 \geq 1 \},$$

$$P_B = \{ x_1 + 3x_2 \leq 8, x_1 \leq 5, x_1 \geq 1, x_2 \geq 1 \}.$$

Questions

1. Show that P_A and P_B are two formulations of the same ILP problem.
2. Compute the vertices of P_A and P_B , and decide which formulation is stronger.
3. Compare the upper bounds given by the LP relaxations of the two formulations.
4. Is P_B an ideal formulation?

Solution

1. In both polyhedra $x_1 \geq 1$ and $x_2 \geq 1$. In P_A , $x_2 = 1$ gives $3x_1 \leq 16$, that is $x_1 \leq 5$, and $x_2 = 2$ gives $x_1 \leq 2$; $x_2 \geq 3$ is impossible. In P_B , $x_2 = 1$ gives $x_1 \leq 5$, and $x_2 = 2$ gives $x_1 \leq 2$; $x_2 \geq 3$ is impossible. Both contain exactly the 7 integer points

$$(1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (1, 2), (2, 2),$$

$$\text{so } P_A \cap \mathbb{Z}^2 = P_B \cap \mathbb{Z}^2 = F.$$

2. The vertices are

$$P_A : (1, 1), (1, \frac{21}{8}), (\frac{16}{3}, 1), \quad P_B : (1, 1), (1, \frac{7}{3}), (5, 1).$$

Every vertex of P_B satisfies the constraints of P_A , so $P_B \subseteq P_A$ (both are convex). The vertex $(1, \frac{21}{8})$ of P_A violates $x_1 + 3x_2 \leq 8$ ($1 \cdot 1 + 3 \cdot \frac{21}{8} = \frac{71}{8} > 8$), so $P_B \subsetneq P_A$: **P_B is stronger.**

Graphically, we have:

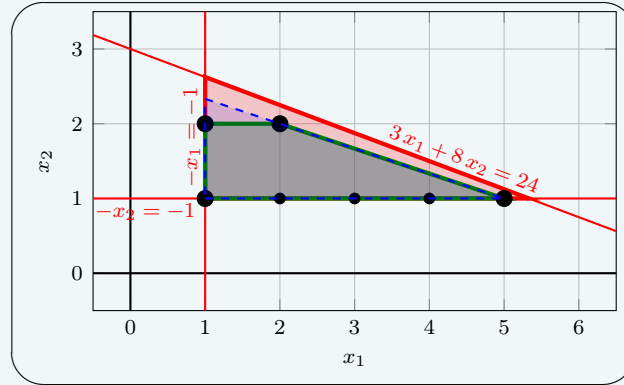


Figure 3.3: The formulation P_A (red), the formulation P_B (blue, dashed) and the convex hull of the integer points (green), with its vertices (black squares).

3. The LP relaxations give $\tilde{x} = (1, \frac{21}{8})$ with $z = \frac{23}{2}$ over P_A , and $\tilde{x} = (1, \frac{7}{3})$ with $z = \frac{31}{3}$ over P_B . The rounded bounds are 11 and 10. The feasible point $(2, 2)$ has value 10, so P_B certifies that it is optimal, while P_A leaves a gap.
4. No: the vertex $(1, \frac{7}{3})$ of P_B is fractional. The convex hull has the vertices $(1, 1)$, $(5, 1)$, $(2, 2)$, $(1, 2)$ and the edges on $x_2 \geq 1$, $x_1 + 3x_2 \leq 8$, $x_2 \leq 2$, $x_1 \geq 1$.

$P_B \subsetneq P_A$: the formulation P_B is stronger and its bound $\lfloor \frac{31}{3} \rfloor = 10$ certifies $z(\text{ILP}) = 10$, but it is not ideal.

Exercise 3.4 — Recognizing an ideal formulation

Consider the three polyhedra (all with $x_1, x_2 \geq 0$)

$$\begin{aligned} P_C &= \{x_1 + x_2 \leq 3, x_1 \leq 2\}, \\ P_D &= \{x_1 + 4x_2 \leq 12, 2x_1 + x_2 \leq 5\}, \\ P_E &= \{x_1 + x_2 \leq 3, 2x_1 + x_2 \leq 5\}. \end{aligned}$$

Questions

1. Show that P_C , P_D and P_E contain the same integer points, so they are formulations of the same ILP problems.
2. Compute the vertices of the three polyhedra. Which formulation is ideal? Order the formulations from the strongest to the weakest.
3. With the objective function $x_1 + 5x_2$, the LP problem over P_D has an integer optimal solution. Does this prove that P_D is ideal? Use the objective function $x_1 + x_2$ to answer.

Solution

1. An integer point of any of the three polyhedra has $x_1 \leq 2$ ($x_1 \leq 2$ in P_C , $2x_1 \leq 5$ in P_D and P_E). For $x_1 = 0, 1, 2$ the largest integer value of x_2 is

$$P_C : 3 - x_1 = 3, 2, 1,$$

$$P_D : \min\{\lfloor (12 - x_1)/4 \rfloor, 5 - 2x_1\} = 3, 2, 1,$$

$$P_E : \min\{3 - x_1, 5 - 2x_1\} = 3, 2, 1.$$

The three formulations have the same 9 integer points

$$(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1).$$

2. The vertices are

$$P_C : (0, 0), (0, 3), (2, 0), (2, 1)$$

$$P_D : (0, 0), (0, 3), (\frac{8}{7}, \frac{19}{7}), (\frac{5}{2}, 0)$$

$$P_E : (0, 0), (0, 3), (2, 1), (\frac{5}{2}, 0)$$

Only P_C has all its vertices integer: $P_C = \text{conv}(F)$ is the **ideal** formulation. Every vertex of P_C belongs to P_E and every vertex of P_E belongs to P_D , so

$$\text{conv}(F) = P_C \subsetneq P_E \subsetneq P_D,$$

and the order from the strongest to the weakest formulation is P_C, P_E, P_D . Graphically, we have:

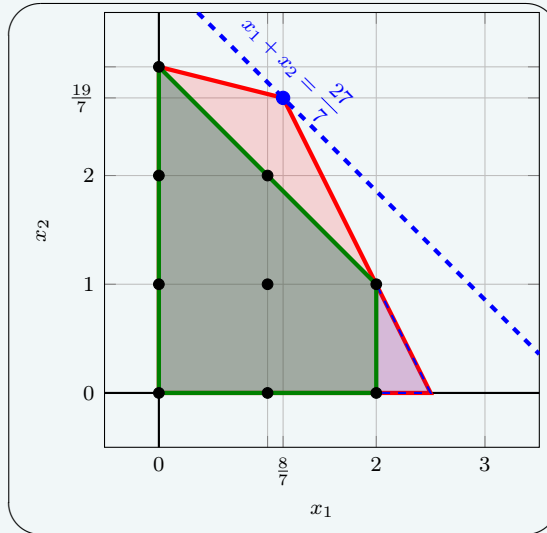


Figure 3.4: The formulations P_D (red), P_E (blue, dashed) and P_C (green), the 9 integer points (black), and the optimal solution $(\frac{8}{7}, \frac{19}{7})$ of the LP problem over P_D with the objective $x_1 + x_2$ (blue), with its objective line (dashed).

3. With $x_1 + 5x_2$ the optimal vertex of P_D is $(0, 3)$, of value 15: it is integer. With $x_1 + x_2$, instead, the optimal vertex of P_D is $(\frac{8}{7}, \frac{19}{7})$, of value $\frac{27}{7}$, while the best integer point has value 3.

An integer LP optimum for one objective function does **not** prove that a formulation is ideal: P_D has the fractional vertex $(\frac{8}{7}, \frac{19}{7})$, optimal for $x_1 + x_2$. Only P_C gives an integer optimal vertex for every objective function.

Exercise 3.5 — An ideal formulation in three dimensions (P_{43})

Consider the following ILP problem in canonical maximization form with $n = 3$ integer variables and $m = 2$ constraints:

$$\begin{array}{rclclcl} \max & x_1 & + & 2x_2 & + & 7x_3 & \\ & x_1 & & & + & 2x_3 & \leq & 2 \\ & & & x_2 & + & 2x_3 & \leq & 2 \\ & x_1, & & x_2, & & x_3 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

Questions

1. Compute all the vertices of the polyhedron P of the LP relaxation.
2. Is the formulation ideal? What does this imply for the LP relaxation with any objective function?
3. Solve the LP relaxation with the given objective function and give $z(\text{ILP})$.

Solution

1. A vertex of $P \subseteq \mathbb{R}^3$ is a feasible point where 3 linearly independent constraints (among the 2 constraints and the 3 non-negativity constraints) hold with equality. Solving all the 3×3 systems and keeping the feasible solutions gives the vertices

$$(0, 0, 0), \quad (0, 0, 1), \quad (0, 2, 0), \quad (2, 0, 0), \quad (2, 2, 0).$$

For example, $x_1 = 0$, $x_1 + 2x_3 = 2$ and $x_2 + 2x_3 = 2$ give $(0, 0, 1)$. The systems that are inconsistent, or whose solution violates a constraint, give no vertex.

2. All the vertices are integer and belong to F . The polyhedron P is bounded, so it is the convex hull of its vertices: $P = \text{conv}(\text{vertices}) \subseteq \text{conv}(F) \subseteq P$, hence $P = \text{conv}(F)$ and the formulation is **ideal**. For every objective function the LP relaxation has an integer optimal vertex, for example:

Objective function	optimal vertex	optimal value
$x_1 + 2x_2 + 7x_3$	$(0, 0, 1)$	7
$x_1 + x_2 + 3x_3$	$(2, 2, 0)$	4
$3x_1 + 2x_2 + 9x_3$	$(2, 2, 0)$	10
$x_1 + x_2 + x_3$	$(2, 2, 0)$	4

and complete enumeration gives the same values.

3. The values of $x_1 + 2x_2 + 7x_3$ at the five vertices are 0, 7, 4, 2, 6: the optimal vertex is $(0, 0, 1)$, with value 7.

The formulation is ideal: the optimal vertex $(0, 0, 1)$ of the LP relaxation is integer, and $z(\text{LP}) = z(\text{ILP}) = 7$.

Valid inequalities

Exercise 4.1 — A valid inequality by rounding (P2)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & x_1 + 3x_2 \\ & -4x_1 + 10x_2 \leq 35 \\ & 3x_1 + x_2 \leq 12 \\ & x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

The optimal solution of its LP relaxation is $\tilde{\mathbf{x}}(\text{LP}) = (\frac{5}{2}, \frac{9}{2})$, with $z(\text{LP}) = 16$.

Questions

1. Using the multipliers $(\frac{3}{34}, \frac{2}{17})$, show that $x_2 \leq 4$ is a valid inequality.
2. Is the inequality tight? Does it cut off the optimal solution of the LP relaxation?
3. Add the inequality to the formulation and compute the new upper bound.

Solution

1. Multiplying the constraints by the non-negative multipliers $(\frac{3}{34}, \frac{2}{17})$ and summing, $\frac{3}{34} \cdot (-4x_1 + 10x_2 \leq 35) + \frac{2}{17} \cdot (3x_1 + x_2 \leq 12)$, we get

$$x_2 \leq \frac{9}{2}.$$

Since $x_j \geq 0$ and the coefficients $0 \geq 0$, $1 \geq 1$ are at least those of x_2 , every feasible solution satisfies $x_2 \leq x_2 \leq \frac{9}{2}$. The left-hand side is an integer, so

$$x_2 \leq \left\lfloor \frac{9}{2} \right\rfloor = 4$$

is a **valid inequality**.

2. The feasible solutions $(2, 4)$ satisfy $x_2 = 4$, so the inequality is **tight**. It **cuts off** $\tilde{\mathbf{x}}(\text{LP}) = (\frac{5}{2}, \frac{9}{2})$, since $0 \cdot \frac{5}{2} + 1 \cdot \frac{9}{2} = \frac{9}{2} > 4$.

3. Graphically, we have:

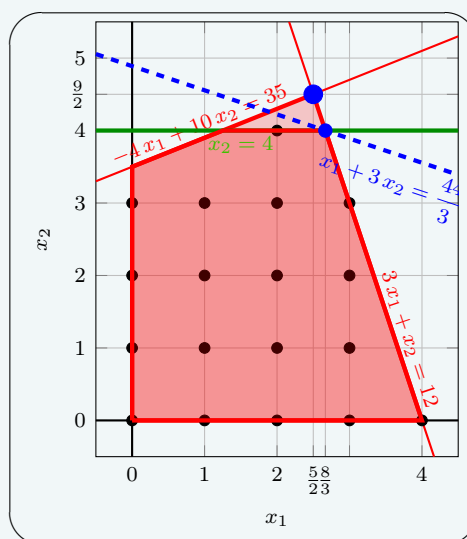


Figure 4.1: The LP relaxation with the valid inequality $x_2 \leq 4$ (green): the optimal solution $(\frac{5}{2}, \frac{9}{2})$ of the original LP relaxation (blue circle) is cut off, and the new optimal solution is blue, with its objective line (dashed).

Formulation	$\tilde{\mathbf{x}}(\text{LP})$	$z(\text{LP})$	upper bound
original	$(\frac{5}{2}, \frac{9}{2})$	16	16
with $x_2 \leq 4$	$(\frac{8}{3}, 4)$	$\frac{44}{3}$	14

$x_2 \leq 4$ is valid and tight, it cuts off $\tilde{\mathbf{x}}(\text{LP})$, and the upper bound improves from 16 to 14 (the optimal value is $z(\text{ILP}) = 14$).

Exercise 4.2 — Valid or not valid (P4)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{aligned}
 \max \quad & 5x_1 + 3x_2 \\
 & 2x_1 + x_2 \leq 14 \\
 & x_1 + 8x_2 \leq 32 \\
 & x_1, x_2 \in \mathbb{Z}_{\geq 0}
 \end{aligned}$$

The optimal solution of its LP relaxation is $\tilde{\mathbf{x}}(\text{LP}) = (\frac{16}{3}, \frac{10}{3})$, with $z(\text{LP}) = \frac{110}{3}$.

Questions

1. Is $x_1 + x_2 \leq 7$ a valid inequality?
2. Show that $x_1 + x_2 \leq 8$ is a valid inequality, using the multipliers $(\frac{7}{15}, \frac{1}{15})$. Does it cut off the optimal solution of the LP relaxation?
3. Add $x_1 + x_2 \leq 8$ to the formulation and solve the new LP relaxation. What can you conclude?

Solution

1. No. The point $(6, 2)$ is feasible, $2 \cdot 6 + 1 \cdot 2 = 14 \leq 14$, $1 \cdot 6 + 8 \cdot 2 = 22 \leq 32$, and it violates the inequality: $6 + 2 = 8 > 7$; the same holds for the feasible point $(5, 3)$. A valid inequality must be satisfied by **all** the feasible solutions.
2. Multiplying the constraints by the non-negative multipliers $(\frac{7}{15}, \frac{1}{15})$ and summing, $\frac{7}{15} \cdot (2x_1 + x_2 \leq 14) + \frac{1}{15} \cdot (x_1 + 8x_2 \leq 32)$, we get

$$x_1 + x_2 \leq \frac{26}{3}.$$

Since $x_j \geq 0$ and the coefficients $1 \geq 1$, $1 \geq 1$ are at least those of $x_1 + x_2$, every feasible solution satisfies $x_1 + x_2 \leq x_1 + x_2 \leq \frac{26}{3}$. The left-hand side is an integer, so

$$x_1 + x_2 \leq \left\lfloor \frac{26}{3} \right\rfloor = 8$$

is a **valid inequality**. It **cuts off** $\tilde{\mathbf{x}}(\text{LP}) = (\frac{16}{3}, \frac{10}{3})$, since $1 \cdot \frac{16}{3} + 1 \cdot \frac{10}{3} = \frac{26}{3} > 8$.

3. Graphically, we have:

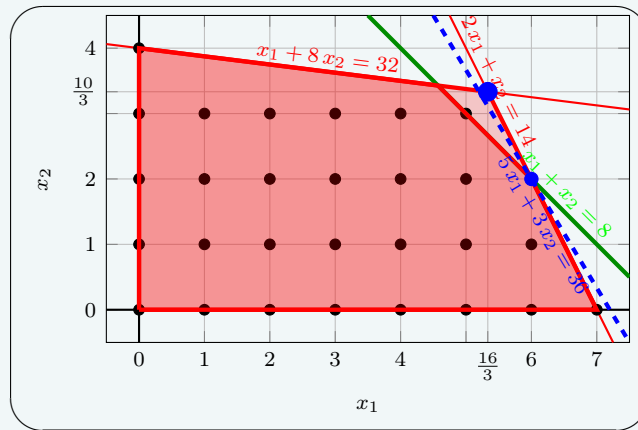


Figure 4.2: The LP relaxation with the valid inequality $x_1 + x_2 \leq 8$ (green): the old optimal solution (blue circle) is cut off and the new optimal solution $(6, 2)$ (blue) is integer.

Formulation	$\tilde{\mathbf{x}}(\text{LP})$	$z(\text{LP})$	upper bound
original	$(\frac{16}{3}, \frac{10}{3})$	$\frac{110}{3}$	36
with $x_1 + x_2 \leq 8$	$(6, 2)$	36	36

The new LP relaxation has the **integer** optimal solution $(6, 2)$: it is an optimal ILP solution, $z(\text{ILP}) = 36$, and the inequality $x_1 + x_2 \leq 7$, which removes it, is not valid.

Exercise 4.3 — Rounding and tightening (P5)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ & 6x_1 + 21x_2 \leq 77 \\ & 8x_1 + 3x_2 \leq 36 \\ & x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

The optimal solution of its LP relaxation is $\tilde{\mathbf{x}}(\text{LP}) = (\frac{7}{2}, \frac{8}{3})$, with $z(\text{LP}) = 15$.

Questions

1. Multiply both constraints by $\frac{1}{7}$, sum them, and derive a valid inequality $2x_1 + 3x_2 \leq \beta$ by rounding. Does it cut off the optimal solution of the LP relaxation?
2. Make the inequality tight with Step 1 of the algorithm of the chapter: decrease β until a feasible integer point lies on the line $2x_1 + 3x_2 = \beta$.
3. Add the tight inequality to the formulation. What is the new upper bound?

Solution

1. Multiplying the constraints by the non-negative multipliers $(\frac{1}{7}, \frac{1}{7})$ and summing, $\frac{1}{7} \cdot (6x_1 + 21x_2 \leq 77) + \frac{1}{7} \cdot (8x_1 + 3x_2 \leq 36)$, we get

$$2x_1 + \frac{24}{7}x_2 \leq \frac{113}{7}.$$

Since $x_j \geq 0$ and the coefficients $2 \geq 2$, $\frac{24}{7} \geq 3$ are at least those of $2x_1 + 3x_2$, every feasible solution satisfies $2x_1 + 3x_2 \leq 2x_1 + \frac{24}{7}x_2 \leq \frac{113}{7}$. The left-hand side is an integer, so

$$2x_1 + 3x_2 \leq \left\lfloor \frac{113}{7} \right\rfloor = 16$$

is a **valid inequality**. It does **not** cut off $\tilde{\mathbf{x}}(\text{LP}) = (\frac{7}{2}, \frac{8}{3})$, since $2 \cdot \frac{7}{2} + 3 \cdot \frac{8}{3} = 15 \leq 16$. The inequality is valid but useless for the bound.

2. For two variables the integer points of the line $2x_1 + 3x_2 = \beta$ with $x_1, x_2 \geq 0$ are finitely many, and we check each of them against the constraints.
 - $\beta = 16$. The integer points of $2x_1 + 3x_2 = 16$ with $x \geq 0$: $(2, 4)$ violates constraint 1 ($6 \cdot 2 + 21 \cdot 4 = 96 > 77$); $(5, 2)$ violates constraint 2 ($8 \cdot 5 + 3 \cdot 2 =$

$46 > 36$); $(8, 0)$ violates constraint 2 ($8 \cdot 8 + 3 \cdot 0 = 64 > 36$). Hence no feasible point: set $\beta = 15$.

- $\beta = 15$. The integer points of $2x_1 + 3x_2 = 15$ with $x \geq 0$: $(0, 5)$ violates constraint 1 ($6 \cdot 0 + 21 \cdot 5 = 105 > 77$); $(3, 3)$ violates constraint 1 ($6 \cdot 3 + 21 \cdot 3 = 81 > 77$); $(6, 1)$ violates constraint 2 ($8 \cdot 6 + 3 \cdot 1 = 51 > 36$). Hence no feasible point: set $\beta = 14$.
- $\beta = 14$. The integer points of $2x_1 + 3x_2 = 14$ with $x \geq 0$: $(1, 4)$ violates constraint 1 ($6 \cdot 1 + 21 \cdot 4 = 90 > 77$); $(4, 2)$ violates constraint 2 ($8 \cdot 4 + 3 \cdot 2 = 38 > 36$); $(7, 0)$ violates constraint 2 ($8 \cdot 7 + 3 \cdot 0 = 56 > 36$). Hence no feasible point: set $\beta = 13$.
- $\beta = 13$. The integer points of $2x_1 + 3x_2 = 13$ with $x \geq 0$: $(2, 3)$ is feasible; $(5, 1)$ violates constraint 2 ($8 \cdot 5 + 3 \cdot 1 = 43 > 36$). Hence a feasible point exists: the inequality is tight. **STOP**.

The tight valid inequality is $2x_1 + 3x_2 \leq 13$. It **cuts off** $\tilde{x}(\text{LP}) = (\frac{7}{2}, \frac{8}{3})$, since $2 \cdot \frac{7}{2} + 3 \cdot \frac{8}{3} = 15 > 13$.

3. The tight inequality has the same coefficients as the objective function, so the new LP relaxation has optimal value $z(\text{LP}) = 13$, attained for example at $(\frac{23}{6}, \frac{16}{9})$.

Formulation	$\tilde{x}(\text{LP})$	$z(\text{LP})$	upper bound
original	$(\frac{7}{2}, \frac{8}{3})$	15	15
with $2x_1 + 3x_2 \leq 13$	$(\frac{23}{6}, \frac{16}{9})$	13	13

The rounding gives $2x_1 + 3x_2 \leq 16$, which does not cut off the LP optimum; the tight version $2x_1 + 3x_2 \leq 13$ gives $\text{UB} = 13$, equal to the value of the feasible point $(2, 3)$: $z(\text{ILP}) = 13$ is certified.

