

SAPIENZA
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Dynamic Programming

Exercises 1 — Dynamic programming

Worked exercises on the chapters of Volume 1, dynamic programming: each exercise is followed by its complete solution.

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Principles

Chapter 1

Principles of dynamic programming

Exercise 1.1 — Optimal substructure and overlapping subproblems: the 0/1 knapsack problem

Consider the 0/1 knapsack problem with $n = 4$ items and capacity $c = 8$:

$$\max\Big\{\sum_{j=1}^{n} p_j x_j : \sum_{j=1}^{n} w_j x_j \le c,\ x_j \in \{0,1\},\ j = 1,\dots,n\Big\},$$

with the data

	$j=1$	$j=2$	$j=3$	$j=4$
p_j	3	5	4	5
w_j	2	4	3	3

For $j \in \{0,\dots,n\}$ and $k \in \{0,\dots,c\}$, let $v(j,k)$ be the optimal value with the items $1,\dots,j$ only and capacity k.

Questions

1. Check the optimal substructure: if an optimal solution of the subproblem (j,k) takes item j, what can be said about its restriction to the items $1,\dots,j-1$? Write the recurrence for $v(j,k)$ and the base cases.
2. Evaluate $v(4,8)$ top-down and list, for each j, the subproblems (j,k) that the recursion generates. Which of them are generated more than once? What property does this show?
3. Fill the lookup table of $v(j,k)$ row by row, storing in $s(j,k)$ whether item j is taken (1) or not (0); a tie is broken in favour of not taking it.
4. Reconstruct an optimal solution from the table s.

Solution

1. Its restriction is a solution of the subproblem $(j-1, k-w_j)$, and it must be optimal: a better one, plus item j, would beat the optimal solution of (j,k). If item j is not taken, the restriction is optimal for $(j-1,k)$. Hence:

$$v(j,k) = \begin{cases} \max\{v(j-1,k),\ p_j + v(j-1,k-w_j)\} & \text{if } j \geq 1,\ w_j \leq k, \\ v(j-1,k) & \text{if } j \geq 1,\ w_j > k, \\ 0 & \text{if } j = 0. \end{cases}$$

2. Each call (j,k) generates $(j-1,k)$ and, if $w_j \leq k$, $(j-1,k-w_j)$:

   - $j=4$: $(4,8)$;
   - $j=3$: $(3,8)$, $(3,5)$;
   - $j=2$: $(2,8)$, $(2,5)$, $(2,5)$, $(2,2)$ (repeated: $(2,5)$);
   - $j=1$: $(1,8)$, $(1,4)$, $(1,5)$, $(1,1)$, $(1,5)$, $(1,1)$, $(1,2)$ (repeated: $(1,1)$, $(1,5)$);
   - $j=0$: $(0,8)$, $(0,6)$, $(0,4)$, $(0,2)$, $(0,5)$, $(0,3)$, $(0,1)$, $(0,5)$, $(0,3)$, $(0,1)$, $(0,2)$, $(0,0)$ (repeated: $(0,1)$, $(0,2)$, $(0,3)$, $(0,5)$);

   The recursion makes 26 calls for 19 distinct subproblems: the subproblems **overlap** (items 3 and 4 have the same weight). A lookup table computes each value once.

3. The tables v and s are:

$v(j,k)$	$k=0$	1	2	3	4	5	6	7	8
$j=0$	0	0	0	0	0	0	0	0	0
$j=1$	0	0	3	3	3	3	3	3	3
$j=2$	0	0	3	3	5	5	8	8	8
$j=3$	0	0	3	4	5	7	8	9	9
$j=4$	0	0	3	5	5	8	9	10	12

$s(j,k)$	$k=0$	1	2	3	4	5	6	7	8
$j=0$	–	–	–	–	–	–	–	–	–
$j=1$	0	0	1	1	1	1	1	1	1
$j=2$	0	0	0	0	1	1	1	1	1
$j=3$	0	0	0	1	0	1	0	1	1
$j=4$	0	0	0	1	0	1	1	1	1

   For example $v(3,7) = \max\{v(2,7),\ 4 + v(2,4)\} = \max\{8,9\} = 9$.

4. Starting from $(4,8)$:

   - $s(4,8) = 1$: item 4 is taken, go to $(3, 8-3) = (3,5)$;
   - $s(3,5) = 1$: item 3 is taken, go to $(2, 5-3) = (2,2)$;
   - $s(2,2) = 0$: item 2 is not taken, go to $(1,2)$;

- $s(1,2) = 1$: item 1 is taken, go to $(0, 2-2) = (0,0)$;

The optimal solution is $\{1,3,4\}$, of weight $2+3+3 = 8 \leq 8$ and profit $3+4+5 = 12 = v(4,8)$. The visited cells:

$v(j,k)$	$k=0$	1	2	3	4	5	6	7	8
$j=0$	0	0	0	0	0	0	0	0	0
$j=1$	0	0	3	3	3	3	3	3	3
$j=2$	0	0	3	3	5	5	8	8	8
$j=3$	0	0	3	4	5	7	8	9	9
$j=4$	0	0	3	5	5	8	9	10	12

$v(4,8) = 12$ with the items $\{1,3,4\}$; the top-down recursion makes 26 calls for 19 distinct subproblems, the lookup table has $(n+1)(c+1) = 45$ cells.

Exercise 1.2 — Writing a recurrence: subset sum via reachable sums

Given $n = 5$ items with integer weights w_j and a capacity $c = 30$, the optimization version of the **subset sum problem** asks for a subset of items of maximum total weight not exceeding c:

$$\max\Big\{\sum_{j=1}^{n} w_j x_j : \sum_{j=1}^{n} w_j x_j \leq c,\ x_j \in \{0,1\},\ j = 1,\dots,n\Big\}.$$

The data are

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
w_j	6	9	11	14	17

For $j \in \{0,\dots,n\}$, let R_j be the set of the **reachable sums**: the totals $k \leq c$ of the subsets of $\{1,\dots,j\}$.

Questions

1. Write a recurrence for R_j, distinguishing whether item j is used, with the base case. How is the optimal value z read from R_n?
2. Compute $R_1,\dots,R_n$. For every reachable sum, store the first index j such that it belongs to R_j.
3. Reconstruct an optimal solution.
4. Take the items greedily by non-increasing weight, as long as they fit. Is the result optimal?

Solution

1. A subset of $\{1, \dots, j\}$ either avoids item j, and then its total is in R_{j-1}, or uses it, and then its total is $k + w_j$ with $k \in R_{j-1}$. Hence:

$$R_j = \begin{cases} R_{j-1} \cup \{k + w_j : \ k \in R_{j-1}, \ k + w_j \le c\} & \text{if } 1 \le j \le n, \\ \{0\} & \text{if } j = 0, \end{cases}$$
$$z = \max\{k : \ k \in R_n\}.$$

   It is the KP with $p_j = w_j$, written with sets of reachable values.

2. The row j lists only the new sums; a sum belongs to R_j if it appears in a row $\le j$:

j	w_j	sums of R_j that are not in R_{j-1}
0	—	0
1	6	6
2	9	9, 15
3	11	11, 17, 20, 26
4	14	14, 23, 25, 29
5	17	28

   So $|R_5| = 13$ and $z = \max R_5 = 29$.

3. A sum k appearing first in R_j uses item j, and $k - w_j$ lies in R_{j-1}:

   - 29 first appears in R_4: item 4 is taken, go to $29 - 14 = 15$;
   - 15 first appears in R_2: item 2 is taken, go to $15 - 9 = 6$;
   - 6 first appears in R_1: item 1 is taken, go to $6 - 6 = 0$;

   The optimal solution is $\{1, 2, 4\}$, of total $6 + 9 + 14 = 29 \le 30$.

4. The greedy choice takes the items $5, 3$, of total $11 + 17 = 28 < 29$: it is not optimal.

$z = 29$ with the items $\{1, 2, 4\}$; there are 13 reachable sums, while the subsets are $2^5 = 32$. The greedy choice by weight gives only 28.

Exercise 1.3 — Reusing a recurrence: two parallel machines via subset sum

In the problem $P2\|C_{\max}$, $n = 6$ jobs with integer processing times p_j are assigned to 2 identical machines, and the **makespan** $C_{\max}$, the largest load of a machine, is minimized:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
p_j	13	11	10	5	4	3

Let $P = \sum_j p_j = 46$, and let L be the load of machine 1.

Questions

1. Express $C_{\max}$ as a function of L. Show that minimizing $C_{\max}$ amounts to the optimization version of the subset sum problem, with capacity $\lfloor P/2 \rfloor$, and write the recurrence of its reachable sums.
2. Compute the reachable sums $R_1, \ldots, R_n$.
3. Reconstruct an optimal assignment and its makespan.
4. Compare with the LPT rule: jobs by non-increasing p_j, each on the least loaded machine.

Solution

1. Machine 2 gets $P-L$, so $C_{\max} = \max\{L,\ P-L\} \geq P/2$. Up to swapping the machines we may assume $L \leq P/2$, and then $C_{\max} = P - L$ is minimum when L is maximum: a subset sum problem with capacity $c = \lfloor 46/2 \rfloor = 23$ and weights p_j:

$$R_j = \begin{cases} R_{j-1} \cup \{k + p_j :\ k \in R_{j-1},\ k + p_j \leq c\} & \text{if } 1 \leq j \leq n, \\ \{0\} & \text{if } j = 0, \end{cases}$$

$$z = P - \max\{k :\ k \in R_n\}.$$

2. The row j lists only the new sums:

j	p_j	sums of R_j that are not in R_{j-1}
0	—	0
1	13	13
2	11	11
3	10	10, 21, 23
4	5	5, 15, 16, 18
5	4	4, 9, 14, 17, 19, 20, 22
6	3	3, 7, 8, 12

   The largest reachable sum is $L = 23$, and the optimal makespan is $z = 46 - 23 = 23$, equal to the lower bound $\lceil P/2 \rceil = 23$.

3. - 23 first appears in R_3: item 3 is taken, go to $23 - 10 = 13$;
   - 13 first appears in R_1: item 1 is taken, go to $13 - 13 = 0$;

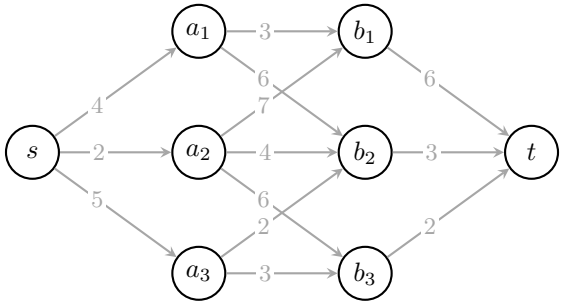
   Machine 1 gets the jobs $\{1, 3\}$ (load $13 + 10 = 23$), machine 2 the jobs $\{2, 4, 5, 6\}$ (load $11 + 5 + 4 + 3 = 23$).

4. LPT gives the machines the jobs $\{1, 4, 5\}$ and $\{2, 3, 6\}$, with loads 22 and 24: makespan $24 > 23$, not optimal.

The optimal makespan is $z = 23$, with the jobs $\{1, 3\}$ on machine 1 and $\{2, 4, 5, 6\}$ on machine 2; LPT gives 24.

Exercise 1.4 — Shortest path in a layered network

We look for a shortest path from s to t in the directed network $G = (N, A)$ below, where every arc $(i, j) \in A$ has a cost c_{ij}:



For every node i, let $v(i)$ be the cost of a shortest path from i to t.

Questions

1. Check the optimal substructure: if a shortest path from s to t passes through node i, what can be said about its part from i to t? Write the recurrence for $v(i)$ and the base case.
2. Which values $v(i)$ are needed by more than one node? What property of DP does this show?
3. Compute the values $v(i)$ backwards, from t to s, storing in $s(i)$ the successor of i on the best path (ties broken by the node listed first).
4. Reconstruct the shortest path and compare its cost with the enumeration of all the paths.

Solution

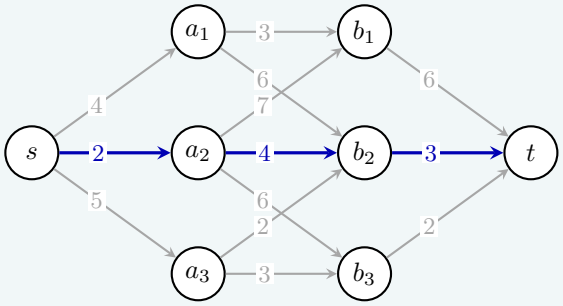
1. The part from i to t must be a shortest path from i to t: a shorter one would give a shorter path from s to t. The first arc out of i goes to some successor j, hence:

$$v(i) = \begin{cases} \min\{c_{ij} + v(j) : (i, j) \in A\} & \text{if } i \neq t, \\ 0 & \text{if } i = t. \end{cases}$$

2. The values $v(b_1)$, $v(b_2)$, $v(b_3)$ are needed by the nodes a_1, a_2, a_3 that point to them: $v(b_1)$ by 2 nodes, $v(b_2)$ by 3 and $v(b_3)$ by 2. The subproblems overlap: with a lookup table each value is computed once.
3. The values, from t backwards:

node i	$c_{ij} + v(j)$ for $(i,j) \in A$	$v(i)$	$s(i)$
t	—	0	—
b_3	$2+0=2$	2	t
b_2	$3+0=3$	3	t
b_1	$6+0=6$	6	t
a_3	$2+3=5$, $3+2=5$	5	b_2
a_2	$7+6=13$, $4+3=7$, $6+2=8$	7	b_2
a_1	$3+6=9$, $6+3=9$	9	b_1
s	$4+9=13$, $2+7=9$, $5+5=10$	9	a_2

4. Following s from s: $s \to a_2 \to b_2 \to t$, of cost $2+4+3=9=v(s)$:



The network has 7 paths from s to t; their enumeration gives the same optimal path, the only one of cost 9.

$v(s) = 9$, with the shortest path $s \to a_2 \to b_2 \to t$; the values of the nodes b_k are computed once and used several times.

Exercise 1.5 — A two-index table: the multiple-choice knapsack problem

In the **multiple-choice knapsack problem** the items are partitioned into classes $\mathscr{J}_1, \ldots, \mathscr{J}_q$, and **exactly one** item per class is chosen, with total weight at most c and maximum total profit:

$$\max\Big\{\sum_{h=1}^{q}\sum_{j\in\mathscr{J}_h} p_j x_j : \sum_{h=1}^{q}\sum_{j\in\mathscr{J}_h} w_j x_j \le c,\ \sum_{j\in\mathscr{J}_h} x_j = 1,\ h=1,\ldots,q,\ x_j\in\{0,1\}\Big\}.$$

Here $q=3$, $c=10$ and

class	$\mathscr{J}_1$			$\mathscr{J}_2$			$\mathscr{J}_3$		
j	1	2	3	4	5	6	7	8	9
w_j	2	3	5	1	4	6	2	3	4
p_j	3	5	8	2	6	9	4	5	7

For $h \in \{0,\ldots,q\}$ and $k \in \{0,\ldots,c\}$, let $v(h,k)$ be the optimal value with the classes $\mathscr{J}_1,\ldots,\mathscr{J}_h$ only and capacity k ($v(h,k) = -\infty$ if there is no feasible choice).

Questions

1. Write the recurrence for $v(h,k)$, deciding which item of class $\mathscr{J}_h$ is chosen, and the base cases. Why does the optimal substructure hold?
2. Fill the table of $v(h,k)$ row by row, storing in $s(h,k)$ the item chosen in class $\mathscr{J}_h$.
3. Reconstruct an optimal solution.
4. Without new computations, give the optimal value for every capacity $k \leq 10$. What is the smallest capacity with a feasible solution?

Solution

1. If item $j \in \mathscr{J}_h$ is chosen in an optimal solution of (h,k), the items of $\mathscr{J}_1, \ldots, \mathscr{J}_{h-1}$ form an optimal solution of $(h-1, k-w_j)$: a better one would improve the whole. Hence:

$$v(h,k) = \begin{cases} \max\{p_j + v(h-1, k-w_j) : \ j \in \mathscr{J}_h, \ w_j \leq k\} & \text{if } h \geq 1, \\ 0 & \text{if } h = 0, \end{cases}$$

   where the maximum of an empty set is $-\infty$.

2. The tables v and s are:

$v(h,k)$	$k=0$	1	2	3	4	5	6	7	8	9	10
$h=0$	0	0	0	0	0	0	0	0	0	0	0
$h=1$	$-\infty$	$-\infty$	3	5	5	8	8	8	8	8	8
$h=2$	$-\infty$	$-\infty$	$-\infty$	5	7	7	10	11	12	14	14
$h=3$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	9	11	12	14	15	17

$s(h,k)$	$k=0$	1	2	3	4	5	6	7	8	9	10
$h=0$	–	–	–	–	–	–	–	–	–	–	–
$h=1$	–	–	1	2	2	3	3	3	3	3	3
$h=2$	–	–	–	4	4	4	4	5	6	5	5
$h=3$	–	–	–	–	–	7	7	8	7	7	9

   For example $v(2,7) = \max\{2 + v(1,6),\ 6 + v(1,3),\ 9 + v(1,1)\} = 11$.

3. Starting from $(3,10)$:
   - $s(3,10) = 9$: item 9 of class $\mathscr{J}_3$, go to $(2, 10-4) = (2,6)$;
   - $s(2,6) = 4$: item 4 of class $\mathscr{J}_2$, go to $(1, 6-1) = (1,5)$;
   - $s(1,5) = 3$: item 3 of class $\mathscr{J}_1$, go to $(0, 5-5) = (0,0)$;

   The optimal solution is $\{3,4,9\}$, of weight $5+1+4 = 10 \leq 10$ and profit $8+2+7 = 17 = v(3,10)$. The visited cells:

$v(h,k)$	$k=0$	1	2	3	4	5	6	7	8	9	10
$h=0$	0	0	0	0	0	0	0	0	0	0	0
$h=1$	$-\infty$	$-\infty$	3	5	5	8	8	8	8	8	8
$h=2$	$-\infty$	$-\infty$	$-\infty$	5	7	7	10	11	12	14	14
$h=3$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	9	11	12	14	15	17

4. The row $h=3$ gives the optimal value for every capacity $k \leq 10$: it is $-\infty$ for $k<5$, and the smallest feasible capacity is 5, the sum of the lightest weights of the classes.

$v(3,10)=17$ with the items $\{3,4,9\}$, one per class; the table has $(q+1)(c+1)=44$ cells, while the choices are 27.

Knapsack problems

Chapter 2

DP for the 0/1 knapsack problem

Exercise 2.1 — The bottom-up DP algorithm for the KP (4 items, $c = 6$)

Consider the following KP instance, where the knapsack has capacity $c = 6$, and the profits and weights associated with the items are:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
p_j	6	7	8	9
w_j	2	3	4	5

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and s.

2. Using the matrix s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

Solution

1. Step-by-step execution of the bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the values of j and k, the cells $v[j][k]$ and $s[j][k]$, and the operation performed. The cells are computed for increasing values of j and, for each fixed value of j, for increasing values of k.

j	k	$v[j][k]$	$s[j][k]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $s[0][0] \leftarrow 0$
0	1	0	0	$v[0][1] \leftarrow 0$, $s[0][1] \leftarrow 0$
0	2	0	0	$v[0][2] \leftarrow 0$, $s[0][2] \leftarrow 0$
0	3	0	0	$v[0][3] \leftarrow 0$, $s[0][3] \leftarrow 0$
0	4	0	0	$v[0][4] \leftarrow 0$, $s[0][4] \leftarrow 0$
0	5	0	0	$v[0][5] \leftarrow 0$, $s[0][5] \leftarrow 0$
0	6	0	0	$v[0][6] \leftarrow 0$, $s[0][6] \leftarrow 0$
1	0	0	0	$k < w_1$: $v[1][0] \leftarrow v[0][0] = 0$, $s[1][0] \leftarrow 0$
1	1	0	0	$k < w_1$: $v[1][1] \leftarrow v[0][1] = 0$, $s[1][1] \leftarrow 0$
1	2	6	1	$p[1] + v[0][0] = 6 + 0 > v[0][2] = 0$: $v[1][2] \leftarrow 6$, $s[1][2] \leftarrow 1$
1	3	6	1	$p[1] + v[0][1] = 6 + 0 > v[0][3] = 0$: $v[1][3] \leftarrow 6$, $s[1][3] \leftarrow 1$
1	4	6	1	$p[1] + v[0][2] = 6 + 0 > v[0][4] = 0$: $v[1][4] \leftarrow 6$, $s[1][4] \leftarrow 1$
1	5	6	1	$p[1] + v[0][3] = 6 + 0 > v[0][5] = 0$: $v[1][5] \leftarrow 6$, $s[1][5] \leftarrow 1$
1	6	6	1	$p[1] + v[0][4] = 6 + 0 > v[0][6] = 0$: $v[1][6] \leftarrow 6$, $s[1][6] \leftarrow 1$
2	0	0	0	$k < w_2$: $v[2][0] \leftarrow v[1][0] = 0$, $s[2][0] \leftarrow 0$
2	1	0	0	$k < w_2$: $v[2][1] \leftarrow v[1][1] = 0$, $s[2][1] \leftarrow 0$
2	2	6	0	$k < w_2$: $v[2][2] \leftarrow v[1][2] = 6$, $s[2][2] \leftarrow 0$
2	3	7	1	$p[2] + v[1][0] = 7 + 0 > v[1][3] = 6$: $v[2][3] \leftarrow 7$, $s[2][3] \leftarrow 1$
2	4	7	1	$p[2] + v[1][1] = 7 + 0 > v[1][4] = 6$: $v[2][4] \leftarrow 7$, $s[2][4] \leftarrow 1$
2	5	13	1	$p[2] + v[1][2] = 7 + 6 > v[1][5] = 6$: $v[2][5] \leftarrow 13$, $s[2][5] \leftarrow 1$
2	6	13	1	$p[2] + v[1][3] = 7 + 6 > v[1][6] = 6$: $v[2][6] \leftarrow 13$, $s[2][6] \leftarrow 1$
3	0	0	0	$k < w_3$: $v[3][0] \leftarrow v[2][0] = 0$, $s[3][0] \leftarrow 0$
3	1	0	0	$k < w_3$: $v[3][1] \leftarrow v[2][1] = 0$, $s[3][1] \leftarrow 0$
3	2	6	0	$k < w_3$: $v[3][2] \leftarrow v[2][2] = 6$, $s[3][2] \leftarrow 0$
3	3	7	0	$k < w_3$: $v[3][3] \leftarrow v[2][3] = 7$, $s[3][3] \leftarrow 0$
3	4	8	1	$p[3] + v[2][0] = 8 + 0 > v[2][4] = 7$: $v[3][4] \leftarrow 8$, $s[3][4] \leftarrow 1$
3	5	13	0	$p[3] + v[2][1] = 8 + 0 \leq v[2][5] = 13$: $v[3][5] \leftarrow 13$, $s[3][5] \leftarrow 0$
3	6	14	1	$p[3] + v[2][2] = 8 + 6 > v[2][6] = 13$: $v[3][6] \leftarrow 14$, $s[3][6] \leftarrow 1$
4	0	0	0	$k < w_4$: $v[4][0] \leftarrow v[3][0] = 0$, $s[4][0] \leftarrow 0$
4	1	0	0	$k < w_4$: $v[4][1] \leftarrow v[3][1] = 0$, $s[4][1] \leftarrow 0$
4	2	6	0	$k < w_4$: $v[4][2] \leftarrow v[3][2] = 6$, $s[4][2] \leftarrow 0$
4	3	7	0	$k < w_4$: $v[4][3] \leftarrow v[3][3] = 7$, $s[4][3] \leftarrow 0$
4	4	8	0	$k < w_4$: $v[4][4] \leftarrow v[3][4] = 8$, $s[4][4] \leftarrow 0$
4	5	13	0	$p[4] + v[3][0] = 9 + 0 \leq v[3][5] = 13$: $v[4][5] \leftarrow 13$, $s[4][5] \leftarrow 0$
4	6	14	0	$p[4] + v[3][1] = 9 + 0 \leq v[3][6] = 14$: $v[4][6] \leftarrow 14$, $s[4][6] \leftarrow 0$

This is the matrix v after the execution of the DP algorithm:

$v[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$j=0$	0	0	0	0	0	0	0
$j=1$	0	0	6	6	6	6	6
$j=2$	0	0	6	7	7	13	13
$j=3$	0	0	6	7	8	13	14
$j=4$	0	0	6	7	8	13	14

This is the matrix s after the execution of the DP algorithm:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$j=0$	0	0	0	0	0	0	0
$j=1$	0	0	1	1	1	1	1
$j=2$	0	0	0	1	1	1	1
$j=3$	0	0	0	0	1	0	1
$j=4$	0	0	0	0	0	0	0

2. We have the following matrix s:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$j=0$	0	0	0	0	0	0	0
$j=1$	0	0	1	1	1	1	1
$j=2$	0	0	0	1	1	1	1
$j=3$	0	0	0	0	1	0	1
$j=4$	0	0	0	0	0	0	0

We can reconstruct an optimal solution by starting from $j=4$ and $k=6$ as follows:

Iteration	j	k	$s[j][k]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	Action
-	-	-	-	0	0	0	0	$k=6$
1	4	6	0	0	0	0	0	no update
2	3	6	1	0	0	1	0	$x[3] \leftarrow 1$, $k \leftarrow 6-4=2$
3	2	2	0	0	0	1	0	no update
4	1	2	1	1	0	1	0	$x[1] \leftarrow 1$, $k \leftarrow 2-2=0$

Thus, the vector

$$(1,0,1,0)$$

is an optimal solution. The optimal value is

$$v[4][6] = 14.$$

Exercise 2.2 — The bottom-up DP algorithm for the KP (4 items, $c = 7$)

Consider the following KP instance, where the knapsack has capacity $c = 7$, and the profits and weights associated with the items are:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
p_j	4	5	10	11
w_j	2	3	5	6

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and s.