Exercise 4.4 — Reaching the convex hull by rounding (P1)

Consider the following ILP problem in canonical maximization form with $n = 2$ integer variables and $m = 1$ constraint:

$$\begin{aligned} \max \quad & x_1 + 8x_2 \\ & 3x_1 + 6x_2 \leq 16 \\ & x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

The optimal solution of its LP relaxation is $\tilde{x}(\text{LP}) = (0, \frac{8}{3})$, with $z(\text{LP}) = \frac{64}{3}$.

Questions

1. Derive a valid inequality $x_2 \leq \beta_1$ with the multiplier $\frac{1}{6}$, and a valid inequality $x_1 +$

- $2x_2 \leq \beta_2$ with the multiplier $\frac{1}{3}$.
2. Compute the upper bound given by the LP relaxation after adding each inequality alone, and after adding both.
 3. Show that the formulation with both inequalities is ideal.

Solution

1. • Multiplying the constraints by the non-negative multipliers $(\frac{1}{6})$ and summing, $\frac{1}{6} \cdot (3x_1 + 6x_2 \leq 16)$, we get

$$\frac{1}{2}x_1 + x_2 \leq \frac{8}{3}.$$

Since $x_j \geq 0$ and the coefficients $\frac{1}{2} \geq 0$, $1 \geq 1$ are at least those of x_2 , every feasible solution satisfies $x_2 \leq \frac{1}{2}x_1 + x_2 \leq \frac{8}{3}$. The left-hand side is an integer, so

$$x_2 \leq \left\lfloor \frac{8}{3} \right\rfloor = 2$$

is a **valid inequality**.

- Multiplying the constraints by the non-negative multipliers $(\frac{1}{3})$ and summing, $\frac{1}{3} \cdot (3x_1 + 6x_2 \leq 16)$, we get

$$x_1 + 2x_2 \leq \frac{16}{3}.$$

Since $x_j \geq 0$ and the coefficients $1 \geq 1$, $2 \geq 2$ are at least those of $x_1 + 2x_2$, every feasible solution satisfies $x_1 + 2x_2 \leq \frac{16}{3}$. The left-hand side is an integer, so

$$x_1 + 2x_2 \leq \left\lfloor \frac{16}{3} \right\rfloor = 5$$

is a **valid inequality**.

2. Graphically, we have:

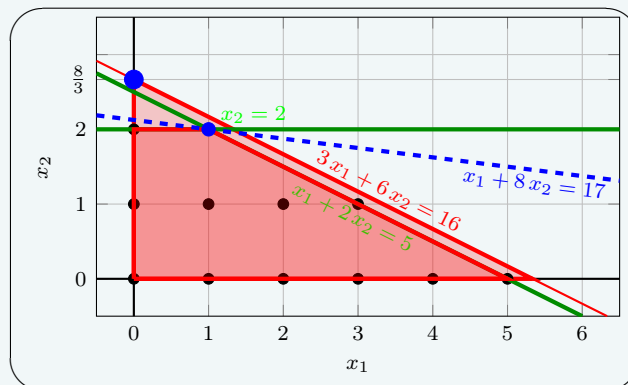


Figure 4.3: The LP relaxation (light red) with the valid inequalities $x_2 \leq 2$ and $x_1 + 2x_2 \leq 5$ (green): the new polyhedron (red) has the integer optimal vertex (1, 2) (blue); the old optimal solution is the blue circle.

Formulation	$\tilde{\mathbf{x}}(\text{LP})$	$z(\text{LP})$	upper bound
original	$(0, \frac{8}{3})$	$\frac{64}{3}$	21
with $x_2 \leq 2$	$(\frac{4}{3}, 2)$	$\frac{52}{3}$	17
with $x_1 + 2x_2 \leq 5$	$(0, \frac{5}{2})$	20	20
with both	$(1, 2)$	17	17

Each inequality alone cuts off $\tilde{\mathbf{x}}(\text{LP})$ but leaves a fractional optimum; together they give the integer optimum $(1, 2)$.

3. The polyhedron with both inequalities has the vertices $(0, 0)$, $(0, 2)$, $(1, 2)$, $(5, 0)$, all integer: it is the convex hull of the feasible solutions, so the formulation is **ideal**.

With $x_2 \leq 2$ and $x_1 + 2x_2 \leq 5$, both obtained by rounding, the LP relaxation gives $z(\text{LP}) = 17 = z(\text{ILP})$ at the integer vertex $(1, 2)$.

Exercise 4.5 — Rounding with three variables (P42)

Consider the following ILP problem in canonical maximization form with $n = 3$ integer variables and $m = 3$ constraints:

$$\begin{aligned}
 \max \quad & 3x_1 + 2x_2 + 4x_3 \\
 & x_1 + x_2 + 2x_3 \leq 4 \\
 & 2x_1 + 3x_3 \leq 5 \\
 & 2x_1 + x_2 + 3x_3 \leq 7 \\
 & x_1, x_2, x_3 \in \mathbb{Z}_{\geq 0}
 \end{aligned}$$

The optimal solution of its LP relaxation is $\tilde{\mathbf{x}}(\text{LP}) = (\frac{5}{2}, \frac{3}{2}, 0)$, with $z(\text{LP}) = \frac{21}{2}$.

Questions

1. Multiply the second constraint by $\frac{1}{2}$, round down its coefficients and its right-hand side, and show that the result is a valid inequality.
2. Does it cut off the optimal solution of the LP relaxation?
3. Add the inequality to the formulation. The new LP relaxation has the optimal solution $(2, 2, 0)$: what can you conclude?

Solution

1. The second constraint multiplied by $\frac{1}{2}$ is $x_1 + \frac{3}{2}x_3 \leq \frac{5}{2}$. Rounding down the coefficients gives $x_1 + x_3$, whose coefficients are not larger. Multiplying the constraints by the non-negative multipliers $(0, \frac{1}{2}, 0)$ and summing, $\frac{1}{2} \cdot (2x_1 + 3x_3 \leq 5)$, we get

$$x_1 + \frac{3}{2}x_3 \leq \frac{5}{2}.$$

Since $x_j \geq 0$ and the coefficients $1 \geq 1$, $0 \geq 0$, $\frac{3}{2} \geq 1$ are at least those of $x_1 + x_3$, every feasible solution satisfies $x_1 + x_3 \leq x_1 + \frac{3}{2}x_3 \leq \frac{5}{2}$. The left-hand side is an integer, so

$$x_1 + x_3 \leq \left\lfloor \frac{5}{2} \right\rfloor = 2$$

is a **valid inequality**.

2. It **cuts off** $\tilde{x}(\text{LP}) = (\frac{5}{2}, \frac{3}{2}, 0)$, since $1 \cdot \frac{5}{2} + 0 \cdot \frac{3}{2} + 1 \cdot 0 = \frac{5}{2} > 2$.
3. The new LP relaxation has the **integer** optimal solution $(2, 2, 0)$, with value 10. By the proposition on the LP relaxation, it is an optimal ILP solution.

The valid inequality $x_1 + x_3 \leq 2$ lowers the upper bound from $\lfloor \frac{21}{2} \rfloor = 10$ to 10 and gives the optimal solution $(2, 2, 0)$, $z(\text{ILP}) = 10$.

Cover inequalities

5.1 Cover separation

Exercise 5.1 — Cover inequalities ($P1$)

Consider a binary problem with one constraint $\sum_j a_j x_j \leq b$, with positive integer coefficients a_j and right-hand side $b = 8$, on the binary variables indexed by the set $\mathcal{J} = \{1, 2, 3, 4\}$ of $n = 4$ variables with the following objective coefficients and constraint coefficients:

j	1	2	3	4
c_j	40	42	25	12
a_j	4	7	5	3

The associated LP is:

$$\begin{array}{llllllll}
 \max & 40x_1 & + & 42x_2 & + & 25x_3 & + & 12x_4 \\
 \text{subject to} & 4x_1 & + & 7x_2 & + & 5x_3 & + & 3x_4 \leq 8 \\
 & x_1 & & & & & & \leq 1 \\
 & & x_2 & & & & & \leq 1 \\
 & & & x_3 & & & & \leq 1 \\
 & & & & x_4 & & & \leq 1 \\
 & x_1, & x_2, & x_3, & x_4 & \geq & 0
 \end{array}$$

An optimal solution $\tilde{\mathbf{x}}$ of LP and its value are:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = \frac{4}{7}, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{and} \quad z(\text{LP}) = 64.$$

Questions

1. List all the cover inequalities and identify the minimal covers.
2. For the point $\tilde{\mathbf{x}}$, write the separation problem ILP_{CSP} and its complementary problem $\text{ILP}_{\text{CSP}}^2$.
3. Apply the greedy heuristic to $\text{ILP}_{\text{CSP}}^2$ and, if possible, find a violated cover inequality.

4. Check whether the greedy solution is optimal.

A subset of indices $C \subseteq \mathcal{J}$ with $\sum_{j \in C} a_j > b$ is a **cover**, and the linear inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

is the **cover inequality** associated with C . Given a point $\tilde{\mathbf{x}}$, a cover inequality violated by $\tilde{\mathbf{x}}$ exists if and only if $z(\text{ILP}_{\text{CSP}}) < 1$, where

$$(\text{ILP}_{\text{CSP}}) \quad \min \left\{ \sum_{j \in \mathcal{J}} (1 - \tilde{x}_j) z_j : \sum_{j \in \mathcal{J}} a_j z_j \geq b + 1, z_j \in \{0, 1\} \forall j \in \mathcal{J} \right\}.$$

Solution

Separation of the cover inequalities violated by $\tilde{\mathbf{x}}$.

- **Cover inequalities**

The covers are the subsets of indices whose total coefficient exceeds $b = 8$:

C	$\sum_{j \in C} a_j$	$ C $	Cover inequality	Minimal
$\{1, 2\}$	11	2	$x_1 + x_2 \leq 1$	yes
$\{1, 3\}$	9	2	$x_1 + x_3 \leq 1$	yes
$\{2, 3\}$	12	2	$x_2 + x_3 \leq 1$	yes
$\{2, 4\}$	10	2	$x_2 + x_4 \leq 1$	yes
$\{1, 2, 3\}$	16	3	$x_1 + x_2 + x_3 \leq 2$	no
$\{1, 2, 4\}$	14	3	$x_1 + x_2 + x_4 \leq 2$	no
$\{1, 3, 4\}$	12	3	$x_1 + x_3 + x_4 \leq 2$	no
$\{2, 3, 4\}$	15	3	$x_2 + x_3 + x_4 \leq 2$	no
$\{1, 2, 3, 4\}$	19	4	$x_1 + x_2 + x_3 + x_4 \leq 3$	no

There are 9 covers, of which 4 are minimal. The cover inequalities of the non-minimal covers are implied by those of the minimal covers and by the bounds $x_j \leq 1$.

- **Separation problem**

The costs $q_j = 1 - \tilde{x}_j$ are $q = (0, \frac{3}{7}, 1, 1)$, and the separation problem ILP_{CSP} is:

$$\begin{aligned} \min \quad & \frac{3}{7} z_2 + z_3 + z_4 \\ \text{subject to} \quad & 4 z_1 + 7 z_2 + 5 z_3 + 3 z_4 \geq 9 \\ & z_1, z_2, z_3, z_4 \in \{0, 1\} \end{aligned}$$

With $y_j = 1 - z_j$, the constant term and the right-hand side of the complementary problem are

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j = \frac{17}{7}, \quad \sum_{j \in \mathcal{J}} a_j - (b + 1) = 19 - 9 = 10,$$

and the complementary problem $\text{ILP}_{\text{CSP}}^2$ is:

$$\begin{array}{llllll} \max & & \frac{3}{7}y_2 & + & y_3 & + & y_4 \\ \text{subject to} & 4y_1 & + & 7y_2 & + & 5y_3 & + & 3y_4 \leq 10 \\ & y_1, & & y_2, & & y_3, & & y_4 \in \{0, 1\} \end{array}$$

with $z(\text{ILP}_{\text{CSP}}) = \frac{17}{7} - z(\text{ILP}_{\text{CSP}}^2)$.

- **Greedy heuristic**

The indices are scanned by non-increasing ratio q_j/a_j , starting from the residual right-hand side 10:

j	q_j	a_j	q_j/a_j	Decision	Residual
4	1	3	$\frac{1}{3}$	$\bar{y}_j = 1$	7
3	1	5	$\frac{1}{5}$	$\bar{y}_j = 1$	2
2	$\frac{3}{7}$	7	$\frac{3}{49}$	$\bar{y}_j = 0$	2
1	0	4	0	$q_j = 0, \bar{y}_j = 0$	2

The greedy solution is $\bar{\mathbf{y}} = (0, 0, 1, 1)$, with value 2. The corresponding cover is $\bar{C} = \{j \in \mathcal{J} : \bar{y}_j = 0\} = \{1, 2\}$, of total coefficient 11 > 8 and cost $\frac{17}{7} - 2 = \frac{3}{7} < 1$. Hence the cover inequality

$$x_1 + x_2 \leq 1$$

is violated by $\tilde{\mathbf{x}}$: $1 + \frac{4}{7} - 1 = \frac{4}{7}$.

- **Optimality check**

Indices with $q_j = 0$ do not change the value of a solution. The non-empty subsets of the indices $\{2, 3, 4\}$ with $q_j > 0$ that satisfy the constraint with right-hand side 10 are: $\{2\}$ (value $\frac{3}{7}$), $\{3\}$ (value 1), $\{4\}$ (value 1), $\{2, 4\}$ (value $\frac{10}{7}$), $\{3, 4\}$ (value 2). Hence $z(\text{ILP}_{\text{CSP}}^2) = 2$, the greedy solution is optimal, $z(\text{ILP}_{\text{CSP}}) = \frac{3}{7} < 1$, and the maximum violation of a cover inequality is $\frac{4}{7}$.

Exercise 5.2 — Cover inequalities ($P2$)

Consider a binary problem with one constraint $\sum_j a_j x_j \leq b$, with positive integer coefficients a_j and right-hand side $b = 15$, on the binary variables indexed by the set $\mathcal{J} = \{1, 2, 3, 4\}$ of $n = 4$ variables with the following objective coefficients and constraint coefficients:

j	1	2	3	4
c_j	24	17	12	6
a_j	10	8	6	5

The associated LP is:

$$\begin{array}{llllll}
\max & 24x_1 & + & 17x_2 & + & 12x_3 & + & 6x_4 \\
\text{subject to} & 10x_1 & + & 8x_2 & + & 6x_3 & + & 5x_4 & \leq & 15 \\
& x_1 & & & & & & & \leq & 1 \\
& & x_2 & & & & & & \leq & 1 \\
& & & x_3 & & & & & \leq & 1 \\
& & & & x_4 & & & & \leq & 1 \\
& x_1, & x_2, & x_3, & x_4 & \geq & 0
\end{array}$$

An optimal solution $\tilde{\mathbf{x}}$ of LP and its value are:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = \frac{5}{8}, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{and} \quad z(\text{LP}) = \frac{277}{8}.$$

Questions

1. List all the cover inequalities and identify the minimal covers.
2. For the point $\tilde{\mathbf{x}}$, write the separation problem ILP_{CSP} and its complementary problem $\text{ILP}_{\text{CSP}}^2$.
3. Apply the greedy heuristic to $\text{ILP}_{\text{CSP}}^2$ and, if possible, find a violated cover inequality.
4. Check whether the greedy solution is optimal.

A subset of indices $C \subseteq \mathcal{J}$ with $\sum_{j \in C} a_j > b$ is a **cover**, and the linear inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

is the **cover inequality** associated with C . Given a point $\tilde{\mathbf{x}}$, a cover inequality violated by $\tilde{\mathbf{x}}$ exists if and only if $z(\text{ILP}_{\text{CSP}}) < 1$, where

$$(\text{ILP}_{\text{CSP}}) \quad \min \left\{ \sum_{j \in \mathcal{J}} (1 - \tilde{x}_j) z_j : \sum_{j \in \mathcal{J}} a_j z_j \geq b + 1, \quad z_j \in \{0, 1\} \quad \forall j \in \mathcal{J} \right\}.$$

Solution

Separation of the cover inequalities violated by $\tilde{\mathbf{x}}$.

- **Cover inequalities**

The covers are the subsets of indices whose total coefficient exceeds $b = 15$:

C	$\sum_{j \in C} a_j$	$ C $	Cover inequality	Minimal
$\{1, 2\}$	18	2	$x_1 + x_2 \leq 1$	yes
$\{1, 3\}$	16	2	$x_1 + x_3 \leq 1$	yes
$\{1, 2, 3\}$	24	3	$x_1 + x_2 + x_3 \leq 2$	no
$\{1, 2, 4\}$	23	3	$x_1 + x_2 + x_4 \leq 2$	no
$\{1, 3, 4\}$	21	3	$x_1 + x_3 + x_4 \leq 2$	no
$\{2, 3, 4\}$	19	3	$x_2 + x_3 + x_4 \leq 2$	yes
$\{1, 2, 3, 4\}$	29	4	$x_1 + x_2 + x_3 + x_4 \leq 3$	no

There are 7 covers, of which 3 are minimal. The cover inequalities of the non-minimal covers are implied by those of the minimal covers and by the bounds $x_j \leq 1$.

- **Separation problem**

The costs $q_j = 1 - \tilde{x}_j$ are $q = (0, \frac{3}{8}, 1, 1)$, and the separation problem ILP_{CSP} is:

$$\begin{aligned}
 \min \quad & \frac{3}{8} z_2 + z_3 + z_4 \\
 \text{subject to} \quad & 10 z_1 + 8 z_2 + 6 z_3 + 5 z_4 \geq 16 \\
 & z_1, z_2, z_3, z_4 \in \{0, 1\}
 \end{aligned}$$

With $y_j = 1 - z_j$, the constant term and the right-hand side of the complementary problem are

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j = \frac{19}{8}, \quad \sum_{j \in \mathcal{J}} a_j - (b + 1) = 29 - 16 = 13,$$

and the complementary problem $\text{ILP}_{\text{CSP}}^2$ is:

$$\begin{aligned}
 \max \quad & \frac{3}{8} y_2 + y_3 + y_4 \\
 \text{subject to} \quad & 10 y_1 + 8 y_2 + 6 y_3 + 5 y_4 \leq 13 \\
 & y_1, y_2, y_3, y_4 \in \{0, 1\}
 \end{aligned}$$

with $z(\text{ILP}_{\text{CSP}}) = \frac{19}{8} - z(\text{ILP}_{\text{CSP}}^2)$.

- **Greedy heuristic**

The indices are scanned by non-increasing ratio q_j/a_j , starting from the residual right-hand side 13:

j	q_j	a_j	q_j/a_j	Decision	Residual
4	1	5	$\frac{1}{5}$	$\bar{y}_j = 1$	8
3	1	6	$\frac{1}{6}$	$\bar{y}_j = 1$	2
2	$\frac{3}{8}$	8	$\frac{3}{64}$	$\bar{y}_j = 0$	2
1	0	10	0	$q_j = 0, \bar{y}_j = 0$	2

The greedy solution is $\bar{\mathbf{y}} = (0, 0, 1, 1)$, with value 2. The corresponding cover is $\bar{C} = \{j \in \mathcal{J} : \bar{y}_j = 0\} = \{1, 2\}$, of total coefficient $18 > 15$ and cost $\frac{19}{8} - 2 = \frac{3}{8} < 1$. Hence the cover inequality

$$x_1 + x_2 \leq 1$$

is violated by $\tilde{\mathbf{x}}$: $1 + \frac{5}{8} - 1 = \frac{5}{8}$.

- **Optimality check**

Indices with $q_j = 0$ do not change the value of a solution. The non-empty subsets of the indices $\{2, 3, 4\}$ with $q_j > 0$ that satisfy the constraint with right-hand side 13 are: $\{2\}$ (value $\frac{3}{8}$), $\{3\}$ (value 1), $\{4\}$ (value 1), $\{2, 4\}$ (value $\frac{11}{8}$), $\{3, 4\}$ (value 2). Hence $z(\text{ILP}_{\text{CSP}}^2) = 2$, the greedy solution is optimal, $z(\text{ILP}_{\text{CSP}}) = \frac{3}{8} < 1$, and the maximum violation of a cover inequality is $\frac{5}{8}$.

Exercise 5.3 — Cover inequalities (P3)

Consider a binary problem with one constraint $\sum_j a_j x_j \leq b$, with positive integer coefficients a_j and right-hand side $b = 12$, on the binary variables indexed by the set $\mathcal{J} = \{1, 2, 3, 4\}$ of $n = 4$ variables with the following objective coefficients and constraint coefficients:

j	1	2	3	4
c_j	38	14	44	19
a_j	5	2	10	6

The associated LP is:

$$\begin{array}{llllll}
 \max & 38x_1 & + & 14x_2 & + & 44x_3 & + & 19x_4 \\
 \text{subject to} & 5x_1 & + & 2x_2 & + & 10x_3 & + & 6x_4 & \leq & 12 \\
 & x_1 & & & & & & & \leq & 1 \\
 & & x_2 & & & & & & \leq & 1 \\
 & & & x_3 & & & & & \leq & 1 \\
 & & & & x_4 & & & & \leq & 1 \\
 & x_1, & x_2, & x_3, & x_4 & \geq & 0
 \end{array}$$

An optimal solution $\tilde{\mathbf{x}}$ of LP and its value are:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 1, \quad \tilde{x}_3 = \frac{1}{2}, \quad \tilde{x}_4 = 0 \quad \text{and} \quad z(\text{LP}) = 74.$$

Questions

1. List all the cover inequalities and identify the minimal covers.
2. For the point $\tilde{\mathbf{x}}$, write the separation problem ILP_{CSP} and its complementary problem $\text{ILP}_{\text{CSP}}^2$.
3. Apply the greedy heuristic to $\text{ILP}_{\text{CSP}}^2$ and, if possible, find a violated cover inequality.
4. Check whether the greedy solution is optimal.

A subset of indices $C \subseteq \mathcal{J}$ with $\sum_{j \in C} a_j > b$ is a **cover**, and the linear inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

is the **cover inequality** associated with C . Given a point $\tilde{\mathbf{x}}$, a cover inequality violated by $\tilde{\mathbf{x}}$ exists if and only if $z(\text{ILP}_{\text{CSP}}) < 1$, where

$$(\text{ILP}_{\text{CSP}}) \quad \min \left\{ \sum_{j \in \mathcal{J}} (1 - \tilde{x}_j) z_j : \sum_{j \in \mathcal{J}} a_j z_j \geq b + 1, z_j \in \{0, 1\} \forall j \in \mathcal{J} \right\}.$$

Solution

Separation of the cover inequalities violated by $\tilde{\mathbf{x}}$.

- **Cover inequalities**

The covers are the subsets of indices whose total coefficient exceeds $b = 12$:

C	$\sum_{j \in C} a_j$	$ C $	Cover inequality	Minimal
$\{1, 3\}$	15	2	$x_1 + x_3 \leq 1$	yes
$\{3, 4\}$	16	2	$x_3 + x_4 \leq 1$	yes
$\{1, 2, 3\}$	17	3	$x_1 + x_2 + x_3 \leq 2$	no
$\{1, 2, 4\}$	13	3	$x_1 + x_2 + x_4 \leq 2$	yes
$\{1, 3, 4\}$	21	3	$x_1 + x_3 + x_4 \leq 2$	no
$\{2, 3, 4\}$	18	3	$x_2 + x_3 + x_4 \leq 2$	no
$\{1, 2, 3, 4\}$	23	4	$x_1 + x_2 + x_3 + x_4 \leq 3$	no

There are 7 covers, of which 3 are minimal. The cover inequalities of the non-minimal covers are implied by those of the minimal covers and by the bounds $x_j \leq 1$.

- **Separation problem**

The costs $q_j = 1 - \tilde{x}_j$ are $q = (0, 0, \frac{1}{2}, 1)$, and the separation problem ILP_{CSP} is:

$$\begin{aligned} \min \quad & \frac{1}{2} z_3 + z_4 \\ \text{subject to} \quad & 5 z_1 + 2 z_2 + 10 z_3 + 6 z_4 \geq 13 \\ & z_1, z_2, z_3, z_4 \in \{0, 1\} \end{aligned}$$

With $y_j = 1 - z_j$, the constant term and the right-hand side of the complementary problem are

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j = \frac{3}{2}, \quad \sum_{j \in \mathcal{J}} a_j - (b + 1) = 23 - 13 = 10,$$

and the complementary problem $\text{ILP}_{\text{CSP}}^2$ is:

$$\begin{aligned} \max \quad & \frac{1}{2} y_3 + y_4 \\ \text{subject to} \quad & 5 y_1 + 2 y_2 + 10 y_3 + 6 y_4 \leq 10 \\ & y_1, y_2, y_3, y_4 \in \{0, 1\} \end{aligned}$$

with $z(\text{ILP}_{\text{CSP}}) = \frac{3}{2} - z(\text{ILP}_{\text{CSP}}^2)$.

- **Greedy heuristic**

The indices are scanned by non-increasing ratio q_j/a_j , starting from the residual right-hand side 10:

j	q_j	a_j	q_j/a_j	Decision	Residual
4	1	6	$\frac{1}{6}$	$\bar{y}_j = 1$	4
3	$\frac{1}{2}$	10	$\frac{1}{20}$	$\bar{y}_j = 0$	4
1	0	5	0	$q_j = 0, \bar{y}_j = 0$	4
2	0	2	0	$q_j = 0, \bar{y}_j = 0$	4

The greedy solution is $\bar{\mathbf{y}} = (0, 0, 0, 1)$, with value 1. The corresponding cover is $\bar{C} = \{j \in \mathcal{J} : \bar{y}_j = 0\} = \{1, 2, 3\}$, of total coefficient 17 > 12 and cost $\frac{3}{2} - 1 = \frac{1}{2} < 1$. Hence the cover inequality

$$x_1 + x_2 + x_3 \leq 2$$

is violated by $\tilde{\mathbf{x}}$: $1 + 1 + \frac{1}{2} - 2 = \frac{1}{2}$.

- **Optimality check**

Indices with $q_j = 0$ do not change the value of a solution. The non-empty subsets of the indices $\{3, 4\}$ with $q_j > 0$ that satisfy the constraint with right-hand side 10 are: $\{3\}$ (value $\frac{1}{2}$), $\{4\}$ (value 1). Hence $z(\text{ILP}_{\text{CSP}}^2) = 1$, the greedy solution is optimal, $z(\text{ILP}_{\text{CSP}}) = \frac{1}{2} < 1$, and the maximum violation of a cover inequality is $\frac{1}{2}$.

The cover $\bar{C}$ is not minimal: removing an index with $q_j = 0$ gives the minimal cover $\{1, 3\}$, with the same cost $\frac{1}{2}$. Its cover inequality $x_1 + x_3 \leq 1$ has the same violation $\frac{1}{2}$ and is stronger.

Exercise 5.4 — Cover inequalities (P4)

Consider a binary problem with one constraint $\sum_j a_j x_j \leq b$, with positive integer coefficients a_j and right-hand side $b = 11$, on the binary variables indexed by the set $\mathcal{J} = \{1, 2, 3, 4\}$ of $n = 4$ variables with the following objective coefficients and constraint coefficients:

j	1	2	3	4
c_j	34	40	31	12
a_j	5	9	9	5

The associated LP is:

$$\begin{array}{rcllclclcl}
\max & 34x_1 & + & 40x_2 & + & 31x_3 & + & 12x_4 & & \\
\text{subject to} & 5x_1 & + & 9x_2 & + & 9x_3 & + & 5x_4 & \leq & 11 \\
& x_1 & & & & & & & \leq & 1 \\
& & & x_2 & & & & & \leq & 1 \\
& & & & & x_3 & & & \leq & 1 \\
& & & & & & & x_4 & \leq & 1 \\
& x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}$$

An optimal solution $\tilde{\mathbf{x}}$ of LP and its value are:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = \frac{2}{3}, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{and} \quad z(\text{LP}) = \frac{182}{3}.$$

Questions

1. List all the cover inequalities and identify the minimal covers.
2. For the point $\tilde{\mathbf{x}}$, write the separation problem ILP_{CSP} and its complementary problem $\text{ILP}_{\text{CSP}}^2$.
3. Apply the greedy heuristic to $\text{ILP}_{\text{CSP}}^2$ and, if possible, find a violated cover inequality.
4. Check whether the greedy solution is optimal.

A subset of indices $C \subseteq \mathcal{J}$ with $\sum_{j \in C} a_j > b$ is a **cover**, and the linear inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

is the **cover inequality** associated with C . Given a point $\tilde{\mathbf{x}}$, a cover inequality violated by $\tilde{\mathbf{x}}$ exists if and only if $z(\text{ILP}_{\text{CSP}}) < 1$, where

$$(\text{ILP}_{\text{CSP}}) \quad \min \left\{ \sum_{j \in \mathcal{J}} (1 - \tilde{x}_j) z_j : \sum_{j \in \mathcal{J}} a_j z_j \geq b + 1, \quad z_j \in \{0, 1\} \quad \forall j \in \mathcal{J} \right\}.$$

Solution

Separation of the cover inequalities violated by $\tilde{\mathbf{x}}$.

- **Cover inequalities**

The covers are the subsets of indices whose total coefficient exceeds $b = 11$:

C	$\sum_{j \in C} a_j$	$ C $	Cover inequality	Minimal
$\{1, 2\}$	14	2	$x_1 + x_2 \leq 1$	yes
$\{1, 3\}$	14	2	$x_1 + x_3 \leq 1$	yes
$\{2, 3\}$	18	2	$x_2 + x_3 \leq 1$	yes
$\{2, 4\}$	14	2	$x_2 + x_4 \leq 1$	yes
$\{3, 4\}$	14	2	$x_3 + x_4 \leq 1$	yes
$\{1, 2, 3\}$	23	3	$x_1 + x_2 + x_3 \leq 2$	no
$\{1, 2, 4\}$	19	3	$x_1 + x_2 + x_4 \leq 2$	no
$\{1, 3, 4\}$	19	3	$x_1 + x_3 + x_4 \leq 2$	no
$\{2, 3, 4\}$	23	3	$x_2 + x_3 + x_4 \leq 2$	no
$\{1, 2, 3, 4\}$	28	4	$x_1 + x_2 + x_3 + x_4 \leq 3$	no

There are 10 covers, of which 5 are minimal. The cover inequalities of the non-minimal covers are implied by those of the minimal covers and by the bounds $x_j \leq 1$.

- **Separation problem**

The costs $q_j = 1 - \tilde{x}_j$ are $q = (0, \frac{1}{3}, 1, 1)$, and the separation problem ILP_{CSP} is:

$$\begin{aligned}
 \min \quad & \frac{1}{3} z_2 + z_3 + z_4 \\
 \text{subject to} \quad & 5z_1 + 9z_2 + 9z_3 + 5z_4 \geq 12 \\
 & z_1, z_2, z_3, z_4 \in \{0, 1\}
 \end{aligned}$$

With $y_j = 1 - z_j$, the constant term and the right-hand side of the complementary problem are

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j = \frac{7}{3}, \quad \sum_{j \in \mathcal{J}} a_j - (b + 1) = 28 - 12 = 16,$$

and the complementary problem $\text{ILP}_{\text{CSP}}^2$ is:

$$\begin{aligned}
 \max \quad & \frac{1}{3} y_2 + y_3 + y_4 \\
 \text{subject to} \quad & 5y_1 + 9y_2 + 9y_3 + 5y_4 \leq 16 \\
 & y_1, y_2, y_3, y_4 \in \{0, 1\}
 \end{aligned}$$

with $z(\text{ILP}_{\text{CSP}}) = \frac{7}{3} - z(\text{ILP}_{\text{CSP}}^2)$.

- **Greedy heuristic**

The indices are scanned by non-increasing ratio q_j/a_j , starting from the residual right-hand side 16:

j	q_j	a_j	q_j/a_j	Decision	Residual
4	1	5	$\frac{1}{5}$	$\bar{y}_j = 1$	11
3	1	9	$\frac{1}{9}$	$\bar{y}_j = 1$	2
2	$\frac{1}{3}$	9	$\frac{1}{27}$	$\bar{y}_j = 0$	2
1	0	5	0	$q_j = 0, \bar{y}_j = 0$	2

The greedy solution is $\bar{\mathbf{y}} = (0, 0, 1, 1)$, with value 2. The corresponding cover is $\bar{C} = \{j \in \mathcal{J} : \bar{y}_j = 0\} = \{1, 2\}$, of total coefficient $14 > 11$ and cost $\frac{7}{3} - 2 = \frac{1}{3} < 1$. Hence the cover inequality

$$x_1 + x_2 \leq 1$$

is violated by $\tilde{\mathbf{x}}$: $1 + \frac{2}{3} - 1 = \frac{2}{3}$.

- **Optimality check**

Indices with $q_j = 0$ do not change the value of a solution. The non-empty subsets of the indices $\{2, 3, 4\}$ with $q_j > 0$ that satisfy the constraint with right-hand side 16 are: $\{2\}$ (value $\frac{1}{3}$), $\{3\}$ (value 1), $\{4\}$ (value 1), $\{2, 4\}$ (value $\frac{4}{3}$), $\{3, 4\}$ (value 2). Hence $z(\text{ILP}_{\text{CSP}}^2) = 2$, the greedy solution is optimal, $z(\text{ILP}_{\text{CSP}}) = \frac{1}{3} < 1$, and the maximum violation of a cover inequality is $\frac{2}{3}$.

Exercise 5.5 — Cover inequalities (P5)

Consider a binary problem with one constraint $\sum_j a_j x_j \leq b$, with positive integer coefficients a_j and right-hand side $b = 12$, on the binary variables indexed by the set $\mathcal{J} = \{1, 2, 3, 4\}$ of $n = 4$ variables with the following objective coefficients and constraint coefficients:

j	1	2	3	4
c_j	41	40	24	11
a_j	3	5	5	5

The associated LP is:

$$\begin{array}{llllllll}
 \max & 41x_1 & + & 40x_2 & + & 24x_3 & + & 11x_4 \\
 \text{subject to} & 3x_1 & + & 5x_2 & + & 5x_3 & + & 5x_4 & \leq & 12 \\
 & x_1 & & & & & & & \leq & 1 \\
 & & x_2 & & & & & & \leq & 1 \\
 & & & x_3 & & & & & \leq & 1 \\
 & & & & x_4 & & & & \leq & 1 \\
 & x_1, & x_2, & x_3, & x_4 & \geq & 0
 \end{array}$$

An optimal solution $\tilde{\mathbf{x}}$ of LP and its value are:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 1, \quad \tilde{x}_3 = \frac{4}{5}, \quad \tilde{x}_4 = 0 \quad \text{and} \quad z(\text{LP}) = \frac{501}{5}.$$

Questions

1. List all the cover inequalities and identify the minimal covers.
2. For the point $\tilde{\mathbf{x}}$, write the separation problem ILP_{CSP} and its complementary problem $\text{ILP}_{\text{CSP}}^2$.
3. Apply the greedy heuristic to $\text{ILP}_{\text{CSP}}^2$ and, if possible, find a violated cover inequality.
4. Check whether the greedy solution is optimal.

A subset of indices $C \subseteq \mathcal{J}$ with $\sum_{j \in C} a_j > b$ is a **cover**, and the linear inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

is the **cover inequality** associated with C . Given a point $\tilde{\mathbf{x}}$, a cover inequality violated by $\tilde{\mathbf{x}}$ exists if and only if $z(\text{ILP}_{\text{CSP}}) < 1$, where

$$(\text{ILP}_{\text{CSP}}) \quad \min \left\{ \sum_{j \in \mathcal{J}} (1 - \tilde{x}_j) z_j : \sum_{j \in \mathcal{J}} a_j z_j \geq b + 1, z_j \in \{0, 1\} \forall j \in \mathcal{J} \right\}.$$

Solution

Separation of the cover inequalities violated by $\tilde{\mathbf{x}}$.

- **Cover inequalities**

The covers are the subsets of indices whose total coefficient exceeds $b = 12$:

C	$\sum_{j \in C} a_j$	$ C $	Cover inequality	Minimal
$\{1, 2, 3\}$	13	3	$x_1 + x_2 + x_3 \leq 2$	yes
$\{1, 2, 4\}$	13	3	$x_1 + x_2 + x_4 \leq 2$	yes
$\{1, 3, 4\}$	13	3	$x_1 + x_3 + x_4 \leq 2$	yes
$\{2, 3, 4\}$	15	3	$x_2 + x_3 + x_4 \leq 2$	yes
$\{1, 2, 3, 4\}$	18	4	$x_1 + x_2 + x_3 + x_4 \leq 3$	no

There are 5 covers, of which 4 are minimal. The cover inequalities of the non-minimal covers are implied by those of the minimal covers and by the bounds $x_j \leq 1$.

- **Separation problem**

The costs $q_j = 1 - \tilde{x}_j$ are $q = (0, 0, \frac{1}{5}, 1)$, and the separation problem ILP_{CSP} is:

$$\begin{aligned} \min \quad & \frac{1}{5} z_3 + z_4 \\ \text{subject to} \quad & 3 z_1 + 5 z_2 + 5 z_3 + 5 z_4 \geq 13 \\ & z_1, z_2, z_3, z_4 \in \{0, 1\} \end{aligned}$$

With $y_j = 1 - z_j$, the constant term and the right-hand side of the complementary problem are

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j = \frac{6}{5}, \quad \sum_{j \in \mathcal{J}} a_j - (b + 1) = 18 - 13 = 5,$$

and the complementary problem $\text{ILP}_{\text{CSP}}^2$ is:

$$\begin{aligned} \max \quad & \frac{1}{5} y_3 + y_4 \\ \text{subject to} \quad & 3 y_1 + 5 y_2 + 5 y_3 + 5 y_4 \leq 5 \\ & y_1, y_2, y_3, y_4 \in \{0, 1\} \end{aligned}$$

with $z(\text{ILP}_{\text{CSP}}) = \frac{6}{5} - z(\text{ILP}_{\text{CSP}}^2)$.

- **Greedy heuristic**

The indices are scanned by non-increasing ratio q_j/a_j , starting from the residual right-hand side 5:

j	q_j	a_j	q_j/a_j	Decision	Residual
4	1	5	$\frac{1}{5}$	$\bar{y}_j = 1$	0
3	$\frac{1}{5}$	5	$\frac{1}{25}$	$\bar{y}_j = 0$	0
1	0	3	0	$q_j = 0, \bar{y}_j = 0$	0
2	0	5	0	$q_j = 0, \bar{y}_j = 0$	0

The greedy solution is $\bar{\mathbf{y}} = (0, 0, 0, 1)$, with value 1. The corresponding cover is $\bar{C} = \{j \in \mathcal{J} : \bar{y}_j = 0\} = \{1, 2, 3\}$, of total coefficient 13 > 12 and cost $\frac{6}{5} - 1 = \frac{1}{5} < 1$. Hence the cover inequality

$$x_1 + x_2 + x_3 \leq 2$$

is violated by $\tilde{\mathbf{x}}$: $1 + 1 + \frac{4}{5} - 2 = \frac{4}{5}$.

- **Optimality check**

Indices with $q_j = 0$ do not change the value of a solution. The non-empty subsets of the indices $\{3, 4\}$ with $q_j > 0$ that satisfy the constraint with right-hand side 5 are: $\{3\}$ (value $\frac{1}{5}$), $\{4\}$ (value 1). Hence $z(\text{ILP}_{\text{CSP}}^2) = 1$, the greedy solution is optimal, $z(\text{ILP}_{\text{CSP}}) = \frac{1}{5} < 1$, and the maximum violation of a cover inequality is $\frac{4}{5}$.

5.2 Cover separation in three dimensions

Exercise 5.6 — Cover inequalities in 3D (P1)

Consider a binary problem with one constraint $\sum_j a_j x_j \leq b$, with positive integer coefficients a_j and right-hand side $b = 8$, on the binary variables indexed by the set $\mathcal{J} = \{1, 2, 3\}$ of $n = 3$ variables with the following objective coefficients and constraint coefficients:

j	1	2	3
c_j	12	4	2
a_j	5	4	3

The associated LP is:

$$\begin{array}{llllll}
 \max & 12x_1 & + & 4x_2 & + & 2x_3 \\
 \text{subject to} & 5x_1 & + & 4x_2 & + & 3x_3 & \leq & 8 \\
 & x_1 & & & & & \leq & 1 \\
 & & & x_2 & & & \leq & 1 \\
 & & & & & x_3 & \leq & 1 \\
 & x_1, & & x_2, & & x_3 & \geq & 0
 \end{array}$$

and the following points of its feasible region:

- (a) $\tilde{\mathbf{x}} = (1, \frac{3}{4}, 0)$,
- (b) $\tilde{\mathbf{x}} = (\frac{1}{5}, 1, 1)$,
- (c) $\tilde{\mathbf{x}} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Questions For each point:

1. write the separation problem ILP_{CSP} and its complementary problem $\text{ILP}_{\text{CSP}}^2$;
2. apply the greedy heuristic to $\text{ILP}_{\text{CSP}}^2$ and, if possible, find a violated cover inequality;
3. check whether the greedy solution is optimal;
4. draw the LP relaxation together with the violated cover inequality, if any.

A subset of indices $C \subseteq \mathcal{J}$ with $\sum_{j \in C} a_j > b$ is a **cover**, and the linear inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

is the **cover inequality** associated with C . Given a point $\tilde{\mathbf{x}}$, a cover inequality violated by $\tilde{\mathbf{x}}$ exists if and only if $z(\text{ILP}_{\text{CSP}}) < 1$, where

$$(\text{ILP}_{\text{CSP}}) \quad \min \left\{ \sum_{j \in \mathcal{J}} (1 - \tilde{x}_j) z_j : \sum_{j \in \mathcal{J}} a_j z_j \geq b + 1, z_j \in \{0, 1\} \forall j \in \mathcal{J} \right\}.$$

Solution

Separation of the cover inequalities violated by the three points.

- **Cover inequalities**

All three points belong to the LP relaxation; the left-hand sides are

$$5 + 4 \cdot \frac{3}{4} + 0 = 8, \quad 5 \cdot \frac{1}{5} + 4 + 3 = 8, \quad 5 \cdot \frac{1}{3} + 4 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3} = 4 \leq 8.$$

The covers are the subsets of indices whose total coefficient exceeds $b = 8$:

C	$\sum_{j \in C} a_j$	$ C $	Cover inequality	Minimal
$\{1, 2\}$	9	2	$x_1 + x_2 \leq 1$	yes
$\{1, 2, 3\}$	12	3	$x_1 + x_2 + x_3 \leq 2$	no

In all three cases, the right-hand side of the complementary problem is $\sum_{j \in \mathcal{J}} a_j - (b + 1) = 12 - 9 = 3$.