2. Using the matrix s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

Solution

1. Step-by-step execution of the bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the values of j and k, the cells $v[j][k]$ and $s[j][k]$, and the operation performed. The cells are computed for increasing values of j and, for each fixed value of j, for increasing values of k.

j	k	$v[j][k]$	$s[j][k]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $s[0][0] \leftarrow 0$
0	1	0	0	$v[0][1] \leftarrow 0$, $s[0][1] \leftarrow 0$
0	2	0	0	$v[0][2] \leftarrow 0$, $s[0][2] \leftarrow 0$
0	3	0	0	$v[0][3] \leftarrow 0$, $s[0][3] \leftarrow 0$
0	4	0	0	$v[0][4] \leftarrow 0$, $s[0][4] \leftarrow 0$
0	5	0	0	$v[0][5] \leftarrow 0$, $s[0][5] \leftarrow 0$
0	6	0	0	$v[0][6] \leftarrow 0$, $s[0][6] \leftarrow 0$
0	7	0	0	$v[0][7] \leftarrow 0$, $s[0][7] \leftarrow 0$
1	0	0	0	$k < w_1$: $v[1][0] \leftarrow v[0][0] = 0$, $s[1][0] \leftarrow 0$
1	1	0	0	$k < w_1$: $v[1][1] \leftarrow v[0][1] = 0$, $s[1][1] \leftarrow 0$

1	2	4	1	$p[1]+v[0][0]=4+0>v[0][2]=0$: $v[1][2]\leftarrow 4$, $s[1][2]\leftarrow 1$
1	3	4	1	$p[1]+v[0][1]=4+0>v[0][3]=0$: $v[1][3]\leftarrow 4$, $s[1][3]\leftarrow 1$
1	4	4	1	$p[1]+v[0][2]=4+0>v[0][4]=0$: $v[1][4]\leftarrow 4$, $s[1][4]\leftarrow 1$
1	5	4	1	$p[1]+v[0][3]=4+0>v[0][5]=0$: $v[1][5]\leftarrow 4$, $s[1][5]\leftarrow 1$
1	6	4	1	$p[1]+v[0][4]=4+0>v[0][6]=0$: $v[1][6]\leftarrow 4$, $s[1][6]\leftarrow 1$
1	7	4	1	$p[1]+v[0][5]=4+0>v[0][7]=0$: $v[1][7]\leftarrow 4$, $s[1][7]\leftarrow 1$
2	0	0	0	$k<w_2$: $v[2][0]\leftarrow v[1][0]=0$, $s[2][0]\leftarrow 0$
2	1	0	0	$k<w_2$: $v[2][1]\leftarrow v[1][1]=0$, $s[2][1]\leftarrow 0$
2	2	4	0	$k<w_2$: $v[2][2]\leftarrow v[1][2]=4$, $s[2][2]\leftarrow 0$
2	3	5	1	$p[2]+v[1][0]=5+0>v[1][3]=4$: $v[2][3]\leftarrow 5$, $s[2][3]\leftarrow 1$
2	4	5	1	$p[2]+v[1][1]=5+0>v[1][4]=4$: $v[2][4]\leftarrow 5$, $s[2][4]\leftarrow 1$
2	5	9	1	$p[2]+v[1][2]=5+4>v[1][5]=4$: $v[2][5]\leftarrow 9$, $s[2][5]\leftarrow 1$
2	6	9	1	$p[2]+v[1][3]=5+4>v[1][6]=4$: $v[2][6]\leftarrow 9$, $s[2][6]\leftarrow 1$
2	7	9	1	$p[2]+v[1][4]=5+4>v[1][7]=4$: $v[2][7]\leftarrow 9$, $s[2][7]\leftarrow 1$
3	0	0	0	$k<w_3$: $v[3][0]\leftarrow v[2][0]=0$, $s[3][0]\leftarrow 0$
3	1	0	0	$k<w_3$: $v[3][1]\leftarrow v[2][1]=0$, $s[3][1]\leftarrow 0$
3	2	4	0	$k<w_3$: $v[3][2]\leftarrow v[2][2]=4$, $s[3][2]\leftarrow 0$
3	3	5	0	$k<w_3$: $v[3][3]\leftarrow v[2][3]=5$, $s[3][3]\leftarrow 0$
3	4	5	0	$k<w_3$: $v[3][4]\leftarrow v[2][4]=5$, $s[3][4]\leftarrow 0$
3	5	10	1	$p[3]+v[2][0]=10+0>v[2][5]=9$: $v[3][5]\leftarrow 10$, $s[3][5]\leftarrow 1$
3	6	10	1	$p[3]+v[2][1]=10+0>v[2][6]=9$: $v[3][6]\leftarrow 10$, $s[3][6]\leftarrow 1$
3	7	14	1	$p[3]+v[2][2]=10+4>v[2][7]=9$: $v[3][7]\leftarrow 14$, $s[3][7]\leftarrow 1$
4	0	0	0	$k<w_4$: $v[4][0]\leftarrow v[3][0]=0$, $s[4][0]\leftarrow 0$
4	1	0	0	$k<w_4$: $v[4][1]\leftarrow v[3][1]=0$, $s[4][1]\leftarrow 0$
4	2	4	0	$k<w_4$: $v[4][2]\leftarrow v[3][2]=4$, $s[4][2]\leftarrow 0$
4	3	5	0	$k<w_4$: $v[4][3]\leftarrow v[3][3]=5$, $s[4][3]\leftarrow 0$
4	4	5	0	$k<w_4$: $v[4][4]\leftarrow v[3][4]=5$, $s[4][4]\leftarrow 0$
4	5	10	0	$k<w_4$: $v[4][5]\leftarrow v[3][5]=10$, $s[4][5]\leftarrow 0$
4	6	11	1	$p[4]+v[3][0]=11+0>v[3][6]=10$: $v[4][6]\leftarrow 11$, $s[4][6]\leftarrow 1$
4	7	14	0	$p[4]+v[3][1]=11+0\leq v[3][7]=14$: $v[4][7]\leftarrow 14$, $s[4][7]\leftarrow 0$

This is the matrix v after the execution of the DP algorithm:

$v[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$j=0$	0	0	0	0	0	0	0	0
$j=1$	0	0	4	4	4	4	4	4
$j=2$	0	0	4	5	5	9	9	9
$j=3$	0	0	4	5	5	10	10	14
$j=4$	0	0	4	5	5	10	11	14

This is the matrix s after the execution of the DP algorithm:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$j=0$	0	0	0	0	0	0	0	0
$j=1$	0	0	1	1	1	1	1	1
$j=2$	0	0	0	1	1	1	1	1
$j=3$	0	0	0	0	0	1	1	1
$j=4$	0	0	0	0	0	0	1	0

2. We have the following matrix s:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$j=0$	0	0	0	0	0	0	0	0
$j=1$	0	0	1	1	1	1	1	1
$j=2$	0	0	0	1	1	1	1	1
$j=3$	0	0	0	0	0	1	1	1
$j=4$	0	0	0	0	0	0	1	0

We can reconstruct an optimal solution by starting from $j=4$ and $k=7$ as follows:

Iteration	j	k	$s[j][k]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	Action
-	-	-	-	0	0	0	0	$k=7$
1	4	7	0	0	0	0	0	no update
2	3	7	1	0	0	1	0	$x[3] \leftarrow 1$, $k \leftarrow 7-5=2$
3	2	2	0	0	0	1	0	no update
4	1	2	1	1	0	1	0	$x[1] \leftarrow 1$, $k \leftarrow 2-2=0$

Thus, the vector

$$(1,0,1,0)$$

is an optimal solution. The optimal value is

$$v[4][7] = 14.$$

Exercise 2.3 — The bottom-up DP algorithm for the KP (5 items, $c = 8$)

Consider the following KP instance, where the knapsack has capacity $c = 8$, and the profits and weights associated with the items are:

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
p_j	3	4	8	8	10
w_j	1	3	4	5	6

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and s.

2. Using the matrix s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

Solution

1. Step-by-step execution of the bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the values of j and k, the cells $v[j][k]$ and $s[j][k]$, and the operation performed. The cells are computed for increasing values of j and, for each fixed value of j, for increasing values of k.

j	k	$v[j][k]$	$s[j][k]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $s[0][0] \leftarrow 0$
0	1	0	0	$v[0][1] \leftarrow 0$, $s[0][1] \leftarrow 0$
0	2	0	0	$v[0][2] \leftarrow 0$, $s[0][2] \leftarrow 0$
0	3	0	0	$v[0][3] \leftarrow 0$, $s[0][3] \leftarrow 0$
0	4	0	0	$v[0][4] \leftarrow 0$, $s[0][4] \leftarrow 0$
0	5	0	0	$v[0][5] \leftarrow 0$, $s[0][5] \leftarrow 0$
0	6	0	0	$v[0][6] \leftarrow 0$, $s[0][6] \leftarrow 0$
0	7	0	0	$v[0][7] \leftarrow 0$, $s[0][7] \leftarrow 0$
0	8	0	0	$v[0][8] \leftarrow 0$, $s[0][8] \leftarrow 0$
1	0	0	0	$k < w_1$: $v[1][0] \leftarrow v[0][0] = 0$, $s[1][0] \leftarrow 0$

1	1	3	1	$p[1]+v[0][0]=3+0>v[0][1]=0$: $v[1][1]\leftarrow 3$, $s[1][1]\leftarrow 1$
1	2	3	1	$p[1]+v[0][1]=3+0>v[0][2]=0$: $v[1][2]\leftarrow 3$, $s[1][2]\leftarrow 1$
1	3	3	1	$p[1]+v[0][2]=3+0>v[0][3]=0$: $v[1][3]\leftarrow 3$, $s[1][3]\leftarrow 1$
1	4	3	1	$p[1]+v[0][3]=3+0>v[0][4]=0$: $v[1][4]\leftarrow 3$, $s[1][4]\leftarrow 1$
1	5	3	1	$p[1]+v[0][4]=3+0>v[0][5]=0$: $v[1][5]\leftarrow 3$, $s[1][5]\leftarrow 1$
1	6	3	1	$p[1]+v[0][5]=3+0>v[0][6]=0$: $v[1][6]\leftarrow 3$, $s[1][6]\leftarrow 1$
1	7	3	1	$p[1]+v[0][6]=3+0>v[0][7]=0$: $v[1][7]\leftarrow 3$, $s[1][7]\leftarrow 1$
1	8	3	1	$p[1]+v[0][7]=3+0>v[0][8]=0$: $v[1][8]\leftarrow 3$, $s[1][8]\leftarrow 1$
2	0	0	0	$k<w_2$: $v[2][0]\leftarrow v[1][0]=0$, $s[2][0]\leftarrow 0$
2	1	3	0	$k<w_2$: $v[2][1]\leftarrow v[1][1]=3$, $s[2][1]\leftarrow 0$
2	2	3	0	$k<w_2$: $v[2][2]\leftarrow v[1][2]=3$, $s[2][2]\leftarrow 0$
2	3	4	1	$p[2]+v[1][0]=4+0>v[1][3]=3$: $v[2][3]\leftarrow 4$, $s[2][3]\leftarrow 1$
2	4	7	1	$p[2]+v[1][1]=4+3>v[1][4]=3$: $v[2][4]\leftarrow 7$, $s[2][4]\leftarrow 1$
2	5	7	1	$p[2]+v[1][2]=4+3>v[1][5]=3$: $v[2][5]\leftarrow 7$, $s[2][5]\leftarrow 1$
2	6	7	1	$p[2]+v[1][3]=4+3>v[1][6]=3$: $v[2][6]\leftarrow 7$, $s[2][6]\leftarrow 1$
2	7	7	1	$p[2]+v[1][4]=4+3>v[1][7]=3$: $v[2][7]\leftarrow 7$, $s[2][7]\leftarrow 1$
2	8	7	1	$p[2]+v[1][5]=4+3>v[1][8]=3$: $v[2][8]\leftarrow 7$, $s[2][8]\leftarrow 1$
3	0	0	0	$k<w_3$: $v[3][0]\leftarrow v[2][0]=0$, $s[3][0]\leftarrow 0$
3	1	3	0	$k<w_3$: $v[3][1]\leftarrow v[2][1]=3$, $s[3][1]\leftarrow 0$
3	2	3	0	$k<w_3$: $v[3][2]\leftarrow v[2][2]=3$, $s[3][2]\leftarrow 0$
3	3	4	0	$k<w_3$: $v[3][3]\leftarrow v[2][3]=4$, $s[3][3]\leftarrow 0$
3	4	8	1	$p[3]+v[2][0]=8+0>v[2][4]=7$: $v[3][4]\leftarrow 8$, $s[3][4]\leftarrow 1$
3	5	11	1	$p[3]+v[2][1]=8+3>v[2][5]=7$: $v[3][5]\leftarrow 11$, $s[3][5]\leftarrow 1$
3	6	11	1	$p[3]+v[2][2]=8+3>v[2][6]=7$: $v[3][6]\leftarrow 11$, $s[3][6]\leftarrow 1$
3	7	12	1	$p[3]+v[2][3]=8+4>v[2][7]=7$: $v[3][7]\leftarrow 12$, $s[3][7]\leftarrow 1$
3	8	15	1	$p[3]+v[2][4]=8+7>v[2][8]=7$: $v[3][8]\leftarrow 15$, $s[3][8]\leftarrow 1$
4	0	0	0	$k<w_4$: $v[4][0]\leftarrow v[3][0]=0$, $s[4][0]\leftarrow 0$
4	1	3	0	$k<w_4$: $v[4][1]\leftarrow v[3][1]=3$, $s[4][1]\leftarrow 0$
4	2	3	0	$k<w_4$: $v[4][2]\leftarrow v[3][2]=3$, $s[4][2]\leftarrow 0$
4	3	4	0	$k<w_4$: $v[4][3]\leftarrow v[3][3]=4$, $s[4][3]\leftarrow 0$
4	4	8	0	$k<w_4$: $v[4][4]\leftarrow v[3][4]=8$, $s[4][4]\leftarrow 0$
4	5	11	0	$p[4]+v[3][0]=8+0\leq v[3][5]=11$: $v[4][5]\leftarrow 11$, $s[4][5]\leftarrow 0$
4	6	11	0	$p[4]+v[3][1]=8+3\leq v[3][6]=11$: $v[4][6]\leftarrow 11$, $s[4][6]\leftarrow 0$
4	7	12	0	$p[4]+v[3][2]=8+3\leq v[3][7]=12$: $v[4][7]\leftarrow 12$, $s[4][7]\leftarrow 0$
4	8	15	0	$p[4]+v[3][3]=8+4\leq v[3][8]=15$: $v[4][8]\leftarrow 15$, $s[4][8]\leftarrow 0$
5	0	0	0	$k<w_5$: $v[5][0]\leftarrow v[4][0]=0$, $s[5][0]\leftarrow 0$
5	1	3	0	$k<w_5$: $v[5][1]\leftarrow v[4][1]=3$, $s[5][1]\leftarrow 0$
5	2	3	0	$k<w_5$: $v[5][2]\leftarrow v[4][2]=3$, $s[5][2]\leftarrow 0$
5	3	4	0	$k<w_5$: $v[5][3]\leftarrow v[4][3]=4$, $s[5][3]\leftarrow 0$
5	4	8	0	$k<w_5$: $v[5][4]\leftarrow v[4][4]=8$, $s[5][4]\leftarrow 0$
5	5	11	0	$k<w_5$: $v[5][5]\leftarrow v[4][5]=11$, $s[5][5]\leftarrow 0$
5	6	11	0	$p[5]+v[4][0]=10+0\leq v[4][6]=11$: $v[5][6]\leftarrow 11$, $s[5][6]\leftarrow 0$
5	7	13	1	$p[5]+v[4][1]=10+3>v[4][7]=12$: $v[5][7]\leftarrow 13$, $s[5][7]\leftarrow 1$
5	8	15	0	$p[5]+v[4][2]=10+3\leq v[4][8]=15$: $v[5][8]\leftarrow 15$, $s[5][8]\leftarrow 0$

This is the matrix v after the execution of the DP algorithm:

$v[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$j=0$	0	0	0	0	0	0	0	0	0
$j=1$	0	3	3	3	3	3	3	3	3
$j=2$	0	3	3	4	7	7	7	7	7
$j=3$	0	3	3	4	8	11	11	12	15
$j=4$	0	3	3	4	8	11	11	12	15
$j=5$	0	3	3	4	8	11	11	13	15

This is the matrix s after the execution of the DP algorithm:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$j=0$	0	0	0	0	0	0	0	0	0
$j=1$	0	1	1	1	1	1	1	1	1
$j=2$	0	0	0	1	1	1	1	1	1
$j=3$	0	0	0	0	1	1	1	1	1
$j=4$	0	0	0	0	0	0	0	0	0
$j=5$	0	0	0	0	0	0	0	1	0

2. We have the following matrix s:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$j=0$	0	0	0	0	0	0	0	0	0
$j=1$	0	1	1	1	1	1	1	1	1
$j=2$	0	0	0	1	1	1	1	1	1
$j=3$	0	0	0	0	1	1	1	1	1
$j=4$	0	0	0	0	0	0	0	0	0
$j=5$	0	0	0	0	0	0	0	1	0

We can reconstruct an optimal solution by starting from $j=5$ and $k=8$ as follows:

Iteration	j	k	$s[j][k]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	Action
-	-	-	-	0	0	0	0	0	$k=8$
1	5	8	0	0	0	0	0	0	no update
2	4	8	0	0	0	0	0	0	no update
3	3	8	1	0	0	1	0	0	$x[3] \leftarrow 1$, $k \leftarrow 8-4=4$
4	2	4	1	0	1	1	0	0	$x[2] \leftarrow 1$, $k \leftarrow 4-3=1$
5	1	1	1	1	1	1	0	0	$x[1] \leftarrow 1$, $k \leftarrow 1-1=0$

Thus, the vector

$$(1,1,1,0,0)$$

is an optimal solution. The optimal value is

$$v[5][8] = 15.$$

Exercise 2.4 — The bottom-up DP algorithm for the KP (5 items, $c = 9$)

Consider the following KP instance, where the knapsack has capacity $c = 9$, and the profits and weights associated with the items are:

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
p_j	5	6	9	10	13
w_j	2	4	5	6	7

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and s.

2. Using the matrix s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

Solution

1. Step-by-step execution of the bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the values of j and k, the cells $v[j][k]$ and $s[j][k]$, and the operation performed. The cells are computed for increasing values of j and, for each fixed value of j, for increasing values of k.

j	k	$v[j][k]$	$s[j][k]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $s[0][0] \leftarrow 0$
0	1	0	0	$v[0][1] \leftarrow 0$, $s[0][1] \leftarrow 0$
0	2	0	0	$v[0][2] \leftarrow 0$, $s[0][2] \leftarrow 0$
0	3	0	0	$v[0][3] \leftarrow 0$, $s[0][3] \leftarrow 0$
0	4	0	0	$v[0][4] \leftarrow 0$, $s[0][4] \leftarrow 0$
0	5	0	0	$v[0][5] \leftarrow 0$, $s[0][5] \leftarrow 0$
0	6	0	0	$v[0][6] \leftarrow 0$, $s[0][6] \leftarrow 0$
0	7	0	0	$v[0][7] \leftarrow 0$, $s[0][7] \leftarrow 0$
0	8	0	0	$v[0][8] \leftarrow 0$, $s[0][8] \leftarrow 0$
0	9	0	0	$v[0][9] \leftarrow 0$, $s[0][9] \leftarrow 0$
1	0	0	0	$k < w_1$: $v[1][0] \leftarrow v[0][0] = 0$, $s[1][0] \leftarrow 0$

1	1	0	0	$k < w_1$: $v[1][1] \leftarrow v[0][1] = 0$, $s[1][1] \leftarrow 0$
1	2	5	1	$p[1] + v[0][0] = 5 + 0 > v[0][2] = 0$: $v[1][2] \leftarrow 5$, $s[1][2] \leftarrow 1$
1	3	5	1	$p[1] + v[0][1] = 5 + 0 > v[0][3] = 0$: $v[1][3] \leftarrow 5$, $s[1][3] \leftarrow 1$
1	4	5	1	$p[1] + v[0][2] = 5 + 0 > v[0][4] = 0$: $v[1][4] \leftarrow 5$, $s[1][4] \leftarrow 1$
1	5	5	1	$p[1] + v[0][3] = 5 + 0 > v[0][5] = 0$: $v[1][5] \leftarrow 5$, $s[1][5] \leftarrow 1$
1	6	5	1	$p[1] + v[0][4] = 5 + 0 > v[0][6] = 0$: $v[1][6] \leftarrow 5$, $s[1][6] \leftarrow 1$
1	7	5	1	$p[1] + v[0][5] = 5 + 0 > v[0][7] = 0$: $v[1][7] \leftarrow 5$, $s[1][7] \leftarrow 1$
1	8	5	1	$p[1] + v[0][6] = 5 + 0 > v[0][8] = 0$: $v[1][8] \leftarrow 5$, $s[1][8] \leftarrow 1$
1	9	5	1	$p[1] + v[0][7] = 5 + 0 > v[0][9] = 0$: $v[1][9] \leftarrow 5$, $s[1][9] \leftarrow 1$
2	0	0	0	$k < w_2$: $v[2][0] \leftarrow v[1][0] = 0$, $s[2][0] \leftarrow 0$
2	1	0	0	$k < w_2$: $v[2][1] \leftarrow v[1][1] = 0$, $s[2][1] \leftarrow 0$
2	2	5	0	$k < w_2$: $v[2][2] \leftarrow v[1][2] = 5$, $s[2][2] \leftarrow 0$
2	3	5	0	$k < w_2$: $v[2][3] \leftarrow v[1][3] = 5$, $s[2][3] \leftarrow 0$
2	4	6	1	$p[2] + v[1][0] = 6 + 0 > v[1][4] = 5$: $v[2][4] \leftarrow 6$, $s[2][4] \leftarrow 1$
2	5	6	1	$p[2] + v[1][1] = 6 + 0 > v[1][5] = 5$: $v[2][5] \leftarrow 6$, $s[2][5] \leftarrow 1$
2	6	11	1	$p[2] + v[1][2] = 6 + 5 > v[1][6] = 5$: $v[2][6] \leftarrow 11$, $s[2][6] \leftarrow 1$
2	7	11	1	$p[2] + v[1][3] = 6 + 5 > v[1][7] = 5$: $v[2][7] \leftarrow 11$, $s[2][7] \leftarrow 1$
2	8	11	1	$p[2] + v[1][4] = 6 + 5 > v[1][8] = 5$: $v[2][8] \leftarrow 11$, $s[2][8] \leftarrow 1$
2	9	11	1	$p[2] + v[1][5] = 6 + 5 > v[1][9] = 5$: $v[2][9] \leftarrow 11$, $s[2][9] \leftarrow 1$
3	0	0	0	$k < w_3$: $v[3][0] \leftarrow v[2][0] = 0$, $s[3][0] \leftarrow 0$
3	1	0	0	$k < w_3$: $v[3][1] \leftarrow v[2][1] = 0$, $s[3][1] \leftarrow 0$
3	2	5	0	$k < w_3$: $v[3][2] \leftarrow v[2][2] = 5$, $s[3][2] \leftarrow 0$
3	3	5	0	$k < w_3$: $v[3][3] \leftarrow v[2][3] = 5$, $s[3][3] \leftarrow 0$
3	4	6	0	$k < w_3$: $v[3][4] \leftarrow v[2][4] = 6$, $s[3][4] \leftarrow 0$
3	5	9	1	$p[3] + v[2][0] = 9 + 0 > v[2][5] = 6$: $v[3][5] \leftarrow 9$, $s[3][5] \leftarrow 1$
3	6	11	0	$p[3] + v[2][1] = 9 + 0 \leq v[2][6] = 11$: $v[3][6] \leftarrow 11$, $s[3][6] \leftarrow 0$
3	7	14	1	$p[3] + v[2][2] = 9 + 5 > v[2][7] = 11$: $v[3][7] \leftarrow 14$, $s[3][7] \leftarrow 1$
3	8	14	1	$p[3] + v[2][3] = 9 + 5 > v[2][8] = 11$: $v[3][8] \leftarrow 14$, $s[3][8] \leftarrow 1$
3	9	15	1	$p[3] + v[2][4] = 9 + 6 > v[2][9] = 11$: $v[3][9] \leftarrow 15$, $s[3][9] \leftarrow 1$
4	0	0	0	$k < w_4$: $v[4][0] \leftarrow v[3][0] = 0$, $s[4][0] \leftarrow 0$
4	1	0	0	$k < w_4$: $v[4][1] \leftarrow v[3][1] = 0$, $s[4][1] \leftarrow 0$
4	2	5	0	$k < w_4$: $v[4][2] \leftarrow v[3][2] = 5$, $s[4][2] \leftarrow 0$
4	3	5	0	$k < w_4$: $v[4][3] \leftarrow v[3][3] = 5$, $s[4][3] \leftarrow 0$
4	4	6	0	$k < w_4$: $v[4][4] \leftarrow v[3][4] = 6$, $s[4][4] \leftarrow 0$
4	5	9	0	$k < w_4$: $v[4][5] \leftarrow v[3][5] = 9$, $s[4][5] \leftarrow 0$
4	6	11	0	$p[4] + v[3][0] = 10 + 0 \leq v[3][6] = 11$: $v[4][6] \leftarrow 11$, $s[4][6] \leftarrow 0$
4	7	14	0	$p[4] + v[3][1] = 10 + 0 \leq v[3][7] = 14$: $v[4][7] \leftarrow 14$, $s[4][7] \leftarrow 0$
4	8	15	1	$p[4] + v[3][2] = 10 + 5 > v[3][8] = 14$: $v[4][8] \leftarrow 15$, $s[4][8] \leftarrow 1$
4	9	15	0	$p[4] + v[3][3] = 10 + 5 \leq v[3][9] = 15$: $v[4][9] \leftarrow 15$, $s[4][9] \leftarrow 0$
5	0	0	0	$k < w_5$: $v[5][0] \leftarrow v[4][0] = 0$, $s[5][0] \leftarrow 0$
5	1	0	0	$k < w_5$: $v[5][1] \leftarrow v[4][1] = 0$, $s[5][1] \leftarrow 0$
5	2	5	0	$k < w_5$: $v[5][2] \leftarrow v[4][2] = 5$, $s[5][2] \leftarrow 0$
5	3	5	0	$k < w_5$: $v[5][3] \leftarrow v[4][3] = 5$, $s[5][3] \leftarrow 0$
5	4	6	0	$k < w_5$: $v[5][4] \leftarrow v[4][4] = 6$, $s[5][4] \leftarrow 0$
5	5	9	0	$k < w_5$: $v[5][5] \leftarrow v[4][5] = 9$, $s[5][5] \leftarrow 0$
5	6	11	0	$k < w_5$: $v[5][6] \leftarrow v[4][6] = 11$, $s[5][6] \leftarrow 0$
5	7	14	0	$p[5] + v[4][0] = 13 + 0 \leq v[4][7] = 14$: $v[5][7] \leftarrow 14$, $s[5][7] \leftarrow 0$
5	8	15	0	$p[5] + v[4][1] = 13 + 0 \leq v[4][8] = 15$: $v[5][8] \leftarrow 15$, $s[5][8] \leftarrow 0$
5	9	18	1	$p[5] + v[4][2] = 13 + 5 > v[4][9] = 15$: $v[5][9] \leftarrow 18$, $s[5][9] \leftarrow 1$

This is the matrix v after the execution of the DP algorithm:

$v[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$j=0$	0	0	0	0	0	0	0	0	0	0
$j=1$	0	0	5	5	5	5	5	5	5	5
$j=2$	0	0	5	5	6	6	11	11	11	11
$j=3$	0	0	5	5	6	9	11	14	14	15
$j=4$	0	0	5	5	6	9	11	14	15	15
$j=5$	0	0	5	5	6	9	11	14	15	18

This is the matrix s after the execution of the DP algorithm:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$j=0$	0	0	0	0	0	0	0	0	0	0
$j=1$	0	0	1	1	1	1	1	1	1	1
$j=2$	0	0	0	0	1	1	1	1	1	1
$j=3$	0	0	0	0	0	1	0	1	1	1
$j=4$	0	0	0	0	0	0	0	0	1	0
$j=5$	0	0	0	0	0	0	0	0	0	1

2. We have the following matrix s:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$j=0$	0	0	0	0	0	0	0	0	0	0
$j=1$	0	0	1	1	1	1	1	1	1	1
$j=2$	0	0	0	0	1	1	1	1	1	1
$j=3$	0	0	0	0	0	1	0	1	1	1
$j=4$	0	0	0	0	0	0	0	0	1	0
$j=5$	0	0	0	0	0	0	0	0	0	1

We can reconstruct an optimal solution by starting from $j=5$ and $k=9$ as follows:

Iteration	j	k	$s[j][k]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	Action
-	-	-	-	0	0	0	0	0	$k=9$
1	5	9	1	0	0	0	0	1	$x[5] \leftarrow 1$, $k \leftarrow 9-7=2$
2	4	2	0	0	0	0	0	1	no update
3	3	2	0	0	0	0	0	1	no update
4	2	2	0	0	0	0	0	1	no update
5	1	2	1	1	0	0	0	1	$x[1] \leftarrow 1$, $k \leftarrow 2-2=0$

Thus, the vector

$$(1,0,0,0,1)$$

is an optimal solution. The optimal value is

$$v[5][9]=18.$$

Exercise 2.5 — The bottom-up DP algorithm for the KP (5 items, $c = 10$)

Consider the following KP instance, where the knapsack has capacity $c = 10$, and the profits and weights associated with the items are:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
p_j	2	7	9	12	14
w_j	1	4	5	7	8

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and s.