- **Case (a): $\tilde{\mathbf{x}} = (1, \frac{3}{4}, 0)$**

a) **Separation problem**

The costs $q_j = 1 - \tilde{x}_j$ are $q = (0, \frac{1}{4}, 1)$. The separation problem ILP_{CSP} (left) and, with $y_j = 1 - z_j$, the complementary problem $\text{ILP}_{\text{CSP}}^2$ (right) are:

$$\begin{aligned} \min \quad & \frac{1}{4} z_2 + z_3 \\ \text{subject to} \quad & 5z_1 + 4z_2 + 3z_3 \geq 9 \\ & z_1, z_2, z_3 \in \{0, 1\} \\ \max \quad & \frac{1}{4} y_2 + y_3 \\ \text{subject to} \quad & 5y_1 + 4y_2 + 3y_3 \leq 3 \\ & y_1, y_2, y_3 \in \{0, 1\} \end{aligned}$$

with $z(\text{ILP}_{\text{CSP}}) = n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{5}{4} - z(\text{ILP}_{\text{CSP}}^2)$.

b) **Greedy heuristic**

The indices are scanned by non-increasing ratio q_j/a_j , starting from the residual right-hand side 3:

j	q_j	a_j	q_j/a_j	Decision	Residual
3	1	3	$\frac{1}{3}$	$\bar{y}_j = 1$	0
2	$\frac{1}{4}$	4	$\frac{1}{16}$	$\bar{y}_j = 0$	0
1	0	5	0	$q_j = 0, \bar{y}_j = 0$	0

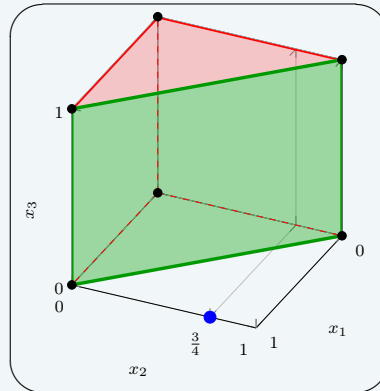
The greedy solution is $\bar{\mathbf{y}} = (0, 0, 1)$, with value 1. The corresponding cover is $\bar{C} = \{j \in \mathcal{J} : \bar{y}_j = 0\} = \{1, 2\}$, of total coefficient $9 > 8$ and cost $\frac{5}{4} - 1 = \frac{1}{4} < 1$. Hence the cover inequality

$$x_1 + x_2 \leq 1$$

is violated by $\tilde{\mathbf{x}}$: $1 + \frac{3}{4} - 1 = \frac{3}{4}$.

c) **Optimality check**

Indices with $q_j = 0$ do not change the value of a solution. The non-empty subsets of the indices $\{2, 3\}$ with $q_j > 0$ that satisfy the constraint with right-hand side 3 are: $\{3\}$ (value 1). Hence $z(\text{ILP}_{\text{CSP}}^2) = 1$, the greedy solution is optimal, $z(\text{ILP}_{\text{CSP}}) = \frac{1}{4} < 1$, and the maximum violation of a cover inequality is $\frac{3}{4}$.



Case (a): the LP relaxation and the cover inequality $x_1 + x_2 \leq 1$.

- **Case (b):** $\tilde{\mathbf{x}} = (\frac{1}{5}, 1, 1)$

a) **Separation problem**

The costs $q_j = 1 - \tilde{x}_j$ are $q = (\frac{4}{5}, 0, 0)$. The separation problem ILP_{CSP} (left) and, with $y_j = 1 - z_j$, the complementary problem $\text{ILP}_{\text{CSP}}^2$ (right) are:

$$\begin{aligned} \min \quad & \frac{4}{5} z_1 \\ \text{subject to} \quad & 5z_1 + 4z_2 + 3z_3 \geq 9 \\ & z_1, \quad z_2, \quad z_3 \in \{0, 1\} \\ \max \quad & \frac{4}{5} y_1 \\ \text{subject to} \quad & 5y_1 + 4y_2 + 3y_3 \leq 3 \\ & y_1, \quad y_2, \quad y_3 \in \{0, 1\} \end{aligned}$$

with $z(\text{ILP}_{\text{CSP}}) = n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{4}{5} - z(\text{ILP}_{\text{CSP}}^2)$.

b) **Greedy heuristic**

The indices are scanned by non-increasing ratio q_j/a_j , starting from the residual right-hand side 3:

j	q_j	a_j	q_j/a_j	Decision	Residual
1	$\frac{4}{5}$	5	$\frac{4}{25}$	$\bar{y}_j = 0$	3
2	0	4	0	$q_j = 0, \bar{y}_j = 0$	3
3	0	3	0	$q_j = 0, \bar{y}_j = 0$	3

The greedy solution is $\bar{\mathbf{y}} = (0, 0, 0)$, with value 0. The corresponding cover is $\bar{C} = \{j \in \mathcal{J} : \bar{y}_j = 0\} = \{1, 2, 3\}$, of total coefficient $12 > 8$ and cost $\frac{4}{5} - 0 = \frac{4}{5} < 1$. Hence the cover inequality

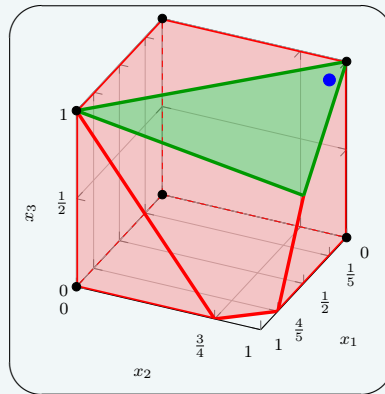
$$x_1 + x_2 + x_3 \leq 2$$

is violated by $\tilde{\mathbf{x}}$: $\frac{1}{5} + 1 + 1 - 2 = \frac{1}{5}$.

c) **Optimality check**

No index with $q_j > 0$ satisfies the constraint with right-hand side 3 alone, hence every feasible solution of $\text{ILP}_{\text{CSP}}^2$ has value 0. Hence $z(\text{ILP}_{\text{CSP}}^2) = 0$, the greedy solution is optimal, $z(\text{ILP}_{\text{CSP}}) = \frac{4}{5} < 1$, and the maximum violation of a cover inequality is $\frac{1}{5}$.

The cover $\bar{C}$ is not minimal: removing an index with $q_j = 0$ gives the minimal cover $\{1, 2\}$, with the same cost $\frac{4}{5}$. Its cover inequality $x_1 + x_2 \leq 1$ has the same violation $\frac{1}{5}$ and is stronger.



Case (b): the LP relaxation and the cover inequality $x_1 + x_2 + x_3 \leq 2$.

• **Case (c):** $\tilde{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

a) **Separation problem**

The costs $q_j = 1 - \tilde{x}_j$ are $q = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$. The separation problem ILP_{CSP} (left) and, with $y_j = 1 - z_j$, the complementary problem $\text{ILP}_{\text{CSP}}^2$ (right) are:

$$\begin{aligned} \min \quad & \frac{2}{3} z_1 + \frac{2}{3} z_2 + \frac{2}{3} z_3 \\ \text{subject to} \quad & 5 z_1 + 4 z_2 + 3 z_3 \geq 9 \\ & z_1, z_2, z_3 \in \{0, 1\} \\ \max \quad & \frac{2}{3} y_1 + \frac{2}{3} y_2 + \frac{2}{3} y_3 \\ \text{subject to} \quad & 5 y_1 + 4 y_2 + 3 y_3 \leq 3 \\ & y_1, y_2, y_3 \in \{0, 1\} \end{aligned}$$

with $z(\text{ILP}_{\text{CSP}}) = n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = 2 - z(\text{ILP}_{\text{CSP}}^2)$.

b) **Greedy heuristic**

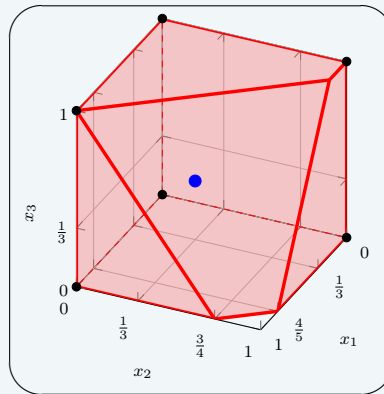
The indices are scanned by non-increasing ratio q_j/a_j , starting from the residual right-hand side 3:

j	q_j	a_j	q_j/a_j	Decision	Residual
3	$\frac{2}{3}$	3	$\frac{2}{9}$	$\bar{y}_j = 1$	0
2	$\frac{2}{3}$	4	$\frac{1}{6}$	$\bar{y}_j = 0$	0
1	$\frac{2}{3}$	5	$\frac{2}{15}$	$\bar{y}_j = 0$	0

The greedy solution is $\bar{y} = (0, 0, 1)$, with value $\frac{2}{3}$. The corresponding cover is $\bar{C} = \{j \in \mathcal{J} : \bar{y}_j = 0\} = \{1, 2\}$, with cost $2 - \frac{2}{3} = \frac{4}{3} \geq 1$: its cover inequality is not violated by $\tilde{x}$.

c) **Optimality check**

Indices with $q_j = 0$ do not change the value of a solution. The non-empty subsets of the indices $\{1, 2, 3\}$ with $q_j > 0$ that satisfy the constraint with right-hand side 3 are: $\{3\}$ (value $\frac{2}{3}$). Hence $z(\text{ILP}_{\text{CSP}}^2) = \frac{2}{3}$, the greedy solution is optimal, and $z(\text{ILP}_{\text{CSP}}) = \frac{4}{3} \geq 1$: **no** cover inequality is violated by $\tilde{x}$.



Case (c): the LP relaxation; no cover inequality is violated by the point.

Exercise 5.7 — Cover inequalities in 3D (P2)

Consider a binary problem with one constraint $\sum_j a_j x_j \leq b$, with positive integer coefficients a_j and right-hand side $b = 11$, on the binary variables indexed by the set $\mathcal{J} = \{1, 2, 3\}$ of $n = 3$ variables with the following objective coefficients and constraint coefficients:

j	1	2	3
c_j	20	11	4
a_j	2	3	9

The associated LP is:

$$\begin{array}{llllll}
 \max & 20x_1 & + & 11x_2 & + & 4x_3 \\
 \text{subject to} & 2x_1 & + & 3x_2 & + & 9x_3 & \leq & 11 \\
 & x_1 & & & & & \leq & 1 \\
 & & & x_2 & & & \leq & 1 \\
 & & & & & x_3 & \leq & 1 \\
 & x_1, & & x_2, & & x_3 & \geq & 0
 \end{array}$$

and the following points of its feasible region:

- (a) $\tilde{\mathbf{x}} = (1, 1, \frac{2}{3})$,
- (b) $\tilde{\mathbf{x}} = (0, \frac{2}{3}, 1)$,
- (c) $\tilde{\mathbf{x}} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Questions For each point:

1. write the separation problem ILP_{CSP} and its complementary problem $\text{ILP}_{\text{CSP}}^2$;
2. apply the greedy heuristic to $\text{ILP}_{\text{CSP}}^2$ and, if possible, find a violated cover inequality;
3. check whether the greedy solution is optimal;
4. draw the LP relaxation together with the violated cover inequality, if any.

A subset of indices $C \subseteq \mathcal{J}$ with $\sum_{j \in C} a_j > b$ is a **cover**, and the linear inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

is the **cover inequality** associated with C . Given a point $\tilde{\mathbf{x}}$, a cover inequality violated by $\tilde{\mathbf{x}}$ exists if and only if $z(\text{ILP}_{\text{CSP}}) < 1$, where

$$(\text{ILP}_{\text{CSP}}) \quad \min \left\{ \sum_{j \in \mathcal{J}} (1 - \tilde{x}_j) z_j : \sum_{j \in \mathcal{J}} a_j z_j \geq b + 1, z_j \in \{0, 1\} \forall j \in \mathcal{J} \right\}.$$

Solution

Separation of the cover inequalities violated by the three points.

- **Cover inequalities**

All three points belong to the LP relaxation; the left-hand sides are

$$2 + 3 + 9 \cdot \frac{2}{3} = 11, \quad 0 + 3 \cdot \frac{2}{3} + 9 = 11, \quad 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3} + 9 \cdot \frac{1}{3} = \frac{14}{3} \leq 11.$$

The covers are the subsets of indices whose total coefficient exceeds $b = 11$:

C	$\sum_{j \in C} a_j$	$ C $	Cover inequality	Minimal
$\{2, 3\}$	12	2	$x_2 + x_3 \leq 1$	yes
$\{1, 2, 3\}$	14	3	$x_1 + x_2 + x_3 \leq 2$	no

In all three cases, the right-hand side of the complementary problem is $\sum_{j \in \mathcal{J}} a_j - (b + 1) = 14 - 12 = 2$.

- **Case (a): $\tilde{x} = (1, 1, \frac{2}{3})$**

a) **Separation problem**

The costs $q_j = 1 - \tilde{x}_j$ are $q = (0, 0, \frac{1}{3})$. The separation problem ILP_{CSP} (left) and, with $y_j = 1 - z_j$, the complementary problem $\text{ILP}_{\text{CSP}}^2$ (right) are:

$$\begin{aligned} \min \quad & \frac{1}{3} z_3 \\ \text{subject to} \quad & 2z_1 + 3z_2 + 9z_3 \geq 12 \\ & z_1, \quad z_2, \quad z_3 \in \{0, 1\} \\ \max \quad & \frac{1}{3} y_3 \\ \text{subject to} \quad & 2y_1 + 3y_2 + 9y_3 \leq 2 \\ & y_1, \quad y_2, \quad y_3 \in \{0, 1\} \end{aligned}$$

$$\text{with } z(\text{ILP}_{\text{CSP}}) = n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{1}{3} - z(\text{ILP}_{\text{CSP}}^2).$$

b) **Greedy heuristic**

The indices are scanned by non-increasing ratio q_j/a_j , starting from the residual right-hand side 2:

j	q_j	a_j	q_j/a_j	Decision	Residual
3	$\frac{1}{3}$	9	$\frac{1}{27}$	$\bar{y}_j = 0$	2
1	0	2	0	$q_j = 0, \bar{y}_j = 0$	2
2	0	3	0	$q_j = 0, \bar{y}_j = 0$	2

The greedy solution is $\bar{\mathbf{y}} = (0, 0, 0)$, with value 0. The corresponding cover is $\bar{C} = \{j \in \mathcal{J} : \bar{y}_j = 0\} = \{1, 2, 3\}$, of total coefficient $14 > 11$ and cost $\frac{1}{3} - 0 = \frac{1}{3} < 1$. Hence the cover inequality

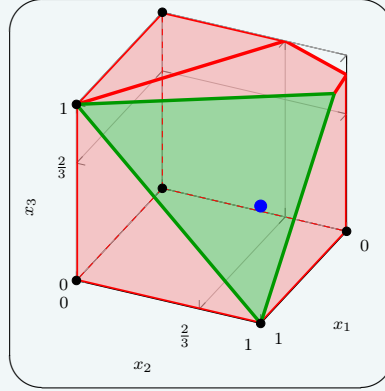
$$x_1 + x_2 + x_3 \leq 2$$

is violated by $\tilde{\mathbf{x}}$: $1 + 1 + \frac{2}{3} - 2 = \frac{2}{3}$.

c) **Optimality check**

No index with $q_j > 0$ satisfies the constraint with right-hand side 2 alone, hence every feasible solution of $\text{ILP}_{\text{CSP}}^2$ has value 0. Hence $z(\text{ILP}_{\text{CSP}}^2) = 0$, the greedy solution is optimal, $z(\text{ILP}_{\text{CSP}}) = \frac{1}{3} < 1$, and the maximum violation of a cover inequality is $\frac{2}{3}$.

The cover $\bar{C}$ is not minimal: removing an index with $q_j = 0$ gives the minimal cover $\{2, 3\}$, with the same cost $\frac{1}{3}$. Its cover inequality $x_2 + x_3 \leq 1$ has the same violation $\frac{2}{3}$ and is stronger.



Case (a): the LP relaxation and the cover inequality $x_1 + x_2 + x_3 \leq 2$.

• **Case (b): $\tilde{x} = (0, \frac{2}{3}, 1)$**

a) **Separation problem**

The costs $q_j = 1 - \tilde{x}_j$ are $q = (1, \frac{1}{3}, 0)$. The separation problem ILP_{CSP} (left) and, with $y_j = 1 - z_j$, the complementary problem $\text{ILP}_{\text{CSP}}^2$ (right) are:

$$\begin{aligned} \min \quad & z_1 + \frac{1}{3} z_2 \\ \text{subject to} \quad & 2z_1 + 3z_2 + 9z_3 \geq 12 \\ & z_1, \quad z_2, \quad z_3 \in \{0, 1\} \\ \max \quad & y_1 + \frac{1}{3} y_2 \\ \text{subject to} \quad & 2y_1 + 3y_2 + 9y_3 \leq 2 \\ & y_1, \quad y_2, \quad y_3 \in \{0, 1\} \end{aligned}$$

with $z(\text{ILP}_{\text{CSP}}) = n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{4}{3} - z(\text{ILP}_{\text{CSP}}^2)$.

b) **Greedy heuristic**

The indices are scanned by non-increasing ratio q_j/a_j , starting from the residual right-hand side 2:

j	q_j	a_j	q_j/a_j	Decision	Residual
1	1	2	$\frac{1}{2}$	$\bar{y}_j = 1$	0
2	$\frac{1}{3}$	3	$\frac{1}{9}$	$\bar{y}_j = 0$	0
3	0	9	0	$q_j = 0, \bar{y}_j = 0$	0

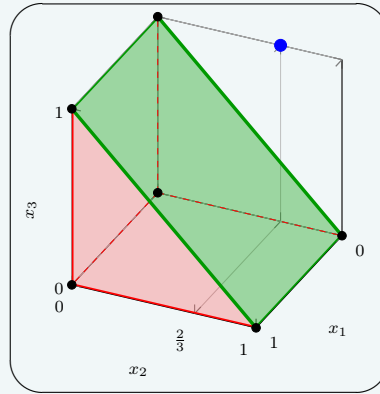
The greedy solution is $\bar{\mathbf{y}} = (1, 0, 0)$, with value 1. The corresponding cover is $\bar{C} = \{j \in \mathcal{J} : \bar{y}_j = 0\} = \{2, 3\}$, of total coefficient $12 > 11$ and cost $\frac{4}{3} - 1 = \frac{1}{3} < 1$. Hence the cover inequality

$$x_2 + x_3 \leq 1$$

is violated by $\tilde{\mathbf{x}}$: $\frac{2}{3} + 1 - 1 = \frac{2}{3}$.

c) **Optimality check**

Indices with $q_j = 0$ do not change the value of a solution. The non-empty subsets of the indices $\{1, 2\}$ with $q_j > 0$ that satisfy the constraint with right-hand side 2 are: $\{1\}$ (value 1). Hence $z(\text{ILP}_{\text{CSP}}^2) = 1$, the greedy solution is optimal, $z(\text{ILP}_{\text{CSP}}) = \frac{1}{3} < 1$, and the maximum violation of a cover inequality is $\frac{2}{3}$.



Case (b): the LP relaxation and the cover inequality $x_2 + x_3 \leq 1$.

• **Case (c):** $\tilde{\mathbf{x}} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

a) **Separation problem**

The costs $q_j = 1 - \tilde{x}_j$ are $q = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$. The separation problem ILP_{CSP} (left) and, with $y_j = 1 - z_j$, the complementary problem $\text{ILP}_{\text{CSP}}^2$ (right) are:

$$\begin{aligned} \min \quad & \frac{2}{3} z_1 + \frac{2}{3} z_2 + \frac{2}{3} z_3 \\ \text{subject to} \quad & 2z_1 + 3z_2 + 9z_3 \geq 12 \\ & z_1, \quad z_2, \quad z_3 \in \{0, 1\} \\ \max \quad & \frac{2}{3} y_1 + \frac{2}{3} y_2 + \frac{2}{3} y_3 \\ \text{subject to} \quad & 2y_1 + 3y_2 + 9y_3 \leq 2 \\ & y_1, \quad y_2, \quad y_3 \in \{0, 1\} \end{aligned}$$

with $z(\text{ILP}_{\text{CSP}}) = n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = 2 - z(\text{ILP}_{\text{CSP}}^2)$.

b) **Greedy heuristic**

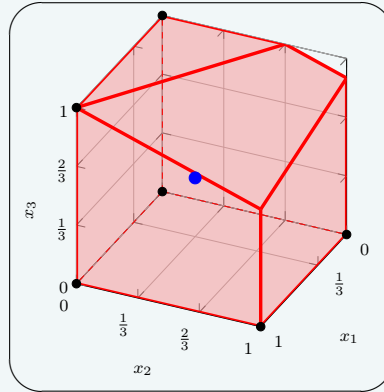
The indices are scanned by non-increasing ratio q_j/a_j , starting from the residual right-hand side 2:

j	q_j	a_j	q_j/a_j	Decision	Residual
1	$\frac{2}{3}$	2	$\frac{1}{3}$	$\bar{y}_j = 1$	0
2	$\frac{2}{3}$	3	$\frac{2}{9}$	$\bar{y}_j = 0$	0
3	$\frac{2}{3}$	9	$\frac{2}{27}$	$\bar{y}_j = 0$	0

The greedy solution is $\bar{\mathbf{y}} = (1, 0, 0)$, with value $\frac{2}{3}$. The corresponding cover is $\bar{C} = \{j \in \mathcal{J} : \bar{y}_j = 0\} = \{2, 3\}$, with cost $2 - \frac{2}{3} = \frac{4}{3} \geq 1$: its cover inequality is not violated by $\tilde{\mathbf{x}}$.

c) **Optimality check**

Indices with $q_j = 0$ do not change the value of a solution. The non-empty subsets of the indices $\{1, 2, 3\}$ with $q_j > 0$ that satisfy the constraint with right-hand side 2 are: $\{1\}$ (value $\frac{2}{3}$). Hence $z(\text{ILP}_{\text{CSP}}^2) = \frac{2}{3}$, the greedy solution is optimal, and $z(\text{ILP}_{\text{CSP}}) = \frac{4}{3} \geq 1$: **no** cover inequality is violated by $\tilde{\mathbf{x}}$.



Case (c): the LP relaxation; no cover inequality is violated by the point.

Exercise 5.8 — Cover inequalities in 3D (P3)

Consider a binary problem with one constraint $\sum_j a_j x_j \leq b$, with positive integer coefficients a_j and right-hand side $b = 9$, on the binary variables indexed by the set $\mathcal{J} = \{1, 2, 3\}$ of $n = 3$ variables with the following objective coefficients and constraint coefficients:

j	1	2	3
c_j	16	19	8
a_j	5	6	7

The associated LP is:

$$\begin{array}{llllll}
 \max & 16x_1 & + & 19x_2 & + & 8x_3 \\
 \text{subject to} & 5x_1 & + & 6x_2 & + & 7x_3 & \leq & 9 \\
 & x_1 & & & & & \leq & 1 \\
 & & & x_2 & & & \leq & 1 \\
 & & & & & x_3 & \leq & 1 \\
 & x_1, & & x_2, & & x_3 & \geq & 0
 \end{array}$$

and the following points of its feasible region:

- (a) $\tilde{\mathbf{x}} = (1, \frac{2}{3}, 0)$,
- (b) $\tilde{\mathbf{x}} = (0, \frac{1}{3}, 1)$,
- (c) $\tilde{\mathbf{x}} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Questions For each point:

1. write the separation problem ILP_{CSP} and its complementary problem $\text{ILP}_{\text{CSP}}^2$;
2. apply the greedy heuristic to $\text{ILP}_{\text{CSP}}^2$ and, if possible, find a violated cover inequality;
3. check whether the greedy solution is optimal;
4. draw the LP relaxation together with the violated cover inequality, if any.