2. Using the matrix s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

Solution

1. Step-by-step execution of the bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the values of j and k, the cells $v[j][k]$ and $s[j][k]$, and the operation performed. The cells are computed for increasing values of j and, for each fixed value of j, for increasing values of k.

j	k	$v[j][k]$	$s[j][k]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $s[0][0] \leftarrow 0$
0	1	0	0	$v[0][1] \leftarrow 0$, $s[0][1] \leftarrow 0$
0	2	0	0	$v[0][2] \leftarrow 0$, $s[0][2] \leftarrow 0$
0	3	0	0	$v[0][3] \leftarrow 0$, $s[0][3] \leftarrow 0$
0	4	0	0	$v[0][4] \leftarrow 0$, $s[0][4] \leftarrow 0$
0	5	0	0	$v[0][5] \leftarrow 0$, $s[0][5] \leftarrow 0$
0	6	0	0	$v[0][6] \leftarrow 0$, $s[0][6] \leftarrow 0$
0	7	0	0	$v[0][7] \leftarrow 0$, $s[0][7] \leftarrow 0$
0	8	0	0	$v[0][8] \leftarrow 0$, $s[0][8] \leftarrow 0$
0	9	0	0	$v[0][9] \leftarrow 0$, $s[0][9] \leftarrow 0$
0	10	0	0	$v[0][10] \leftarrow 0$, $s[0][10] \leftarrow 0$

1	0	0	0	$k < w_1$: $v[1][0] \leftarrow v[0][0] = 0$, $s[1][0] \leftarrow 0$
1	1	2	1	$p[1] + v[0][0] = 2 + 0 > v[0][1] = 0$: $v[1][1] \leftarrow 2$, $s[1][1] \leftarrow 1$
1	2	2	1	$p[1] + v[0][1] = 2 + 0 > v[0][2] = 0$: $v[1][2] \leftarrow 2$, $s[1][2] \leftarrow 1$
1	3	2	1	$p[1] + v[0][2] = 2 + 0 > v[0][3] = 0$: $v[1][3] \leftarrow 2$, $s[1][3] \leftarrow 1$
1	4	2	1	$p[1] + v[0][3] = 2 + 0 > v[0][4] = 0$: $v[1][4] \leftarrow 2$, $s[1][4] \leftarrow 1$
1	5	2	1	$p[1] + v[0][4] = 2 + 0 > v[0][5] = 0$: $v[1][5] \leftarrow 2$, $s[1][5] \leftarrow 1$
1	6	2	1	$p[1] + v[0][5] = 2 + 0 > v[0][6] = 0$: $v[1][6] \leftarrow 2$, $s[1][6] \leftarrow 1$
1	7	2	1	$p[1] + v[0][6] = 2 + 0 > v[0][7] = 0$: $v[1][7] \leftarrow 2$, $s[1][7] \leftarrow 1$
1	8	2	1	$p[1] + v[0][7] = 2 + 0 > v[0][8] = 0$: $v[1][8] \leftarrow 2$, $s[1][8] \leftarrow 1$
1	9	2	1	$p[1] + v[0][8] = 2 + 0 > v[0][9] = 0$: $v[1][9] \leftarrow 2$, $s[1][9] \leftarrow 1$
1	10	2	1	$p[1] + v[0][9] = 2 + 0 > v[0][10] = 0$: $v[1][10] \leftarrow 2$, $s[1][10] \leftarrow 1$
2	0	0	0	$k < w_2$: $v[2][0] \leftarrow v[1][0] = 0$, $s[2][0] \leftarrow 0$
2	1	2	0	$k < w_2$: $v[2][1] \leftarrow v[1][1] = 2$, $s[2][1] \leftarrow 0$
2	2	2	0	$k < w_2$: $v[2][2] \leftarrow v[1][2] = 2$, $s[2][2] \leftarrow 0$
2	3	2	0	$k < w_2$: $v[2][3] \leftarrow v[1][3] = 2$, $s[2][3] \leftarrow 0$
2	4	7	1	$p[2] + v[1][0] = 7 + 0 > v[1][4] = 2$: $v[2][4] \leftarrow 7$, $s[2][4] \leftarrow 1$
2	5	9	1	$p[2] + v[1][1] = 7 + 2 > v[1][5] = 2$: $v[2][5] \leftarrow 9$, $s[2][5] \leftarrow 1$
2	6	9	1	$p[2] + v[1][2] = 7 + 2 > v[1][6] = 2$: $v[2][6] \leftarrow 9$, $s[2][6] \leftarrow 1$
2	7	9	1	$p[2] + v[1][3] = 7 + 2 > v[1][7] = 2$: $v[2][7] \leftarrow 9$, $s[2][7] \leftarrow 1$
2	8	9	1	$p[2] + v[1][4] = 7 + 2 > v[1][8] = 2$: $v[2][8] \leftarrow 9$, $s[2][8] \leftarrow 1$
2	9	9	1	$p[2] + v[1][5] = 7 + 2 > v[1][9] = 2$: $v[2][9] \leftarrow 9$, $s[2][9] \leftarrow 1$
2	10	9	1	$p[2] + v[1][6] = 7 + 2 > v[1][10] = 2$: $v[2][10] \leftarrow 9$, $s[2][10] \leftarrow 1$
3	0	0	0	$k < w_3$: $v[3][0] \leftarrow v[2][0] = 0$, $s[3][0] \leftarrow 0$
3	1	2	0	$k < w_3$: $v[3][1] \leftarrow v[2][1] = 2$, $s[3][1] \leftarrow 0$
3	2	2	0	$k < w_3$: $v[3][2] \leftarrow v[2][2] = 2$, $s[3][2] \leftarrow 0$
3	3	2	0	$k < w_3$: $v[3][3] \leftarrow v[2][3] = 2$, $s[3][3] \leftarrow 0$
3	4	7	0	$k < w_3$: $v[3][4] \leftarrow v[2][4] = 7$, $s[3][4] \leftarrow 0$
3	5	9	0	$p[3] + v[2][0] = 9 + 0 \leq v[2][5] = 9$: $v[3][5] \leftarrow 9$, $s[3][5] \leftarrow 0$
3	6	11	1	$p[3] + v[2][1] = 9 + 2 > v[2][6] = 9$: $v[3][6] \leftarrow 11$, $s[3][6] \leftarrow 1$
3	7	11	1	$p[3] + v[2][2] = 9 + 2 > v[2][7] = 9$: $v[3][7] \leftarrow 11$, $s[3][7] \leftarrow 1$
3	8	11	1	$p[3] + v[2][3] = 9 + 2 > v[2][8] = 9$: $v[3][8] \leftarrow 11$, $s[3][8] \leftarrow 1$
3	9	16	1	$p[3] + v[2][4] = 9 + 7 > v[2][9] = 9$: $v[3][9] \leftarrow 16$, $s[3][9] \leftarrow 1$
3	10	18	1	$p[3] + v[2][5] = 9 + 9 > v[2][10] = 9$: $v[3][10] \leftarrow 18$, $s[3][10] \leftarrow 1$
4	0	0	0	$k < w_4$: $v[4][0] \leftarrow v[3][0] = 0$, $s[4][0] \leftarrow 0$
4	1	2	0	$k < w_4$: $v[4][1] \leftarrow v[3][1] = 2$, $s[4][1] \leftarrow 0$
4	2	2	0	$k < w_4$: $v[4][2] \leftarrow v[3][2] = 2$, $s[4][2] \leftarrow 0$
4	3	2	0	$k < w_4$: $v[4][3] \leftarrow v[3][3] = 2$, $s[4][3] \leftarrow 0$
4	4	7	0	$k < w_4$: $v[4][4] \leftarrow v[3][4] = 7$, $s[4][4] \leftarrow 0$
4	5	9	0	$k < w_4$: $v[4][5] \leftarrow v[3][5] = 9$, $s[4][5] \leftarrow 0$
4	6	11	0	$k < w_4$: $v[4][6] \leftarrow v[3][6] = 11$, $s[4][6] \leftarrow 0$
4	7	12	1	$p[4] + v[3][0] = 12 + 0 > v[3][7] = 11$: $v[4][7] \leftarrow 12$, $s[4][7] \leftarrow 1$
4	8	14	1	$p[4] + v[3][1] = 12 + 2 > v[3][8] = 11$: $v[4][8] \leftarrow 14$, $s[4][8] \leftarrow 1$
4	9	16	0	$p[4] + v[3][2] = 12 + 2 \leq v[3][9] = 16$: $v[4][9] \leftarrow 16$, $s[4][9] \leftarrow 0$
4	10	18	0	$p[4] + v[3][3] = 12 + 2 \leq v[3][10] = 18$: $v[4][10] \leftarrow 18$, $s[4][10] \leftarrow 0$
5	0	0	0	$k < w_5$: $v[5][0] \leftarrow v[4][0] = 0$, $s[5][0] \leftarrow 0$
5	1	2	0	$k < w_5$: $v[5][1] \leftarrow v[4][1] = 2$, $s[5][1] \leftarrow 0$
5	2	2	0	$k < w_5$: $v[5][2] \leftarrow v[4][2] = 2$, $s[5][2] \leftarrow 0$
5	3	2	0	$k < w_5$: $v[5][3] \leftarrow v[4][3] = 2$, $s[5][3] \leftarrow 0$
5	4	7	0	$k < w_5$: $v[5][4] \leftarrow v[4][4] = 7$, $s[5][4] \leftarrow 0$

5	5	9	0	$k < w_5$: $v[5][5] \leftarrow v[4][5] = 9$, $s[5][5] \leftarrow 0$
5	6	11	0	$k < w_5$: $v[5][6] \leftarrow v[4][6] = 11$, $s[5][6] \leftarrow 0$
5	7	12	0	$k < w_5$: $v[5][7] \leftarrow v[4][7] = 12$, $s[5][7] \leftarrow 0$
5	8	14	0	$p[5] + v[4][0] = 14 + 0 \leq v[4][8] = 14$: $v[5][8] \leftarrow 14$, $s[5][8] \leftarrow 0$
5	9	16	0	$p[5] + v[4][1] = 14 + 2 \leq v[4][9] = 16$: $v[5][9] \leftarrow 16$, $s[5][9] \leftarrow 0$
5	10	18	0	$p[5] + v[4][2] = 14 + 2 \leq v[4][10] = 18$: $v[5][10] \leftarrow 18$, $s[5][10] \leftarrow 0$

This is the matrix v after the execution of the DP algorithm:

$v[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
$j=0$	0	0	0	0	0	0	0	0	0	0	0
$j=1$	0	2	2	2	2	2	2	2	2	2	2
$j=2$	0	2	2	2	7	9	9	9	9	9	9
$j=3$	0	2	2	2	7	9	11	11	11	16	18
$j=4$	0	2	2	2	7	9	11	12	14	16	18
$j=5$	0	2	2	2	7	9	11	12	14	16	18

This is the matrix s after the execution of the DP algorithm:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
$j=0$	0	0	0	0	0	0	0	0	0	0	0
$j=1$	0	1	1	1	1	1	1	1	1	1	1
$j=2$	0	0	0	0	1	1	1	1	1	1	1
$j=3$	0	0	0	0	0	0	1	1	1	1	1
$j=4$	0	0	0	0	0	0	0	1	1	0	0
$j=5$	0	0	0	0	0	0	0	0	0	0	0

2. We have the following matrix s:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
$j=0$	0	0	0	0	0	0	0	0	0	0	0
$j=1$	0	1	1	1	1	1	1	1	1	1	1
$j=2$	0	0	0	0	1	1	1	1	1	1	1
$j=3$	0	0	0	0	0	0	1	1	1	1	1
$j=4$	0	0	0	0	0	0	0	1	1	0	0
$j=5$	0	0	0	0	0	0	0	0	0	0	0

We can reconstruct an optimal solution by starting from $j=5$ and $k=10$ as follows:

Iteration	j	k	$s[j][k]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	Action
-	-	-	-	0	0	0	0	0	$k=10$
1	5	10	0	0	0	0	0	0	no update
2	4	10	0	0	0	0	0	0	no update
3	3	10	1	0	0	1	0	0	$x[3] \leftarrow 1$, $k \leftarrow 10-5=5$
4	2	5	1	0	1	1	0	0	$x[2] \leftarrow 1$, $k \leftarrow 5-4=1$
5	1	1	1	1	1	1	0	0	$x[1] \leftarrow 1$, $k \leftarrow 1-1=0$

Thus, the vector

$$(1,1,1,0,0)$$

is an optimal solution. The optimal value is

$$v[5][10] = 18.$$

Chapter 3

DP for the unbounded knapsack problem

Exercise 3.1 — The two DP algorithms for the UKP (4 item types, $c = 7$)

Consider the following UKP instance, where the knapsack has capacity $c = 7$, each item type is available in infinitely many copies, and the profits and weights associated with the item types are:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
p_j	4	5	10	11
w_j	2	3	5	6

Questions

1. Show the execution of the first bottom-up DP algorithm by reporting, for each iteration, the values of the vector v and the values of the vector s.

2. Using the vector s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

3. Show the execution of the second bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and s.

4. Using the matrix s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

Solution

1. Step-by-step execution of the first bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the values of k and j, the cells $v[k]$ and $s[k]$, the current value of m, the current value of t, and the operation performed.

k	j	$v[k]$	$s[k]$	m	t	Action
–	–	0	–	–	–	$v[0] \leftarrow 0$
1	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
1	1	–	–	0	0	$w[1] = 2 > k = 1$: no update
1	2	–	–	0	0	$w[2] = 3 > k = 1$: no update
1	3	–	–	0	0	$w[3] = 5 > k = 1$: no update
1	4	–	–	0	0	$w[4] = 6 > k = 1$: no update
1	–	0	0	0	0	$v[1] \leftarrow 0$, $s[1] \leftarrow 0$
2	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
2	1	–	–	4	1	$p[1] + v[0] = 4 + 0 = 4 > m = 0$: $m \leftarrow 4$, $t \leftarrow 1$
2	2	–	–	4	1	$w[2] = 3 > k = 2$: no update
2	3	–	–	4	1	$w[3] = 5 > k = 2$: no update
2	4	–	–	4	1	$w[4] = 6 > k = 2$: no update
2	–	4	1	4	1	$v[2] \leftarrow 4$, $s[2] \leftarrow 1$
3	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
3	1	–	–	4	1	$p[1] + v[1] = 4 + 0 = 4 > m = 0$: $m \leftarrow 4$, $t \leftarrow 1$
3	2	–	–	5	2	$p[2] + v[0] = 5 + 0 = 5 > m = 4$: $m \leftarrow 5$, $t \leftarrow 2$
3	3	–	–	5	2	$w[3] = 5 > k = 3$: no update
3	4	–	–	5	2	$w[4] = 6 > k = 3$: no update
3	–	5	2	5	2	$v[3] \leftarrow 5$, $s[3] \leftarrow 2$
4	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
4	1	–	–	8	1	$p[1] + v[2] = 4 + 4 = 8 > m = 0$: $m \leftarrow 8$, $t \leftarrow 1$
4	2	–	–	8	1	$p[2] + v[1] = 5 + 0 = 5 \leq m = 8$: no update
4	3	–	–	8	1	$w[3] = 5 > k = 4$: no update
4	4	–	–	8	1	$w[4] = 6 > k = 4$: no update
4	–	8	1	8	1	$v[4] \leftarrow 8$, $s[4] \leftarrow 1$
5	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
5	1	–	–	9	1	$p[1] + v[3] = 4 + 5 = 9 > m = 0$: $m \leftarrow 9$, $t \leftarrow 1$
5	2	–	–	9	1	$p[2] + v[2] = 5 + 4 = 9 \leq m = 9$: no update
5	3	–	–	10	3	$p[3] + v[0] = 10 + 0 = 10 > m = 9$: $m \leftarrow 10$, $t \leftarrow 3$
5	4	–	–	10	3	$w[4] = 6 > k = 5$: no update
5	–	10	3	10	3	$v[5] \leftarrow 10$, $s[5] \leftarrow 3$
6	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
6	1	–	–	12	1	$p[1] + v[4] = 4 + 8 = 12 > m = 0$: $m \leftarrow 12$, $t \leftarrow 1$
6	2	–	–	12	1	$p[2] + v[3] = 5 + 5 = 10 \leq m = 12$: no update
6	3	–	–	12	1	$p[3] + v[1] = 10 + 0 = 10 \leq m = 12$: no update
6	4	–	–	12	1	$p[4] + v[0] = 11 + 0 = 11 \leq m = 12$: no update
6	–	12	1	12	1	$v[6] \leftarrow 12$, $s[6] \leftarrow 1$
7	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
7	1	–	–	14	1	$p[1] + v[5] = 4 + 10 = 14 > m = 0$: $m \leftarrow 14$, $t \leftarrow 1$
7	2	–	–	14	1	$p[2] + v[4] = 5 + 8 = 13 \leq m = 14$: no update
7	3	–	–	14	1	$p[3] + v[2] = 10 + 4 = 14 \leq m = 14$: no update
7	4	–	–	14	1	$p[4] + v[1] = 11 + 0 = 11 \leq m = 14$: no update
7	–	14	1	14	1	$v[7] \leftarrow 14$, $s[7] \leftarrow 1$

This is the vector v after the execution of the first DP algorithm:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$v[k]$	0	0	4	5	8	10	12	14

This is the vector s after the execution of the first DP algorithm:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$s[k]$	0	0	1	2	1	3	1	1

2. We have the following vector s:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$s[k]$	0	0	1	2	1	3	1	1

We can reconstruct an optimal solution by starting from $k = 7$ as follows:

Iteration	k	$s[k]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	Action
-	-	-	0	0	0	0	$k = 7$
1	7	1	1	0	0	0	$x[1] \leftarrow 1$, $k \leftarrow 7 - 2 = 5$
2	5	3	1	0	1	0	$x[3] \leftarrow 1$, $k \leftarrow 5 - 5 = 0$

Thus, the vector

$$(1, 0, 1, 0)$$

is an optimal solution. The optimal value is

$$v[7] = 14.$$

3. Step-by-step execution of the second bottom-up DP algorithm

For each step of the DP algorithm, the table shows the values of j and k, the cells $v[j][k]$ and $s[j][k]$, and the operation performed. All entries in row $j = 0$ are initialized to 0.

j	k	$v[j][k]$	$s[j][k]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $s[0][0] \leftarrow 0$
0	1	0	0	$v[0][1] \leftarrow 0$, $s[0][1] \leftarrow 0$
0	2	0	0	$v[0][2] \leftarrow 0$, $s[0][2] \leftarrow 0$
0	3	0	0	$v[0][3] \leftarrow 0$, $s[0][3] \leftarrow 0$
0	4	0	0	$v[0][4] \leftarrow 0$, $s[0][4] \leftarrow 0$
0	5	0	0	$v[0][5] \leftarrow 0$, $s[0][5] \leftarrow 0$
0	6	0	0	$v[0][6] \leftarrow 0$, $s[0][6] \leftarrow 0$
0	7	0	0	$v[0][7] \leftarrow 0$, $s[0][7] \leftarrow 0$
1	0	0	0	$k < w[1]$: $v[1][0] \leftarrow v[0][0] = 0$, $s[1][0] \leftarrow 0$
1	1	0	0	$k < w[1]$: $v[1][1] \leftarrow v[0][1] = 0$, $s[1][1] \leftarrow 0$
1	2	4	1	$p[1] + v[1][0] = 4 + 0 > v[0][2] = 0$: $v[1][2] \leftarrow 4$, $s[1][2] \leftarrow 1$
1	3	4	1	$p[1] + v[1][1] = 4 + 0 > v[0][3] = 0$: $v[1][3] \leftarrow 4$, $s[1][3] \leftarrow 1$
1	4	8	1	$p[1] + v[1][2] = 4 + 4 > v[0][4] = 0$: $v[1][4] \leftarrow 8$, $s[1][4] \leftarrow 1$
1	5	8	1	$p[1] + v[1][3] = 4 + 4 > v[0][5] = 0$: $v[1][5] \leftarrow 8$, $s[1][5] \leftarrow 1$
1	6	12	1	$p[1] + v[1][4] = 4 + 8 > v[0][6] = 0$: $v[1][6] \leftarrow 12$, $s[1][6] \leftarrow 1$
1	7	12	1	$p[1] + v[1][5] = 4 + 8 > v[0][7] = 0$: $v[1][7] \leftarrow 12$, $s[1][7] \leftarrow 1$
2	0	0	0	$k < w[2]$: $v[2][0] \leftarrow v[1][0] = 0$, $s[2][0] \leftarrow 0$
2	1	0	0	$k < w[2]$: $v[2][1] \leftarrow v[1][1] = 0$, $s[2][1] \leftarrow 0$
2	2	4	0	$k < w[2]$: $v[2][2] \leftarrow v[1][2] = 4$, $s[2][2] \leftarrow 0$
2	3	5	1	$p[2] + v[2][0] = 5 + 0 > v[1][3] = 4$: $v[2][3] \leftarrow 5$, $s[2][3] \leftarrow 1$
2	4	8	0	$p[2] + v[2][1] = 5 + 0 \leq v[1][4] = 8$: $v[2][4] \leftarrow 8$, $s[2][4] \leftarrow 0$
2	5	9	1	$p[2] + v[2][2] = 5 + 4 > v[1][5] = 8$: $v[2][5] \leftarrow 9$, $s[2][5] \leftarrow 1$
2	6	12	0	$p[2] + v[2][3] = 5 + 5 \leq v[1][6] = 12$: $v[2][6] \leftarrow 12$, $s[2][6] \leftarrow 0$
2	7	13	1	$p[2] + v[2][4] = 5 + 8 > v[1][7] = 12$: $v[2][7] \leftarrow 13$, $s[2][7] \leftarrow 1$
3	0	0	0	$k < w[3]$: $v[3][0] \leftarrow v[2][0] = 0$, $s[3][0] \leftarrow 0$
3	1	0	0	$k < w[3]$: $v[3][1] \leftarrow v[2][1] = 0$, $s[3][1] \leftarrow 0$
3	2	4	0	$k < w[3]$: $v[3][2] \leftarrow v[2][2] = 4$, $s[3][2] \leftarrow 0$
3	3	5	0	$k < w[3]$: $v[3][3] \leftarrow v[2][3] = 5$, $s[3][3] \leftarrow 0$
3	4	8	0	$k < w[3]$: $v[3][4] \leftarrow v[2][4] = 8$, $s[3][4] \leftarrow 0$
3	5	10	1	$p[3] + v[3][0] = 10 + 0 > v[2][5] = 9$: $v[3][5] \leftarrow 10$, $s[3][5] \leftarrow 1$
3	6	12	0	$p[3] + v[3][1] = 10 + 0 \leq v[2][6] = 12$: $v[3][6] \leftarrow 12$, $s[3][6] \leftarrow 0$
3	7	14	1	$p[3] + v[3][2] = 10 + 4 > v[2][7] = 13$: $v[3][7] \leftarrow 14$, $s[3][7] \leftarrow 1$
4	0	0	0	$k < w[4]$: $v[4][0] \leftarrow v[3][0] = 0$, $s[4][0] \leftarrow 0$
4	1	0	0	$k < w[4]$: $v[4][1] \leftarrow v[3][1] = 0$, $s[4][1] \leftarrow 0$
4	2	4	0	$k < w[4]$: $v[4][2] \leftarrow v[3][2] = 4$, $s[4][2] \leftarrow 0$
4	3	5	0	$k < w[4]$: $v[4][3] \leftarrow v[3][3] = 5$, $s[4][3] \leftarrow 0$
4	4	8	0	$k < w[4]$: $v[4][4] \leftarrow v[3][4] = 8$, $s[4][4] \leftarrow 0$
4	5	10	0	$k < w[4]$: $v[4][5] \leftarrow v[3][5] = 10$, $s[4][5] \leftarrow 0$
4	6	12	0	$p[4] + v[4][0] = 11 + 0 \leq v[3][6] = 12$: $v[4][6] \leftarrow 12$, $s[4][6] \leftarrow 0$
4	7	14	0	$p[4] + v[4][1] = 11 + 0 \leq v[3][7] = 14$: $v[4][7] \leftarrow 14$, $s[4][7] \leftarrow 0$

This is the matrix v after the execution of the second DP algorithm:

$v[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$j=0$	0	0	0	0	0	0	0	0
$j=1$	0	0	4	4	8	8	12	12
$j=2$	0	0	4	5	8	9	12	13
$j=3$	0	0	4	5	8	10	12	14
$j=4$	0	0	4	5	8	10	12	14

This is the matrix s after the execution of the second DP algorithm:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$j=0$	0	0	0	0	0	0	0	0
$j=1$	0	0	1	1	1	1	1	1
$j=2$	0	0	0	1	0	1	0	1
$j=3$	0	0	0	0	0	1	0	1
$j=4$	0	0	0	0	0	0	0	0

4. We have the following matrix s:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$j=0$	0	0	0	0	0	0	0	0
$j=1$	0	0	1	1	1	1	1	1
$j=2$	0	0	0	1	0	1	0	1
$j=3$	0	0	0	0	0	1	0	1
$j=4$	0	0	0	0	0	0	0	0

We can reconstruct an optimal solution by starting from $j=4$ and $k=7$ as follows:

Iteration	j	k	$s[j][k]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	Action
-	-	-	-	0	0	0	0	$j=4, k=7$
1	4	7	0	0	0	0	0	$j \leftarrow 3$
2	3	7	1	0	0	1	0	$x[3] \leftarrow 1, k \leftarrow 7-5=2$
3	3	2	0	0	0	1	0	$j \leftarrow 2$
4	2	2	0	0	0	1	0	$j \leftarrow 1$
5	1	2	1	1	0	1	0	$x[1] \leftarrow 1, k \leftarrow 2-2=0$

Thus, the vector

$$(1,0,1,0)$$

is an optimal solution. The optimal value is

$$v[4][7]=14.$$

Exercise 3.2 — The two DP algorithms for the UKP (5 item types, $c = 8$)

Consider the following UKP instance, where the knapsack has capacity $c = 8$, each item type is available in infinitely many copies, and the profits and weights associated with the item types are:

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
p_j	2	5	7	8	10
w_j	1	2	3	4	5

Questions

1. Show the execution of the first bottom-up DP algorithm by reporting, for each iteration, the values of the vector v and the values of the vector s.

2. Using the vector s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

3. Show the execution of the second bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and s.

4. Using the matrix s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

Solution

1. Step-by-step execution of the first bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the values of k and j, the cells $v[k]$ and $s[k]$, the current value of m, the current value of t, and the operation performed.

k	j	$v[k]$	$s[k]$	m	t	Action
–	–	0	–	–	–	$v[0] \leftarrow 0$
1	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
1	1	–	–	2	1	$p[1] + v[0] = 2 + 0 = 2 > m = 0$: $m \leftarrow 2$, $t \leftarrow 1$

1	2	–	–	2	1	$w[2] = 2 > k = 1$: no update
1	3	–	–	2	1	$w[3] = 3 > k = 1$: no update
1	4	–	–	2	1	$w[4] = 4 > k = 1$: no update
1	5	–	–	2	1	$w[5] = 5 > k = 1$: no update
1	–	2	1	2	1	$v[1] \leftarrow 2$, $s[1] \leftarrow 1$
2	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
2	1	–	–	4	1	$p[1] + v[1] = 2 + 2 = 4 > m = 0$: $m \leftarrow 4$, $t \leftarrow 1$
2	2	–	–	5	2	$p[2] + v[0] = 5 + 0 = 5 > m = 4$: $m \leftarrow 5$, $t \leftarrow 2$
2	3	–	–	5	2	$w[3] = 3 > k = 2$: no update
2	4	–	–	5	2	$w[4] = 4 > k = 2$: no update
2	5	–	–	5	2	$w[5] = 5 > k = 2$: no update
2	–	5	2	5	2	$v[2] \leftarrow 5$, $s[2] \leftarrow 2$
3	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
3	1	–	–	7	1	$p[1] + v[2] = 2 + 5 = 7 > m = 0$: $m \leftarrow 7$, $t \leftarrow 1$
3	2	–	–	7	1	$p[2] + v[1] = 5 + 2 = 7 \leq m = 7$: no update
3	3	–	–	7	1	$p[3] + v[0] = 7 + 0 = 7 \leq m = 7$: no update
3	4	–	–	7	1	$w[4] = 4 > k = 3$: no update
3	5	–	–	7	1	$w[5] = 5 > k = 3$: no update
3	–	7	1	7	1	$v[3] \leftarrow 7$, $s[3] \leftarrow 1$
4	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
4	1	–	–	9	1	$p[1] + v[3] = 2 + 7 = 9 > m = 0$: $m \leftarrow 9$, $t \leftarrow 1$
4	2	–	–	10	2	$p[2] + v[2] = 5 + 5 = 10 > m = 9$: $m \leftarrow 10$, $t \leftarrow 2$
4	3	–	–	10	2	$p[3] + v[1] = 7 + 2 = 9 \leq m = 10$: no update
4	4	–	–	10	2	$p[4] + v[0] = 8 + 0 = 8 \leq m = 10$: no update
4	5	–	–	10	2	$w[5] = 5 > k = 4$: no update
4	–	10	2	10	2	$v[4] \leftarrow 10$, $s[4] \leftarrow 2$
5	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
5	1	–	–	12	1	$p[1] + v[4] = 2 + 10 = 12 > m = 0$: $m \leftarrow 12$, $t \leftarrow 1$
5	2	–	–	12	1	$p[2] + v[3] = 5 + 7 = 12 \leq m = 12$: no update
5	3	–	–	12	1	$p[3] + v[2] = 7 + 5 = 12 \leq m = 12$: no update
5	4	–	–	12	1	$p[4] + v[1] = 8 + 2 = 10 \leq m = 12$: no update
5	5	–	–	12	1	$p[5] + v[0] = 10 + 0 = 10 \leq m = 12$: no update
5	–	12	1	12	1	$v[5] \leftarrow 12$, $s[5] \leftarrow 1$
6	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
6	1	–	–	14	1	$p[1] + v[5] = 2 + 12 = 14 > m = 0$: $m \leftarrow 14$, $t \leftarrow 1$
6	2	–	–	15	2	$p[2] + v[4] = 5 + 10 = 15 > m = 14$: $m \leftarrow 15$, $t \leftarrow 2$
6	3	–	–	15	2	$p[3] + v[3] = 7 + 7 = 14 \leq m = 15$: no update
6	4	–	–	15	2	$p[4] + v[2] = 8 + 5 = 13 \leq m = 15$: no update
6	5	–	–	15	2	$p[5] + v[1] = 10 + 2 = 12 \leq m = 15$: no update
6	–	15	2	15	2	$v[6] \leftarrow 15$, $s[6] \leftarrow 2$
7	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
7	1	–	–	17	1	$p[1] + v[6] = 2 + 15 = 17 > m = 0$: $m \leftarrow 17$, $t \leftarrow 1$
7	2	–	–	17	1	$p[2] + v[5] = 5 + 12 = 17 \leq m = 17$: no update
7	3	–	–	17	1	$p[3] + v[4] = 7 + 10 = 17 \leq m = 17$: no update
7	4	–	–	17	1	$p[4] + v[3] = 8 + 7 = 15 \leq m = 17$: no update
7	5	–	–	17	1	$p[5] + v[2] = 10 + 5 = 15 \leq m = 17$: no update
7	–	17	1	17	1	$v[7] \leftarrow 17$, $s[7] \leftarrow 1$

8	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
8	1	–	–	19	1	$p[1] + v[7] = 2 + 17 = 19 > m = 0$: $m \leftarrow 19$, $t \leftarrow 1$
8	2	–	–	20	2	$p[2] + v[6] = 5 + 15 = 20 > m = 19$: $m \leftarrow 20$, $t \leftarrow 2$
8	3	–	–	20	2	$p[3] + v[5] = 7 + 12 = 19 \leq m = 20$: no update
8	4	–	–	20	2	$p[4] + v[4] = 8 + 10 = 18 \leq m = 20$: no update
8	5	–	–	20	2	$p[5] + v[3] = 10 + 7 = 17 \leq m = 20$: no update
8	–	20	2	20	2	$v[8] \leftarrow 20$, $s[8] \leftarrow 2$

This is the vector v after the execution of the first DP algorithm:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$v[k]$	0	2	5	7	10	12	15	17	20

This is the vector s after the execution of the first DP algorithm:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$s[k]$	0	1	2	1	2	1	2	1	2

2. We have the following vector s:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$s[k]$	0	1	2	1	2	1	2	1	2

We can reconstruct an optimal solution by starting from $k = 8$ as follows:

Iteration	k	$s[k]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	Action
-	-	-	0	0	0	0	0	$k = 8$
1	8	2	0	1	0	0	0	$x[2] \leftarrow 1$, $k \leftarrow 8 - 2 = 6$
2	6	2	0	2	0	0	0	$x[2] \leftarrow 2$, $k \leftarrow 6 - 2 = 4$
3	4	2	0	3	0	0	0	$x[2] \leftarrow 3$, $k \leftarrow 4 - 2 = 2$
4	2	2	0	4	0	0	0	$x[2] \leftarrow 4$, $k \leftarrow 2 - 2 = 0$

Thus, the vector

$$(0, 4, 0, 0, 0)$$

is an optimal solution. The optimal value is

$$v[8] = 20.$$

3. Step-by-step execution of the second bottom-up DP algorithm

For each step of the DP algorithm, the table shows the values of j and k, the cells $v[j][k]$ and $s[j][k]$, and the operation performed. All entries in row $j = 0$ are initialized to 0.

j	k	$v[j][k]$	$s[j][k]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $s[0][0] \leftarrow 0$
0	1	0	0	$v[0][1] \leftarrow 0$, $s[0][1] \leftarrow 0$
0	2	0	0	$v[0][2] \leftarrow 0$, $s[0][2] \leftarrow 0$
0	3	0	0	$v[0][3] \leftarrow 0$, $s[0][3] \leftarrow 0$
0	4	0	0	$v[0][4] \leftarrow 0$, $s[0][4] \leftarrow 0$
0	5	0	0	$v[0][5] \leftarrow 0$, $s[0][5] \leftarrow 0$
0	6	0	0	$v[0][6] \leftarrow 0$, $s[0][6] \leftarrow 0$
0	7	0	0	$v[0][7] \leftarrow 0$, $s[0][7] \leftarrow 0$
0	8	0	0	$v[0][8] \leftarrow 0$, $s[0][8] \leftarrow 0$
1	0	0	0	$k < w[1]$: $v[1][0] \leftarrow v[0][0] = 0$, $s[1][0] \leftarrow 0$
1	1	2	1	$p[1] + v[1][0] = 2 + 0 > v[0][1] = 0$: $v[1][1] \leftarrow 2$, $s[1][1] \leftarrow 1$
1	2	4	1	$p[1] + v[1][1] = 2 + 2 > v[0][2] = 0$: $v[1][2] \leftarrow 4$, $s[1][2] \leftarrow 1$
1	3	6	1	$p[1] + v[1][2] = 2 + 4 > v[0][3] = 0$: $v[1][3] \leftarrow 6$, $s[1][3] \leftarrow 1$
1	4	8	1	$p[1] + v[1][3] = 2 + 6 > v[0][4] = 0$: $v[1][4] \leftarrow 8$, $s[1][4] \leftarrow 1$
1	5	10	1	$p[1] + v[1][4] = 2 + 8 > v[0][5] = 0$: $v[1][5] \leftarrow 10$, $s[1][5] \leftarrow 1$
1	6	12	1	$p[1] + v[1][5] = 2 + 10 > v[0][6] = 0$: $v[1][6] \leftarrow 12$, $s[1][6] \leftarrow 1$
1	7	14	1	$p[1] + v[1][6] = 2 + 12 > v[0][7] = 0$: $v[1][7] \leftarrow 14$, $s[1][7] \leftarrow 1$
1	8	16	1	$p[1] + v[1][7] = 2 + 14 > v[0][8] = 0$: $v[1][8] \leftarrow 16$, $s[1][8] \leftarrow 1$
2	0	0	0	$k < w[2]$: $v[2][0] \leftarrow v[1][0] = 0$, $s[2][0] \leftarrow 0$
2	1	2	0	$k < w[2]$: $v[2][1] \leftarrow v[1][1] = 2$, $s[2][1] \leftarrow 0$
2	2	5	1	$p[2] + v[2][0] = 5 + 0 > v[1][2] = 4$: $v[2][2] \leftarrow 5$, $s[2][2] \leftarrow 1$
2	3	7	1	$p[2] + v[2][1] = 5 + 2 > v[1][3] = 6$: $v[2][3] \leftarrow 7$, $s[2][3] \leftarrow 1$
2	4	10	1	$p[2] + v[2][2] = 5 + 5 > v[1][4] = 8$: $v[2][4] \leftarrow 10$, $s[2][4] \leftarrow 1$
2	5	12	1	$p[2] + v[2][3] = 5 + 7 > v[1][5] = 10$: $v[2][5] \leftarrow 12$, $s[2][5] \leftarrow 1$
2	6	15	1	$p[2] + v[2][4] = 5 + 10 > v[1][6] = 12$: $v[2][6] \leftarrow 15$, $s[2][6] \leftarrow 1$
2	7	17	1	$p[2] + v[2][5] = 5 + 12 > v[1][7] = 14$: $v[2][7] \leftarrow 17$, $s[2][7] \leftarrow 1$
2	8	20	1	$p[2] + v[2][6] = 5 + 15 > v[1][8] = 16$: $v[2][8] \leftarrow 20$, $s[2][8] \leftarrow 1$
3	0	0	0	$k < w[3]$: $v[3][0] \leftarrow v[2][0] = 0$, $s[3][0] \leftarrow 0$
3	1	2	0	$k < w[3]$: $v[3][1] \leftarrow v[2][1] = 2$, $s[3][1] \leftarrow 0$
3	2	5	0	$k < w[3]$: $v[3][2] \leftarrow v[2][2] = 5$, $s[3][2] \leftarrow 0$
3	3	7	0	$p[3] + v[3][0] = 7 + 0 \leq v[2][3] = 7$: $v[3][3] \leftarrow 7$, $s[3][3] \leftarrow 0$
3	4	10	0	$p[3] + v[3][1] = 7 + 2 \leq v[2][4] = 10$: $v[3][4] \leftarrow 10$, $s[3][4] \leftarrow 0$
3	5	12	0	$p[3] + v[3][2] = 7 + 5 \leq v[2][5] = 12$: $v[3][5] \leftarrow 12$, $s[3][5] \leftarrow 0$
3	6	15	0	$p[3] + v[3][3] = 7 + 7 \leq v[2][6] = 15$: $v[3][6] \leftarrow 15$, $s[3][6] \leftarrow 0$
3	7	17	0	$p[3] + v[3][4] = 7 + 10 \leq v[2][7] = 17$: $v[3][7] \leftarrow 17$, $s[3][7] \leftarrow 0$
3	8	20	0	$p[3] + v[3][5] = 7 + 12 \leq v[2][8] = 20$: $v[3][8] \leftarrow 20$, $s[3][8] \leftarrow 0$
4	0	0	0	$k < w[4]$: $v[4][0] \leftarrow v[3][0] = 0$, $s[4][0] \leftarrow 0$
4	1	2	0	$k < w[4]$: $v[4][1] \leftarrow v[3][1] = 2$, $s[4][1] \leftarrow 0$
4	2	5	0	$k < w[4]$: $v[4][2] \leftarrow v[3][2] = 5$, $s[4][2] \leftarrow 0$
4	3	7	0	$k < w[4]$: $v[4][3] \leftarrow v[3][3] = 7$, $s[4][3] \leftarrow 0$
4	4	10	0	$p[4] + v[4][0] = 8 + 0 \leq v[3][4] = 10$: $v[4][4] \leftarrow 10$, $s[4][4] \leftarrow 0$
4	5	12	0	$p[4] + v[4][1] = 8 + 2 \leq v[3][5] = 12$: $v[4][5] \leftarrow 12$, $s[4][5] \leftarrow 0$
4	6	15	0	$p[4] + v[4][2] = 8 + 5 \leq v[3][6] = 15$: $v[4][6] \leftarrow 15$, $s[4][6] \leftarrow 0$
4	7	17	0	$p[4] + v[4][3] = 8 + 7 \leq v[3][7] = 17$: $v[4][7] \leftarrow 17$, $s[4][7] \leftarrow 0$
4	8	20	0	$p[4] + v[4][4] = 8 + 10 \leq v[3][8] = 20$: $v[4][8] \leftarrow 20$, $s[4][8] \leftarrow 0$
5	0	0	0	$k < w[5]$: $v[5][0] \leftarrow v[4][0] = 0$, $s[5][0] \leftarrow 0$
5	1	2	0	$k < w[5]$: $v[5][1] \leftarrow v[4][1] = 2$, $s[5][1] \leftarrow 0$
5	2	5	0	$k < w[5]$: $v[5][2] \leftarrow v[4][2] = 5$, $s[5][2] \leftarrow 0$
5	3	7	0	$k < w[5]$: $v[5][3] \leftarrow v[4][3] = 7$, $s[5][3] \leftarrow 0$

5	4	10	0	$k < w[5]$: $v[5][4] \leftarrow v[4][4] = 10$, $s[5][4] \leftarrow 0$
5	5	12	0	$p[5] + v[5][0] = 10 + 0 \leq v[4][5] = 12$: $v[5][5] \leftarrow 12$, $s[5][5] \leftarrow 0$
5	6	15	0	$p[5] + v[5][1] = 10 + 2 \leq v[4][6] = 15$: $v[5][6] \leftarrow 15$, $s[5][6] \leftarrow 0$
5	7	17	0	$p[5] + v[5][2] = 10 + 5 \leq v[4][7] = 17$: $v[5][7] \leftarrow 17$, $s[5][7] \leftarrow 0$
5	8	20	0	$p[5] + v[5][3] = 10 + 7 \leq v[4][8] = 20$: $v[5][8] \leftarrow 20$, $s[5][8] \leftarrow 0$

This is the matrix v after the execution of the second DP algorithm:

$v[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$j=0$	0	0	0	0	0	0	0	0	0
$j=1$	0	2	4	6	8	10	12	14	16
$j=2$	0	2	5	7	10	12	15	17	20
$j=3$	0	2	5	7	10	12	15	17	20
$j=4$	0	2	5	7	10	12	15	17	20
$j=5$	0	2	5	7	10	12	15	17	20

This is the matrix s after the execution of the second DP algorithm:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$j=0$	0	0	0	0	0	0	0	0	0
$j=1$	0	1	1	1	1	1	1	1	1
$j=2$	0	0	1	1	1	1	1	1	1
$j=3$	0	0	0	0	0	0	0	0	0
$j=4$	0	0	0	0	0	0	0	0	0
$j=5$	0	0	0	0	0	0	0	0	0

4. We have the following matrix s:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$j=0$	0	0	0	0	0	0	0	0	0
$j=1$	0	1	1	1	1	1	1	1	1
$j=2$	0	0	1	1	1	1	1	1	1
$j=3$	0	0	0	0	0	0	0	0	0
$j=4$	0	0	0	0	0	0	0	0	0
$j=5$	0	0	0	0	0	0	0	0	0

We can reconstruct an optimal solution by starting from $j=5$ and $k=8$ as follows:

Iteration	j	k	$s[j][k]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	Action
-	-	-	-	0	0	0	0	0	$j=5, k=8$
1	5	8	0	0	0	0	0	0	$j \leftarrow 4$
2	4	8	0	0	0	0	0	0	$j \leftarrow 3$
3	3	8	0	0	0	0	0	0	$j \leftarrow 2$
4	2	8	1	0	1	0	0	0	$x[2] \leftarrow 1, k \leftarrow 8-2=6$
5	2	6	1	0	2	0	0	0	$x[2] \leftarrow 2, k \leftarrow 6-2=4$
6	2	4	1	0	3	0	0	0	$x[2] \leftarrow 3, k \leftarrow 4-2=2$
7	2	2	1	0	4	0	0	0	$x[2] \leftarrow 4, k \leftarrow 2-2=0$

Thus, the vector

$$(0, 4, 0, 0, 0)$$

is an optimal solution. The optimal value is

$$v[5][8] = 20.$$

Exercise 3.3 — The two DP algorithms for the UKP (4 item types, $c = 10$)

Consider the following UKP instance, where the knapsack has capacity $c = 10$, each item type is available in infinitely many copies, and the profits and weights associated with the item types are:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
p_j	6	10	12	13
w_j	3	4	6	7

Questions

1. Show the execution of the first bottom-up DP algorithm by reporting, for each iteration, the values of the vector v and the values of the vector s.

2. Using the vector s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

3. Show the execution of the second bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and s.

4. Using the matrix s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

Solution

1. Step-by-step execution of the first bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the values of k and j, the cells $v[k]$

and $s[k]$, the current value of m, the current value of t, and the operation performed.

k	j	$v[k]$	$s[k]$	m	t	Action
–	–	0	–	–	–	$v[0] \leftarrow 0$
1	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
1	1	–	–	0	0	$w[1] = 3 > k = 1$: no update
1	2	–	–	0	0	$w[2] = 4 > k = 1$: no update
1	3	–	–	0	0	$w[3] = 6 > k = 1$: no update
1	4	–	–	0	0	$w[4] = 7 > k = 1$: no update
1	–	0	0	0	0	$v[1] \leftarrow 0$, $s[1] \leftarrow 0$
2	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
2	1	–	–	0	0	$w[1] = 3 > k = 2$: no update
2	2	–	–	0	0	$w[2] = 4 > k = 2$: no update
2	3	–	–	0	0	$w[3] = 6 > k = 2$: no update
2	4	–	–	0	0	$w[4] = 7 > k = 2$: no update
2	–	0	0	0	0	$v[2] \leftarrow 0$, $s[2] \leftarrow 0$
3	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
3	1	–	–	6	1	$p[1] + v[0] = 6 + 0 = 6 > m = 0$: $m \leftarrow 6$, $t \leftarrow 1$
3	2	–	–	6	1	$w[2] = 4 > k = 3$: no update
3	3	–	–	6	1	$w[3] = 6 > k = 3$: no update
3	4	–	–	6	1	$w[4] = 7 > k = 3$: no update
3	–	6	1	6	1	$v[3] \leftarrow 6$, $s[3] \leftarrow 1$
4	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
4	1	–	–	6	1	$p[1] + v[1] = 6 + 0 = 6 > m = 0$: $m \leftarrow 6$, $t \leftarrow 1$
4	2	–	–	10	2	$p[2] + v[0] = 10 + 0 = 10 > m = 6$: $m \leftarrow 10$, $t \leftarrow 2$
4	3	–	–	10	2	$w[3] = 6 > k = 4$: no update
4	4	–	–	10	2	$w[4] = 7 > k = 4$: no update
4	–	10	2	10	2	$v[4] \leftarrow 10$, $s[4] \leftarrow 2$
5	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
5	1	–	–	6	1	$p[1] + v[2] = 6 + 0 = 6 > m = 0$: $m \leftarrow 6$, $t \leftarrow 1$
5	2	–	–	10	2	$p[2] + v[1] = 10 + 0 = 10 > m = 6$: $m \leftarrow 10$, $t \leftarrow 2$
5	3	–	–	10	2	$w[3] = 6 > k = 5$: no update
5	4	–	–	10	2	$w[4] = 7 > k = 5$: no update
5	–	10	2	10	2	$v[5] \leftarrow 10$, $s[5] \leftarrow 2$
6	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
6	1	–	–	12	1	$p[1] + v[3] = 6 + 6 = 12 > m = 0$: $m \leftarrow 12$, $t \leftarrow 1$
6	2	–	–	12	1	$p[2] + v[2] = 10 + 0 = 10 \leq m = 12$: no update
6	3	–	–	12	1	$p[3] + v[0] = 12 + 0 = 12 \leq m = 12$: no update
6	4	–	–	12	1	$w[4] = 7 > k = 6$: no update
6	–	12	1	12	1	$v[6] \leftarrow 12$, $s[6] \leftarrow 1$
7	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
7	1	–	–	16	1	$p[1] + v[4] = 6 + 10 = 16 > m = 0$: $m \leftarrow 16$, $t \leftarrow 1$
7	2	–	–	16	1	$p[2] + v[3] = 10 + 6 = 16 \leq m = 16$: no update
7	3	–	–	16	1	$p[3] + v[1] = 12 + 0 = 12 \leq m = 16$: no update
7	4	–	–	16	1	$p[4] + v[0] = 13 + 0 = 13 \leq m = 16$: no update
7	–	16	1	16	1	$v[7] \leftarrow 16$, $s[7] \leftarrow 1$
8	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$

8	1	–	–	16	1	$p[1] + v[5] = 6 + 10 = 16 > m = 0$: $m \leftarrow 16$, $t \leftarrow 1$
8	2	–	–	20	2	$p[2] + v[4] = 10 + 10 = 20 > m = 16$: $m \leftarrow 20$, $t \leftarrow 2$
8	3	–	–	20	2	$p[3] + v[2] = 12 + 0 = 12 \leq m = 20$: no update
8	4	–	–	20	2	$p[4] + v[1] = 13 + 0 = 13 \leq m = 20$: no update
8	–	20	2	20	2	$v[8] \leftarrow 20$, $s[8] \leftarrow 2$
9	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
9	1	–	–	18	1	$p[1] + v[6] = 6 + 12 = 18 > m = 0$: $m \leftarrow 18$, $t \leftarrow 1$
9	2	–	–	20	2	$p[2] + v[5] = 10 + 10 = 20 > m = 18$: $m \leftarrow 20$, $t \leftarrow 2$
9	3	–	–	20	2	$p[3] + v[3] = 12 + 6 = 18 \leq m = 20$: no update
9	4	–	–	20	2	$p[4] + v[2] = 13 + 0 = 13 \leq m = 20$: no update
9	–	20	2	20	2	$v[9] \leftarrow 20$, $s[9] \leftarrow 2$
10	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
10	1	–	–	22	1	$p[1] + v[7] = 6 + 16 = 22 > m = 0$: $m \leftarrow 22$, $t \leftarrow 1$
10	2	–	–	22	1	$p[2] + v[6] = 10 + 12 = 22 \leq m = 22$: no update
10	3	–	–	22	1	$p[3] + v[4] = 12 + 10 = 22 \leq m = 22$: no update
10	4	–	–	22	1	$p[4] + v[3] = 13 + 6 = 19 \leq m = 22$: no update
10	–	22	1	22	1	$v[10] \leftarrow 22$, $s[10] \leftarrow 1$

This is the vector v after the execution of the first DP algorithm:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
$v[k]$	0	0	0	6	10	10	12	16	20	20	22

This is the vector s after the execution of the first DP algorithm:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
$s[k]$	0	0	0	1	2	2	1	1	2	2	1

2. We have the following vector s:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
$s[k]$	0	0	0	1	2	2	1	1	2	2	1

We can reconstruct an optimal solution by starting from $k = 10$ as follows:

Iteration	k	$s[k]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	Action
-	-	-	0	0	0	0	$k = 10$
1	10	1	1	0	0	0	$x[1] \leftarrow 1$, $k \leftarrow 10 - 3 = 7$
2	7	1	2	0	0	0	$x[1] \leftarrow 2$, $k \leftarrow 7 - 3 = 4$
3	4	2	2	1	0	0	$x[2] \leftarrow 1$, $k \leftarrow 4 - 4 = 0$

Thus, the vector

$$(2, 1, 0, 0)$$

is an optimal solution. The optimal value is

$$v[10] = 22.$$

3. Step-by-step execution of the second bottom-up DP algorithm

For each step of the DP algorithm, the table shows the values of j and k, the cells $v[j][k]$ and $s[j][k]$, and the operation performed. All entries in row $j = 0$ are initialized to 0.

j	k	$v[j][k]$	$s[j][k]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $s[0][0] \leftarrow 0$
0	1	0	0	$v[0][1] \leftarrow 0$, $s[0][1] \leftarrow 0$
0	2	0	0	$v[0][2] \leftarrow 0$, $s[0][2] \leftarrow 0$
0	3	0	0	$v[0][3] \leftarrow 0$, $s[0][3] \leftarrow 0$
0	4	0	0	$v[0][4] \leftarrow 0$, $s[0][4] \leftarrow 0$
0	5	0	0	$v[0][5] \leftarrow 0$, $s[0][5] \leftarrow 0$
0	6	0	0	$v[0][6] \leftarrow 0$, $s[0][6] \leftarrow 0$
0	7	0	0	$v[0][7] \leftarrow 0$, $s[0][7] \leftarrow 0$
0	8	0	0	$v[0][8] \leftarrow 0$, $s[0][8] \leftarrow 0$
0	9	0	0	$v[0][9] \leftarrow 0$, $s[0][9] \leftarrow 0$
0	10	0	0	$v[0][10] \leftarrow 0$, $s[0][10] \leftarrow 0$
1	0	0	0	$k < w[1]$: $v[1][0] \leftarrow v[0][0] = 0$, $s[1][0] \leftarrow 0$
1	1	0	0	$k < w[1]$: $v[1][1] \leftarrow v[0][1] = 0$, $s[1][1] \leftarrow 0$
1	2	0	0	$k < w[1]$: $v[1][2] \leftarrow v[0][2] = 0$, $s[1][2] \leftarrow 0$
1	3	6	1	$p[1] + v[1][0] = 6 + 0 > v[0][3] = 0$: $v[1][3] \leftarrow 6$, $s[1][3] \leftarrow 1$
1	4	6	1	$p[1] + v[1][1] = 6 + 0 > v[0][4] = 0$: $v[1][4] \leftarrow 6$, $s[1][4] \leftarrow 1$
1	5	6	1	$p[1] + v[1][2] = 6 + 0 > v[0][5] = 0$: $v[1][5] \leftarrow 6$, $s[1][5] \leftarrow 1$
1	6	12	1	$p[1] + v[1][3] = 6 + 6 > v[0][6] = 0$: $v[1][6] \leftarrow 12$, $s[1][6] \leftarrow 1$
1	7	12	1	$p[1] + v[1][4] = 6 + 6 > v[0][7] = 0$: $v[1][7] \leftarrow 12$, $s[1][7] \leftarrow 1$
1	8	12	1	$p[1] + v[1][5] = 6 + 6 > v[0][8] = 0$: $v[1][8] \leftarrow 12$, $s[1][8] \leftarrow 1$
1	9	18	1	$p[1] + v[1][6] = 6 + 12 > v[0][9] = 0$: $v[1][9] \leftarrow 18$, $s[1][9] \leftarrow 1$
1	10	18	1	$p[1] + v[1][7] = 6 + 12 > v[0][10] = 0$: $v[1][10] \leftarrow 18$, $s[1][10] \leftarrow 1$
2	0	0	0	$k < w[2]$: $v[2][0] \leftarrow v[1][0] = 0$, $s[2][0] \leftarrow 0$
2	1	0	0	$k < w[2]$: $v[2][1] \leftarrow v[1][1] = 0$, $s[2][1] \leftarrow 0$
2	2	0	0	$k < w[2]$: $v[2][2] \leftarrow v[1][2] = 0$, $s[2][2] \leftarrow 0$
2	3	6	0	$k < w[2]$: $v[2][3] \leftarrow v[1][3] = 6$, $s[2][3] \leftarrow 0$
2	4	10	1	$p[2] + v[2][0] = 10 + 0 > v[1][4] = 6$: $v[2][4] \leftarrow 10$, $s[2][4] \leftarrow 1$
2	5	10	1	$p[2] + v[2][1] = 10 + 0 > v[1][5] = 6$: $v[2][5] \leftarrow 10$, $s[2][5] \leftarrow 1$
2	6	12	0	$p[2] + v[2][2] = 10 + 0 \leq v[1][6] = 12$: $v[2][6] \leftarrow 12$, $s[2][6] \leftarrow 0$
2	7	16	1	$p[2] + v[2][3] = 10 + 6 > v[1][7] = 12$: $v[2][7] \leftarrow 16$, $s[2][7] \leftarrow 1$
2	8	20	1	$p[2] + v[2][4] = 10 + 10 > v[1][8] = 12$: $v[2][8] \leftarrow 20$, $s[2][8] \leftarrow 1$
2	9	20	1	$p[2] + v[2][5] = 10 + 10 > v[1][9] = 18$: $v[2][9] \leftarrow 20$, $s[2][9] \leftarrow 1$
2	10	22	1	$p[2] + v[2][6] = 10 + 12 > v[1][10] = 18$: $v[2][10] \leftarrow 22$, $s[2][10] \leftarrow 1$
3	0	0	0	$k < w[3]$: $v[3][0] \leftarrow v[2][0] = 0$, $s[3][0] \leftarrow 0$
3	1	0	0	$k < w[3]$: $v[3][1] \leftarrow v[2][1] = 0$, $s[3][1] \leftarrow 0$
3	2	0	0	$k < w[3]$: $v[3][2] \leftarrow v[2][2] = 0$, $s[3][2] \leftarrow 0$
3	3	6	0	$k < w[3]$: $v[3][3] \leftarrow v[2][3] = 6$, $s[3][3] \leftarrow 0$
3	4	10	0	$k < w[3]$: $v[3][4] \leftarrow v[2][4] = 10$, $s[3][4] \leftarrow 0$
3	5	10	0	$k < w[3]$: $v[3][5] \leftarrow v[2][5] = 10$, $s[3][5] \leftarrow 0$
3	6	12	0	$p[3] + v[3][0] = 12 + 0 \leq v[2][6] = 12$: $v[3][6] \leftarrow 12$, $s[3][6] \leftarrow 0$
3	7	16	0	$p[3] + v[3][1] = 12 + 0 \leq v[2][7] = 16$: $v[3][7] \leftarrow 16$, $s[3][7] \leftarrow 0$
3	8	20	0	$p[3] + v[3][2] = 12 + 0 \leq v[2][8] = 20$: $v[3][8] \leftarrow 20$, $s[3][8] \leftarrow 0$
3	9	20	0	$p[3] + v[3][3] = 12 + 6 \leq v[2][9] = 20$: $v[3][9] \leftarrow 20$, $s[3][9] \leftarrow 0$
3	10	22	0	$p[3] + v[3][4] = 12 + 10 \leq v[2][10] = 22$: $v[3][10] \leftarrow 22$, $s[3][10] \leftarrow 0$
4	0	0	0	$k < w[4]$: $v[4][0] \leftarrow v[3][0] = 0$, $s[4][0] \leftarrow 0$
4	1	0	0	$k < w[4]$: $v[4][1] \leftarrow v[3][1] = 0$, $s[4][1] \leftarrow 0$
4	2	0	0	$k < w[4]$: $v[4][2] \leftarrow v[3][2] = 0$, $s[4][2] \leftarrow 0$
4	3	6	0	$k < w[4]$: $v[4][3] \leftarrow v[3][3] = 6$, $s[4][3] \leftarrow 0$
4	4	10	0	$k < w[4]$: $v[4][4] \leftarrow v[3][4] = 10$, $s[4][4] \leftarrow 0$

4	5	10	0	$k < w[4]$: $v[4][5] \leftarrow v[3][5] = 10$, $s[4][5] \leftarrow 0$
4	6	12	0	$k < w[4]$: $v[4][6] \leftarrow v[3][6] = 12$, $s[4][6] \leftarrow 0$
4	7	16	0	$p[4] + v[4][0] = 13 + 0 \leq v[3][7] = 16$: $v[4][7] \leftarrow 16$, $s[4][7] \leftarrow 0$
4	8	20	0	$p[4] + v[4][1] = 13 + 0 \leq v[3][8] = 20$: $v[4][8] \leftarrow 20$, $s[4][8] \leftarrow 0$
4	9	20	0	$p[4] + v[4][2] = 13 + 0 \leq v[3][9] = 20$: $v[4][9] \leftarrow 20$, $s[4][9] \leftarrow 0$
4	10	22	0	$p[4] + v[4][3] = 13 + 6 \leq v[3][10] = 22$: $v[4][10] \leftarrow 22$, $s[4][10] \leftarrow 0$

This is the matrix v after the execution of the second DP algorithm:

$v[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
$j=0$	0	0	0	0	0	0	0	0	0	0	0
$j=1$	0	0	0	6	6	6	12	12	12	18	18
$j=2$	0	0	0	6	10	10	12	16	20	20	22
$j=3$	0	0	0	6	10	10	12	16	20	20	22
$j=4$	0	0	0	6	10	10	12	16	20	20	22

This is the matrix s after the execution of the second DP algorithm:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
$j=0$	0	0	0	0	0	0	0	0	0	0	0
$j=1$	0	0	0	1	1	1	1	1	1	1	1
$j=2$	0	0	0	0	1	1	0	1	1	1	1
$j=3$	0	0	0	0	0	0	0	0	0	0	0
$j=4$	0	0	0	0	0	0	0	0	0	0	0

4. We have the following matrix s:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
$j=0$	0	0	0	0	0	0	0	0	0	0	0
$j=1$	0	0	0	1	1	1	1	1	1	1	1
$j=2$	0	0	0	0	1	1	0	1	1	1	1
$j=3$	0	0	0	0	0	0	0	0	0	0	0
$j=4$	0	0	0	0	0	0	0	0	0	0	0

We can reconstruct an optimal solution by starting from $j=4$ and $k=10$ as follows:

Iteration	j	k	$s[j][k]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	Action
-	-	-	-	0	0	0	0	$j=4$, $k=10$
1	4	10	0	0	0	0	0	$j \leftarrow 3$
2	3	10	0	0	0	0	0	$j \leftarrow 2$
3	2	10	1	0	1	0	0	$x[2] \leftarrow 1$, $k \leftarrow 10-4=6$
4	2	6	0	0	1	0	0	$j \leftarrow 1$
5	1	6	1	1	1	0	0	$x[1] \leftarrow 1$, $k \leftarrow 6-3=3$
6	1	3	1	2	1	0	0	$x[1] \leftarrow 2$, $k \leftarrow 3-3=0$

Thus, the vector

$$(2, 1, 0, 0)$$

is an optimal solution. The optimal value is

$$v[4][10] = 22.$$

The optimal solution is not unique: $(0, 1, 1, 0)$, with total weight $4 + 6 = 10$ and total profit $10 + 12 = 22$, is also optimal. Both DP algorithms find $(2, 1, 0, 0)$, because they update the stored value only when it strictly improves.

Exercise 3.4 — The two DP algorithms for the UKP (4 item types, $c = 9$)

Consider the following UKP instance, where the knapsack has capacity $c = 9$, each item type is available in infinitely many copies, and the profits and weights associated with the item types are:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
p_j	5	6	14	18
w_j	2	3	6	7

Questions

1. Show the execution of the first bottom-up DP algorithm by reporting, for each iteration, the values of the vector v and the values of the vector s.