A subset of indices $C \subseteq \mathcal{J}$ with $\sum_{j \in C} a_j > b$ is a **cover**, and the linear inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

is the **cover inequality** associated with C . Given a point $\tilde{\mathbf{x}}$, a cover inequality violated by $\tilde{\mathbf{x}}$ exists if and only if $z(\text{ILP}_{\text{CSP}}) < 1$, where

$$(\text{ILP}_{\text{CSP}}) \quad \min \left\{ \sum_{j \in \mathcal{J}} (1 - \tilde{x}_j) z_j : \sum_{j \in \mathcal{J}} a_j z_j \geq b + 1, z_j \in \{0, 1\} \forall j \in \mathcal{J} \right\}.$$

Solution

Separation of the cover inequalities violated by the three points.

- **Cover inequalities**

All three points belong to the LP relaxation; the left-hand sides are

$$5 + 6 \cdot \frac{2}{3} + 0 = 9, \quad 0 + 6 \cdot \frac{1}{3} + 7 = 9, \quad 5 \cdot \frac{1}{3} + 6 \cdot \frac{1}{3} + 7 \cdot \frac{1}{3} = 6 \leq 9.$$

The covers are the subsets of indices whose total coefficient exceeds $b = 9$:

C	$\sum_{j \in C} a_j$	$ C $	Cover inequality	Minimal
$\{1, 2\}$	11	2	$x_1 + x_2 \leq 1$	yes
$\{1, 3\}$	12	2	$x_1 + x_3 \leq 1$	yes
$\{2, 3\}$	13	2	$x_2 + x_3 \leq 1$	yes
$\{1, 2, 3\}$	18	3	$x_1 + x_2 + x_3 \leq 2$	no

In all three cases, the right-hand side of the complementary problem is $\sum_{j \in \mathcal{J}} a_j - (b + 1) = 18 - 10 = 8$.

• **Case (a):** $\tilde{x} = (1, \frac{2}{3}, 0)$

a) **Separation problem**

The costs $q_j = 1 - \tilde{x}_j$ are $q = (0, \frac{1}{3}, 1)$. The separation problem ILP_{CSP} (left) and, with $y_j = 1 - z_j$, the complementary problem $\text{ILP}_{\text{CSP}}^2$ (right) are:

$$\begin{aligned} \min \quad & \frac{1}{3} z_2 + z_3 \\ \text{subject to} \quad & 5z_1 + 6z_2 + 7z_3 \geq 10 \\ & z_1, \quad z_2, \quad z_3 \in \{0, 1\} \\ \max \quad & \frac{1}{3} y_2 + y_3 \\ \text{subject to} \quad & 5y_1 + 6y_2 + 7y_3 \leq 8 \\ & y_1, \quad y_2, \quad y_3 \in \{0, 1\} \end{aligned}$$

with $z(\text{ILP}_{\text{CSP}}) = n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{4}{3} - z(\text{ILP}_{\text{CSP}}^2)$.

b) **Greedy heuristic**

The indices are scanned by non-increasing ratio q_j/a_j , starting from the residual right-hand side 8:

j	q_j	a_j	q_j/a_j	Decision	Residual
3	1	7	$\frac{1}{7}$	$\bar{y}_j = 1$	1
2	$\frac{1}{3}$	6	$\frac{1}{18}$	$\bar{y}_j = 0$	1
1	0	5	0	$q_j = 0, \bar{y}_j = 0$	1

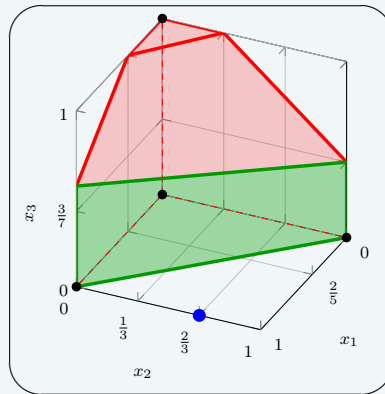
The greedy solution is $\bar{y} = (0, 0, 1)$, with value 1. The corresponding cover is $\bar{C} = \{j \in \mathcal{J} : \bar{y}_j = 0\} = \{1, 2\}$, of total coefficient $11 > 9$ and cost $\frac{4}{3} - 1 = \frac{1}{3} < 1$. Hence the cover inequality

$$x_1 + x_2 \leq 1$$

is violated by $\tilde{x}$: $1 + \frac{2}{3} - 1 = \frac{2}{3}$.

c) **Optimality check**

Indices with $q_j = 0$ do not change the value of a solution. The non-empty subsets of the indices $\{2, 3\}$ with $q_j > 0$ that satisfy the constraint with right-hand side 8 are: $\{2\}$ (value $\frac{1}{3}$), $\{3\}$ (value 1). Hence $z(\text{ILP}_{\text{CSP}}^2) = 1$, the greedy solution is optimal, $z(\text{ILP}_{\text{CSP}}) = \frac{1}{3} < 1$, and the maximum violation of a cover inequality is $\frac{2}{3}$.



Case (a): the LP relaxation and the cover inequality $x_1 + x_2 \leq 1$.

• **Case (b):** $\tilde{x} = (0, \frac{1}{3}, 1)$

a) **Separation problem**

The costs $q_j = 1 - \tilde{x}_j$ are $q = (1, \frac{2}{3}, 0)$. The separation problem ILP_{CSP} (left) and, with $y_j = 1 - z_j$, the complementary problem $\text{ILP}_{\text{CSP}}^2$ (right) are:

$$\begin{aligned} \min \quad & z_1 + \frac{2}{3} z_2 \\ \text{subject to} \quad & 5z_1 + 6z_2 + 7z_3 \geq 10 \\ & z_1, \quad z_2, \quad z_3 \in \{0, 1\} \\ \max \quad & y_1 + \frac{2}{3} y_2 \\ \text{subject to} \quad & 5y_1 + 6y_2 + 7y_3 \leq 8 \\ & y_1, \quad y_2, \quad y_3 \in \{0, 1\} \end{aligned}$$

with $z(\text{ILP}_{\text{CSP}}) = n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{5}{3} - z(\text{ILP}_{\text{CSP}}^2)$.

b) **Greedy heuristic**

The indices are scanned by non-increasing ratio q_j/a_j , starting from the residual right-hand side 8:

j	q_j	a_j	q_j/a_j	Decision	Residual
1	1	5	$\frac{1}{5}$	$\bar{y}_j = 1$	3
2	$\frac{2}{3}$	6	$\frac{1}{9}$	$\bar{y}_j = 0$	3
3	0	7	0	$q_j = 0, \bar{y}_j = 0$	3

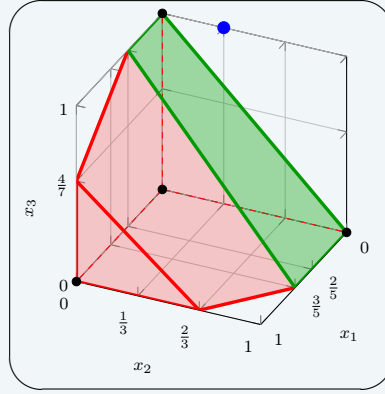
The greedy solution is $\bar{y} = (1, 0, 0)$, with value 1. The corresponding cover is $\bar{C} = \{j \in \mathcal{J} : \bar{y}_j = 0\} = \{2, 3\}$, of total coefficient $13 > 9$ and cost $\frac{5}{3} - 1 = \frac{2}{3} < 1$. Hence the cover inequality

$$x_2 + x_3 \leq 1$$

is violated by $\tilde{x}$: $\frac{1}{3} + 1 - 1 = \frac{1}{3}$.

c) **Optimality check**

Indices with $q_j = 0$ do not change the value of a solution. The non-empty subsets of the indices $\{1, 2\}$ with $q_j > 0$ that satisfy the constraint with right-hand side 8 are: $\{1\}$ (value 1), $\{2\}$ (value $\frac{2}{3}$). Hence $z(\text{ILP}_{\text{CSP}}^2) = 1$, the greedy solution is optimal, $z(\text{ILP}_{\text{CSP}}) = \frac{2}{3} < 1$, and the maximum violation of a cover inequality is $\frac{1}{3}$.



Case (b): the LP relaxation and the cover inequality $x_2 + x_3 \leq 1$.

• **Case (c):** $\tilde{\mathbf{x}} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

a) **Separation problem**

The costs $q_j = 1 - \tilde{x}_j$ are $q = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$. The separation problem ILP_{CSP} (left) and, with $y_j = 1 - z_j$, the complementary problem $\text{ILP}_{\text{CSP}}^2$ (right) are:

$$\begin{aligned} \min \quad & \frac{2}{3} z_1 + \frac{2}{3} z_2 + \frac{2}{3} z_3 \\ \text{subject to} \quad & 5z_1 + 6z_2 + 7z_3 \geq 10 \\ & z_1, z_2, z_3 \in \{0, 1\} \\ \max \quad & \frac{2}{3} y_1 + \frac{2}{3} y_2 + \frac{2}{3} y_3 \\ \text{subject to} \quad & 5y_1 + 6y_2 + 7y_3 \leq 8 \\ & y_1, y_2, y_3 \in \{0, 1\} \end{aligned}$$

with $z(\text{ILP}_{\text{CSP}}) = n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = 2 - z(\text{ILP}_{\text{CSP}}^2)$.

b) **Greedy heuristic**

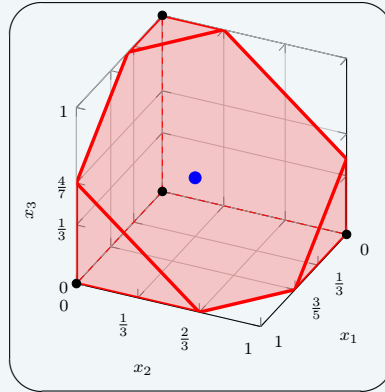
The indices are scanned by non-increasing ratio q_j/a_j , starting from the residual right-hand side 8:

j	q_j	a_j	q_j/a_j	Decision	Residual
1	$\frac{2}{3}$	5	$\frac{2}{15}$	$\bar{y}_j = 1$	3
2	$\frac{2}{3}$	6	$\frac{1}{9}$	$\bar{y}_j = 0$	3
3	$\frac{2}{3}$	7	$\frac{2}{21}$	$\bar{y}_j = 0$	3

The greedy solution is $\bar{\mathbf{y}} = (1, 0, 0)$, with value $\frac{2}{3}$. The corresponding cover is $\bar{C} = \{j \in \mathcal{J} : \bar{y}_j = 0\} = \{2, 3\}$, with cost $2 - \frac{2}{3} = \frac{4}{3} \geq 1$: its cover inequality is not violated by $\tilde{\mathbf{x}}$.

c) **Optimality check**

Indices with $q_j = 0$ do not change the value of a solution. The non-empty subsets of the indices $\{1, 2, 3\}$ with $q_j > 0$ that satisfy the constraint with right-hand side 8 are: $\{1\}$ (value $\frac{2}{3}$), $\{2\}$ (value $\frac{2}{3}$), $\{3\}$ (value $\frac{2}{3}$). Hence $z(\text{ILP}_{\text{CSP}}^2) = \frac{2}{3}$, the greedy solution is optimal, and $z(\text{ILP}_{\text{CSP}}) = \frac{4}{3} \geq 1$: **no** cover inequality is violated by $\tilde{\mathbf{x}}$.



Case (c): the LP relaxation; no cover inequality is violated by the point.

Exercise 5.9 — Cover inequalities in 3D ($P4$)

Consider a binary problem with one constraint $\sum_j a_j x_j \leq b$, with positive integer coefficients a_j and right-hand side $b = 5$, on the binary variables indexed by the set $\mathcal{J} = \{1, 2, 3\}$ of $n = 3$ variables with the following objective coefficients and constraint coefficients:

j	1	2	3
c_j	14	17	8
a_j	2	5	5

The associated LP is:

$$\begin{aligned}
 \max \quad & 14x_1 + 17x_2 + 8x_3 \\
 \text{subject to} \quad & 2x_1 + 5x_2 + 5x_3 \leq 5 \\
 & x_1 \leq 1 \\
 & x_2 \leq 1 \\
 & x_3 \leq 1 \\
 & x_1, x_2, x_3 \geq 0
 \end{aligned}$$

and the following points of its feasible region:

- (a) $\tilde{\mathbf{x}} = (1, \frac{3}{5}, 0)$,
- (b) $\tilde{\mathbf{x}} = (1, 0, \frac{3}{5})$,
- (c) $\tilde{\mathbf{x}} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Questions For each point:

1. write the separation problem ILP_{CSP} and its complementary problem $\text{ILP}_{\text{CSP}}^2$;
2. apply the greedy heuristic to $\text{ILP}_{\text{CSP}}^2$ and, if possible, find a violated cover inequality;
3. check whether the greedy solution is optimal;
4. draw the LP relaxation together with the violated cover inequality, if any.

A subset of indices $C \subseteq \mathcal{J}$ with $\sum_{j \in C} a_j > b$ is a **cover**, and the linear inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

is the **cover inequality** associated with C . Given a point $\tilde{x}$, a cover inequality violated by $\tilde{x}$ exists if and only if $z(\text{ILP}_{\text{CSP}}) < 1$, where

$$(\text{ILP}_{\text{CSP}}) \quad \min \left\{ \sum_{j \in \mathcal{J}} (1 - \tilde{x}_j) z_j : \sum_{j \in \mathcal{J}} a_j z_j \geq b + 1, z_j \in \{0, 1\} \forall j \in \mathcal{J} \right\}.$$

Solution

Separation of the cover inequalities violated by the three points.

- **Cover inequalities**

All three points belong to the LP relaxation; the left-hand sides are

$$2 + 5 \cdot \frac{3}{5} + 0 = 5, \quad 2 + 0 + 5 \cdot \frac{3}{5} = 5, \quad 2 \cdot \frac{1}{3} + 5 \cdot \frac{1}{3} + 5 \cdot \frac{1}{3} = 4 \leq 5.$$

The covers are the subsets of indices whose total coefficient exceeds $b = 5$:

C	$\sum_{j \in C} a_j$	$ C $	Cover inequality	Minimal
$\{1, 2\}$	7	2	$x_1 + x_2 \leq 1$	yes
$\{1, 3\}$	7	2	$x_1 + x_3 \leq 1$	yes
$\{2, 3\}$	10	2	$x_2 + x_3 \leq 1$	yes
$\{1, 2, 3\}$	12	3	$x_1 + x_2 + x_3 \leq 2$	no

In all three cases, the right-hand side of the complementary problem is $\sum_{j \in \mathcal{J}} a_j - (b + 1) = 12 - 6 = 6$.

- **Case (a): $\tilde{x} = (1, \frac{3}{5}, 0)$**

a) **Separation problem**

The costs $q_j = 1 - \tilde{x}_j$ are $q = (0, \frac{2}{5}, 1)$. The separation problem ILP_{CSP} (left) and, with $y_j = 1 - z_j$, the complementary problem $\text{ILP}_{\text{CSP}}^2$ (right) are:

$$\begin{aligned} & \min && \frac{2}{5} z_2 + z_3 \\ & \text{subject to} && 2 z_1 + 5 z_2 + 5 z_3 \geq 6 \\ & && z_1, z_2, z_3 \in \{0, 1\} \\ & \max && \frac{2}{5} y_2 + y_3 \\ & \text{subject to} && 2 y_1 + 5 y_2 + 5 y_3 \leq 6 \\ & && y_1, y_2, y_3 \in \{0, 1\} \end{aligned}$$

$$\text{with } z(\text{ILP}_{\text{CSP}}) = n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{7}{5} - z(\text{ILP}_{\text{CSP}}^2).$$

b) **Greedy heuristic**

The indices are scanned by non-increasing ratio q_j/a_j , starting from the residual right-hand side 6:

j	q_j	a_j	q_j/a_j	Decision	Residual
3	1	5	$\frac{1}{5}$	$\bar{y}_j = 1$	1
2	$\frac{2}{5}$	5	$\frac{2}{25}$	$\bar{y}_j = 0$	1
1	0	2	0	$q_j = 0, \bar{y}_j = 0$	1

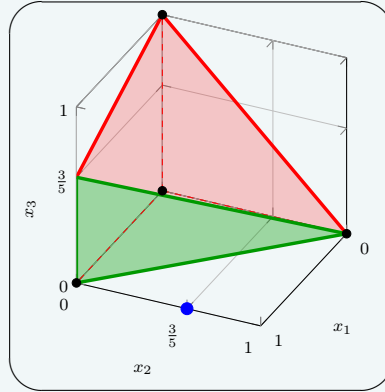
The greedy solution is $\bar{\mathbf{y}} = (0, 0, 1)$, with value 1. The corresponding cover is $\bar{C} = \{j \in \mathcal{J} : \bar{y}_j = 0\} = \{1, 2\}$, of total coefficient $7 > 5$ and cost $\frac{7}{5} - 1 = \frac{2}{5} < 1$. Hence the cover inequality

$$x_1 + x_2 \leq 1$$

is violated by $\tilde{\mathbf{x}}$: $1 + \frac{3}{5} - 1 = \frac{3}{5}$.

c) **Optimality check**

Indices with $q_j = 0$ do not change the value of a solution. The non-empty subsets of the indices $\{2, 3\}$ with $q_j > 0$ that satisfy the constraint with right-hand side 6 are: $\{2\}$ (value $\frac{2}{5}$), $\{3\}$ (value 1). Hence $z(\text{ILP}_{\text{CSP}}^2) = 1$, the greedy solution is optimal, $z(\text{ILP}_{\text{CSP}}) = \frac{2}{5} < 1$, and the maximum violation of a cover inequality is $\frac{3}{5}$.



Case (a): the LP relaxation and the cover inequality $x_1 + x_2 \leq 1$.

• **Case (b): $\tilde{\mathbf{x}} = (1, 0, \frac{3}{5})$**

a) **Separation problem**

The costs $q_j = 1 - \tilde{x}_j$ are $q = (0, 1, \frac{2}{5})$. The separation problem ILP_{CSP} (left) and, with $y_j = 1 - z_j$, the complementary problem $\text{ILP}_{\text{CSP}}^2$ (right) are:

$$\begin{aligned}
 & \min && z_2 + \frac{2}{5} z_3 \\
 & \text{subject to} && 2z_1 + 5z_2 + 5z_3 \geq 6 \\
 & && z_1, \quad z_2, \quad z_3 \in \{0, 1\} \\
 & \max && y_2 + \frac{2}{5} y_3 \\
 & \text{subject to} && 2y_1 + 5y_2 + 5y_3 \leq 6 \\
 & && y_1, \quad y_2, \quad y_3 \in \{0, 1\}
 \end{aligned}$$

with $z(\text{ILP}_{\text{CSP}}) = n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{7}{5} - z(\text{ILP}_{\text{CSP}}^2)$.

b) **Greedy heuristic**

The indices are scanned by non-increasing ratio q_j/a_j , starting from the residual right-hand side 6:

j	q_j	a_j	q_j/a_j	Decision	Residual
2	1	5	$\frac{1}{5}$	$\bar{y}_j = 1$	1
3	$\frac{2}{5}$	5	$\frac{2}{25}$	$\bar{y}_j = 0$	1
1	0	2	0	$q_j = 0, \bar{y}_j = 0$	1

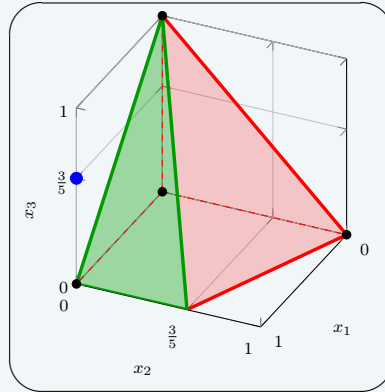
The greedy solution is $\bar{\mathbf{y}} = (0, 1, 0)$, with value 1. The corresponding cover is $\bar{C} = \{j \in \mathcal{J} : \bar{y}_j = 0\} = \{1, 3\}$, of total coefficient $7 > 5$ and cost $\frac{7}{5} - 1 = \frac{2}{5} < 1$. Hence the cover inequality

$$x_1 + x_3 \leq 1$$

is violated by $\tilde{\mathbf{x}}$: $1 + \frac{3}{5} - 1 = \frac{3}{5}$.

c) **Optimality check**

Indices with $q_j = 0$ do not change the value of a solution. The non-empty subsets of the indices $\{2, 3\}$ with $q_j > 0$ that satisfy the constraint with right-hand side 6 are: $\{2\}$ (value 1), $\{3\}$ (value $\frac{2}{5}$). Hence $z(\text{ILP}_{\text{CSP}}^2) = 1$, the greedy solution is optimal, $z(\text{ILP}_{\text{CSP}}) = \frac{2}{5} < 1$, and the maximum violation of a cover inequality is $\frac{3}{5}$.



Case (b): the LP relaxation and the cover inequality $x_1 + x_3 \leq 1$.