2. Using the vector s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

3. Show the execution of the second bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and s.

4. Using the matrix s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

Solution

1. Step-by-step execution of the first bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the values of k and j, the cells $v[k]$ and $s[k]$, the current value of m, the current value of t, and the operation performed.

k	j	$v[k]$	$s[k]$	m	t	Action
–	–	0	–	–	–	$v[0] \leftarrow 0$
1	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
1	1	–	–	0	0	$w[1] = 2 > k = 1$: no update
1	2	–	–	0	0	$w[2] = 3 > k = 1$: no update
1	3	–	–	0	0	$w[3] = 6 > k = 1$: no update
1	4	–	–	0	0	$w[4] = 7 > k = 1$: no update
1	–	0	0	0	0	$v[1] \leftarrow 0$, $s[1] \leftarrow 0$
2	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
2	1	–	–	5	1	$p[1] + v[0] = 5 + 0 = 5 > m = 0$: $m \leftarrow 5$, $t \leftarrow 1$
2	2	–	–	5	1	$w[2] = 3 > k = 2$: no update
2	3	–	–	5	1	$w[3] = 6 > k = 2$: no update
2	4	–	–	5	1	$w[4] = 7 > k = 2$: no update
2	–	5	1	5	1	$v[2] \leftarrow 5$, $s[2] \leftarrow 1$
3	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
3	1	–	–	5	1	$p[1] + v[1] = 5 + 0 = 5 > m = 0$: $m \leftarrow 5$, $t \leftarrow 1$
3	2	–	–	6	2	$p[2] + v[0] = 6 + 0 = 6 > m = 5$: $m \leftarrow 6$, $t \leftarrow 2$
3	3	–	–	6	2	$w[3] = 6 > k = 3$: no update
3	4	–	–	6	2	$w[4] = 7 > k = 3$: no update
3	–	6	2	6	2	$v[3] \leftarrow 6$, $s[3] \leftarrow 2$
4	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
4	1	–	–	10	1	$p[1] + v[2] = 5 + 5 = 10 > m = 0$: $m \leftarrow 10$, $t \leftarrow 1$
4	2	–	–	10	1	$p[2] + v[1] = 6 + 0 = 6 \leq m = 10$: no update
4	3	–	–	10	1	$w[3] = 6 > k = 4$: no update
4	4	–	–	10	1	$w[4] = 7 > k = 4$: no update
4	–	10	1	10	1	$v[4] \leftarrow 10$, $s[4] \leftarrow 1$
5	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
5	1	–	–	11	1	$p[1] + v[3] = 5 + 6 = 11 > m = 0$: $m \leftarrow 11$, $t \leftarrow 1$
5	2	–	–	11	1	$p[2] + v[2] = 6 + 5 = 11 \leq m = 11$: no update
5	3	–	–	11	1	$w[3] = 6 > k = 5$: no update
5	4	–	–	11	1	$w[4] = 7 > k = 5$: no update
5	–	11	1	11	1	$v[5] \leftarrow 11$, $s[5] \leftarrow 1$
6	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
6	1	–	–	15	1	$p[1] + v[4] = 5 + 10 = 15 > m = 0$: $m \leftarrow 15$, $t \leftarrow 1$
6	2	–	–	15	1	$p[2] + v[3] = 6 + 6 = 12 \leq m = 15$: no update
6	3	–	–	15	1	$p[3] + v[0] = 14 + 0 = 14 \leq m = 15$: no update
6	4	–	–	15	1	$w[4] = 7 > k = 6$: no update
6	–	15	1	15	1	$v[6] \leftarrow 15$, $s[6] \leftarrow 1$
7	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
7	1	–	–	16	1	$p[1] + v[5] = 5 + 11 = 16 > m = 0$: $m \leftarrow 16$, $t \leftarrow 1$
7	2	–	–	16	1	$p[2] + v[4] = 6 + 10 = 16 \leq m = 16$: no update
7	3	–	–	16	1	$p[3] + v[1] = 14 + 0 = 14 \leq m = 16$: no update
7	4	–	–	18	4	$p[4] + v[0] = 18 + 0 = 18 > m = 16$: $m \leftarrow 18$, $t \leftarrow 4$
7	–	18	4	18	4	$v[7] \leftarrow 18$, $s[7] \leftarrow 4$
8	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
8	1	–	–	20	1	$p[1] + v[6] = 5 + 15 = 20 > m = 0$: $m \leftarrow 20$, $t \leftarrow 1$

8	2	–	–	20	1	$p[2] + v[5] = 6 + 11 = 17 \leq m = 20$: no update
8	3	–	–	20	1	$p[3] + v[2] = 14 + 5 = 19 \leq m = 20$: no update
8	4	–	–	20	1	$p[4] + v[1] = 18 + 0 = 18 \leq m = 20$: no update
8	–	20	1	20	1	$v[8] \leftarrow 20$, $s[8] \leftarrow 1$
9	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
9	1	–	–	23	1	$p[1] + v[7] = 5 + 18 = 23 > m = 0$: $m \leftarrow 23$, $t \leftarrow 1$
9	2	–	–	23	1	$p[2] + v[6] = 6 + 15 = 21 \leq m = 23$: no update
9	3	–	–	23	1	$p[3] + v[3] = 14 + 6 = 20 \leq m = 23$: no update
9	4	–	–	23	1	$p[4] + v[2] = 18 + 5 = 23 \leq m = 23$: no update
9	–	23	1	23	1	$v[9] \leftarrow 23$, $s[9] \leftarrow 1$

This is the vector v after the execution of the first DP algorithm:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$v[k]$	0	0	5	6	10	11	15	18	20	23

This is the vector s after the execution of the first DP algorithm:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$s[k]$	0	0	1	2	1	1	1	4	1	1

2. We have the following vector s:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$s[k]$	0	0	1	2	1	1	1	4	1	1

We can reconstruct an optimal solution by starting from $k = 9$ as follows:

Iteration	k	$s[k]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	Action
-	-	-	0	0	0	0	$k = 9$
1	9	1	1	0	0	0	$x[1] \leftarrow 1$, $k \leftarrow 9 - 2 = 7$
2	7	4	1	0	0	1	$x[4] \leftarrow 1$, $k \leftarrow 7 - 7 = 0$

Thus, the vector

$$(1, 0, 0, 1)$$

is an optimal solution. The optimal value is

$$v[9] = 23.$$

3. Step-by-step execution of the second bottom-up DP algorithm

For each step of the DP algorithm, the table shows the values of j and k, the cells $v[j][k]$ and $s[j][k]$, and the operation performed. All entries in row $j = 0$ are initialized to 0.

j	k	$v[j][k]$	$s[j][k]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $s[0][0] \leftarrow 0$
0	1	0	0	$v[0][1] \leftarrow 0$, $s[0][1] \leftarrow 0$
0	2	0	0	$v[0][2] \leftarrow 0$, $s[0][2] \leftarrow 0$
0	3	0	0	$v[0][3] \leftarrow 0$, $s[0][3] \leftarrow 0$
0	4	0	0	$v[0][4] \leftarrow 0$, $s[0][4] \leftarrow 0$
0	5	0	0	$v[0][5] \leftarrow 0$, $s[0][5] \leftarrow 0$
0	6	0	0	$v[0][6] \leftarrow 0$, $s[0][6] \leftarrow 0$
0	7	0	0	$v[0][7] \leftarrow 0$, $s[0][7] \leftarrow 0$
0	8	0	0	$v[0][8] \leftarrow 0$, $s[0][8] \leftarrow 0$
0	9	0	0	$v[0][9] \leftarrow 0$, $s[0][9] \leftarrow 0$
1	0	0	0	$k < w[1]$: $v[1][0] \leftarrow v[0][0] = 0$, $s[1][0] \leftarrow 0$
1	1	0	0	$k < w[1]$: $v[1][1] \leftarrow v[0][1] = 0$, $s[1][1] \leftarrow 0$
1	2	5	1	$p[1] + v[1][0] = 5 + 0 > v[0][2] = 0$: $v[1][2] \leftarrow 5$, $s[1][2] \leftarrow 1$
1	3	5	1	$p[1] + v[1][1] = 5 + 0 > v[0][3] = 0$: $v[1][3] \leftarrow 5$, $s[1][3] \leftarrow 1$
1	4	10	1	$p[1] + v[1][2] = 5 + 5 > v[0][4] = 0$: $v[1][4] \leftarrow 10$, $s[1][4] \leftarrow 1$
1	5	10	1	$p[1] + v[1][3] = 5 + 5 > v[0][5] = 0$: $v[1][5] \leftarrow 10$, $s[1][5] \leftarrow 1$
1	6	15	1	$p[1] + v[1][4] = 5 + 10 > v[0][6] = 0$: $v[1][6] \leftarrow 15$, $s[1][6] \leftarrow 1$
1	7	15	1	$p[1] + v[1][5] = 5 + 10 > v[0][7] = 0$: $v[1][7] \leftarrow 15$, $s[1][7] \leftarrow 1$
1	8	20	1	$p[1] + v[1][6] = 5 + 15 > v[0][8] = 0$: $v[1][8] \leftarrow 20$, $s[1][8] \leftarrow 1$
1	9	20	1	$p[1] + v[1][7] = 5 + 15 > v[0][9] = 0$: $v[1][9] \leftarrow 20$, $s[1][9] \leftarrow 1$
2	0	0	0	$k < w[2]$: $v[2][0] \leftarrow v[1][0] = 0$, $s[2][0] \leftarrow 0$
2	1	0	0	$k < w[2]$: $v[2][1] \leftarrow v[1][1] = 0$, $s[2][1] \leftarrow 0$
2	2	5	0	$k < w[2]$: $v[2][2] \leftarrow v[1][2] = 5$, $s[2][2] \leftarrow 0$
2	3	6	1	$p[2] + v[2][0] = 6 + 0 > v[1][3] = 5$: $v[2][3] \leftarrow 6$, $s[2][3] \leftarrow 1$
2	4	10	0	$p[2] + v[2][1] = 6 + 0 \leq v[1][4] = 10$: $v[2][4] \leftarrow 10$, $s[2][4] \leftarrow 0$
2	5	11	1	$p[2] + v[2][2] = 6 + 5 > v[1][5] = 10$: $v[2][5] \leftarrow 11$, $s[2][5] \leftarrow 1$
2	6	15	0	$p[2] + v[2][3] = 6 + 6 \leq v[1][6] = 15$: $v[2][6] \leftarrow 15$, $s[2][6] \leftarrow 0$
2	7	16	1	$p[2] + v[2][4] = 6 + 10 > v[1][7] = 15$: $v[2][7] \leftarrow 16$, $s[2][7] \leftarrow 1$
2	8	20	0	$p[2] + v[2][5] = 6 + 11 \leq v[1][8] = 20$: $v[2][8] \leftarrow 20$, $s[2][8] \leftarrow 0$
2	9	21	1	$p[2] + v[2][6] = 6 + 15 > v[1][9] = 20$: $v[2][9] \leftarrow 21$, $s[2][9] \leftarrow 1$
3	0	0	0	$k < w[3]$: $v[3][0] \leftarrow v[2][0] = 0$, $s[3][0] \leftarrow 0$
3	1	0	0	$k < w[3]$: $v[3][1] \leftarrow v[2][1] = 0$, $s[3][1] \leftarrow 0$
3	2	5	0	$k < w[3]$: $v[3][2] \leftarrow v[2][2] = 5$, $s[3][2] \leftarrow 0$
3	3	6	0	$k < w[3]$: $v[3][3] \leftarrow v[2][3] = 6$, $s[3][3] \leftarrow 0$
3	4	10	0	$k < w[3]$: $v[3][4] \leftarrow v[2][4] = 10$, $s[3][4] \leftarrow 0$
3	5	11	0	$k < w[3]$: $v[3][5] \leftarrow v[2][5] = 11$, $s[3][5] \leftarrow 0$
3	6	15	0	$p[3] + v[3][0] = 14 + 0 \leq v[2][6] = 15$: $v[3][6] \leftarrow 15$, $s[3][6] \leftarrow 0$
3	7	16	0	$p[3] + v[3][1] = 14 + 0 \leq v[2][7] = 16$: $v[3][7] \leftarrow 16$, $s[3][7] \leftarrow 0$
3	8	20	0	$p[3] + v[3][2] = 14 + 5 \leq v[2][8] = 20$: $v[3][8] \leftarrow 20$, $s[3][8] \leftarrow 0$
3	9	21	0	$p[3] + v[3][3] = 14 + 6 \leq v[2][9] = 21$: $v[3][9] \leftarrow 21$, $s[3][9] \leftarrow 0$
4	0	0	0	$k < w[4]$: $v[4][0] \leftarrow v[3][0] = 0$, $s[4][0] \leftarrow 0$
4	1	0	0	$k < w[4]$: $v[4][1] \leftarrow v[3][1] = 0$, $s[4][1] \leftarrow 0$
4	2	5	0	$k < w[4]$: $v[4][2] \leftarrow v[3][2] = 5$, $s[4][2] \leftarrow 0$
4	3	6	0	$k < w[4]$: $v[4][3] \leftarrow v[3][3] = 6$, $s[4][3] \leftarrow 0$
4	4	10	0	$k < w[4]$: $v[4][4] \leftarrow v[3][4] = 10$, $s[4][4] \leftarrow 0$
4	5	11	0	$k < w[4]$: $v[4][5] \leftarrow v[3][5] = 11$, $s[4][5] \leftarrow 0$
4	6	15	0	$k < w[4]$: $v[4][6] \leftarrow v[3][6] = 15$, $s[4][6] \leftarrow 0$
4	7	18	1	$p[4] + v[4][0] = 18 + 0 > v[3][7] = 16$: $v[4][7] \leftarrow 18$, $s[4][7] \leftarrow 1$
4	8	20	0	$p[4] + v[4][1] = 18 + 0 \leq v[3][8] = 20$: $v[4][8] \leftarrow 20$, $s[4][8] \leftarrow 0$

4	9	23	1	$p[4] + v[4][2] = 18 + 5 > v[3][9] = 21$: $v[4][9] \leftarrow 23$, $s[4][9] \leftarrow 1$

This is the matrix v after the execution of the second DP algorithm:

$v[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$j=0$	0	0	0	0	0	0	0	0	0	0
$j=1$	0	0	5	5	10	10	15	15	20	20
$j=2$	0	0	5	6	10	11	15	16	20	21
$j=3$	0	0	5	6	10	11	15	16	20	21
$j=4$	0	0	5	6	10	11	15	18	20	23

This is the matrix s after the execution of the second DP algorithm:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$j=0$	0	0	0	0	0	0	0	0	0	0
$j=1$	0	0	1	1	1	1	1	1	1	1
$j=2$	0	0	0	1	0	1	0	1	0	1
$j=3$	0	0	0	0	0	0	0	0	0	0
$j=4$	0	0	0	0	0	0	0	1	0	1

4. We have the following matrix s:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$j=0$	0	0	0	0	0	0	0	0	0	0
$j=1$	0	0	1	1	1	1	1	1	1	1
$j=2$	0	0	0	1	0	1	0	1	0	1
$j=3$	0	0	0	0	0	0	0	0	0	0
$j=4$	0	0	0	0	0	0	0	1	0	1

We can reconstruct an optimal solution by starting from $j=4$ and $k=9$ as follows:

Iteration	j	k	$s[j][k]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	Action
-	-	-	-	0	0	0	0	$j=4$, $k=9$
1	4	9	1	0	0	0	1	$x[4] \leftarrow 1$, $k \leftarrow 9-7=2$
2	4	2	0	0	0	0	1	$j \leftarrow 3$
3	3	2	0	0	0	0	1	$j \leftarrow 2$
4	2	2	0	0	0	0	1	$j \leftarrow 1$
5	1	2	1	1	0	0	1	$x[1] \leftarrow 1$, $k \leftarrow 2-2=0$

Thus, the vector

$$(1,0,0,1)$$

is an optimal solution. The optimal value is

$$v[4][9] = 23.$$

Exercise 3.5 — The two DP algorithms for the UKP (5 item types, $c = 11$)

Consider the following UKP instance, where the knapsack has capacity $c = 11$, each item type is available in infinitely many copies, and the profits and weights associated with the item types are:

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
p_j	3	9	10	16	17
w_j	2	5	6	9	10

Questions

1. Show the execution of the first bottom-up DP algorithm by reporting, for each iteration, the values of the vector v and the values of the vector s.

2. Using the vector s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

3. Show the execution of the second bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and s.

4. Using the matrix s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

Solution

1. Step-by-step execution of the first bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the values of k and j, the cells $v[k]$ and $s[k]$, the current value of m, the current value of t, and the operation performed.

k	j	$v[k]$	$s[k]$	m	t	Action
–	–	0	–	–	–	$v[0] \leftarrow 0$
1	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
1	1	–	–	0	0	$w[1] = 2 > k = 1$: no update

1	2	–	–	0	0	$w[2] = 5 > k = 1$: no update
1	3	–	–	0	0	$w[3] = 6 > k = 1$: no update
1	4	–	–	0	0	$w[4] = 9 > k = 1$: no update
1	5	–	–	0	0	$w[5] = 10 > k = 1$: no update
1	–	0	0	0	0	$v[1] \leftarrow 0$, $s[1] \leftarrow 0$
2	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
2	1	–	–	3	1	$p[1] + v[0] = 3 + 0 = 3 > m = 0$: $m \leftarrow 3$, $t \leftarrow 1$
2	2	–	–	3	1	$w[2] = 5 > k = 2$: no update
2	3	–	–	3	1	$w[3] = 6 > k = 2$: no update
2	4	–	–	3	1	$w[4] = 9 > k = 2$: no update
2	5	–	–	3	1	$w[5] = 10 > k = 2$: no update
2	–	3	1	3	1	$v[2] \leftarrow 3$, $s[2] \leftarrow 1$
3	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
3	1	–	–	3	1	$p[1] + v[1] = 3 + 0 = 3 > m = 0$: $m \leftarrow 3$, $t \leftarrow 1$
3	2	–	–	3	1	$w[2] = 5 > k = 3$: no update
3	3	–	–	3	1	$w[3] = 6 > k = 3$: no update
3	4	–	–	3	1	$w[4] = 9 > k = 3$: no update
3	5	–	–	3	1	$w[5] = 10 > k = 3$: no update
3	–	3	1	3	1	$v[3] \leftarrow 3$, $s[3] \leftarrow 1$
4	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
4	1	–	–	6	1	$p[1] + v[2] = 3 + 3 = 6 > m = 0$: $m \leftarrow 6$, $t \leftarrow 1$
4	2	–	–	6	1	$w[2] = 5 > k = 4$: no update
4	3	–	–	6	1	$w[3] = 6 > k = 4$: no update
4	4	–	–	6	1	$w[4] = 9 > k = 4$: no update
4	5	–	–	6	1	$w[5] = 10 > k = 4$: no update
4	–	6	1	6	1	$v[4] \leftarrow 6$, $s[4] \leftarrow 1$
5	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
5	1	–	–	6	1	$p[1] + v[3] = 3 + 3 = 6 > m = 0$: $m \leftarrow 6$, $t \leftarrow 1$
5	2	–	–	9	2	$p[2] + v[0] = 9 + 0 = 9 > m = 6$: $m \leftarrow 9$, $t \leftarrow 2$
5	3	–	–	9	2	$w[3] = 6 > k = 5$: no update
5	4	–	–	9	2	$w[4] = 9 > k = 5$: no update
5	5	–	–	9	2	$w[5] = 10 > k = 5$: no update
5	–	9	2	9	2	$v[5] \leftarrow 9$, $s[5] \leftarrow 2$
6	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
6	1	–	–	9	1	$p[1] + v[4] = 3 + 6 = 9 > m = 0$: $m \leftarrow 9$, $t \leftarrow 1$
6	2	–	–	9	1	$p[2] + v[1] = 9 + 0 = 9 \leq m = 9$: no update
6	3	–	–	10	3	$p[3] + v[0] = 10 + 0 = 10 > m = 9$: $m \leftarrow 10$, $t \leftarrow 3$
6	4	–	–	10	3	$w[4] = 9 > k = 6$: no update
6	5	–	–	10	3	$w[5] = 10 > k = 6$: no update
6	–	10	3	10	3	$v[6] \leftarrow 10$, $s[6] \leftarrow 3$
7	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
7	1	–	–	12	1	$p[1] + v[5] = 3 + 9 = 12 > m = 0$: $m \leftarrow 12$, $t \leftarrow 1$
7	2	–	–	12	1	$p[2] + v[2] = 9 + 3 = 12 \leq m = 12$: no update
7	3	–	–	12	1	$p[3] + v[1] = 10 + 0 = 10 \leq m = 12$: no update
7	4	–	–	12	1	$w[4] = 9 > k = 7$: no update
7	5	–	–	12	1	$w[5] = 10 > k = 7$: no update
7	–	12	1	12	1	$v[7] \leftarrow 12$, $s[7] \leftarrow 1$
8	–	–	–	0	0	$m \leftarrow 0$, $t \leftarrow 0$
8	1	–	–	13	1	$p[1] + v[6] = 3 + 10 = 13 > m = 0$: $m \leftarrow 13$, $t \leftarrow 1$

8	2	–	–	13	1	$p[2]+v[3]=9+3=12\leq m=13$: no update
8	3	–	–	13	1	$p[3]+v[2]=10+3=13\leq m=13$: no update
8	4	–	–	13	1	$w[4]=9>k=8$: no update
8	5	–	–	13	1	$w[5]=10>k=8$: no update
8	–	13	1	13	1	$v[8]\leftarrow 13$, $s[8]\leftarrow 1$
9	–	–	–	0	0	$m\leftarrow 0$, $t\leftarrow 0$
9	1	–	–	15	1	$p[1]+v[7]=3+12=15>m=0$: $m\leftarrow 15$, $t\leftarrow 1$
9	2	–	–	15	1	$p[2]+v[4]=9+6=15\leq m=15$: no update
9	3	–	–	15	1	$p[3]+v[3]=10+3=13\leq m=15$: no update
9	4	–	–	16	4	$p[4]+v[0]=16+0=16>m=15$: $m\leftarrow 16$, $t\leftarrow 4$
9	5	–	–	16	4	$w[5]=10>k=9$: no update
9	–	16	4	16	4	$v[9]\leftarrow 16$, $s[9]\leftarrow 4$
10	–	–	–	0	0	$m\leftarrow 0$, $t\leftarrow 0$
10	1	–	–	16	1	$p[1]+v[8]=3+13=16>m=0$: $m\leftarrow 16$, $t\leftarrow 1$
10	2	–	–	18	2	$p[2]+v[5]=9+9=18>m=16$: $m\leftarrow 18$, $t\leftarrow 2$
10	3	–	–	18	2	$p[3]+v[4]=10+6=16\leq m=18$: no update
10	4	–	–	18	2	$p[4]+v[1]=16+0=16\leq m=18$: no update
10	5	–	–	18	2	$p[5]+v[0]=17+0=17\leq m=18$: no update
10	–	18	2	18	2	$v[10]\leftarrow 18$, $s[10]\leftarrow 2$
11	–	–	–	0	0	$m\leftarrow 0$, $t\leftarrow 0$
11	1	–	–	19	1	$p[1]+v[9]=3+16=19>m=0$: $m\leftarrow 19$, $t\leftarrow 1$
11	2	–	–	19	1	$p[2]+v[6]=9+10=19\leq m=19$: no update
11	3	–	–	19	1	$p[3]+v[5]=10+9=19\leq m=19$: no update
11	4	–	–	19	1	$p[4]+v[2]=16+3=19\leq m=19$: no update
11	5	–	–	19	1	$p[5]+v[1]=17+0=17\leq m=19$: no update
11	–	19	1	19	1	$v[11]\leftarrow 19$, $s[11]\leftarrow 1$

This is the vector v after the execution of the first DP algorithm:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$	$k=11$
$v[k]$	0	0	3	3	6	9	10	12	13	16	18	19

This is the vector s after the execution of the first DP algorithm:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$	$k=11$
$s[k]$	0	0	1	1	1	2	3	1	1	4	2	1

2. We have the following vector s:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$	$k=11$
$s[k]$	0	0	1	1	1	2	3	1	1	4	2	1

We can reconstruct an optimal solution by starting from $k = 11$ as follows:

Iteration	k	$s[k]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	Action
-	-	-	0	0	0	0	0	$k = 11$
1	11	1	1	0	0	0	0	$x[1] \leftarrow 1$, $k \leftarrow 11 - 2 = 9$
2	9	4	1	0	0	1	0	$x[4] \leftarrow 1$, $k \leftarrow 9 - 9 = 0$

Thus, the vector

$$(1, 0, 0, 1, 0)$$

is an optimal solution. The optimal value is

$$v[11] = 19.$$

3. Step-by-step execution of the second bottom-up DP algorithm

For each step of the DP algorithm, the table shows the values of j and k, the cells $v[j][k]$ and $s[j][k]$, and the operation performed. All entries in row $j = 0$ are initialized to 0.

j	k	$v[j][k]$	$s[j][k]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $s[0][0] \leftarrow 0$
0	1	0	0	$v[0][1] \leftarrow 0$, $s[0][1] \leftarrow 0$
0	2	0	0	$v[0][2] \leftarrow 0$, $s[0][2] \leftarrow 0$
0	3	0	0	$v[0][3] \leftarrow 0$, $s[0][3] \leftarrow 0$
0	4	0	0	$v[0][4] \leftarrow 0$, $s[0][4] \leftarrow 0$
0	5	0	0	$v[0][5] \leftarrow 0$, $s[0][5] \leftarrow 0$
0	6	0	0	$v[0][6] \leftarrow 0$, $s[0][6] \leftarrow 0$
0	7	0	0	$v[0][7] \leftarrow 0$, $s[0][7] \leftarrow 0$
0	8	0	0	$v[0][8] \leftarrow 0$, $s[0][8] \leftarrow 0$
0	9	0	0	$v[0][9] \leftarrow 0$, $s[0][9] \leftarrow 0$
0	10	0	0	$v[0][10] \leftarrow 0$, $s[0][10] \leftarrow 0$
0	11	0	0	$v[0][11] \leftarrow 0$, $s[0][11] \leftarrow 0$
1	0	0	0	$k < w[1]$: $v[1][0] \leftarrow v[0][0] = 0$, $s[1][0] \leftarrow 0$
1	1	0	0	$k < w[1]$: $v[1][1] \leftarrow v[0][1] = 0$, $s[1][1] \leftarrow 0$
1	2	3	1	$p[1] + v[1][0] = 3 + 0 > v[0][2] = 0$: $v[1][2] \leftarrow 3$, $s[1][2] \leftarrow 1$
1	3	3	1	$p[1] + v[1][1] = 3 + 0 > v[0][3] = 0$: $v[1][3] \leftarrow 3$, $s[1][3] \leftarrow 1$
1	4	6	1	$p[1] + v[1][2] = 3 + 3 > v[0][4] = 0$: $v[1][4] \leftarrow 6$, $s[1][4] \leftarrow 1$
1	5	6	1	$p[1] + v[1][3] = 3 + 3 > v[0][5] = 0$: $v[1][5] \leftarrow 6$, $s[1][5] \leftarrow 1$
1	6	9	1	$p[1] + v[1][4] = 3 + 6 > v[0][6] = 0$: $v[1][6] \leftarrow 9$, $s[1][6] \leftarrow 1$
1	7	9	1	$p[1] + v[1][5] = 3 + 6 > v[0][7] = 0$: $v[1][7] \leftarrow 9$, $s[1][7] \leftarrow 1$
1	8	12	1	$p[1] + v[1][6] = 3 + 9 > v[0][8] = 0$: $v[1][8] \leftarrow 12$, $s[1][8] \leftarrow 1$
1	9	12	1	$p[1] + v[1][7] = 3 + 9 > v[0][9] = 0$: $v[1][9] \leftarrow 12$, $s[1][9] \leftarrow 1$
1	10	15	1	$p[1] + v[1][8] = 3 + 12 > v[0][10] = 0$: $v[1][10] \leftarrow 15$, $s[1][10] \leftarrow 1$
1	11	15	1	$p[1] + v[1][9] = 3 + 12 > v[0][11] = 0$: $v[1][11] \leftarrow 15$, $s[1][11] \leftarrow 1$
2	0	0	0	$k < w[2]$: $v[2][0] \leftarrow v[1][0] = 0$, $s[2][0] \leftarrow 0$
2	1	0	0	$k < w[2]$: $v[2][1] \leftarrow v[1][1] = 0$, $s[2][1] \leftarrow 0$
2	2	3	0	$k < w[2]$: $v[2][2] \leftarrow v[1][2] = 3$, $s[2][2] \leftarrow 0$
2	3	3	0	$k < w[2]$: $v[2][3] \leftarrow v[1][3] = 3$, $s[2][3] \leftarrow 0$
2	4	6	0	$k < w[2]$: $v[2][4] \leftarrow v[1][4] = 6$, $s[2][4] \leftarrow 0$
2	5	9	1	$p[2] + v[2][0] = 9 + 0 > v[1][5] = 6$: $v[2][5] \leftarrow 9$, $s[2][5] \leftarrow 1$
2	6	9	0	$p[2] + v[2][1] = 9 + 0 \leq v[1][6] = 9$: $v[2][6] \leftarrow 9$, $s[2][6] \leftarrow 0$
2	7	12	1	$p[2] + v[2][2] = 9 + 3 > v[1][7] = 9$: $v[2][7] \leftarrow 12$, $s[2][7] \leftarrow 1$
2	8	12	0	$p[2] + v[2][3] = 9 + 3 \leq v[1][8] = 12$: $v[2][8] \leftarrow 12$, $s[2][8] \leftarrow 0$
2	9	15	1	$p[2] + v[2][4] = 9 + 6 > v[1][9] = 12$: $v[2][9] \leftarrow 15$, $s[2][9] \leftarrow 1$
2	10	18	1	$p[2] + v[2][5] = 9 + 9 > v[1][10] = 15$: $v[2][10] \leftarrow 18$, $s[2][10] \leftarrow 1$
2	11	18	1	$p[2] + v[2][6] = 9 + 9 > v[1][11] = 15$: $v[2][11] \leftarrow 18$, $s[2][11] \leftarrow 1$
3	0	0	0	$k < w[3]$: $v[3][0] \leftarrow v[2][0] = 0$, $s[3][0] \leftarrow 0$
3	1	0	0	$k < w[3]$: $v[3][1] \leftarrow v[2][1] = 0$, $s[3][1] \leftarrow 0$
3	2	3	0	$k < w[3]$: $v[3][2] \leftarrow v[2][2] = 3$, $s[3][2] \leftarrow 0$
3	3	3	0	$k < w[3]$: $v[3][3] \leftarrow v[2][3] = 3$, $s[3][3] \leftarrow 0$
3	4	6	0	$k < w[3]$: $v[3][4] \leftarrow v[2][4] = 6$, $s[3][4] \leftarrow 0$
3	5	9	0	$k < w[3]$: $v[3][5] \leftarrow v[2][5] = 9$, $s[3][5] \leftarrow 0$
3	6	10	1	$p[3] + v[3][0] = 10 + 0 > v[2][6] = 9$: $v[3][6] \leftarrow 10$, $s[3][6] \leftarrow 1$
3	7	12	0	$p[3] + v[3][1] = 10 + 0 \leq v[2][7] = 12$: $v[3][7] \leftarrow 12$, $s[3][7] \leftarrow 0$
3	8	13	1	$p[3] + v[3][2] = 10 + 3 > v[2][8] = 12$: $v[3][8] \leftarrow 13$, $s[3][8] \leftarrow 1$
3	9	15	0	$p[3] + v[3][3] = 10 + 3 \leq v[2][9] = 15$: $v[3][9] \leftarrow 15$, $s[3][9] \leftarrow 0$
3	10	18	0	$p[3] + v[3][4] = 10 + 6 \leq v[2][10] = 18$: $v[3][10] \leftarrow 18$, $s[3][10] \leftarrow 0$
3	11	19	1	$p[3] + v[3][5] = 10 + 9 > v[2][11] = 18$: $v[3][11] \leftarrow 19$, $s[3][11] \leftarrow 1$
4	0	0	0	$k < w[4]$: $v[4][0] \leftarrow v[3][0] = 0$, $s[4][0] \leftarrow 0$

4	1	0	0	$k < w[4]$: $v[4][1] \leftarrow v[3][1] = 0$, $s[4][1] \leftarrow 0$
4	2	3	0	$k < w[4]$: $v[4][2] \leftarrow v[3][2] = 3$, $s[4][2] \leftarrow 0$
4	3	3	0	$k < w[4]$: $v[4][3] \leftarrow v[3][3] = 3$, $s[4][3] \leftarrow 0$
4	4	6	0	$k < w[4]$: $v[4][4] \leftarrow v[3][4] = 6$, $s[4][4] \leftarrow 0$
4	5	9	0	$k < w[4]$: $v[4][5] \leftarrow v[3][5] = 9$, $s[4][5] \leftarrow 0$
4	6	10	0	$k < w[4]$: $v[4][6] \leftarrow v[3][6] = 10$, $s[4][6] \leftarrow 0$
4	7	12	0	$k < w[4]$: $v[4][7] \leftarrow v[3][7] = 12$, $s[4][7] \leftarrow 0$
4	8	13	0	$k < w[4]$: $v[4][8] \leftarrow v[3][8] = 13$, $s[4][8] \leftarrow 0$
4	9	16	1	$p[4] + v[4][0] = 16 + 0 > v[3][9] = 15$: $v[4][9] \leftarrow 16$, $s[4][9] \leftarrow 1$
4	10	18	0	$p[4] + v[4][1] = 16 + 0 \leq v[3][10] = 18$: $v[4][10] \leftarrow 18$, $s[4][10] \leftarrow 0$
4	11	19	0	$p[4] + v[4][2] = 16 + 3 \leq v[3][11] = 19$: $v[4][11] \leftarrow 19$, $s[4][11] \leftarrow 0$
5	0	0	0	$k < w[5]$: $v[5][0] \leftarrow v[4][0] = 0$, $s[5][0] \leftarrow 0$
5	1	0	0	$k < w[5]$: $v[5][1] \leftarrow v[4][1] = 0$, $s[5][1] \leftarrow 0$
5	2	3	0	$k < w[5]$: $v[5][2] \leftarrow v[4][2] = 3$, $s[5][2] \leftarrow 0$
5	3	3	0	$k < w[5]$: $v[5][3] \leftarrow v[4][3] = 3$, $s[5][3] \leftarrow 0$
5	4	6	0	$k < w[5]$: $v[5][4] \leftarrow v[4][4] = 6$, $s[5][4] \leftarrow 0$
5	5	9	0	$k < w[5]$: $v[5][5] \leftarrow v[4][5] = 9$, $s[5][5] \leftarrow 0$
5	6	10	0	$k < w[5]$: $v[5][6] \leftarrow v[4][6] = 10$, $s[5][6] \leftarrow 0$
5	7	12	0	$k < w[5]$: $v[5][7] \leftarrow v[4][7] = 12$, $s[5][7] \leftarrow 0$
5	8	13	0	$k < w[5]$: $v[5][8] \leftarrow v[4][8] = 13$, $s[5][8] \leftarrow 0$
5	9	16	0	$k < w[5]$: $v[5][9] \leftarrow v[4][9] = 16$, $s[5][9] \leftarrow 0$
5	10	18	0	$p[5] + v[5][0] = 17 + 0 \leq v[4][10] = 18$: $v[5][10] \leftarrow 18$, $s[5][10] \leftarrow 0$
5	11	19	0	$p[5] + v[5][1] = 17 + 0 \leq v[4][11] = 19$: $v[5][11] \leftarrow 19$, $s[5][11] \leftarrow 0$

This is the matrix v after the execution of the second DP algorithm:

$v[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$	$k=11$
$j=0$	0	0	0	0	0	0	0	0	0	0	0	0
$j=1$	0	0	3	3	6	6	9	9	12	12	15	15
$j=2$	0	0	3	3	6	9	9	12	12	15	18	18
$j=3$	0	0	3	3	6	9	10	12	13	15	18	19
$j=4$	0	0	3	3	6	9	10	12	13	16	18	19
$j=5$	0	0	3	3	6	9	10	12	13	16	18	19

This is the matrix s after the execution of the second DP algorithm:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$	$k=11$
$j=0$	0	0	0	0	0	0	0	0	0	0	0	0
$j=1$	0	0	1	1	1	1	1	1	1	1	1	1
$j=2$	0	0	0	0	0	1	0	1	0	1	1	1
$j=3$	0	0	0	0	0	0	1	0	1	0	0	1
$j=4$	0	0	0	0	0	0	0	0	0	1	0	0
$j=5$	0	0	0	0	0	0	0	0	0	0	0	0

4. We have the following matrix s:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$	$k=11$
$j=0$	0	0	0	0	0	0	0	0	0	0	0	0
$j=1$	0	0	1	1	1	1	1	1	1	1	1	1
$j=2$	0	0	0	0	0	1	0	1	0	1	1	1
$j=3$	0	0	0	0	0	0	1	0	1	0	0	1
$j=4$	0	0	0	0	0	0	0	0	0	1	0	0
$j=5$	0	0	0	0	0	0	0	0	0	0	0	0

We can reconstruct an optimal solution by starting from $j=5$ and $k=11$ as follows:

Iteration	j	k	$s[j][k]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	Action
-	-	-	-	0	0	0	0	0	$j=5, k=11$
1	5	11	0	0	0	0	0	0	$j \leftarrow 4$
2	4	11	0	0	0	0	0	0	$j \leftarrow 3$
3	3	11	1	0	0	1	0	0	$x[3] \leftarrow 1, k \leftarrow 11-6=5$
4	3	5	0	0	0	1	0	0	$j \leftarrow 2$
5	2	5	1	0	1	1	0	0	$x[2] \leftarrow 1, k \leftarrow 5-5=0$

Thus, the vector

$$(0, 1, 1, 0, 0)$$

is an optimal solution. The optimal value is

$$v[5][11] = 19.$$

The optimal solution is not unique, and the two DP algorithms find different ones: the first finds $(1, 0, 0, 1, 0)$, with total weight $2 + 9 = 11$ and total profit $3 + 16 = 19$; the second finds $(0, 1, 1, 0, 0)$, with total weight $5+6 = 11$ and total profit $9+10 = 19$. These are the only two optimal solutions.

Chapter 4

DP for the bounded knapsack problem

Exercise 4.1

Consider the following BKP instance, where the knapsack has capacity $c = 6$, and the profits, weights, and bounds associated with the item types are:

	$j = 1$	$j = 2$
p_j	4	5
w_j	2	3
b_j	2	1

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and s.

2. Using the matrix s, reconstruct an optimal solution in integer notation (x_1, x_2), and report the optimal value.

Solution

1. Step-by-step execution of the bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the values of j and k, the cells $v[j][k]$ and $s[j][k]$, and the operation performed.

j	k	$v[j][k]$	$s[j][k]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $s[0][0] \leftarrow 0$
0	1	0	0	$v[0][1] \leftarrow 0$, $s[0][1] \leftarrow 0$
0	2	0	0	$v[0][2] \leftarrow 0$, $s[0][2] \leftarrow 0$
0	3	0	0	$v[0][3] \leftarrow 0$, $s[0][3] \leftarrow 0$
0	4	0	0	$v[0][4] \leftarrow 0$, $s[0][4] \leftarrow 0$
0	5	0	0	$v[0][5] \leftarrow 0$, $s[0][5] \leftarrow 0$
0	6	0	0	$v[0][6] \leftarrow 0$, $s[0][6] \leftarrow 0$
1	0	0	0	$\max\{v[0][0]\} = 0$
1	1	0	0	$\max\{v[0][1]\} = 0$
1	2	4	1	$\max\{v[0][2], p[1]+v[0][0]\} = \max\{0, 4\} = 4$
1	3	4	1	$\max\{v[0][3], p[1]+v[0][1]\} = \max\{0, 4\} = 4$
1	4	8	2	$\max\{v[0][4], p[1]+v[0][2], 2p[1]+v[0][0]\} = \max\{0, 4, 8\} = 8$
1	5	8	2	$\max\{v[0][5], p[1]+v[0][3], 2p[1]+v[0][1]\} = \max\{0, 4, 8\} = 8$
1	6	8	2	$\max\{v[0][6], p[1]+v[0][4], 2p[1]+v[0][2]\} = \max\{0, 4, 8\} = 8$
2	0	0	0	$\max\{v[1][0]\} = 0$
2	1	0	0	$\max\{v[1][1]\} = 0$
2	2	4	0	$\max\{v[1][2]\} = 4$
2	3	5	1	$\max\{v[1][3], p[2]+v[1][0]\} = \max\{4, 5\} = 5$
2	4	8	0	$\max\{v[1][4], p[2]+v[1][1]\} = \max\{8, 5\} = 8$
2	5	9	1	$\max\{v[1][5], p[2]+v[1][2]\} = \max\{8, 9\} = 9$
2	6	9	1	$\max\{v[1][6], p[2]+v[1][3]\} = \max\{8, 9\} = 9$

This is the matrix v after the execution of the DP algorithm:

$v[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$j=0$	0	0	0	0	0	0	0
$j=1$	0	0	4	4	8	8	8
$j=2$	0	0	4	5	8	9	9

This is the matrix s after the execution of the DP algorithm:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$j=0$	0	0	0	0	0	0	0
$j=1$	0	0	1	1	2	2	2
$j=2$	0	0	0	1	0	1	1

2. We have the following matrix s:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$j=0$	0	0	0	0	0	0	0
$j=1$	0	0	1	1	2	2	2
$j=2$	0	0	0	1	0	1	1

We can reconstruct an optimal solution by starting from $j=2$ and $k=6$ as follows:

Iteration	j	k	$s[j][k]$	$x[1]$	$x[2]$	Action
-	-	-	-	0	0	$k=6$
1	2	6	1	0	1	$x[2] \leftarrow 1$, $k \leftarrow 6-1\cdot 3=3$
2	1	3	1	1	1	$x[1] \leftarrow 1$, $k \leftarrow 3-1\cdot 2=1$

Thus, the vector

$$(1,1)$$

is an optimal solution. The optimal value is

$$v[2][6]=9.$$

Exercise 4.2

Consider the following BKP instance, where the knapsack has capacity $c = 7$, and the profits, weights, and bounds associated with the item types are:

	$j = 1$	$j = 2$
p_j	3	8
w_j	1	4
b_j	3	1

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and s.

2. Using the matrix s, reconstruct an optimal solution in integer notation (x_1, x_2), and report the optimal value.

Solution

1. Step-by-step execution of the bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the values of j and k, the cells $v[j][k]$ and $s[j][k]$, and the operation performed.

j	k	$v[j][k]$	$s[j][k]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $s[0][0] \leftarrow 0$
0	1	0	0	$v[0][1] \leftarrow 0$, $s[0][1] \leftarrow 0$
0	2	0	0	$v[0][2] \leftarrow 0$, $s[0][2] \leftarrow 0$
0	3	0	0	$v[0][3] \leftarrow 0$, $s[0][3] \leftarrow 0$
0	4	0	0	$v[0][4] \leftarrow 0$, $s[0][4] \leftarrow 0$
0	5	0	0	$v[0][5] \leftarrow 0$, $s[0][5] \leftarrow 0$
0	6	0	0	$v[0][6] \leftarrow 0$, $s[0][6] \leftarrow 0$
0	7	0	0	$v[0][7] \leftarrow 0$, $s[0][7] \leftarrow 0$
1	0	0	0	$\max\{v[0][0]\} = 0$
1	1	3	1	$\max\{v[0][1],\ p[1]+v[0][0]\} = \max\{0, 3\} = 3$
1	2	6	2	$\max\{v[0][2],\ p[1]+v[0][1],\ 2p[1]+v[0][0]\} = \max\{0, 3, 6\} = 6$

1	3	9	3	$\max\{v[0][3],\ p[1]+v[0][2],\ 2p[1]+v[0][1],\ 3p[1]+v[0][0]\} = \max\{0,3,6,9\} = 9$
1	4	9	3	$\max\{v[0][4],\ p[1]+v[0][3],\ 2p[1]+v[0][2],\ 3p[1]+v[0][1]\} = \max\{0,3,6,9\} = 9$
1	5	9	3	$\max\{v[0][5],\ p[1]+v[0][4],\ 2p[1]+v[0][3],\ 3p[1]+v[0][2]\} = \max\{0,3,6,9\} = 9$
1	6	9	3	$\max\{v[0][6],\ p[1]+v[0][5],\ 2p[1]+v[0][4],\ 3p[1]+v[0][3]\} = \max\{0,3,6,9\} = 9$
1	7	9	3	$\max\{v[0][7],\ p[1]+v[0][6],\ 2p[1]+v[0][5],\ 3p[1]+v[0][4]\} = \max\{0,3,6,9\} = 9$
2	0	0	0	$\max\{v[1][0]\} = 0$
2	1	3	0	$\max\{v[1][1]\} = 3$
2	2	6	0	$\max\{v[1][2]\} = 6$
2	3	9	0	$\max\{v[1][3]\} = 9$
2	4	9	0	$\max\{v[1][4],\ p[2]+v[1][0]\} = \max\{9,8\} = 9$
2	5	11	1	$\max\{v[1][5],\ p[2]+v[1][1]\} = \max\{9,11\} = 11$
2	6	14	1	$\max\{v[1][6],\ p[2]+v[1][2]\} = \max\{9,14\} = 14$
2	7	17	1	$\max\{v[1][7],\ p[2]+v[1][3]\} = \max\{9,17\} = 17$

This is the matrix v after the execution of the DP algorithm:

$v[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$j=0$	0	0	0	0	0	0	0	0
$j=1$	0	3	6	9	9	9	9	9
$j=2$	0	3	6	9	9	11	14	17

This is the matrix s after the execution of the DP algorithm:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$j=0$	0	0	0	0	0	0	0	0
$j=1$	0	1	2	3	3	3	3	3
$j=2$	0	0	0	0	0	1	1	1

2. We have the following matrix s:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$j=0$	0	0	0	0	0	0	0	0
$j=1$	0	1	2	3	3	3	3	3
$j=2$	0	0	0	0	0	1	1	1

We can reconstruct an optimal solution by starting from $j=2$ and $k=7$ as follows:

Iteration	j	k	$s[j][k]$	$x[1]$	$x[2]$	Action
-	-	-	-	0	0	$k=7$
1	2	7	1	0	1	$x[2] \leftarrow 1$, $k \leftarrow 7-1\cdot 4=3$
2	1	3	3	3	1	$x[1] \leftarrow 3$, $k \leftarrow 3-3\cdot 1=0$

Thus, the vector

$$(3,1)$$

is an optimal solution. The optimal value is

$$v[2][7]=17.$$

Exercise 4.3

Consider the following BKP instance, where the knapsack has capacity $c = 5$, and the profits, weights, and bounds associated with the item types are:

	$j = 1$	$j = 2$	$j = 3$
p_j	2	3	4
w_j	1	2	3
b_j	1	1	1

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and s.

2. Using the matrix s, reconstruct an optimal solution in integer notation (x_1, x_2, x_3), and report the optimal value.

Solution

1. Step-by-step execution of the bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the values of j and k, the cells $v[j][k]$ and $s[j][k]$, and the operation performed.

j	k	$v[j][k]$	$s[j][k]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $s[0][0] \leftarrow 0$
0	1	0	0	$v[0][1] \leftarrow 0$, $s[0][1] \leftarrow 0$
0	2	0	0	$v[0][2] \leftarrow 0$, $s[0][2] \leftarrow 0$
0	3	0	0	$v[0][3] \leftarrow 0$, $s[0][3] \leftarrow 0$
0	4	0	0	$v[0][4] \leftarrow 0$, $s[0][4] \leftarrow 0$
0	5	0	0	$v[0][5] \leftarrow 0$, $s[0][5] \leftarrow 0$
1	0	0	0	$\max\{v[0][0]\} = 0$
1	1	2	1	$\max\{v[0][1],\ p[1]+v[0][0]\} = \max\{0, 2\} = 2$
1	2	2	1	$\max\{v[0][2],\ p[1]+v[0][1]\} = \max\{0, 2\} = 2$
1	3	2	1	$\max\{v[0][3],\ p[1]+v[0][2]\} = \max\{0, 2\} = 2$
1	4	2	1	$\max\{v[0][4],\ p[1]+v[0][3]\} = \max\{0, 2\} = 2$

1	5	2	1	$\max\{v[0][5],\ p[1]+v[0][4]\} = \max\{0, 2\} = 2$
2	0	0	0	$\max\{v[1][0]\} = 0$
2	1	2	0	$\max\{v[1][1]\} = 2$
2	2	3	1	$\max\{v[1][2],\ p[2]+v[1][0]\} = \max\{2, 3\} = 3$
2	3	5	1	$\max\{v[1][3],\ p[2]+v[1][1]\} = \max\{2, 5\} = 5$
2	4	5	1	$\max\{v[1][4],\ p[2]+v[1][2]\} = \max\{2, 5\} = 5$
2	5	5	1	$\max\{v[1][5],\ p[2]+v[1][3]\} = \max\{2, 5\} = 5$
3	0	0	0	$\max\{v[2][0]\} = 0$
3	1	2	0	$\max\{v[2][1]\} = 2$
3	2	3	0	$\max\{v[2][2]\} = 3$
3	3	5	0	$\max\{v[2][3],\ p[3]+v[2][0]\} = \max\{5, 4\} = 5$
3	4	6	1	$\max\{v[2][4],\ p[3]+v[2][1]\} = \max\{5, 6\} = 6$
3	5	7	1	$\max\{v[2][5],\ p[3]+v[2][2]\} = \max\{5, 7\} = 7$

This is the matrix v after the execution of the DP algorithm:

$v[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$j=0$	0	0	0	0	0	0
$j=1$	0	2	2	2	2	2
$j=2$	0	2	3	5	5	5
$j=3$	0	2	3	5	6	7

This is the matrix s after the execution of the DP algorithm:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$j=0$	0	0	0	0	0	0
$j=1$	0	1	1	1	1	1
$j=2$	0	0	1	1	1	1
$j=3$	0	0	0	0	1	1

2. We have the following matrix s:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$j=0$	0	0	0	0	0	0
$j=1$	0	1	1	1	1	1
$j=2$	0	0	1	1	1	1
$j=3$	0	0	0	0	1	1

We can reconstruct an optimal solution by starting from $j=3$ and $k=5$ as follows:

Iteration	j	k	$s[j][k]$	$x[1]$	$x[2]$	$x[3]$	Action
-	-	-	-	0	0	0	$k=5$
1	3	5	1	0	0	1	$x[3] \leftarrow 1$, $k \leftarrow 5-1\cdot 3=2$
2	2	2	1	0	1	1	$x[2] \leftarrow 1$, $k \leftarrow 2-1\cdot 2=0$
3	1	0	0	0	1	1	$x[1] \leftarrow 0$, $k \leftarrow 0-0\cdot 1=0$

Thus, the vector

$$(0,1,1)$$

is an optimal solution. The optimal value is

$$v[3][5]=7.$$

Exercise 4.4

Consider the following BKP instance, where the knapsack has capacity $c = 8$, and the profits, weights, and bounds associated with the item types are:

	$j = 1$	$j = 2$
p_j	5	9
w_j	2	5
b_j	3	1

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and s.

2. Using the matrix s, reconstruct an optimal solution in integer notation (x_1, x_2), and report the optimal value.

Solution

1. Step-by-step execution of the bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the values of j and k, the cells $v[j][k]$ and $s[j][k]$, and the operation performed.

j	k	$v[j][k]$	$s[j][k]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $s[0][0] \leftarrow 0$
0	1	0	0	$v[0][1] \leftarrow 0$, $s[0][1] \leftarrow 0$
0	2	0	0	$v[0][2] \leftarrow 0$, $s[0][2] \leftarrow 0$
0	3	0	0	$v[0][3] \leftarrow 0$, $s[0][3] \leftarrow 0$
0	4	0	0	$v[0][4] \leftarrow 0$, $s[0][4] \leftarrow 0$
0	5	0	0	$v[0][5] \leftarrow 0$, $s[0][5] \leftarrow 0$
0	6	0	0	$v[0][6] \leftarrow 0$, $s[0][6] \leftarrow 0$
0	7	0	0	$v[0][7] \leftarrow 0$, $s[0][7] \leftarrow 0$
0	8	0	0	$v[0][8] \leftarrow 0$, $s[0][8] \leftarrow 0$
1	0	0	0	$\max\{v[0][0]\} = 0$
1	1	0	0	$\max\{v[0][1]\} = 0$

1	2	5	1	$\max\{v[0][2],\ p[1]+v[0][0]\} = \max\{0,5\} = 5$
1	3	5	1	$\max\{v[0][3],\ p[1]+v[0][1]\} = \max\{0,5\} = 5$
1	4	10	2	$\max\{v[0][4],\ p[1]+v[0][2],\ 2p[1]+v[0][0]\} = \max\{0,5,10\} = 10$
1	5	10	2	$\max\{v[0][5],\ p[1]+v[0][3],\ 2p[1]+v[0][1]\} = \max\{0,5,10\} = 10$
1	6	15	3	$\max\{v[0][6],\ p[1]+v[0][4],\ 2p[1]+v[0][2],\ 3p[1]+v[0][0]\} = \max\{0,5,10,15\} = 15$
1	7	15	3	$\max\{v[0][7],\ p[1]+v[0][5],\ 2p[1]+v[0][3],\ 3p[1]+v[0][1]\} = \max\{0,5,10,15\} = 15$
1	8	15	3	$\max\{v[0][8],\ p[1]+v[0][6],\ 2p[1]+v[0][4],\ 3p[1]+v[0][2]\} = \max\{0,5,10,15\} = 15$
2	0	0	0	$\max\{v[1][0]\} = 0$
2	1	0	0	$\max\{v[1][1]\} = 0$
2	2	5	0	$\max\{v[1][2]\} = 5$
2	3	5	0	$\max\{v[1][3]\} = 5$
2	4	10	0	$\max\{v[1][4]\} = 10$
2	5	10	0	$\max\{v[1][5],\ p[2]+v[1][0]\} = \max\{10,9\} = 10$
2	6	15	0	$\max\{v[1][6],\ p[2]+v[1][1]\} = \max\{15,9\} = 15$
2	7	15	0	$\max\{v[1][7],\ p[2]+v[1][2]\} = \max\{15,14\} = 15$
2	8	15	0	$\max\{v[1][8],\ p[2]+v[1][3]\} = \max\{15,14\} = 15$

This is the matrix v after the execution of the DP algorithm:

$v[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$j=0$	0	0	0	0	0	0	0	0	0
$j=1$	0	0	5	5	10	10	15	15	15
$j=2$	0	0	5	5	10	10	15	15	15

This is the matrix s after the execution of the DP algorithm:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$j=0$	0	0	0	0	0	0	0	0	0
$j=1$	0	0	1	1	2	2	3	3	3
$j=2$	0	0	0	0	0	0	0	0	0

2. We have the following matrix s:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$j=0$	0	0	0	0	0	0	0	0	0
$j=1$	0	0	1	1	2	2	3	3	3
$j=2$	0	0	0	0	0	0	0	0	0

We can reconstruct an optimal solution by starting from $j=2$ and $k=8$ as follows:

Iteration	j	k	$s[j][k]$	$x[1]$	$x[2]$	Action
-	-	-	-	0	0	$k=8$
1	2	8	0	0	0	$x[2] \leftarrow 0$, $k \leftarrow 8-0\cdot 5=8$
2	1	8	3	3	0	$x[1] \leftarrow 3$, $k \leftarrow 8-3\cdot 2=2$

Thus, the vector

$$(3,0)$$

is an optimal solution. The optimal value is

$$v[2][8]=15.$$

Exercise 4.5

Consider the following BKP instance, where the knapsack has capacity $c = 9$, and the profits, weights, and bounds associated with the item types are:

	$j = 1$	$j = 2$	$j = 3$
p_j	4	5	3
w_j	3	4	2
b_j	2	1	2

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and s.

2. Using the matrix s, reconstruct an optimal solution in integer notation (x_1, x_2, x_3), and report the optimal value.