• **Case (c):** $\tilde{\mathbf{x}} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

a) **Separation problem**

The costs $q_j = 1 - \tilde{x}_j$ are $q = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$. The separation problem ILP_{CSP} (left) and, with $y_j = 1 - z_j$, the complementary problem $\text{ILP}_{\text{CSP}}^2$ (right) are:

$$\begin{aligned} \min \quad & \frac{2}{3} z_1 + \frac{2}{3} z_2 + \frac{2}{3} z_3 \\ \text{subject to} \quad & 2 z_1 + 5 z_2 + 5 z_3 \geq 6 \\ & z_1, \quad z_2, \quad z_3 \in \{0, 1\} \end{aligned}$$

$$\begin{aligned} \max \quad & \frac{2}{3} y_1 + \frac{2}{3} y_2 + \frac{2}{3} y_3 \\ \text{subject to} \quad & 2 y_1 + 5 y_2 + 5 y_3 \leq 6 \\ & y_1, \quad y_2, \quad y_3 \in \{0, 1\} \end{aligned}$$

with $z(\text{ILP}_{\text{CSP}}) = n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = 2 - z(\text{ILP}_{\text{CSP}}^2)$.

b) **Greedy heuristic**

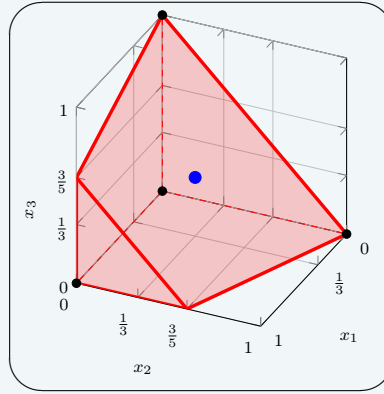
The indices are scanned by non-increasing ratio q_j/a_j , starting from the residual right-hand side 6:

j	q_j	a_j	q_j/a_j	Decision	Residual
1	$\frac{2}{3}$	2	$\frac{1}{3}$	$\bar{y}_j = 1$	4
2	$\frac{2}{3}$	5	$\frac{2}{15}$	$\bar{y}_j = 0$	4
3	$\frac{2}{3}$	5	$\frac{2}{15}$	$\bar{y}_j = 0$	4

The greedy solution is $\bar{\mathbf{y}} = (1, 0, 0)$, with value $\frac{2}{3}$. The corresponding cover is $\bar{C} = \{j \in \mathcal{J} : \bar{y}_j = 0\} = \{2, 3\}$, with cost $2 - \frac{2}{3} = \frac{4}{3} \geq 1$: its cover inequality is not violated by $\tilde{\mathbf{x}}$.

c) **Optimality check**

Indices with $q_j = 0$ do not change the value of a solution. The non-empty subsets of the indices $\{1, 2, 3\}$ with $q_j > 0$ that satisfy the constraint with right-hand side 6 are: $\{1\}$ (value $\frac{2}{3}$), $\{2\}$ (value $\frac{2}{3}$), $\{3\}$ (value $\frac{2}{3}$). Hence $z(\text{ILP}_{\text{CSP}}^2) = \frac{2}{3}$, the greedy solution is optimal, and $z(\text{ILP}_{\text{CSP}}) = \frac{4}{3} \geq 1$: **no** cover inequality is violated by $\tilde{\mathbf{x}}$.



Case (c): the LP relaxation; no cover inequality is violated by the point.

Exercise 5.10 — Cover inequalities in 3D (P5)

Consider a binary problem with one constraint $\sum_j a_j x_j \leq b$, with positive integer coefficients a_j and right-hand side $b = 10$, on the binary variables indexed by the set $\mathcal{J} = \{1, 2, 3\}$ of $n = 3$ variables with the following objective coefficients and constraint coefficients:

j	1	2	3
c_j	13	10	3
a_j	5	4	6

The associated LP is:

$$\begin{array}{llllll}
 \max & 13x_1 & + & 10x_2 & + & 3x_3 \\
 \text{subject to} & 5x_1 & + & 4x_2 & + & 6x_3 \leq 10 \\
 & x_1 & & & & \leq 1 \\
 & & & x_2 & & \leq 1 \\
 & & & & & x_3 \leq 1 \\
 & x_1, & & x_2, & & x_3 \geq 0
 \end{array}$$

and the following points of its feasible region:

- (a) $\tilde{\mathbf{x}} = (1, 1, \frac{1}{6})$,
- (b) $\tilde{\mathbf{x}} = (\frac{4}{5}, 0, 1)$,
- (c) $\tilde{\mathbf{x}} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Questions For each point:

1. write the separation problem ILP_{CSP} and its complementary problem $\text{ILP}_{\text{CSP}}^2$;
2. apply the greedy heuristic to $\text{ILP}_{\text{CSP}}^2$ and, if possible, find a violated cover inequality;
3. check whether the greedy solution is optimal;
4. draw the LP relaxation together with the violated cover inequality, if any.

A subset of indices $C \subseteq \mathcal{J}$ with $\sum_{j \in C} a_j > b$ is a **cover**, and the linear inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

is the **cover inequality** associated with C . Given a point $\tilde{\mathbf{x}}$, a cover inequality violated by $\tilde{\mathbf{x}}$ exists if and only if $z(\text{ILP}_{\text{CSP}}) < 1$, where

$$(\text{ILP}_{\text{CSP}}) \quad \min \left\{ \sum_{j \in \mathcal{J}} (1 - \tilde{x}_j) z_j : \sum_{j \in \mathcal{J}} a_j z_j \geq b + 1, z_j \in \{0, 1\} \forall j \in \mathcal{J} \right\}.$$

Solution

Separation of the cover inequalities violated by the three points.

- **Cover inequalities**

All three points belong to the LP relaxation; the left-hand sides are

$$5 + 4 + 6 \cdot \frac{1}{6} = 10, \quad 5 \cdot \frac{4}{5} + 0 + 6 = 10, \quad 5 \cdot \frac{1}{3} + 4 \cdot \frac{1}{3} + 6 \cdot \frac{1}{3} = 5 \leq 10.$$

The covers are the subsets of indices whose total coefficient exceeds $b = 10$:

C	$\sum_{j \in C} a_j$	$ C $	Cover inequality	Minimal
$\{1, 3\}$	11	2	$x_1 + x_3 \leq 1$	yes
$\{1, 2, 3\}$	15	3	$x_1 + x_2 + x_3 \leq 2$	no

In all three cases, the right-hand side of the complementary problem is $\sum_{j \in \mathcal{J}} a_j - (b + 1) = 15 - 11 = 4$.

• **Case (a):** $\tilde{x} = (1, 1, \frac{1}{6})$

a) **Separation problem**

The costs $q_j = 1 - \tilde{x}_j$ are $q = (0, 0, \frac{5}{6})$. The separation problem ILP_{CSP} (left) and, with $y_j = 1 - z_j$, the complementary problem $\text{ILP}_{\text{CSP}}^2$ (right) are:

$$\begin{aligned} \min \quad & \frac{5}{6} z_3 \\ \text{subject to} \quad & 5z_1 + 4z_2 + 6z_3 \geq 11 \\ & z_1, \quad z_2, \quad z_3 \in \{0, 1\} \\ \max \quad & \frac{5}{6} y_3 \\ \text{subject to} \quad & 5y_1 + 4y_2 + 6y_3 \leq 4 \\ & y_1, \quad y_2, \quad y_3 \in \{0, 1\} \end{aligned}$$

with $z(\text{ILP}_{\text{CSP}}) = n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{5}{6} - z(\text{ILP}_{\text{CSP}}^2)$.

b) **Greedy heuristic**

The indices are scanned by non-increasing ratio q_j/a_j , starting from the residual right-hand side 4:

j	q_j	a_j	q_j/a_j	Decision	Residual
3	$\frac{5}{6}$	6	$\frac{5}{36}$	$\bar{y}_j = 0$	4
1	0	5	0	$q_j = 0, \bar{y}_j = 0$	4
2	0	4	0	$q_j = 0, \bar{y}_j = 0$	4

The greedy solution is $\bar{y} = (0, 0, 0)$, with value 0. The corresponding cover is $\bar{C} = \{j \in \mathcal{J} : \bar{y}_j = 0\} = \{1, 2, 3\}$, of total coefficient $15 > 10$ and cost $\frac{5}{6} - 0 = \frac{5}{6} < 1$. Hence the cover inequality

$$x_1 + x_2 + x_3 \leq 2$$

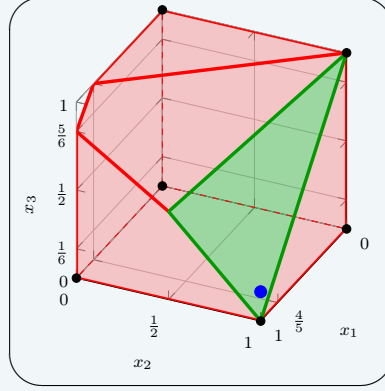
is violated by $\tilde{x}$: $1 + 1 + \frac{1}{6} - 2 = \frac{1}{6}$.

c) **Optimality check**

No index with $q_j > 0$ satisfies the constraint with right-hand side 4 alone, hence every feasible solution of $\text{ILP}_{\text{CSP}}^2$ has value 0. Hence $z(\text{ILP}_{\text{CSP}}^2) = 0$, the greedy

solution is optimal, $z(\text{ILP}_{\text{CSP}}) = \frac{5}{6} < 1$, and the maximum violation of a cover inequality is $\frac{1}{6}$.

The cover $\bar{C}$ is not minimal: removing an index with $q_j = 0$ gives the minimal cover $\{1, 3\}$, with the same cost $\frac{5}{6}$. Its cover inequality $x_1 + x_3 \leq 1$ has the same violation $\frac{1}{6}$ and is stronger.



Case (a): the LP relaxation and the cover inequality $x_1 + x_2 + x_3 \leq 2$.

• **Case (b):** $\tilde{x} = (\frac{4}{5}, 0, 1)$

a) **Separation problem**

The costs $q_j = 1 - \tilde{x}_j$ are $q = (\frac{1}{5}, 1, 0)$. The separation problem ILP_{CSP} (left) and, with $y_j = 1 - z_j$, the complementary problem $\text{ILP}_{\text{CSP}}^2$ (right) are:

$$\begin{aligned} \min \quad & \frac{1}{5} z_1 + z_2 \\ \text{subject to} \quad & 5z_1 + 4z_2 + 6z_3 \geq 11 \\ & z_1, \quad z_2, \quad z_3 \in \{0, 1\} \\ \max \quad & \frac{1}{5} y_1 + y_2 \\ \text{subject to} \quad & 5y_1 + 4y_2 + 6y_3 \leq 4 \\ & y_1, \quad y_2, \quad y_3 \in \{0, 1\} \end{aligned}$$

with $z(\text{ILP}_{\text{CSP}}) = n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{6}{5} - z(\text{ILP}_{\text{CSP}}^2)$.

b) **Greedy heuristic**

The indices are scanned by non-increasing ratio q_j/a_j , starting from the residual right-hand side 4:

j	q_j	a_j	q_j/a_j	Decision	Residual
2	1	4	$\frac{1}{4}$	$\bar{y}_j = 1$	0
1	$\frac{1}{5}$	5	$\frac{1}{25}$	$\bar{y}_j = 0$	0
3	0	6	0	$q_j = 0, \bar{y}_j = 0$	0

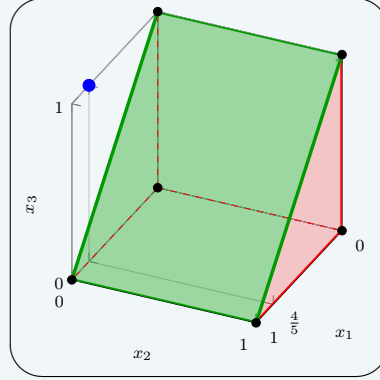
The greedy solution is $\bar{y} = (0, 1, 0)$, with value 1. The corresponding cover is $\bar{C} = \{j \in \mathcal{J} : \bar{y}_j = 0\} = \{1, 3\}$, of total coefficient $11 > 10$ and cost $\frac{6}{5} - 1 = \frac{1}{5} < 1$. Hence the cover inequality

$$x_1 + x_3 \leq 1$$

is violated by $\tilde{\mathbf{x}}$: $\frac{4}{5} + 1 - 1 = \frac{4}{5}$.

c) **Optimality check**

Indices with $q_j = 0$ do not change the value of a solution. The non-empty subsets of the indices $\{1, 2\}$ with $q_j > 0$ that satisfy the constraint with right-hand side 4 are: $\{2\}$ (value 1). Hence $z(\text{ILP}_{\text{CSP}}^2) = 1$, the greedy solution is optimal, $z(\text{ILP}_{\text{CSP}}) = \frac{1}{5} < 1$, and the maximum violation of a cover inequality is $\frac{4}{5}$.



Case (b): the LP relaxation and the cover inequality $x_1 + x_3 \leq 1$.

• **Case (c): $\tilde{\mathbf{x}} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$**

a) **Separation problem**

The costs $q_j = 1 - \tilde{x}_j$ are $q = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$. The separation problem ILP_{CSP} (left) and, with $y_j = 1 - z_j$, the complementary problem $\text{ILP}_{\text{CSP}}^2$ (right) are:

$$\begin{aligned} \min \quad & \frac{2}{3} z_1 + \frac{2}{3} z_2 + \frac{2}{3} z_3 \\ \text{subject to} \quad & 5z_1 + 4z_2 + 6z_3 \geq 11 \\ & z_1, z_2, z_3 \in \{0, 1\} \\ \max \quad & \frac{2}{3} y_1 + \frac{2}{3} y_2 + \frac{2}{3} y_3 \\ \text{subject to} \quad & 5y_1 + 4y_2 + 6y_3 \leq 4 \\ & y_1, y_2, y_3 \in \{0, 1\} \end{aligned}$$

with $z(\text{ILP}_{\text{CSP}}) = n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = 2 - z(\text{ILP}_{\text{CSP}}^2)$.

b) **Greedy heuristic**

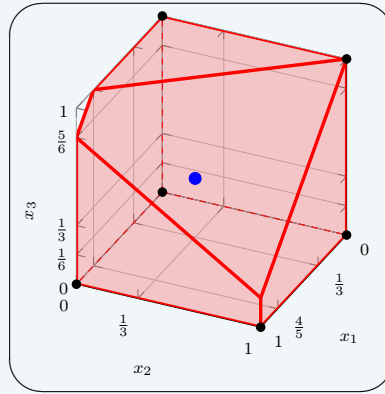
The indices are scanned by non-increasing ratio q_j/a_j , starting from the residual right-hand side 4:

j	q_j	a_j	q_j/a_j	Decision	Residual
2	$\frac{2}{3}$	4	$\frac{1}{6}$	$\bar{y}_j = 1$	0
1	$\frac{2}{3}$	5	$\frac{2}{15}$	$\bar{y}_j = 0$	0
3	$\frac{2}{3}$	6	$\frac{1}{9}$	$\bar{y}_j = 0$	0

The greedy solution is $\bar{\mathbf{y}} = (0, 1, 0)$, with value $\frac{2}{3}$. The corresponding cover is $\bar{C} = \{j \in \mathcal{J} : \bar{y}_j = 0\} = \{1, 3\}$, with cost $2 - \frac{2}{3} = \frac{4}{3} \geq 1$: its cover inequality is not violated by $\tilde{\mathbf{x}}$.

c) **Optimality check**

Indices with $q_j = 0$ do not change the value of a solution. The non-empty subsets of the indices $\{1, 2, 3\}$ with $q_j > 0$ that satisfy the constraint with right-hand side 4 are: $\{2\}$ (value $\frac{2}{3}$). Hence $z(\text{ILP}_{\text{CSP}}^2) = \frac{2}{3}$, the greedy solution is optimal, and $z(\text{ILP}_{\text{CSP}}) = \frac{4}{3} \geq 1$: **no** cover inequality is violated by $\tilde{x}$.



Case (c): the LP relaxation; no cover inequality is violated by the point.

5.3 Closure and separation

Exercise 5.11 — Cover closure by separation ($P1$)

Consider a binary problem with one constraint $\sum_j a_j x_j \leq b$, with positive integer coefficients a_j and right-hand side $b = 9$, on the binary variables indexed by the set $\mathcal{J} = \{1, 2, 3, 4\}$ of $n = 4$ variables with the following objective coefficients and constraint coefficients:

j	1	2	3	4
c_j	22	33	9	13
a_j	5	9	3	6

The associated LP is:

$$\begin{array}{llllll}
 \max & 22x_1 & + & 33x_2 & + & 9x_3 & + & 13x_4 \\
 \text{subject to} & 5x_1 & + & 9x_2 & + & 3x_3 & + & 6x_4 & \leq & 9 \\
 & x_1 & & & & & & & \leq & 1 \\
 & & x_2 & & & & & & \leq & 1 \\
 & & & x_3 & & & & & \leq & 1 \\
 & & & & x_4 & & & & \leq & 1 \\
 & x_1, & x_2, & x_3, & x_4 & \geq & 0
 \end{array}$$

An optimal solution $\tilde{x}$ of LP and its value are:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = \frac{4}{9}, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{and} \quad z(\text{LP}) = \frac{110}{3}.$$

Questions

1. Starting from $\tilde{\mathbf{x}}$, separate the cover inequalities by solving the complementary problem $\text{ILP}_{\text{CSP}}^2$ to optimality. Add the maximally violated cover inequality to the LP, solve it again, and repeat until no cover inequality is violated. If $\text{ILP}_{\text{CSP}}^2$ has several optimal solutions, choose the one that selects the largest number of variables.
2. What is the optimal value over the cover closure? Is the final solution integer?

The **cover closure** is the polyhedron obtained by adding to LP all the cover inequalities $\sum_{j \in C} x_j \leq |C| - 1$, $C \in \mathcal{C}$. Given a point $\tilde{\mathbf{x}}$, let $\tilde{y}_1, \dots, \tilde{y}_n$ be an optimal solution of the complementary problem

$$(\text{ILP}_{\text{CSP}}^2) \quad \max \left\{ \sum_{j \in \mathcal{J}} (1 - \tilde{x}_j) y_j : \sum_{j \in \mathcal{J}} a_j y_j \leq \sum_{j \in \mathcal{J}} a_j - (b + 1), y_j \in \{0, 1\} \forall j \in \mathcal{J} \right\}.$$

If $n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) < 1$, the cover $\tilde{C} = \{j \in \mathcal{J} : \tilde{y}_j = 0\}$ is associated with the cover inequality maximally violated by $\tilde{\mathbf{x}}$; otherwise, no cover inequality is violated by $\tilde{\mathbf{x}}$.

Solution

Separation of the cover inequalities with the complementary problem; in all the iterations, the right-hand side of $\text{ILP}_{\text{CSP}}^2$ is $\sum_{j \in \mathcal{J}} a_j - (b + 1) = 23 - 10 = 13$.

- **Iteration 1**

The optimal solution of LP is $\tilde{\mathbf{x}} = (1, \frac{4}{9}, 0, 0)$, with value $z(\text{LP}) = \frac{110}{3}$.

The costs are $q = (0, \frac{5}{9}, 1, 1)$, the constant term is $n - \sum_{j \in \mathcal{J}} \tilde{x}_j = \frac{23}{9}$, and the complementary problem is:

$$\begin{array}{llllll} \max & & \frac{5}{9} y_2 & + & y_3 & + & y_4 \\ \text{subject to} & 5 y_1 & + & 9 y_2 & + & 3 y_3 & + & 6 y_4 \leq 13 \\ & y_1, & & y_2, & & y_3, & & y_4 \in \{0, 1\} \end{array}$$

The maximal subsets of indices that satisfy the constraint with right-hand side 13 are: $\{1, 3\}$ (value 1), $\{1, 4\}$ (value 1), $\{2, 3\}$ (value $\frac{14}{9}$), $\{3, 4\}$ (value 2). An optimal solution of $\text{ILP}_{\text{CSP}}^2$ is $\tilde{y}_1 = 0$, $\tilde{y}_2 = 0$, $\tilde{y}_3 = 1$, $\tilde{y}_4 = 1$, with value $z(\text{ILP}_{\text{CSP}}^2) = 2$:

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{23}{9} - 2 = \frac{5}{9} < 1.$$

Accordingly, $\tilde{C} = \{j \in \mathcal{J} : \tilde{y}_j = 0\} = \{1, 2\}$ and the cover inequality

$$x_1 + x_2 \leq 1$$

is maximally violated by $\tilde{\mathbf{x}}$: $1 + \frac{4}{9} - 1 = \frac{4}{9}$.

- **Iteration 2**

Adding the cover inequality $x_1 + x_2 \leq 1$, we obtain the following LP, denoted LP_1 :

$$\begin{array}{llllll}
\max & 22x_1 & + & 33x_2 & + & 9x_3 & + & 13x_4 \\
\text{subject to} & 5x_1 & + & 9x_2 & + & 3x_3 & + & 6x_4 \leq 9 \\
& x_1 & + & x_2 & & & & \leq 1 \\
& & & x_1 & & & & \leq 1 \\
& & & & x_2 & & & \leq 1 \\
& & & & & x_3 & & \leq 1 \\
& & & & & & x_4 & \leq 1 \\
& x_1, & & x_2, & & x_3, & & x_4 \geq 0
\end{array}$$

An optimal solution of LP_1 is $\tilde{\mathbf{x}} = (\frac{3}{4}, \frac{1}{4}, 1, 0)$, with value $z(LP_1) = \frac{135}{4}$.

The costs are $q = (\frac{1}{4}, \frac{3}{4}, 0, 1)$, the constant term is $n - \sum_{j \in \mathcal{J}} \tilde{x}_j = 2$, and the complementary problem is:

$$\begin{array}{llllll}
\max & \frac{1}{4}y_1 & + & \frac{3}{4}y_2 & & + & y_4 \\
\text{subject to} & 5y_1 & + & 9y_2 & + & 3y_3 & + & 6y_4 \leq 13 \\
& y_1, & & y_2, & & y_3, & & y_4 \in \{0, 1\}
\end{array}$$

The maximal subsets of indices that satisfy the constraint with right-hand side 13 are: $\{1, 3\}$ (value $\frac{1}{4}$), $\{1, 4\}$ (value $\frac{5}{4}$), $\{2, 3\}$ (value $\frac{3}{4}$), $\{3, 4\}$ (value 1). An optimal solution of ILP_{CSP}^2 is $\tilde{y}_1 = 1$, $\tilde{y}_2 = 0$, $\tilde{y}_3 = 0$, $\tilde{y}_4 = 1$, with value $z(ILP_{CSP}^2) = \frac{5}{4}$:

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(ILP_{CSP}^2) = 2 - \frac{5}{4} = \frac{3}{4} < 1.$$

Accordingly, $\tilde{C} = \{j \in \mathcal{J} : \tilde{y}_j = 0\} = \{2, 3\}$ and the cover inequality

$$x_2 + x_3 \leq 1$$

is maximally violated by $\tilde{\mathbf{x}}$: $\frac{1}{4} + 1 - 1 = \frac{1}{4}$.