Solution

1. Step-by-step execution of the bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the values of j and k, the cells $v[j][k]$ and $s[j][k]$, and the operation performed.

j	k	$v[j][k]$	$s[j][k]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $s[0][0] \leftarrow 0$
0	1	0	0	$v[0][1] \leftarrow 0$, $s[0][1] \leftarrow 0$
0	2	0	0	$v[0][2] \leftarrow 0$, $s[0][2] \leftarrow 0$
0	3	0	0	$v[0][3] \leftarrow 0$, $s[0][3] \leftarrow 0$
0	4	0	0	$v[0][4] \leftarrow 0$, $s[0][4] \leftarrow 0$
0	5	0	0	$v[0][5] \leftarrow 0$, $s[0][5] \leftarrow 0$
0	6	0	0	$v[0][6] \leftarrow 0$, $s[0][6] \leftarrow 0$
0	7	0	0	$v[0][7] \leftarrow 0$, $s[0][7] \leftarrow 0$
0	8	0	0	$v[0][8] \leftarrow 0$, $s[0][8] \leftarrow 0$
0	9	0	0	$v[0][9] \leftarrow 0$, $s[0][9] \leftarrow 0$
1	0	0	0	$\max\{v[0][0]\} = 0$

1	1	0	0	$\max\{v[0][1]\} = 0$
1	2	0	0	$\max\{v[0][2]\} = 0$
1	3	4	1	$\max\{v[0][3], p[1]+v[0][0]\} = \max\{0, 4\} = 4$
1	4	4	1	$\max\{v[0][4], p[1]+v[0][1]\} = \max\{0, 4\} = 4$
1	5	4	1	$\max\{v[0][5], p[1]+v[0][2]\} = \max\{0, 4\} = 4$
1	6	8	2	$\max\{v[0][6], p[1]+v[0][3], 2p[1]+v[0][0]\} = \max\{0, 4, 8\} = 8$
1	7	8	2	$\max\{v[0][7], p[1]+v[0][4], 2p[1]+v[0][1]\} = \max\{0, 4, 8\} = 8$
1	8	8	2	$\max\{v[0][8], p[1]+v[0][5], 2p[1]+v[0][2]\} = \max\{0, 4, 8\} = 8$
1	9	8	2	$\max\{v[0][9], p[1]+v[0][6], 2p[1]+v[0][3]\} = \max\{0, 4, 8\} = 8$
2	0	0	0	$\max\{v[1][0]\} = 0$
2	1	0	0	$\max\{v[1][1]\} = 0$
2	2	0	0	$\max\{v[1][2]\} = 0$
2	3	4	0	$\max\{v[1][3]\} = 4$
2	4	5	1	$\max\{v[1][4], p[2]+v[1][0]\} = \max\{4, 5\} = 5$
2	5	5	1	$\max\{v[1][5], p[2]+v[1][1]\} = \max\{4, 5\} = 5$
2	6	8	0	$\max\{v[1][6], p[2]+v[1][2]\} = \max\{8, 5\} = 8$
2	7	9	1	$\max\{v[1][7], p[2]+v[1][3]\} = \max\{8, 9\} = 9$
2	8	9	1	$\max\{v[1][8], p[2]+v[1][4]\} = \max\{8, 9\} = 9$
2	9	9	1	$\max\{v[1][9], p[2]+v[1][5]\} = \max\{8, 9\} = 9$
3	0	0	0	$\max\{v[2][0]\} = 0$
3	1	0	0	$\max\{v[2][1]\} = 0$
3	2	3	1	$\max\{v[2][2], p[3]+v[2][0]\} = \max\{0, 3\} = 3$
3	3	4	0	$\max\{v[2][3], p[3]+v[2][1]\} = \max\{4, 3\} = 4$
3	4	6	2	$\max\{v[2][4], p[3]+v[2][2], 2p[3]+v[2][0]\} = \max\{5, 3, 6\} = 6$
3	5	7	1	$\max\{v[2][5], p[3]+v[2][3], 2p[3]+v[2][1]\} = \max\{5, 7, 6\} = 7$
3	6	8	0	$\max\{v[2][6], p[3]+v[2][4], 2p[3]+v[2][2]\} = \max\{8, 8, 6\} = 8$
3	7	10	2	$\max\{v[2][7], p[3]+v[2][5], 2p[3]+v[2][3]\} = \max\{9, 8, 10\} = 10$
3	8	11	1	$\max\{v[2][8], p[3]+v[2][6], 2p[3]+v[2][4]\} = \max\{9, 11, 11\} = 11$
3	9	12	1	$\max\{v[2][9], p[3]+v[2][7], 2p[3]+v[2][5]\} = \max\{9, 12, 11\} = 12$

This is the matrix v after the execution of the DP algorithm:

$v[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$j=0$	0	0	0	0	0	0	0	0	0	0
$j=1$	0	0	0	4	4	4	8	8	8	8
$j=2$	0	0	0	4	5	5	8	9	9	9
$j=3$	0	0	3	4	6	7	8	10	11	12

This is the matrix s after the execution of the DP algorithm:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$j=0$	0	0	0	0	0	0	0	0	0	0
$j=1$	0	0	0	1	1	1	2	2	2	2
$j=2$	0	0	0	0	1	1	0	1	1	1
$j=3$	0	0	1	0	2	1	0	2	1	1

2. We have the following matrix s:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$j=0$	0	0	0	0	0	0	0	0	0	0
$j=1$	0	0	0	1	1	1	2	2	2	2
$j=2$	0	0	0	0	1	1	0	1	1	1
$j=3$	0	0	1	0	2	1	0	2	1	1

We can reconstruct an optimal solution by starting from $j=3$ and $k=9$ as follows:

Iteration	j	k	$s[j][k]$	$x[1]$	$x[2]$	$x[3]$	Action
-	-	-	-	0	0	0	$k=9$
1	3	9	1	0	0	1	$x[3] \leftarrow 1$, $k \leftarrow 9-1\cdot 2=7$
2	2	7	1	0	1	1	$x[2] \leftarrow 1$, $k \leftarrow 7-1\cdot 4=3$
3	1	3	1	1	1	1	$x[1] \leftarrow 1$, $k \leftarrow 3-1\cdot 3=0$

Thus, the vector

$$(1,1,1)$$

is an optimal solution. The optimal value is

$$v[3][9]=12.$$

Chapter 5

DP for the subset sum problem

Recurrence used in these exercises

The DP algorithm of the chapter «DP for the subset sum problem»: $r(j,k) \in \{true, false\}$ tells whether the target value k is the total weight of a subset of the items $1, 2, \ldots, j$, and

$$r(j,k) = \begin{cases} true & \text{if } k = 0, \\ false & \text{if } j = 0 \text{ and } k \geq 1, \\ r(j-1,k) & \text{if } j,k \geq 1 \text{ and } w_j > k, \\ r(j-1,k) \vee r(j-1,k-w_j) & \text{if } j,k \geq 1 \text{ and } w_j \leq k. \end{cases}$$

The answer is $r(n,c)$. The matrix s stores $s[j][k] = 1$ if item j is used, that is, if $r[j-1][k]$ is *false* and $r[j-1][k-w_j]$ is *true*; a subset is reconstructed from $j = n$ and $k = c$ with $x_j \leftarrow s[j][k]$, $k \leftarrow k - s[j][k]\, w_j$.

Exercise 5.1 — Exercise 1: Subset Sum Problem

Consider the following SSP instance, with target value $c = 7$ and 3 items with weights:

	$j=1$	$j=2$	$j=3$
w_j	2	3	5

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each step, the cells $r[j][k]$ and $s[j][k]$; then write the matrices r and s.
2. Give the answer to the SSP instance and, if it is yes, reconstruct from the matrix s a subset of total weight c, as a binary vector (x_1, x_2, x_3).
3. Using the last row of the matrix r, list all the target values $k \in \{1, 2, \ldots, 7\}$ that are reachable with the 3 items.

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. Step-by-step execution of the bottom-up DP algorithm. The row $j = 0$ is $r[0][0] = \mathrm{T}$ and $r[0][k] = \mathrm{F}$ for $k \geq 1$, with $s[0][k] = 0$; T stands for *true* and F for *false*. Item j is used ($s[j][k] = 1$) only if $r[j-1][k]$ is F.

j	k	$r[j][k]$	$s[j][k]$	Action
1	0	T	0	$w_1 = 2 > 0$: $r[0][0] = \mathrm{T}$
1	1	F	0	$w_1 = 2 > 1$: $r[0][1] = \mathrm{F}$
1	2	T	1	$r[0][2] \vee r[0][0] = \mathrm{F} \vee \mathrm{T} = \mathrm{T}$
1	3	F	0	$r[0][3] \vee r[0][1] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
1	4	F	0	$r[0][4] \vee r[0][2] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
1	5	F	0	$r[0][5] \vee r[0][3] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
1	6	F	0	$r[0][6] \vee r[0][4] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
1	7	F	0	$r[0][7] \vee r[0][5] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
2	0	T	0	$w_2 = 3 > 0$: $r[1][0] = \mathrm{T}$
2	1	F	0	$w_2 = 3 > 1$: $r[1][1] = \mathrm{F}$
2	2	T	0	$w_2 = 3 > 2$: $r[1][2] = \mathrm{T}$
2	3	T	1	$r[1][3] \vee r[1][0] = \mathrm{F} \vee \mathrm{T} = \mathrm{T}$
2	4	F	0	$r[1][4] \vee r[1][1] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
2	5	T	1	$r[1][5] \vee r[1][2] = \mathrm{F} \vee \mathrm{T} = \mathrm{T}$
2	6	F	0	$r[1][6] \vee r[1][3] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
2	7	F	0	$r[1][7] \vee r[1][4] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$

j	k	$r[j][k]$	$s[j][k]$	Action
3	0	T	0	$w_3 = 5 > 0$: $r[2][0] = \mathrm{T}$
3	1	F	0	$w_3 = 5 > 1$: $r[2][1] = \mathrm{F}$
3	2	T	0	$w_3 = 5 > 2$: $r[2][2] = \mathrm{T}$
3	3	T	0	$w_3 = 5 > 3$: $r[2][3] = \mathrm{T}$
3	4	F	0	$w_3 = 5 > 4$: $r[2][4] = \mathrm{F}$
3	5	T	0	$r[2][5] \vee r[2][0] = \mathrm{T} \vee \mathrm{T} = \mathrm{T}$
3	6	F	0	$r[2][6] \vee r[2][1] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
3	7	T	1	$r[2][7] \vee r[2][2] = \mathrm{F} \vee \mathrm{T} = \mathrm{T}$

This is the matrix r after the execution of the DP algorithm:

$r[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$j=0$	T	F	F	F	F	F	F	F
$j=1$	T	F	T	F	F	F	F	F
$j=2$	T	F	T	T	F	T	F	F
$j=3$	T	F	T	T	F	T	F	T

This is the matrix s after the execution of the DP algorithm:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$j=0$	0	0	0	0	0	0	0	0
$j=1$	0	0	1	0	0	0	0	0
$j=2$	0	0	0	1	0	1	0	0
$j=3$	0	0	0	0	0	0	0	1

2. Since $r[3][7] = \mathrm{T}$, the answer is **yes**. Starting from $j = 3$ and $k = 7$:

Iteration	j	k	$s[j][k]$	$x[1]$	$x[2]$	$x[3]$	Action
–	–	–	–	0	0	0	$k = 7$
1	3	7	1	0	0	1	$x[3] \leftarrow 1$, $k \leftarrow 7 - 5 = 2$
2	2	2	0	0	0	1	$x[2] \leftarrow 0$, $k = 2$
3	1	2	1	1	0	1	$x[1] \leftarrow 1$, $k \leftarrow 2 - 2 = 0$

   The vector $(1, 0, 1)$ selects the subset $\{1, 3\}$, of total weight $2 + 5 = 7 = c$. It is the only subset with total weight 7.

3. The last row $r[3][\cdot]$ is T exactly for $k \in \{2, 3, 5, 7\}$: these are the reachable target values. The other values in $\{1, 2, \ldots, 7\}$ are not the total weight of any subset.

Exercise 5.2 — Exercise 2: Subset Sum Problem

Consider the following SSP instance, with target value $c = 9$ and 3 items with weights:

	$j=1$	$j=2$	$j=3$
w_j	3	4	6

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each step, the cells $r[j][k]$ and $s[j][k]$; then write the matrices r and s.

2. Give the answer to the SSP instance and, if it is yes, reconstruct from the matrix s a subset of total weight c, as a binary vector (x_1, x_2, x_3).

3. Using the last row of the matrix r, list all the target values $k \in \{1, 2, \ldots, 9\}$ that are reachable with the 3 items.

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. Step-by-step execution of the bottom-up DP algorithm. The row $j = 0$ is $r[0][0] = \mathrm{T}$ and $r[0][k] = \mathrm{F}$ for $k \geq 1$, with $s[0][k] = 0$; T stands for *true* and F for *false*. Item j is used ($s[j][k] = 1$) only if $r[j-1][k]$ is F.

j	k	$r[j][k]$	$s[j][k]$	Action
1	0	T	0	$w_1 = 3 > 0$: $r[0][0] = \mathrm{T}$
1	1	F	0	$w_1 = 3 > 1$: $r[0][1] = \mathrm{F}$
1	2	F	0	$w_1 = 3 > 2$: $r[0][2] = \mathrm{F}$
1	3	T	1	$r[0][3] \vee r[0][0] = \mathrm{F} \vee \mathrm{T} = \mathrm{T}$
1	4	F	0	$r[0][4] \vee r[0][1] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
1	5	F	0	$r[0][5] \vee r[0][2] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
1	6	F	0	$r[0][6] \vee r[0][3] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
1	7	F	0	$r[0][7] \vee r[0][4] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
1	8	F	0	$r[0][8] \vee r[0][5] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
1	9	F	0	$r[0][9] \vee r[0][6] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
2	0	T	0	$w_2 = 4 > 0$: $r[1][0] = \mathrm{T}$
2	1	F	0	$w_2 = 4 > 1$: $r[1][1] = \mathrm{F}$
2	2	F	0	$w_2 = 4 > 2$: $r[1][2] = \mathrm{F}$
2	3	T	0	$w_2 = 4 > 3$: $r[1][3] = \mathrm{T}$
2	4	T	1	$r[1][4] \vee r[1][0] = \mathrm{F} \vee \mathrm{T} = \mathrm{T}$
2	5	F	0	$r[1][5] \vee r[1][1] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
2	6	F	0	$r[1][6] \vee r[1][2] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
2	7	T	1	$r[1][7] \vee r[1][3] = \mathrm{F} \vee \mathrm{T} = \mathrm{T}$
2	8	F	0	$r[1][8] \vee r[1][4] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
2	9	F	0	$r[1][9] \vee r[1][5] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$

j	k	$r[j][k]$	$s[j][k]$	Action
3	0	T	0	$w_3 = 6 > 0$: $r[2][0] = \mathrm{T}$
3	1	F	0	$w_3 = 6 > 1$: $r[2][1] = \mathrm{F}$
3	2	F	0	$w_3 = 6 > 2$: $r[2][2] = \mathrm{F}$
3	3	T	0	$w_3 = 6 > 3$: $r[2][3] = \mathrm{T}$
3	4	T	0	$w_3 = 6 > 4$: $r[2][4] = \mathrm{T}$
3	5	F	0	$w_3 = 6 > 5$: $r[2][5] = \mathrm{F}$
3	6	T	1	$r[2][6] \vee r[2][0] = \mathrm{F} \vee \mathrm{T} = \mathrm{T}$
3	7	T	0	$r[2][7] \vee r[2][1] = \mathrm{T} \vee \mathrm{F} = \mathrm{T}$
3	8	F	0	$r[2][8] \vee r[2][2] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
3	9	T	1	$r[2][9] \vee r[2][3] = \mathrm{F} \vee \mathrm{T} = \mathrm{T}$

This is the matrix r after the execution of the DP algorithm:

$r[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$j=0$	T	F	F	F	F	F	F	F	F	F
$j=1$	T	F	F	T	F	F	F	F	F	F
$j=2$	T	F	F	T	T	F	F	T	F	F
$j=3$	T	F	F	T	T	F	T	T	F	T

This is the matrix s after the execution of the DP algorithm:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$j=0$	0	0	0	0	0	0	0	0	0	0
$j=1$	0	0	0	1	0	0	0	0	0	0
$j=2$	0	0	0	0	1	0	0	1	0	0
$j=3$	0	0	0	0	0	0	1	0	0	1

2. Since $r[3][9] = \mathrm{T}$, the answer is **yes**. Starting from $j = 3$ and $k = 9$:

Iteration	j	k	$s[j][k]$	$x[1]$	$x[2]$	$x[3]$	Action
–	–	–	–	0	0	0	$k = 9$
1	3	9	1	0	0	1	$x[3] \leftarrow 1$, $k \leftarrow 9 - 6 = 3$
2	2	3	0	0	0	1	$x[2] \leftarrow 0$, $k = 3$
3	1	3	1	1	0	1	$x[1] \leftarrow 1$, $k \leftarrow 3 - 3 = 0$

   The vector $(1, 0, 1)$ selects the subset $\{1, 3\}$, of total weight $3 + 6 = 9 = c$. It is the only subset with total weight 9.

3. The last row $r[3][\cdot]$ is T exactly for $k \in \{3, 4, 6, 7, 9\}$: these are the reachable target values. The other values in $\{1, 2, \ldots, 9\}$ are not the total weight of any subset.

Exercise 5.3 — Exercise 3: Subset Sum Problem

Consider the following SSP instance, with target value $c = 7$ and 3 items with weights:

	$j = 1$	$j = 2$	$j = 3$
w_j	2	4	6

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each step, the cells $r[j][k]$ and $s[j][k]$; then write the matrices r and s.

2. Give the answer to the SSP instance and, if it is yes, reconstruct from the matrix s a subset of total weight c, as a binary vector (x_1, x_2, x_3).

3. Using the last row of the matrix r, list all the target values $k \in \{1, 2, \ldots, 7\}$ that are reachable with the 3 items.

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. Step-by-step execution of the bottom-up DP algorithm. The row $j = 0$ is $r[0][0] = \mathrm{T}$ and $r[0][k] = \mathrm{F}$ for $k \geq 1$, with $s[0][k] = 0$; T stands for *true* and F for *false*. Item j is used ($s[j][k] = 1$) only if $r[j-1][k]$ is F.

j	k	$r[j][k]$	$s[j][k]$	Action
1	0	T	0	$w_1 = 2 > 0$: $r[0][0] = \mathrm{T}$
1	1	F	0	$w_1 = 2 > 1$: $r[0][1] = \mathrm{F}$
1	2	T	1	$r[0][2] \vee r[0][0] = \mathrm{F} \vee \mathrm{T} = \mathrm{T}$
1	3	F	0	$r[0][3] \vee r[0][1] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
1	4	F	0	$r[0][4] \vee r[0][2] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
1	5	F	0	$r[0][5] \vee r[0][3] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
1	6	F	0	$r[0][6] \vee r[0][4] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
1	7	F	0	$r[0][7] \vee r[0][5] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
2	0	T	0	$w_2 = 4 > 0$: $r[1][0] = \mathrm{T}$
2	1	F	0	$w_2 = 4 > 1$: $r[1][1] = \mathrm{F}$
2	2	T	0	$w_2 = 4 > 2$: $r[1][2] = \mathrm{T}$
2	3	F	0	$w_2 = 4 > 3$: $r[1][3] = \mathrm{F}$
2	4	T	1	$r[1][4] \vee r[1][0] = \mathrm{F} \vee \mathrm{T} = \mathrm{T}$
2	5	F	0	$r[1][5] \vee r[1][1] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
2	6	T	1	$r[1][6] \vee r[1][2] = \mathrm{F} \vee \mathrm{T} = \mathrm{T}$
2	7	F	0	$r[1][7] \vee r[1][3] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$

j	k	$r[j][k]$	$s[j][k]$	Action
3	0	T	0	$w_3 = 6 > 0$: $r[2][0] = \mathrm{T}$
3	1	F	0	$w_3 = 6 > 1$: $r[2][1] = \mathrm{F}$
3	2	T	0	$w_3 = 6 > 2$: $r[2][2] = \mathrm{T}$
3	3	F	0	$w_3 = 6 > 3$: $r[2][3] = \mathrm{F}$
3	4	T	0	$w_3 = 6 > 4$: $r[2][4] = \mathrm{T}$
3	5	F	0	$w_3 = 6 > 5$: $r[2][5] = \mathrm{F}$
3	6	T	0	$r[2][6] \vee r[2][0] = \mathrm{T} \vee \mathrm{T} = \mathrm{T}$
3	7	F	0	$r[2][7] \vee r[2][1] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$

This is the matrix r after the execution of the DP algorithm:

$r[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$j=0$	T	F	F	F	F	F	F	F
$j=1$	T	F	T	F	F	F	F	F
$j=2$	T	F	T	F	T	F	T	F
$j=3$	T	F	T	F	T	F	T	F

This is the matrix s after the execution of the DP algorithm:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$j=0$	0	0	0	0	0	0	0	0
$j=1$	0	0	1	0	0	0	0	0
$j=2$	0	0	0	0	1	0	1	0
$j=3$	0	0	0	0	0	0	0	0

2. Since $r[3][7] = \mathrm{F}$, the answer is **no**: no subset of the items has total weight 7, and there is nothing to reconstruct. Indeed, all the weights $w = (2, 4, 6)$ are even, so every reachable target value is even, while $c = 7$ is odd.

3. The last row $r[3][\cdot]$ is T exactly for $k \in \{2, 4, 6\}$: these are the reachable target values. The other values in $\{1, 2, \ldots, 7\}$ are not the total weight of any subset.

Exercise 5.4 — Exercise 4: Subset Sum Problem

Consider the following SSP instance, with target value $c = 8$ and 4 items with weights:

	$j=1$	$j=2$	$j=3$	$j=4$
w_j	1	4	5	7

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each step, the cells $r[j][k]$ and $s[j][k]$; then write the matrices r and s.
2. Give the answer to the SSP instance and, if it is yes, reconstruct from the matrix s a subset of total weight c, as a binary vector (x_1, x_2, x_3, x_4).
3. Using the last row of the matrix r, list all the target values $k \in \{1, 2, \ldots, 8\}$ that are reachable with the 4 items.

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. Step-by-step execution of the bottom-up DP algorithm. The row $j = 0$ is $r[0][0] = \text{T}$ and $r[0][k] = \text{F}$ for $k \geq 1$, with $s[0][k] = 0$; T stands for *true* and F for *false*. Item j is used ($s[j][k] = 1$) only if $r[j-1][k]$ is F.

j	k	$r[j][k]$	$s[j][k]$	Action
1	0	T	0	$w_1 = 1 > 0$: $r[0][0] = \text{T}$
1	1	T	1	$r[0][1] \vee r[0][0] = \text{F} \vee \text{T} = \text{T}$
1	2	F	0	$r[0][2] \vee r[0][1] = \text{F} \vee \text{F} = \text{F}$
1	3	F	0	$r[0][3] \vee r[0][2] = \text{F} \vee \text{F} = \text{F}$
1	4	F	0	$r[0][4] \vee r[0][3] = \text{F} \vee \text{F} = \text{F}$
1	5	F	0	$r[0][5] \vee r[0][4] = \text{F} \vee \text{F} = \text{F}$
1	6	F	0	$r[0][6] \vee r[0][5] = \text{F} \vee \text{F} = \text{F}$
1	7	F	0	$r[0][7] \vee r[0][6] = \text{F} \vee \text{F} = \text{F}$
1	8	F	0	$r[0][8] \vee r[0][7] = \text{F} \vee \text{F} = \text{F}$
2	0	T	0	$w_2 = 4 > 0$: $r[1][0] = \text{T}$
2	1	T	0	$w_2 = 4 > 1$: $r[1][1] = \text{T}$
2	2	F	0	$w_2 = 4 > 2$: $r[1][2] = \text{F}$
2	3	F	0	$w_2 = 4 > 3$: $r[1][3] = \text{F}$
2	4	T	1	$r[1][4] \vee r[1][0] = \text{F} \vee \text{T} = \text{T}$
2	5	T	1	$r[1][5] \vee r[1][1] = \text{F} \vee \text{T} = \text{T}$
2	6	F	0	$r[1][6] \vee r[1][2] = \text{F} \vee \text{F} = \text{F}$
2	7	F	0	$r[1][7] \vee r[1][3] = \text{F} \vee \text{F} = \text{F}$
2	8	F	0	$r[1][8] \vee r[1][4] = \text{F} \vee \text{F} = \text{F}$

j	k	$r[j][k]$	$s[j][k]$	Action
3	0	T	0	$w_3 = 5 > 0$: $r[2][0] = \mathrm{T}$
3	1	T	0	$w_3 = 5 > 1$: $r[2][1] = \mathrm{T}$
3	2	F	0	$w_3 = 5 > 2$: $r[2][2] = \mathrm{F}$
3	3	F	0	$w_3 = 5 > 3$: $r[2][3] = \mathrm{F}$
3	4	T	0	$w_3 = 5 > 4$: $r[2][4] = \mathrm{T}$
3	5	T	0	$r[2][5] \vee r[2][0] = \mathrm{T} \vee \mathrm{T} = \mathrm{T}$
3	6	T	1	$r[2][6] \vee r[2][1] = \mathrm{F} \vee \mathrm{T} = \mathrm{T}$
3	7	F	0	$r[2][7] \vee r[2][2] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
3	8	F	0	$r[2][8] \vee r[2][3] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
4	0	T	0	$w_4 = 7 > 0$: $r[3][0] = \mathrm{T}$
4	1	T	0	$w_4 = 7 > 1$: $r[3][1] = \mathrm{T}$
4	2	F	0	$w_4 = 7 > 2$: $r[3][2] = \mathrm{F}$
4	3	F	0	$w_4 = 7 > 3$: $r[3][3] = \mathrm{F}$
4	4	T	0	$w_4 = 7 > 4$: $r[3][4] = \mathrm{T}$
4	5	T	0	$w_4 = 7 > 5$: $r[3][5] = \mathrm{T}$
4	6	T	0	$w_4 = 7 > 6$: $r[3][6] = \mathrm{T}$
4	7	T	1	$r[3][7] \vee r[3][0] = \mathrm{F} \vee \mathrm{T} = \mathrm{T}$
4	8	T	1	$r[3][8] \vee r[3][1] = \mathrm{F} \vee \mathrm{T} = \mathrm{T}$

This is the matrix r after the execution of the DP algorithm:

$r[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$j=0$	T	F	F	F	F	F	F	F	F
$j=1$	T	T	F	F	F	F	F	F	F
$j=2$	T	T	F	F	T	T	F	F	F
$j=3$	T	T	F	F	T	T	T	F	F
$j=4$	T	T	F	F	T	T	T	T	T

This is the matrix s after the execution of the DP algorithm:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$j=0$	0	0	0	0	0	0	0	0	0
$j=1$	0	1	0	0	0	0	0	0	0
$j=2$	0	0	0	0	1	1	0	0	0
$j=3$	0	0	0	0	0	0	1	0	0
$j=4$	0	0	0	0	0	0	0	1	1

2. Since $r[4][8] = \mathrm{T}$, the answer is **yes**. Starting from $j = 4$ and $k = 8$:

Iteration	j	k	$s[j][k]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	Action
–	–	–	–	0	0	0	0	$k = 8$
1	4	8	1	0	0	0	1	$x[4] \leftarrow 1$, $k \leftarrow 8 - 7 = 1$
2	3	1	0	0	0	0	1	$x[3] \leftarrow 0$, $k = 1$
3	2	1	0	0	0	0	1	$x[2] \leftarrow 0$, $k = 1$
4	1	1	1	1	0	0	1	$x[1] \leftarrow 1$, $k \leftarrow 1 - 1 = 0$

The vector $(1, 0, 0, 1)$ selects the subset $\{1, 4\}$, of total weight $1 + 7 = 8 = c$. It is the only subset with total weight 8.

3. The last row $r[4][\cdot]$ is T exactly for $k \in \{1, 4, 5, 6, 7, 8\}$: these are the reachable target values. The other values in $\{1, 2, \ldots, 8\}$ are not the total weight of any subset.

Exercise 5.5 — Exercise 5: Subset Sum Problem

Consider the following SSP instance, with target value $c = 8$ and 4 items with weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
w_j	2	5	6	7

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each step, the cells $r[j][k]$ and $s[j][k]$; then write the matrices r and s.
2. Give the answer to the SSP instance and, if it is yes, reconstruct from the matrix s a subset of total weight c, as a binary vector (x_1, x_2, x_3, x_4).
3. Using the last row of the matrix r, list all the target values $k \in \{1, 2, \ldots, 8\}$ that are reachable with the 4 items.

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. Step-by-step execution of the bottom-up DP algorithm. The row $j = 0$ is $r[0][0] = \text{T}$ and $r[0][k] = \text{F}$ for $k \geq 1$, with $s[0][k] = 0$; T stands for *true* and F for *false*. Item j is used ($s[j][k] = 1$) only if $r[j-1][k]$ is F.

j	k	$r[j][k]$	$s[j][k]$	Action
1	0	T	0	$w_1 = 2 > 0$: $r[0][0] = \mathrm{T}$
1	1	F	0	$w_1 = 2 > 1$: $r[0][1] = \mathrm{F}$
1	2	T	1	$r[0][2] \vee r[0][0] = \mathrm{F} \vee \mathrm{T} = \mathrm{T}$
1	3	F	0	$r[0][3] \vee r[0][1] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
1	4	F	0	$r[0][4] \vee r[0][2] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
1	5	F	0	$r[0][5] \vee r[0][3] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
1	6	F	0	$r[0][6] \vee r[0][4] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
1	7	F	0	$r[0][7] \vee r[0][5] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
1	8	F	0	$r[0][8] \vee r[0][6] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
2	0	T	0	$w_2 = 5 > 0$: $r[1][0] = \mathrm{T}$
2	1	F	0	$w_2 = 5 > 1$: $r[1][1] = \mathrm{F}$
2	2	T	0	$w_2 = 5 > 2$: $r[1][2] = \mathrm{T}$
2	3	F	0	$w_2 = 5 > 3$: $r[1][3] = \mathrm{F}$
2	4	F	0	$w_2 = 5 > 4$: $r[1][4] = \mathrm{F}$
2	5	T	1	$r[1][5] \vee r[1][0] = \mathrm{F} \vee \mathrm{T} = \mathrm{T}$
2	6	F	0	$r[1][6] \vee r[1][1] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$
2	7	T	1	$r[1][7] \vee r[1][2] = \mathrm{F} \vee \mathrm{T} = \mathrm{T}$
2	8	F	0	$r[1][8] \vee r[1][3] = \mathrm{F} \vee \mathrm{F} = \mathrm{F}$

j	k	$r[j][k]$	$s[j][k]$	Action
3	0	T	0	$w_3 = 6 > 0$: $r[2][0] = \mathrm{T}$
3	1	F	0	$w_3 = 6 > 1$: $r[2][1] = \mathrm{F}$
3	2	T	0	$w_3 = 6 > 2$: $r[2][2] = \mathrm{T}$
3	3	F	0	$w_3 = 6 > 3$: $r[2][3] = \mathrm{F}$
3	4	F	0	$w_3 = 6 > 4$: $r[2][4] = \mathrm{F}$
3	5	T	0	$w_3 = 6 > 5$: $r[2][5] = \mathrm{T}$
3	6	T	1	$r[2][6] \vee r[2][0] = \mathrm{F} \vee \mathrm{T} = \mathrm{T}$
3	7	T	0	$r[2][7] \vee r[2][1] = \mathrm{T} \vee \mathrm{F} = \mathrm{T}$
3	8	T	1	$r[2][8] \vee r[2][2] = \mathrm{F} \vee \mathrm{T} = \mathrm{T}$
4	0	T	0	$w_4 = 7 > 0$: $r[3][0] = \mathrm{T}$
4	1	F	0	$w_4 = 7 > 1$: $r[3][1] = \mathrm{F}$
4	2	T	0	$w_4 = 7 > 2$: $r[3][2] = \mathrm{T}$
4	3	F	0	$w_4 = 7 > 3$: $r[3][3] = \mathrm{F}$
4	4	F	0	$w_4 = 7 > 4$: $r[3][4] = \mathrm{F}$
4	5	T	0	$w_4 = 7 > 5$: $r[3][5] = \mathrm{T}$
4	6	T	0	$w_4 = 7 > 6$: $r[3][6] = \mathrm{T}$
4	7	T	0	$r[3][7] \vee r[3][0] = \mathrm{T} \vee \mathrm{T} = \mathrm{T}$
4	8	T	0	$r[3][8] \vee r[3][1] = \mathrm{T} \vee \mathrm{F} = \mathrm{T}$

This is the matrix r after the execution of the DP algorithm:

$r[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$j=0$	T	F	F	F	F	F	F	F	F
$j=1$	T	F	T	F	F	F	F	F	F
$j=2$	T	F	T	F	F	T	F	T	F
$j=3$	T	F	T	F	F	T	T	T	T
$j=4$	T	F	T	F	F	T	T	T	T

This is the matrix s after the execution of the DP algorithm:

$s[j][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$j=0$	0	0	0	0	0	0	0	0	0
$j=1$	0	0	1	0	0	0	0	0	0
$j=2$	0	0	0	0	0	1	0	1	0
$j=3$	0	0	0	0	0	0	1	0	1
$j=4$	0	0	0	0	0	0	0	0	0

2. Since $r[4][8] = \mathrm{T}$, the answer is **yes**. Starting from $j=4$ and $k=8$:

Iteration	j	k	$s[j][k]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	Action
–	–	–	–	0	0	0	0	$k=8$
1	4	8	0	0	0	0	0	$x[4] \leftarrow 0$, $k=8$
2	3	8	1	0	0	1	0	$x[3] \leftarrow 1$, $k \leftarrow 8-6=2$
3	2	2	0	0	0	1	0	$x[2] \leftarrow 0$, $k=2$
4	1	2	1	1	0	1	0	$x[1] \leftarrow 1$, $k \leftarrow 2-2=0$

The vector $(1,0,1,0)$ selects the subset $\{1,3\}$, of total weight $2+6=8=c$. It is the only subset with total weight 8.