- **Iteration 3**

Adding the cover inequality $x_2 + x_3 \leq 1$, we obtain the following LP, denoted LP_2 :

$$\begin{array}{llllll}
\max & 22x_1 & + & 33x_2 & + & 9x_3 & + & 13x_4 \\
\text{subject to} & 5x_1 & + & 9x_2 & + & 3x_3 & + & 6x_4 \leq 9 \\
& x_1 & + & x_2 & & & & \leq 1 \\
& & & x_2 & + & x_3 & & \leq 1 \\
& & & & & x_1 & & \leq 1 \\
& & & & x_2 & & & \leq 1 \\
& & & & & x_3 & & \leq 1 \\
& & & & & & x_4 & \leq 1 \\
& x_1, & & x_2, & & x_3, & & x_4 \geq 0
\end{array}$$

An optimal solution of LP_2 is $\tilde{\mathbf{x}} = (1, 0, 1, \frac{1}{6})$, with value $z(LP_2) = \frac{199}{6}$.

The costs are $q = (0, 1, 0, \frac{5}{6})$, the constant term is $n - \sum_{j \in \mathcal{J}} \tilde{x}_j = \frac{11}{6}$, and the complementary problem is:

$$\begin{array}{llllll} \max & & y_2 & & + & \frac{5}{6} y_4 \\ \text{subject to} & 5 y_1 & + & 9 y_2 & + & 3 y_3 & + & 6 y_4 & \leq & 13 \\ & y_1, & & y_2, & & y_3, & & y_4 & \in & \{0, 1\} \end{array}$$

The maximal subsets of indices that satisfy the constraint with right-hand side 13 are: $\{1, 3\}$ (value 0), $\{1, 4\}$ (value $\frac{5}{6}$), $\{2, 3\}$ (value 1), $\{3, 4\}$ (value $\frac{5}{6}$). An optimal solution of ILP_{CSP}^2 is $\tilde{y}_1 = 0$, $\tilde{y}_2 = 1$, $\tilde{y}_3 = 1$, $\tilde{y}_4 = 0$, with value $z(ILP_{CSP}^2) = 1$:

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(ILP_{CSP}^2) = \frac{11}{6} - 1 = \frac{5}{6} < 1.$$

Accordingly, $\tilde{C} = \{j \in \mathcal{J} : \tilde{y}_j = 0\} = \{1, 4\}$ and the cover inequality

$$x_1 + x_4 \leq 1$$

is maximally violated by $\tilde{\mathbf{x}}$: $1 + \frac{1}{6} - 1 = \frac{1}{6}$.

- **Iteration 4**

Adding the cover inequality $x_1 + x_4 \leq 1$, we obtain the following LP, denoted LP_3 :

$$\begin{array}{llllllll} \max & 22 x_1 & + & 33 x_2 & + & 9 x_3 & + & 13 x_4 \\ \text{subject to} & 5 x_1 & + & 9 x_2 & + & 3 x_3 & + & 6 x_4 & \leq & 9 \\ & x_1 & + & x_2 & & & & & \leq & 1 \\ & & & x_2 & + & x_3 & & & \leq & 1 \\ & x_1 & & & & & + & x_4 & \leq & 1 \\ & x_1 & & & & & & & \leq & 1 \\ & & & x_2 & & & & & \leq & 1 \\ & & & & & x_3 & & & \leq & 1 \\ & & & & & & & x_4 & \leq & 1 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

An optimal solution of LP_3 is $\tilde{\mathbf{x}} = (\frac{6}{7}, \frac{1}{7}, \frac{6}{7}, \frac{1}{7})$, with value $z(LP_3) = \frac{232}{7}$.

The costs are $q = (\frac{1}{7}, \frac{6}{7}, \frac{1}{7}, \frac{6}{7})$, the constant term is $n - \sum_{j \in \mathcal{J}} \tilde{x}_j = 2$, and the complementary problem is:

$$\begin{array}{llllll} \max & \frac{1}{7} y_1 & + & \frac{6}{7} y_2 & + & \frac{1}{7} y_3 & + & \frac{6}{7} y_4 \\ \text{subject to} & 5 y_1 & + & 9 y_2 & + & 3 y_3 & + & 6 y_4 & \leq & 13 \\ & y_1, & & y_2, & & y_3, & & y_4 & \in & \{0, 1\} \end{array}$$

The maximal subsets of indices that satisfy the constraint with right-hand side 13 are: $\{1, 3\}$ (value $\frac{2}{7}$), $\{1, 4\}$ (value 1), $\{2, 3\}$ (value 1), $\{3, 4\}$ (value 1). The optimal value of $\text{ILP}_{\text{CSP}}^2$ is $z(\text{ILP}_{\text{CSP}}^2) = 1$:

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = 2 - 1 = 1 \geq 1.$$

No cover inequality is violated by $\tilde{\mathbf{x}} = (\frac{6}{7}, \frac{1}{7}, \frac{6}{7}, \frac{1}{7})$: the separation stops after 3 cover inequalities. The optimal value over the cover closure is

$$z(\text{LP}^{\text{cov}}) = z(\text{LP}_3) = \frac{332}{7} < z(\text{LP}) = \frac{110}{3},$$

and $\tilde{\mathbf{x}}$ is **fractional**: the cover closure does not coincide with the convex hull of the integer solutions.

Exercise 5.12 — Cover closure by separation ($P2$)

Consider a binary problem with one constraint $\sum_j a_j x_j \leq b$, with positive integer coefficients a_j and right-hand side $b = 17$, on the binary variables indexed by the set $\mathcal{J} = \{1, 2, 3, 4\}$ of $n = 4$ variables with the following objective coefficients and constraint coefficients:

j	1	2	3	4
c_j	29	29	27	21
a_j	5	10	10	8

The associated LP is:

$$\begin{array}{llllll}
 \max & 29x_1 & + & 29x_2 & + & 27x_3 & + & 21x_4 \\
 \text{subject to} & 5x_1 & + & 10x_2 & + & 10x_3 & + & 8x_4 & \leq & 17 \\
 & x_1 & & & & & & & \leq & 1 \\
 & & x_2 & & & & & & \leq & 1 \\
 & & & x_3 & & & & & \leq & 1 \\
 & & & & x_4 & & & \leq & 1 \\
 & x_1, & x_2, & x_3, & x_4 & \geq & 0
 \end{array}$$

An optimal solution $\tilde{\mathbf{x}}$ of LP and its value are:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 1, \quad \tilde{x}_3 = \frac{1}{5}, \quad \tilde{x}_4 = 0 \quad \text{and} \quad z(\text{LP}) = \frac{317}{5}.$$

Questions

1. Starting from $\tilde{\mathbf{x}}$, separate the cover inequalities by solving the complementary problem $\text{ILP}_{\text{CSP}}^2$ to optimality. Add the maximally violated cover inequality to the LP, solve it again, and repeat until no cover inequality is violated. If $\text{ILP}_{\text{CSP}}^2$ has several optimal solutions, choose the one that selects the largest number of variables.

2. What is the optimal value over the cover closure? Is the final solution integer?

The **cover closure** is the polyhedron obtained by adding to LP all the cover inequalities $\sum_{j \in C} x_j \leq |C| - 1$, $C \in \mathcal{C}$. Given a point $\tilde{\mathbf{x}}$, let $\tilde{y}_1, \dots, \tilde{y}_n$ be an optimal solution of the complementary problem

$$(\text{ILP}_{\text{CSP}}^2) \quad \max \left\{ \sum_{j \in \mathcal{J}} (1 - \tilde{x}_j) y_j : \sum_{j \in \mathcal{J}} a_j y_j \leq \sum_{j \in \mathcal{J}} a_j - (b + 1), y_j \in \{0, 1\} \forall j \in \mathcal{J} \right\}.$$

If $n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) < 1$, the cover $\tilde{C} = \{j \in \mathcal{J} : \tilde{y}_j = 0\}$ is associated with the cover inequality maximally violated by $\tilde{\mathbf{x}}$; otherwise, no cover inequality is violated by $\tilde{\mathbf{x}}$.

Solution

Separation of the cover inequalities with the complementary problem; in all the iterations, the right-hand side of $\text{ILP}_{\text{CSP}}^2$ is $\sum_{j \in \mathcal{J}} a_j - (b + 1) = 33 - 18 = 15$.

- **Iteration 1**

The optimal solution of LP is $\tilde{\mathbf{x}} = (1, 1, \frac{1}{5}, 0)$, with value $z(\text{LP}) = \frac{317}{5}$.

The costs are $q = (0, 0, \frac{4}{5}, 1)$, the constant term is $n - \sum_{j \in \mathcal{J}} \tilde{x}_j = \frac{9}{5}$, and the complementary problem is:

$$\begin{array}{llllll} \max & & & \frac{4}{5} y_3 & + & y_4 \\ \text{subject to} & 5 y_1 & + & 10 y_2 & + & 10 y_3 & + & 8 y_4 & \leq & 15 \\ & y_1, & & y_2, & & y_3, & & y_4 & \in & \{0, 1\} \end{array}$$

The maximal subsets of indices that satisfy the constraint with right-hand side 15 are: $\{1, 2\}$ (value 0), $\{1, 3\}$ (value $\frac{4}{5}$), $\{1, 4\}$ (value 1). An optimal solution of $\text{ILP}_{\text{CSP}}^2$ is $\tilde{y}_1 = 1$, $\tilde{y}_2 = 0$, $\tilde{y}_3 = 0$, $\tilde{y}_4 = 1$, with value $z(\text{ILP}_{\text{CSP}}^2) = 1$:

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{9}{5} - 1 = \frac{4}{5} < 1.$$

Accordingly, $\tilde{C} = \{j \in \mathcal{J} : \tilde{y}_j = 0\} = \{2, 3\}$ and the cover inequality

$$x_2 + x_3 \leq 1$$

is maximally violated by $\tilde{\mathbf{x}}$: $1 + \frac{1}{5} - 1 = \frac{1}{5}$.

- **Iteration 2**

Adding the cover inequality $x_2 + x_3 \leq 1$, we obtain the following LP, denoted LP_1 :

$$\begin{array}{llllll}
\max & 29x_1 & + & 29x_2 & + & 27x_3 & + & 21x_4 \\
\text{subject to} & 5x_1 & + & 10x_2 & + & 10x_3 & + & 8x_4 \leq 17 \\
& & & x_2 & + & x_3 & & \leq 1 \\
& x_1 & & & & & & \leq 1 \\
& & x_2 & & & & & \leq 1 \\
& & & x_3 & & & & \leq 1 \\
& & & & x_4 & & & \leq 1 \\
& x_1, & x_2, & x_3, & x_4 & \geq & 0
\end{array}$$

An optimal solution of LP_1 is $\tilde{\mathbf{x}} = (1, 1, 0, \frac{1}{4})$, with value $z(\text{LP}_1) = \frac{253}{4}$.

The costs are $q = (0, 0, 1, \frac{3}{4})$, the constant term is $n - \sum_{j \in \mathcal{J}} \tilde{x}_j = \frac{7}{4}$, and the complementary problem is:

$$\begin{array}{llllll}
\max & & & y_3 & + & \frac{3}{4}y_4 \\
\text{subject to} & 5y_1 & + & 10y_2 & + & 10y_3 & + & 8y_4 \leq 15 \\
& y_1, & y_2, & y_3, & y_4 & \in & \{0, 1\}
\end{array}$$

The maximal subsets of indices that satisfy the constraint with right-hand side 15 are: $\{1, 2\}$ (value 0), $\{1, 3\}$ (value 1), $\{1, 4\}$ (value $\frac{3}{4}$). An optimal solution of $\text{ILP}_{\text{CSP}}^2$ is $\tilde{y}_1 = 1, \tilde{y}_2 = 0, \tilde{y}_3 = 1, \tilde{y}_4 = 0$, with value $z(\text{ILP}_{\text{CSP}}^2) = 1$:

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{7}{4} - 1 = \frac{3}{4} < 1.$$

Accordingly, $\tilde{C} = \{j \in \mathcal{J} : \tilde{y}_j = 0\} = \{2, 4\}$ and the cover inequality

$$x_2 + x_4 \leq 1$$

is maximally violated by $\tilde{\mathbf{x}}$: $1 + \frac{1}{4} - 1 = \frac{1}{4}$.

- **Iteration 3**

Adding the cover inequality $x_2 + x_4 \leq 1$, we obtain the following LP, denoted LP_2 :

$$\begin{array}{llllll}
\max & 29x_1 & + & 29x_2 & + & 27x_3 & + & 21x_4 \\
\text{subject to} & 5x_1 & + & 10x_2 & + & 10x_3 & + & 8x_4 \leq 17 \\
& & & x_2 & + & x_3 & & \leq 1 \\
& & & x_2 & & & + & x_4 \leq 1 \\
& x_1 & & & & & & \leq 1 \\
& & x_2 & & & & & \leq 1 \\
& & & x_3 & & & & \leq 1 \\
& & & & x_4 & & & \leq 1 \\
& x_1, & x_2, & x_3, & x_4 & \geq & 0
\end{array}$$

An optimal solution of LP_2 is $\tilde{x} = (1, \frac{3}{4}, \frac{1}{4}, \frac{1}{4})$, with value $z(LP_2) = \frac{251}{4}$.

The costs are $q = (0, \frac{1}{4}, \frac{3}{4}, \frac{3}{4})$, the constant term is $n - \sum_{j \in \mathcal{J}} \tilde{x}_j = \frac{7}{4}$, and the complementary problem is:

$$\begin{array}{llllll} \max & & \frac{1}{4} y_2 & + & \frac{3}{4} y_3 & + & \frac{3}{4} y_4 \\ \text{subject to} & 5 y_1 & + & 10 y_2 & + & 10 y_3 & + & 8 y_4 & \leq & 15 \\ & y_1, & & y_2, & & y_3, & & y_4 & \in & \{0, 1\} \end{array}$$

The maximal subsets of indices that satisfy the constraint with right-hand side 15 are: $\{1, 2\}$ (value $\frac{1}{4}$), $\{1, 3\}$ (value $\frac{3}{4}$), $\{1, 4\}$ (value $\frac{3}{4}$). The optimal value of ILP_{CSP}^2 is $z(ILP_{CSP}^2) = \frac{3}{4}$:

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(ILP_{CSP}^2) = \frac{7}{4} - \frac{3}{4} = 1 \geq 1.$$

No cover inequality is violated by $\tilde{x} = (1, \frac{3}{4}, \frac{1}{4}, \frac{1}{4})$: the separation stops after 2 cover inequalities. The optimal value over the cover closure is

$$z(LP^{cov}) = z(LP_2) = \frac{251}{4} < z(LP) = \frac{317}{5},$$

and $\tilde{x}$ is **fractional**: the cover closure does not coincide with the convex hull of the integer solutions.

Exercise 5.13 — Cover closure by separation ($P3$)

Consider a binary problem with one constraint $\sum_j a_j x_j \leq b$, with positive integer coefficients a_j and right-hand side $b = 15$, on the binary variables indexed by the set $\mathcal{J} = \{1, 2, 3, 4\}$ of $n = 4$ variables with the following objective coefficients and constraint coefficients:

j	1	2	3	4
c_j	30	8	9	6
a_j	5	2	9	9

The associated LP is:

$$\begin{array}{llllll} \max & 30 x_1 & + & 8 x_2 & + & 9 x_3 & + & 6 x_4 \\ \text{subject to} & 5 x_1 & + & 2 x_2 & + & 9 x_3 & + & 9 x_4 & \leq & 15 \\ & x_1 & & & & & & & \leq & 1 \\ & & x_2 & & & & & & \leq & 1 \\ & & & x_3 & & & & & \leq & 1 \\ & & & & x_4 & & & & \leq & 1 \\ & x_1, & x_2, & x_3, & x_4 & \geq & 0 \end{array}$$

An optimal solution $\tilde{\mathbf{x}}$ of LP and its value are:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 1, \quad \tilde{x}_3 = \frac{8}{9}, \quad \tilde{x}_4 = 0 \quad \text{and} \quad z(\text{LP}) = 46.$$

Questions

1. Starting from $\tilde{\mathbf{x}}$, separate the cover inequalities by solving the complementary problem $\text{ILP}_{\text{CSP}}^2$ to optimality. Add the maximally violated cover inequality to the LP, solve it again, and repeat until no cover inequality is violated. If $\text{ILP}_{\text{CSP}}^2$ has several optimal solutions, choose the one that selects the largest number of variables.
2. What is the optimal value over the cover closure? Is the final solution integer?

The **cover closure** is the polyhedron obtained by adding to LP all the cover inequalities $\sum_{j \in C} x_j \leq |C| - 1$, $C \in \mathcal{C}$. Given a point $\tilde{\mathbf{x}}$, let $\tilde{y}_1, \dots, \tilde{y}_n$ be an optimal solution of the complementary problem

$$(\text{ILP}_{\text{CSP}}^2) \quad \max \left\{ \sum_{j \in \mathcal{J}} (1 - \tilde{x}_j) y_j : \sum_{j \in \mathcal{J}} a_j y_j \leq \sum_{j \in \mathcal{J}} a_j - (b + 1), y_j \in \{0, 1\} \forall j \in \mathcal{J} \right\}.$$

If $n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) < 1$, the cover $\tilde{C} = \{j \in \mathcal{J} : \tilde{y}_j = 0\}$ is associated with the cover inequality maximally violated by $\tilde{\mathbf{x}}$; otherwise, no cover inequality is violated by $\tilde{\mathbf{x}}$.

Solution

Separation of the cover inequalities with the complementary problem; in all the iterations, the right-hand side of $\text{ILP}_{\text{CSP}}^2$ is $\sum_{j \in \mathcal{J}} a_j - (b + 1) = 25 - 16 = 9$.

• Iteration 1

The optimal solution of LP is $\tilde{\mathbf{x}} = (1, 1, \frac{8}{9}, 0)$, with value $z(\text{LP}) = 46$.

The costs are $q = (0, 0, \frac{1}{9}, 1)$, the constant term is $n - \sum_{j \in \mathcal{J}} \tilde{x}_j = \frac{10}{9}$, and the complementary problem is:

$$\begin{array}{llllll} \max & & & \frac{1}{9} y_3 & + & y_4 \\ \text{subject to} & 5 y_1 & + & 2 y_2 & + & 9 y_3 & + & 9 y_4 & \leq & 9 \\ & y_1, & & y_2, & & y_3, & & y_4 & \in & \{0, 1\} \end{array}$$

The maximal subsets of indices that satisfy the constraint with right-hand side 9 are: $\{3\}$ (value $\frac{1}{9}$), $\{4\}$ (value 1), $\{1, 2\}$ (value 0). An optimal solution of $\text{ILP}_{\text{CSP}}^2$ is $\tilde{y}_1 = 0$, $\tilde{y}_2 = 0$, $\tilde{y}_3 = 0$, $\tilde{y}_4 = 1$, with value $z(\text{ILP}_{\text{CSP}}^2) = 1$:

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{10}{9} - 1 = \frac{1}{9} < 1.$$

Accordingly, $\tilde{C} = \{j \in \mathcal{J} : \tilde{y}_j = 0\} = \{1, 2, 3\}$ and the cover inequality

$$x_1 + x_2 + x_3 \leq 2$$

is maximally violated by $\tilde{\mathbf{x}}$: $1 + 1 + \frac{8}{9} - 2 = \frac{8}{9}$.

- **Iteration 2**

Adding the cover inequality $x_1 + x_2 + x_3 \leq 2$, we obtain the following LP, denoted LP₁:

$$\begin{array}{llllll}
 \max & 30x_1 & + & 8x_2 & + & 9x_3 & + & 6x_4 \\
 \text{subject to} & 5x_1 & + & 2x_2 & + & 9x_3 & + & 9x_4 & \leq & 15 \\
 & x_1 & + & x_2 & + & x_3 & & & \leq & 2 \\
 & x_1 & & & & & & & \leq & 1 \\
 & & & x_2 & & & & & \leq & 1 \\
 & & & & & x_3 & & & \leq & 1 \\
 & & & & & & & x_4 & \leq & 1 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
 \end{array}$$

An optimal solution of LP₁ is $\tilde{\mathbf{x}} = (1, 1, 0, \frac{8}{9})$, with value $z(\text{LP}_1) = \frac{130}{3}$.

The costs are $q = (0, 0, 1, \frac{1}{9})$, the constant term is $n - \sum_{j \in \mathcal{J}} \tilde{x}_j = \frac{10}{9}$, and the complementary problem is:

$$\begin{array}{llllll}
 \max & & & y_3 & + & \frac{1}{9}y_4 \\
 \text{subject to} & 5y_1 & + & 2y_2 & + & 9y_3 & + & 9y_4 & \leq & 9 \\
 & y_1, & & y_2, & & y_3, & & y_4 & \in & \{0, 1\}
 \end{array}$$

The maximal subsets of indices that satisfy the constraint with right-hand side 9 are: $\{3\}$ (value 1), $\{4\}$ (value $\frac{1}{9}$), $\{1, 2\}$ (value 0). An optimal solution of ILP_{CSP}² is $\tilde{y}_1 = 0$, $\tilde{y}_2 = 0$, $\tilde{y}_3 = 1$, $\tilde{y}_4 = 0$, with value $z(\text{ILP}_{\text{CSP}}^2) = 1$:

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{10}{9} - 1 = \frac{1}{9} < 1.$$

Accordingly, $\tilde{C} = \{j \in \mathcal{J} : \tilde{y}_j = 0\} = \{1, 2, 4\}$ and the cover inequality

$$x_1 + x_2 + x_4 \leq 2$$

is maximally violated by $\tilde{\mathbf{x}}$: $1 + 1 + \frac{8}{9} - 2 = \frac{8}{9}$.

- **Iteration 3**

Adding the cover inequality $x_1 + x_2 + x_4 \leq 2$, we obtain the following LP, denoted LP₂:

$$\begin{array}{llllll}
\max & 30x_1 & + & 8x_2 & + & 9x_3 & + & 6x_4 \\
\text{subject to} & 5x_1 & + & 2x_2 & + & 9x_3 & + & 9x_4 & \leq & 15 \\
& x_1 & + & x_2 & + & x_3 & & & \leq & 2 \\
& x_1 & + & x_2 & & & + & x_4 & \leq & 2 \\
& x_1 & & & & & & & \leq & 1 \\
& & & x_2 & & & & & \leq & 1 \\
& & & & & x_3 & & & \leq & 1 \\
& & & & & & & x_4 & \leq & 1 \\
& x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}$$

An optimal solution of LP_2 is $\tilde{\mathbf{x}} = (1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, with value $z(\text{LP}_2) = \frac{83}{2}$.

The costs are $q = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, the constant term is $n - \sum_{j \in \mathcal{J}} \tilde{x}_j = \frac{3}{2}$, and the complementary problem is:

$$\begin{array}{llllll}
\max & & \frac{1}{2}y_2 & + & \frac{1}{2}y_3 & + & \frac{1}{2}y_4 \\
\text{subject to} & 5y_1 & + & 2y_2 & + & 9y_3 & + & 9y_4 & \leq & 9 \\
& y_1, & & y_2, & & y_3, & & y_4 & \in & \{0, 1\}
\end{array}$$

The maximal subsets of indices that satisfy the constraint with right-hand side 9 are: $\{3\}$ (value $\frac{1}{2}$), $\{4\}$ (value $\frac{1}{2}$), $\{1, 2\}$ (value $\frac{1}{2}$). The optimal value of $\text{ILP}_{\text{CSP}}^2$ is $z(\text{ILP}_{\text{CSP}}^2) = \frac{1}{2}$:

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{3}{2} - \frac{1}{2} = 1 \geq 1.$$

No cover inequality is violated by $\tilde{\mathbf{x}} = (1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$: the separation stops after 2 cover inequalities. The optimal value over the cover closure is

$$z(\text{LP}^{\text{cov}}) = z(\text{LP}_2) = \frac{83}{2} < z(\text{LP}) = 46,$$

and $\tilde{\mathbf{x}}$ is **fractional**: the cover closure does not coincide with the convex hull of the integer solutions.

Exercise 5.14 — Cover closure by separation (P_4)

Consider a binary problem with one constraint $\sum_j a_j x_j \leq b$, with positive integer coefficients a_j and right-hand side $b = 9$, on the binary variables indexed by the set $\mathcal{J} = \{1, 2, 3, 4\}$ of $n = 4$ variables with the following objective coefficients and constraint coefficients:

j	1	2	3	4
c_j	31	13	5	9
a_j	8	6	3	8

The associated LP is:

$$\begin{array}{rcllcl}
\max & 31x_1 & + & 13x_2 & + & 5x_3 & + & 9x_4 \\
\text{subject to} & 8x_1 & + & 6x_2 & + & 3x_3 & + & 8x_4 & \leq & 9 \\
& x_1 & & & & & & & \leq & 1 \\
& & & x_2 & & & & & \leq & 1 \\
& & & & & x_3 & & & \leq & 1 \\
& & & & & & x_4 & \leq & 1 \\
& x_1, & x_2, & x_3, & x_4 & \geq & 0
\end{array}$$

An optimal solution $\tilde{\mathbf{x}}$ of LP and its value are:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = \frac{1}{6}, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{and} \quad z(\text{LP}) = \frac{199}{6}.$$

Questions

1. Starting from $\tilde{\mathbf{x}}$, separate the cover inequalities by solving the complementary problem $\text{ILP}_{\text{CSP}}^2$ to optimality. Add the maximally violated cover inequality to the LP, solve it again, and repeat until no cover inequality is violated. If $\text{ILP}_{\text{CSP}}^2$ has several optimal solutions, choose the one that selects the largest number of variables.
2. What is the optimal value over the cover closure? Is the final solution integer?

The **cover closure** is the polyhedron obtained by adding to LP all the cover inequalities $\sum_{j \in C} x_j \leq |C| - 1$, $C \in \mathcal{C}$. Given a point $\tilde{\mathbf{x}}$, let $\tilde{y}_1, \dots, \tilde{y}_n$ be an optimal solution of the complementary problem

$$(\text{ILP}_{\text{CSP}}^2) \quad \max \left\{ \sum_{j \in \mathcal{J}} (1 - \tilde{x}_j) y_j : \sum_{j \in \mathcal{J}} a_j y_j \leq \sum_{j \in \mathcal{J}} a_j - (b + 1), y_j \in \{0, 1\} \forall j \in \mathcal{J} \right\}.$$

If $n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) < 1$, the cover $\tilde{C} = \{j \in \mathcal{J} : \tilde{y}_j = 0\}$ is associated with the cover inequality maximally violated by $\tilde{\mathbf{x}}$; otherwise, no cover inequality is violated by $\tilde{\mathbf{x}}$.

Solution

Separation of the cover inequalities with the complementary problem; in all the iterations, the right-hand side of $\text{ILP}_{\text{CSP}}^2$ is $\sum_{j \in \mathcal{J}} a_j - (b + 1) = 25 - 10 = 15$.

• Iteration 1

The optimal solution of LP is $\tilde{\mathbf{x}} = (1, \frac{1}{6}, 0, 0)$, with value $z(\text{LP}) = \frac{199}{6}$.

The costs are $q = (0, \frac{5}{6}, 1, 1)$, the constant term is $n - \sum_{j \in \mathcal{J}} \tilde{x}_j = \frac{17}{6}$, and the complementary problem is:

$$\begin{array}{ll}
\max & \frac{5}{6} y_2 + y_3 + y_4 \\
\text{subject to} & 8 y_1 + 6 y_2 + 3 y_3 + 8 y_4 \leq 15 \\
& y_1, y_2, y_3, y_4 \in \{0, 1\}
\end{array}$$

The maximal subsets of indices that satisfy the constraint with right-hand side 15 are: $\{1, 2\}$ (value $\frac{5}{6}$), $\{1, 3\}$ (value 1), $\{2, 3\}$ (value $\frac{11}{6}$), $\{2, 4\}$ (value $\frac{11}{6}$), $\{3, 4\}$ (value 2). An optimal solution of $\text{ILP}_{\text{CSP}}^2$ is $\tilde{y}_1 = 0$, $\tilde{y}_2 = 0$, $\tilde{y}_3 = 1$, $\tilde{y}_4 = 1$, with value $z(\text{ILP}_{\text{CSP}}^2) = 2$:

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{17}{6} - 2 = \frac{5}{6} < 1.$$

Accordingly, $\tilde{C} = \{j \in \mathcal{J} : \tilde{y}_j = 0\} = \{1, 2\}$ and the cover inequality

$$x_1 + x_2 \leq 1$$

is maximally violated by $\tilde{\mathbf{x}}$: $1 + \frac{1}{6} - 1 = \frac{1}{6}$.

• **Iteration 2**

Adding the cover inequality $x_1 + x_2 \leq 1$, we obtain the following LP, denoted LP_1 :

$$\begin{array}{ll}
\max & 31 x_1 + 13 x_2 + 5 x_3 + 9 x_4 \\
\text{subject to} & 8 x_1 + 6 x_2 + 3 x_3 + 8 x_4 \leq 9 \\
& x_1 + x_2 \leq 1 \\
& x_1 \leq 1 \\
& x_2 \leq 1 \\
& x_3 \leq 1 \\
& x_4 \leq 1 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{array}$$

An optimal solution of LP_1 is $\tilde{\mathbf{x}} = (1, 0, \frac{1}{3}, 0)$, with value $z(\text{LP}_1) = \frac{98}{3}$.

The costs are $q = (0, 1, \frac{2}{3}, 1)$, the constant term is $n - \sum_{j \in \mathcal{J}} \tilde{x}_j = \frac{8}{3}$, and the complementary problem is:

$$\begin{array}{ll}
\max & y_2 + \frac{2}{3} y_3 + y_4 \\
\text{subject to} & 8 y_1 + 6 y_2 + 3 y_3 + 8 y_4 \leq 15 \\
& y_1, y_2, y_3, y_4 \in \{0, 1\}
\end{array}$$

The maximal subsets of indices that satisfy the constraint with right-hand side 15 are: $\{1, 2\}$ (value 1), $\{1, 3\}$ (value $\frac{2}{3}$), $\{2, 3\}$ (value $\frac{5}{3}$), $\{2, 4\}$ (value 2), $\{3, 4\}$ (value $\frac{5}{3}$). An optimal solution of $\text{ILP}_{\text{CSP}}^2$ is $\tilde{y}_1 = 0$, $\tilde{y}_2 = 1$, $\tilde{y}_3 = 0$, $\tilde{y}_4 = 1$, with value $z(\text{ILP}_{\text{CSP}}^2) = 2$:

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{8}{3} - 2 = \frac{2}{3} < 1.$$

Accordingly, $\tilde{C} = \{j \in \mathcal{J} : \tilde{y}_j = 0\} = \{1, 3\}$ and the cover inequality

$$x_1 + x_3 \leq 1$$

is maximally violated by $\tilde{\mathbf{x}}$: $1 + \frac{1}{3} - 1 = \frac{1}{3}$.

- **Iteration 3**

Adding the cover inequality $x_1 + x_3 \leq 1$, we obtain the following LP, denoted LP₂:

$$\begin{array}{llllll} \max & 31x_1 & + & 13x_2 & + & 5x_3 & + & 9x_4 \\ \text{subject to} & 8x_1 & + & 6x_2 & + & 3x_3 & + & 8x_4 & \leq & 9 \\ & x_1 & + & x_2 & & & & & \leq & 1 \\ & x_1 & & & + & x_3 & & & \leq & 1 \\ & x_1 & & & & & & & \leq & 1 \\ & & & x_2 & & & & & \leq & 1 \\ & & & & & x_3 & & & \leq & 1 \\ & & & & & & & x_4 & \leq & 1 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

An optimal solution of LP₂ is $\tilde{\mathbf{x}} = (1, 0, 0, \frac{1}{8})$, with value $z(\text{LP}_2) = \frac{257}{8}$.

The costs are $q = (0, 1, 1, \frac{7}{8})$, the constant term is $n - \sum_{j \in \mathcal{J}} \tilde{x}_j = \frac{23}{8}$, and the complementary problem is:

$$\begin{array}{llllll} \max & & y_2 & + & y_3 & + & \frac{7}{8}y_4 \\ \text{subject to} & 8y_1 & + & 6y_2 & + & 3y_3 & + & 8y_4 & \leq & 15 \\ & y_1, & & y_2, & & y_3, & & y_4 & \in & \{0, 1\} \end{array}$$

The maximal subsets of indices that satisfy the constraint with right-hand side 15 are: $\{1, 2\}$ (value 1), $\{1, 3\}$ (value 1), $\{2, 3\}$ (value 2), $\{2, 4\}$ (value $\frac{15}{8}$), $\{3, 4\}$ (value $\frac{15}{8}$). An optimal solution of ILP_{CSP}² is $\tilde{y}_1 = 0$, $\tilde{y}_2 = 1$, $\tilde{y}_3 = 1$, $\tilde{y}_4 = 0$, with value $z(\text{ILP}_{\text{CSP}}^2) = 2$:

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{23}{8} - 2 = \frac{7}{8} < 1.$$

Accordingly, $\tilde{C} = \{j \in \mathcal{J} : \tilde{y}_j = 0\} = \{1, 4\}$ and the cover inequality

$$x_1 + x_4 \leq 1$$

is maximally violated by $\tilde{\mathbf{x}}$: $1 + \frac{1}{8} - 1 = \frac{1}{8}$.

- **Iteration 4**

Adding the cover inequality $x_1 + x_4 \leq 1$, we obtain the following LP, denoted LP₃:

$$\begin{array}{rcll}
\max & 31x_1 & + & 13x_2 & + & 5x_3 & + & 9x_4 \\
\text{subject to} & 8x_1 & + & 6x_2 & + & 3x_3 & + & 8x_4 & \leq & 9 \\
& x_1 & + & x_2 & & & & & \leq & 1 \\
& x_1 & & & + & x_3 & & & \leq & 1 \\
& x_1 & & & & & + & x_4 & \leq & 1 \\
& x_1 & & & & & & & \leq & 1 \\
& & & x_2 & & & & & \leq & 1 \\
& & & & & x_3 & & & \leq & 1 \\
& & & & & & & x_4 & \leq & 1 \\
& x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}$$

An optimal solution of LP_3 is $\tilde{\mathbf{x}} = (1, 0, 0, 0)$, with value $z(LP_3) = 31$.

The costs are $q = (0, 1, 1, 1)$, the constant term is $n - \sum_{j \in \mathcal{J}} \tilde{x}_j = 3$, and the complementary problem is:

$$\begin{array}{rcll}
\max & & y_2 & + & y_3 & + & y_4 \\
\text{subject to} & 8y_1 & + & 6y_2 & + & 3y_3 & + & 8y_4 & \leq & 15 \\
& y_1, & & y_2, & & y_3, & & y_4 & \in & \{0, 1\}
\end{array}$$

The maximal subsets of indices that satisfy the constraint with right-hand side 15 are: $\{1, 2\}$ (value 1), $\{1, 3\}$ (value 1), $\{2, 3\}$ (value 2), $\{2, 4\}$ (value 2), $\{3, 4\}$ (value 2). The optimal value of ILP_{CSP}^2 is $z(ILP_{CSP}^2) = 2$:

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(ILP_{CSP}^2) = 3 - 2 = 1 \geq 1.$$

No cover inequality is violated by $\tilde{\mathbf{x}} = (1, 0, 0, 0)$: the separation stops after 3 cover inequalities. The optimal value over the cover closure is

$$z(LP^{\text{cov}}) = z(LP_3) = 31 < z(LP) = \frac{199}{6},$$

and $\tilde{\mathbf{x}}$ is **integer**, hence it is also optimal for the binary problem.

Exercise 5.15 — Cover closure by separation ($P5$)

Consider a binary problem with one constraint $\sum_j a_j x_j \leq b$, with positive integer coefficients a_j and right-hand side $b = 12$, on the binary variables indexed by the set $\mathcal{J} = \{1, 2, 3, 4\}$ of $n = 4$ variables with the following objective coefficients and constraint coefficients:

j	1	2	3	4
c_j	38	16	14	4
a_j	7	4	7	6

The associated LP is:

$$\begin{array}{llllll}
 \max & 38x_1 & + & 16x_2 & + & 14x_3 & + & 4x_4 \\
 \text{subject to} & 7x_1 & + & 4x_2 & + & 7x_3 & + & 6x_4 & \leq & 12 \\
 & x_1 & & & & & & & \leq & 1 \\
 & & & x_2 & & & & & \leq & 1 \\
 & & & & & x_3 & & & \leq & 1 \\
 & & & & & & & x_4 & \leq & 1 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
 \end{array}$$

An optimal solution $\tilde{\mathbf{x}}$ of LP and its value are:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 1, \quad \tilde{x}_3 = \frac{1}{7}, \quad \tilde{x}_4 = 0 \quad \text{and} \quad z(\text{LP}) = 56.$$

Questions

1. Starting from $\tilde{\mathbf{x}}$, separate the cover inequalities by solving the complementary problem $\text{ILP}_{\text{CSP}}^2$ to optimality. Add the maximally violated cover inequality to the LP, solve it again, and repeat until no cover inequality is violated. If $\text{ILP}_{\text{CSP}}^2$ has several optimal solutions, choose the one that selects the largest number of variables.
2. What is the optimal value over the cover closure? Is the final solution integer?

The **cover closure** is the polyhedron obtained by adding to LP all the cover inequalities $\sum_{j \in C} x_j \leq |C| - 1$, $C \in \mathcal{C}$. Given a point $\tilde{\mathbf{x}}$, let $\tilde{y}_1, \dots, \tilde{y}_n$ be an optimal solution of the complementary problem

$$(\text{ILP}_{\text{CSP}}^2) \quad \max \left\{ \sum_{j \in \mathcal{J}} (1 - \tilde{x}_j) y_j : \sum_{j \in \mathcal{J}} a_j y_j \leq \sum_{j \in \mathcal{J}} a_j - (b + 1), y_j \in \{0, 1\} \forall j \in \mathcal{J} \right\}.$$

If $n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) < 1$, the cover $\tilde{C} = \{j \in \mathcal{J} : \tilde{y}_j = 0\}$ is associated with the cover inequality maximally violated by $\tilde{\mathbf{x}}$; otherwise, no cover inequality is violated by $\tilde{\mathbf{x}}$.

Solution

Separation of the cover inequalities with the complementary problem; in all the iterations, the right-hand side of $\text{ILP}_{\text{CSP}}^2$ is $\sum_{j \in \mathcal{J}} a_j - (b + 1) = 24 - 13 = 11$.

- **Iteration 1**

The optimal solution of LP is $\tilde{\mathbf{x}} = (1, 1, \frac{1}{7}, 0)$, with value $z(\text{LP}) = 56$.

The costs are $q = (0, 0, \frac{6}{7}, 1)$, the constant term is $n - \sum_{j \in \mathcal{J}} \tilde{x}_j = \frac{13}{7}$, and the complementary problem is:

$$\begin{array}{ll}
\max & \frac{6}{7} y_3 + y_4 \\
\text{subject to} & 7 y_1 + 4 y_2 + 7 y_3 + 6 y_4 \leq 11 \\
& y_1, y_2, y_3, y_4 \in \{0, 1\}
\end{array}$$

The maximal subsets of indices that satisfy the constraint with right-hand side 11 are: $\{1, 2\}$ (value 0), $\{2, 3\}$ (value $\frac{6}{7}$), $\{2, 4\}$ (value 1). An optimal solution of $\text{ILP}_{\text{CSP}}^2$ is $\tilde{y}_1 = 0, \tilde{y}_2 = 1, \tilde{y}_3 = 0, \tilde{y}_4 = 1$, with value $z(\text{ILP}_{\text{CSP}}^2) = 1$:

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{13}{7} - 1 = \frac{6}{7} < 1.$$

Accordingly, $\tilde{C} = \{j \in \mathcal{J} : \tilde{y}_j = 0\} = \{1, 3\}$ and the cover inequality

$$x_1 + x_3 \leq 1$$

is maximally violated by $\tilde{\mathbf{x}}$: $1 + \frac{1}{7} - 1 = \frac{1}{7}$.

- **Iteration 2**

Adding the cover inequality $x_1 + x_3 \leq 1$, we obtain the following LP, denoted LP_1 :

$$\begin{array}{ll}
\max & 38 x_1 + 16 x_2 + 14 x_3 + 4 x_4 \\
\text{subject to} & 7 x_1 + 4 x_2 + 7 x_3 + 6 x_4 \leq 12 \\
& x_1 + x_3 \leq 1 \\
& x_1 \leq 1 \\
& x_2 \leq 1 \\
& x_3 \leq 1 \\
& x_4 \leq 1 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{array}$$

An optimal solution of LP_1 is $\tilde{\mathbf{x}} = (1, 1, 0, \frac{1}{6})$, with value $z(\text{LP}_1) = \frac{164}{3}$.

The costs are $q = (0, 0, 1, \frac{5}{6})$, the constant term is $n - \sum_{j \in \mathcal{J}} \tilde{x}_j = \frac{11}{6}$, and the complementary problem is:

$$\begin{array}{ll}
\max & y_3 + \frac{5}{6} y_4 \\
\text{subject to} & 7 y_1 + 4 y_2 + 7 y_3 + 6 y_4 \leq 11 \\
& y_1, y_2, y_3, y_4 \in \{0, 1\}
\end{array}$$

The maximal subsets of indices that satisfy the constraint with right-hand side 11 are: $\{1, 2\}$ (value 0), $\{2, 3\}$ (value 1), $\{2, 4\}$ (value $\frac{5}{6}$). An optimal solution of $\text{ILP}_{\text{CSP}}^2$ is $\tilde{y}_1 = 0, \tilde{y}_2 = 1, \tilde{y}_3 = 1, \tilde{y}_4 = 0$, with value $z(\text{ILP}_{\text{CSP}}^2) = 1$:

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{11}{6} - 1 = \frac{5}{6} < 1.$$

Accordingly, $\tilde{C} = \{j \in \mathcal{J} : \tilde{y}_j = 0\} = \{1, 4\}$ and the cover inequality

$$x_1 + x_4 \leq 1$$

is maximally violated by $\tilde{\mathbf{x}}$: $1 + \frac{1}{6} - 1 = \frac{1}{6}$.

- **Iteration 3**

Adding the cover inequality $x_1 + x_4 \leq 1$, we obtain the following LP, denoted LP_2 :

$$\begin{array}{llllllll}
 \max & 38x_1 & + & 16x_2 & + & 14x_3 & + & 4x_4 \\
 \text{subject to} & 7x_1 & + & 4x_2 & + & 7x_3 & + & 6x_4 \leq 12 \\
 & x_1 & & & + & x_3 & & \leq 1 \\
 & x_1 & & & & & + & x_4 \leq 1 \\
 & x_1 & & & & & & \leq 1 \\
 & & & x_2 & & & & \leq 1 \\
 & & & & & x_3 & & \leq 1 \\
 & & & & & & x_4 & \leq 1 \\
 & x_1, & & x_2, & & x_3, & & x_4 \geq 0
 \end{array}$$

An optimal solution of LP_2 is $\tilde{\mathbf{x}} = (1, 1, 0, 0)$, with value $z(LP_2) = 54$.

The costs are $q = (0, 0, 1, 1)$, the constant term is $n - \sum_{j \in \mathcal{J}} \tilde{x}_j = 2$, and the complementary problem is:

$$\begin{array}{llllllll}
 \max & & & & y_3 & + & & y_4 \\
 \text{subject to} & 7y_1 & + & 4y_2 & + & 7y_3 & + & 6y_4 \leq 11 \\
 & y_1, & & y_2, & & y_3, & & y_4 \in \{0, 1\}
 \end{array}$$

The maximal subsets of indices that satisfy the constraint with right-hand side 11 are: $\{1, 2\}$ (value 0), $\{2, 3\}$ (value 1), $\{2, 4\}$ (value 1). The optimal value of ILP_{CSP}^2 is $z(ILP_{CSP}^2) = 1$:

$$n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(ILP_{CSP}^2) = 2 - 1 = 1 \geq 1.$$

No cover inequality is violated by $\tilde{\mathbf{x}} = (1, 1, 0, 0)$: the separation stops after 2 cover inequalities. The optimal value over the cover closure is

$$z(LP^{\text{cov}}) = z(LP_2) = 54 < z(LP) = 56,$$

and $\tilde{\mathbf{x}}$ is **integer**, hence it is also optimal for the binary problem.