3. The last row $r[4][\cdot]$ is T exactly for $k \in \{2,5,6,7,8\}$: these are the reachable target values. The other values in $\{1,2,\ldots,8\}$ are not the total weight of any subset.

Chapter 6

DP for the multiple-choice knapsack problem

Recurrence used in these exercises

The DP algorithm of the chapter «DP for the multiple-choice knapsack problem»: $v(h,k)$ is the optimal value with the classes $\mathscr{J}_1,\dots,\mathscr{J}_h$ and capacity k, with $v(h,k)=-\infty$ if no feasible solution exists, and

$$v(h,k)=\begin{cases} 0 & \text{if } h=0,\\ -\infty & \text{if } h\geq 1 \text{ and } \forall j\in\mathscr{J}_h\colon\ w_j>k,\\ \max\limits_{j\in\mathscr{J}_h\colon w_j\leq k}\{p_j+v(h-1,k-w_j)\} & \text{if } h\geq 1 \text{ and } \exists j\in\mathscr{J}_h\colon\ w_j\leq k. \end{cases}$$

with $p_j+(-\infty)=-\infty$. The matrix s stores the item $s[h][k]$ chosen in class h (0 if $v[h][k]=-\infty$); an optimal solution is reconstructed from $h=q$ and $k=c$ with $j\leftarrow s[h][k]$, $x_j\leftarrow 1$, $k\leftarrow k-w_j$.

Exercise 6.1 — Exercise 1: Multiple-Choice Knapsack Problem

Consider the following MCKP instance, with a knapsack of capacity $c=6$ and 4 items partitioned into $q=2$ classes: $\mathscr{J}_1=\{1,2\}$, $\mathscr{J}_2=\{3,4\}$. The profits and weights are:

	$j=1$	$j=2$	$j=3$	$j=4$
p_j	4	6	3	6
w_j	2	4	1	3

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each step, the cells $v[h][k]$ and $s[h][k]$; then write the matrices v and s.
2. Using the matrix s, reconstruct an optimal solution, one item per class, and report the optimal value.
3. Using the last row of the matrix v, find the smallest capacity for which the instance is feasible, and compare it with $\sum_{h=1}^{q}\min_{j\in\mathscr{J}_h} w_j$.

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. Step-by-step execution of the bottom-up DP algorithm. The row $h = 0$ is $v[0][k] = 0$ and $s[0][k] = 0$ for every k; $s[h][k] = 0$ when $v[h][k] = -\infty$. In case of ties the item met first in $\mathscr{J}_h$ is kept.

h	k	$v[h][k]$	$s[h][k]$	Action
1	0	$-\infty$	0	no item of $\mathscr{J}_1$ fits: $-\infty$
1	1	$-\infty$	0	no item of $\mathscr{J}_1$ fits: $-\infty$
1	2	4	1	$\max\{p_1+v[0][0]\} = \max\{4\} = 4$
1	3	4	1	$\max\{p_1+v[0][1]\} = \max\{4\} = 4$
1	4	6	2	$\max\{p_1+v[0][2],\ p_2+v[0][0]\} = \max\{4,6\} = 6$
1	5	6	2	$\max\{p_1+v[0][3],\ p_2+v[0][1]\} = \max\{4,6\} = 6$
1	6	6	2	$\max\{p_1+v[0][4],\ p_2+v[0][2]\} = \max\{4,6\} = 6$
2	0	$-\infty$	0	no item of $\mathscr{J}_2$ fits: $-\infty$
2	1	$-\infty$	0	$\max\{p_3+v[1][0]\} = \max\{-\infty\} = -\infty$
2	2	$-\infty$	0	$\max\{p_3+v[1][1]\} = \max\{-\infty\} = -\infty$
2	3	7	3	$\max\{p_3+v[1][2],\ p_4+v[1][0]\} = \max\{7,-\infty\} = 7$
2	4	7	3	$\max\{p_3+v[1][3],\ p_4+v[1][1]\} = \max\{7,-\infty\} = 7$
2	5	10	4	$\max\{p_3+v[1][4],\ p_4+v[1][2]\} = \max\{9,10\} = 10$
2	6	10	4	$\max\{p_3+v[1][5],\ p_4+v[1][3]\} = \max\{9,10\} = 10$

   This is the matrix v after the execution of the DP algorithm:

$v[h][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$h=0$	0	0	0	0	0	0	0
$h=1$	$-\infty$	$-\infty$	4	4	6	6	6
$h=2$	$-\infty$	$-\infty$	$-\infty$	7	7	10	10

   This is the matrix s after the execution of the DP algorithm:

$s[h][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$h=0$	0	0	0	0	0	0	0
$h=1$	0	0	1	1	2	2	2
$h=2$	0	0	0	3	3	4	4

2. Starting from $h = 2$ and $k = 6$:

Iteration	h	k	$j = s[h][k]$	Selection	Capacity
–	–	–	–	–	$k = 6$
1	2	6	4	$x[4] \leftarrow 1$	$k \leftarrow 6 - 3 = 3$
2	1	3	1	$x[1] \leftarrow 1$	$k \leftarrow 3 - 2 = 1$

   The optimal solution is $S = \{1, 4\}$, that is, the vector $x = (1, 0, 0, 1)$. Its total weight is $2 + 3 = 5 \leq c$, and the optimal value is $v[2][6] = 4 + 6 = 10$.

3. The last row $v[2][\cdot]$ is $-\infty$ for $k < 3$ and finite from $k = 3$ on, so the smallest capacity for which the instance is feasible is 3. It equals $\sum_{h=1}^{2} \min_{j \in \mathscr{J}_h} w_j = 2 + 1 = 3$: the lightest item of every class must fit.

Exercise 6.2 — Exercise 2: Multiple-Choice Knapsack Problem

Consider the following MCKP instance, with a knapsack of capacity $c = 8$ and 5 items partitioned into $q = 2$ classes: $\mathscr{J}_1 = \{1, 2\}$, $\mathscr{J}_2 = \{3, 4, 5\}$. The profits and weights are:

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
p_j	6	9	2	5	6
w_j	2	4	1	3	5

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each step, the cells $v[h][k]$ and $s[h][k]$; then write the matrices v and s.
2. Using the matrix s, reconstruct an optimal solution, one item per class, and report the optimal value.
3. Using the last row of the matrix v, find the smallest capacity for which the instance is feasible, and compare it with $\sum_{h=1}^{q} \min_{j \in \mathscr{J}_h} w_j$.

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. Step-by-step execution of the bottom-up DP algorithm. The row $h = 0$ is $v[0][k] = 0$ and $s[0][k] = 0$ for every k; $s[h][k] = 0$ when $v[h][k] = -\infty$. In case of ties the item met first in $\mathscr{J}_h$ is kept.

h	k	$v[h][k]$	$s[h][k]$	Action
1	0	$-\infty$	0	no item of $\mathscr{J}_1$ fits: $-\infty$
1	1	$-\infty$	0	no item of $\mathscr{J}_1$ fits: $-\infty$
1	2	6	1	$\max\{p_1{+}v[0][0]\} = \max\{6\} = 6$
1	3	6	1	$\max\{p_1{+}v[0][1]\} = \max\{6\} = 6$
1	4	9	2	$\max\{p_1{+}v[0][2],\ p_2{+}v[0][0]\} = \max\{6, 9\} = 9$
1	5	9	2	$\max\{p_1{+}v[0][3],\ p_2{+}v[0][1]\} = \max\{6, 9\} = 9$
1	6	9	2	$\max\{p_1{+}v[0][4],\ p_2{+}v[0][2]\} = \max\{6, 9\} = 9$
1	7	9	2	$\max\{p_1{+}v[0][5],\ p_2{+}v[0][3]\} = \max\{6, 9\} = 9$
1	8	9	2	$\max\{p_1{+}v[0][6],\ p_2{+}v[0][4]\} = \max\{6, 9\} = 9$
2	0	$-\infty$	0	no item of $\mathscr{J}_2$ fits: $-\infty$
2	1	$-\infty$	0	$\max\{p_3{+}v[1][0]\} = \max\{-\infty\} = -\infty$
2	2	$-\infty$	0	$\max\{p_3{+}v[1][1]\} = \max\{-\infty\} = -\infty$
2	3	8	3	$\max\{p_3{+}v[1][2],\ p_4{+}v[1][0]\} = \max\{8, -\infty\} = 8$
2	4	8	3	$\max\{p_3{+}v[1][3],\ p_4{+}v[1][1]\} = \max\{8, -\infty\} = 8$
2	5	11	3	$\max\{p_3{+}v[1][4],\ p_4{+}v[1][2],\ p_5{+}v[1][0]\} = \max\{11, 11, -\infty\} = 11$
2	6	11	3	$\max\{p_3{+}v[1][5],\ p_4{+}v[1][3],\ p_5{+}v[1][1]\} = \max\{11, 11, -\infty\} = 11$
2	7	14	4	$\max\{p_3{+}v[1][6],\ p_4{+}v[1][4],\ p_5{+}v[1][2]\} = \max\{11, 14, 12\} = 14$
2	8	14	4	$\max\{p_3{+}v[1][7],\ p_4{+}v[1][5],\ p_5{+}v[1][3]\} = \max\{11, 14, 12\} = 14$

This is the matrix v after the execution of the DP algorithm:

$v[h][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$h=0$	0	0	0	0	0	0	0	0	0
$h=1$	$-\infty$	$-\infty$	6	6	9	9	9	9	9
$h=2$	$-\infty$	$-\infty$	$-\infty$	8	8	11	11	14	14

This is the matrix s after the execution of the DP algorithm:

$s[h][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$h=0$	0	0	0	0	0	0	0	0	0
$h=1$	0	0	1	1	2	2	2	2	2
$h=2$	0	0	0	3	3	3	3	4	4

2. Starting from $h=2$ and $k=8$:

Iteration	h	k	$j=s[h][k]$	Selection	Capacity
–	–	–	–	–	$k=8$
1	2	8	4	$x[4]\leftarrow 1$	$k\leftarrow 8-3=5$
2	1	5	2	$x[2]\leftarrow 1$	$k\leftarrow 5-4=1$

The optimal solution is $S=\{2,4\}$, that is, the vector $x=(0,1,0,1,0)$. Its total weight is $4+3=7\leq c$, and the optimal value is $v[2][8]=9+5=14$.

3. The last row $v[2][\cdot]$ is $-\infty$ for $k<3$ and finite from $k=3$ on, so the smallest capacity for which the instance is feasible is 3. It equals $\sum_{h=1}^{2}\min_{j\in\mathscr{J}_h} w_j = 2+1=3$: the lightest item of every class must fit.

Exercise 6.3 — Exercise 3: Multiple-Choice Knapsack Problem

Consider the following MCKP instance, with a knapsack of capacity $c=8$ and 6 items partitioned into $q=3$ classes: $\mathscr{J}_1=\{1,2\}$, $\mathscr{J}_2=\{3,4\}$, $\mathscr{J}_3=\{5,6\}$. The profits and weights are:

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
p_j	2	5	3	7	4	6
w_j	2	5	1	2	4	5

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each step, the cells $v[h][k]$ and $s[h][k]$; then write the matrices v and s.
2. Using the matrix s, reconstruct an optimal solution, one item per class, and report the optimal value.
3. Using the last row of the matrix v, find the smallest capacity for which the instance is feasible, and compare it with $\sum_{h=1}^{q}\min_{j\in\mathscr{J}_h} w_j$.

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. Step-by-step execution of the bottom-up DP algorithm. The row $h = 0$ is $v[0][k] = 0$ and $s[0][k] = 0$ for every k; $s[h][k] = 0$ when $v[h][k] = -\infty$. In case of ties the item met first in $\mathscr{J}_h$ is kept.

h	k	$v[h][k]$	$s[h][k]$	Action
1	0	$-\infty$	0	no item of $\mathscr{J}_1$ fits: $-\infty$
1	1	$-\infty$	0	no item of $\mathscr{J}_1$ fits: $-\infty$
1	2	2	1	$\max\{p_1+v[0][0]\} = \max\{2\} = 2$
1	3	2	1	$\max\{p_1+v[0][1]\} = \max\{2\} = 2$
1	4	2	1	$\max\{p_1+v[0][2]\} = \max\{2\} = 2$
1	5	5	2	$\max\{p_1+v[0][3], p_2+v[0][0]\} = \max\{2,5\} = 5$
1	6	5	2	$\max\{p_1+v[0][4], p_2+v[0][1]\} = \max\{2,5\} = 5$
1	7	5	2	$\max\{p_1+v[0][5], p_2+v[0][2]\} = \max\{2,5\} = 5$
1	8	5	2	$\max\{p_1+v[0][6], p_2+v[0][3]\} = \max\{2,5\} = 5$
2	0	$-\infty$	0	no item of $\mathscr{J}_2$ fits: $-\infty$
2	1	$-\infty$	0	$\max\{p_3+v[1][0]\} = \max\{-\infty\} = -\infty$
2	2	$-\infty$	0	$\max\{p_3+v[1][1], p_4+v[1][0]\} = \max\{-\infty,-\infty\} = -\infty$
2	3	5	3	$\max\{p_3+v[1][2], p_4+v[1][1]\} = \max\{5,-\infty\} = 5$
2	4	9	4	$\max\{p_3+v[1][3], p_4+v[1][2]\} = \max\{5,9\} = 9$
2	5	9	4	$\max\{p_3+v[1][4], p_4+v[1][3]\} = \max\{5,9\} = 9$
2	6	9	4	$\max\{p_3+v[1][5], p_4+v[1][4]\} = \max\{8,9\} = 9$
2	7	12	4	$\max\{p_3+v[1][6], p_4+v[1][5]\} = \max\{8,12\} = 12$
2	8	12	4	$\max\{p_3+v[1][7], p_4+v[1][6]\} = \max\{8,12\} = 12$

h	k	$v[h][k]$	$s[h][k]$	Action
3	0	$-\infty$	0	no item of $\mathscr{J}_3$ fits: $-\infty$
3	1	$-\infty$	0	no item of $\mathscr{J}_3$ fits: $-\infty$
3	2	$-\infty$	0	no item of $\mathscr{J}_3$ fits: $-\infty$
3	3	$-\infty$	0	no item of $\mathscr{J}_3$ fits: $-\infty$
3	4	$-\infty$	0	$\max\{p_5+v[2][0]\} = \max\{-\infty\} = -\infty$
3	5	$-\infty$	0	$\max\{p_5+v[2][1], p_6+v[2][0]\} = \max\{-\infty,-\infty\} = -\infty$
3	6	$-\infty$	0	$\max\{p_5+v[2][2], p_6+v[2][1]\} = \max\{-\infty,-\infty\} = -\infty$
3	7	9	5	$\max\{p_5+v[2][3], p_6+v[2][2]\} = \max\{9,-\infty\} = 9$
3	8	13	5	$\max\{p_5+v[2][4], p_6+v[2][3]\} = \max\{13,11\} = 13$

This is the matrix v after the execution of the DP algorithm:

$v[h][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$h=0$	0	0	0	0	0	0	0	0	0
$h=1$	$-\infty$	$-\infty$	2	2	2	5	5	5	5
$h=2$	$-\infty$	$-\infty$	$-\infty$	5	9	9	9	12	12
$h=3$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	9	13

This is the matrix s after the execution of the DP algorithm:

$s[h][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$h=0$	0	0	0	0	0	0	0	0	0
$h=1$	0	0	1	1	1	2	2	2	2
$h=2$	0	0	0	3	4	4	4	4	4
$h=3$	0	0	0	0	0	0	0	5	5

2. Starting from $h=3$ and $k=8$:

Iteration	h	k	$j=s[h][k]$	Selection	Capacity
–	–	–	–	–	$k=8$
1	3	8	5	$x[5]\leftarrow 1$	$k\leftarrow 8-4=4$
2	2	4	4	$x[4]\leftarrow 1$	$k\leftarrow 4-2=2$
3	1	2	1	$x[1]\leftarrow 1$	$k\leftarrow 2-2=0$

   The optimal solution is $S=\{1,4,5\}$, that is, the vector $x=(1,0,0,1,1,0)$. Its total weight is $2+2+4=8\leq c$, and the optimal value is $v[3][8]=2+7+4=13$.

3. The last row $v[3][\cdot]$ is $-\infty$ for $k<7$ and finite from $k=7$ on, so the smallest capacity for which the instance is feasible is 7. It equals $\sum_{h=1}^{3}\min_{j\in\mathscr{J}_h} w_j = 2+1+4=7$: the lightest item of every class must fit.

Exercise 6.4 — Exercise 4: Multiple-Choice Knapsack Problem

Consider the following MCKP instance, with a knapsack of capacity $c=7$ and 5 items partitioned into $q=2$ classes: $\mathscr{J}_1=\{1,2,3\}$, $\mathscr{J}_2=\{4,5\}$. The profits and weights are:

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
p_j	3	7	8	3	9
w_j	2	3	4	4	5

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each step, the cells $v[h][k]$ and $s[h][k]$; then write the matrices v and s.
2. Using the matrix s, reconstruct an optimal solution, one item per class, and report the optimal value.
3. Using the last row of the matrix v, find the smallest capacity for which the instance is feasible, and compare it with $\sum_{h=1}^{q}\min_{j\in\mathscr{J}_h} w_j$.

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. Step-by-step execution of the bottom-up DP algorithm. The row $h=0$ is $v[0][k]=0$ and $s[0][k]=0$ for every k; $s[h][k]=0$ when $v[h][k]=-\infty$. In case of ties the item

met first in $\mathscr{J}_h$ is kept.

h	k	$v[h][k]$	$s[h][k]$	Action
1	0	$-\infty$	0	no item of $\mathscr{J}_1$ fits: $-\infty$
1	1	$-\infty$	0	no item of $\mathscr{J}_1$ fits: $-\infty$
1	2	3	1	$\max\{p_1+v[0][0]\} = \max\{3\} = 3$
1	3	7	2	$\max\{p_1+v[0][1], p_2+v[0][0]\} = \max\{3,7\} = 7$
1	4	8	3	$\max\{p_1+v[0][2], p_2+v[0][1], p_3+v[0][0]\} = \max\{3,7,8\} = 8$
1	5	8	3	$\max\{p_1+v[0][3], p_2+v[0][2], p_3+v[0][1]\} = \max\{3,7,8\} = 8$
1	6	8	3	$\max\{p_1+v[0][4], p_2+v[0][3], p_3+v[0][2]\} = \max\{3,7,8\} = 8$
1	7	8	3	$\max\{p_1+v[0][5], p_2+v[0][4], p_3+v[0][3]\} = \max\{3,7,8\} = 8$
2	0	$-\infty$	0	no item of $\mathscr{J}_2$ fits: $-\infty$
2	1	$-\infty$	0	no item of $\mathscr{J}_2$ fits: $-\infty$
2	2	$-\infty$	0	no item of $\mathscr{J}_2$ fits: $-\infty$
2	3	$-\infty$	0	no item of $\mathscr{J}_2$ fits: $-\infty$
2	4	$-\infty$	0	$\max\{p_4+v[1][0]\} = \max\{-\infty\} = -\infty$
2	5	$-\infty$	0	$\max\{p_4+v[1][1], p_5+v[1][0]\} = \max\{-\infty,-\infty\} = -\infty$
2	6	6	4	$\max\{p_4+v[1][2], p_5+v[1][1]\} = \max\{6,-\infty\} = 6$
2	7	12	5	$\max\{p_4+v[1][3], p_5+v[1][2]\} = \max\{10,12\} = 12$

This is the matrix v after the execution of the DP algorithm:

$v[h][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$h=0$	0	0	0	0	0	0	0	0
$h=1$	$-\infty$	$-\infty$	3	7	8	8	8	8
$h=2$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	6	12

This is the matrix s after the execution of the DP algorithm:

$s[h][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$h=0$	0	0	0	0	0	0	0	0
$h=1$	0	0	1	2	3	3	3	3
$h=2$	0	0	0	0	0	0	4	5

2. Starting from $h=2$ and $k=7$:

Iteration	h	k	$j = s[h][k]$	Selection	Capacity
–	–	–	–	–	$k=7$
1	2	7	5	$x[5] \leftarrow 1$	$k \leftarrow 7-5=2$
2	1	2	1	$x[1] \leftarrow 1$	$k \leftarrow 2-2=0$

The optimal solution is $S = \{1,5\}$, that is, the vector $x = (1,0,0,0,1)$. Its total weight is $2+5=7 \leq c$, and the optimal value is $v[2][7] = 3+9 = 12$.

3. The last row $v[2][\cdot]$ is $-\infty$ for $k<6$ and finite from $k=6$ on, so the smallest capacity for which the instance is feasible is 6. It equals $\sum_{h=1}^{2} \min_{j\in\mathscr{J}_h} w_j = 2+4 = 6$: the lightest item of every class must fit.

Exercise 6.5 — Exercise 5: Multiple-Choice Knapsack Problem

Consider the following MCKP instance, with a knapsack of capacity $c = 9$ and 6 items partitioned into $q = 3$ classes: $\mathscr{J}_1 = \{1, 2\}$, $\mathscr{J}_2 = \{3, 4\}$, $\mathscr{J}_3 = \{5, 6\}$. The profits and weights are:

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
p_j	3	6	2	7	3	6
w_j	1	4	1	3	4	5

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each step, the cells $v[h][k]$ and $s[h][k]$; then write the matrices v and s.
2. Using the matrix s, reconstruct an optimal solution, one item per class, and report the optimal value.
3. Using the last row of the matrix v, find the smallest capacity for which the instance is feasible, and compare it with $\sum_{h=1}^{q} \min_{j \in \mathscr{J}_h} w_j$.

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. Step-by-step execution of the bottom-up DP algorithm. The row $h = 0$ is $v[0][k] = 0$ and $s[0][k] = 0$ for every k; $s[h][k] = 0$ when $v[h][k] = -\infty$. In case of ties the item met first in $\mathscr{J}_h$ is kept.

h	k	$v[h][k]$	$s[h][k]$	Action
1	0	$-\infty$	0	no item of $\mathscr{J}_1$ fits: $-\infty$
1	1	3	1	$\max\{p_1+v[0][0]\} = \max\{3\} = 3$
1	2	3	1	$\max\{p_1+v[0][1]\} = \max\{3\} = 3$
1	3	3	1	$\max\{p_1+v[0][2]\} = \max\{3\} = 3$
1	4	6	2	$\max\{p_1+v[0][3],\ p_2+v[0][0]\} = \max\{3,6\} = 6$
1	5	6	2	$\max\{p_1+v[0][4],\ p_2+v[0][1]\} = \max\{3,6\} = 6$
1	6	6	2	$\max\{p_1+v[0][5],\ p_2+v[0][2]\} = \max\{3,6\} = 6$
1	7	6	2	$\max\{p_1+v[0][6],\ p_2+v[0][3]\} = \max\{3,6\} = 6$
1	8	6	2	$\max\{p_1+v[0][7],\ p_2+v[0][4]\} = \max\{3,6\} = 6$
1	9	6	2	$\max\{p_1+v[0][8],\ p_2+v[0][5]\} = \max\{3,6\} = 6$
2	0	$-\infty$	0	no item of $\mathscr{J}_2$ fits: $-\infty$
2	1	$-\infty$	0	$\max\{p_3+v[1][0]\} = \max\{-\infty\} = -\infty$
2	2	5	3	$\max\{p_3+v[1][1]\} = \max\{5\} = 5$
2	3	5	3	$\max\{p_3+v[1][2],\ p_4+v[1][0]\} = \max\{5,-\infty\} = 5$
2	4	10	4	$\max\{p_3+v[1][3],\ p_4+v[1][1]\} = \max\{5,10\} = 10$
2	5	10	4	$\max\{p_3+v[1][4],\ p_4+v[1][2]\} = \max\{8,10\} = 10$
2	6	10	4	$\max\{p_3+v[1][5],\ p_4+v[1][3]\} = \max\{8,10\} = 10$
2	7	13	4	$\max\{p_3+v[1][6],\ p_4+v[1][4]\} = \max\{8,13\} = 13$
2	8	13	4	$\max\{p_3+v[1][7],\ p_4+v[1][5]\} = \max\{8,13\} = 13$
2	9	13	4	$\max\{p_3+v[1][8],\ p_4+v[1][6]\} = \max\{8,13\} = 13$

h	k	$v[h][k]$	$s[h][k]$	Action
3	0	$-\infty$	0	no item of $\mathscr{J}_3$ fits: $-\infty$
3	1	$-\infty$	0	no item of $\mathscr{J}_3$ fits: $-\infty$
3	2	$-\infty$	0	no item of $\mathscr{J}_3$ fits: $-\infty$
3	3	$-\infty$	0	no item of $\mathscr{J}_3$ fits: $-\infty$
3	4	$-\infty$	0	$\max\{p_5+v[2][0]\} = \max\{-\infty\} = -\infty$
3	5	$-\infty$	0	$\max\{p_5+v[2][1],\ p_6+v[2][0]\} = \max\{-\infty,-\infty\} = -\infty$
3	6	8	5	$\max\{p_5+v[2][2],\ p_6+v[2][1]\} = \max\{8,-\infty\} = 8$
3	7	11	6	$\max\{p_5+v[2][3],\ p_6+v[2][2]\} = \max\{8,11\} = 11$
3	8	13	5	$\max\{p_5+v[2][4],\ p_6+v[2][3]\} = \max\{13,11\} = 13$
3	9	16	6	$\max\{p_5+v[2][5],\ p_6+v[2][4]\} = \max\{13,16\} = 16$

This is the matrix v after the execution of the DP algorithm:

$v[h][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$h=0$	0	0	0	0	0	0	0	0	0	0
$h=1$	$-\infty$	3	3	3	6	6	6	6	6	6
$h=2$	$-\infty$	$-\infty$	5	5	10	10	10	13	13	13
$h=3$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	8	11	13	16

This is the matrix s after the execution of the DP algorithm:

$s[h][k]$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$h=0$	0	0	0	0	0	0	0	0	0	0
$h=1$	0	1	1	1	2	2	2	2	2	2
$h=2$	0	0	3	3	4	4	4	4	4	4
$h=3$	0	0	0	0	0	0	5	6	5	6

2. Starting from $h = 3$ and $k = 9$:

Iteration	h	k	$j = s[h][k]$	Selection	Capacity
–	–	–	–	–	$k = 9$
1	3	9	6	$x[6] \leftarrow 1$	$k \leftarrow 9 - 5 = 4$
2	2	4	4	$x[4] \leftarrow 1$	$k \leftarrow 4 - 3 = 1$
3	1	1	1	$x[1] \leftarrow 1$	$k \leftarrow 1 - 1 = 0$

   The optimal solution is $S = \{1, 4, 6\}$, that is, the vector $x = (1, 0, 0, 1, 0, 1)$. Its total weight is $1 + 3 + 5 = 9 \leq c$, and the optimal value is $v[3][9] = 3 + 7 + 6 = 16$.

3. The last row $v[3][\cdot]$ is $-\infty$ for $k < 6$ and finite from $k = 6$ on, so the smallest capacity for which the instance is feasible is 6. It equals $\sum_{h=1}^{3} \min_{j \in \mathcal{J}_h} w_j = 1 + 1 + 4 = 6$: the lightest item of every class must fit.

DP for the multidimensional knapsack problem

Recurrence used in these exercises

The DP algorithm of the chapter «DP for the multidimensional knapsack problem», with m dimensions, the vector $\boldsymbol{k} = (k_1, \ldots, k_m)$ of the capacities and the weight vector $\boldsymbol{w}_j = (w_{1j}, \ldots, w_{mj})$ of item j:

$$v(j, \boldsymbol{k}) = \begin{cases} 0 & \text{if } j = 0, \\ v(j-1, \boldsymbol{k}) & \text{if } j \geq 1 \text{ and } w_{dj} > k_d \text{ for some } d, \\ \max \left\{ v(j-1, \boldsymbol{k}), \right. & \text{if } j \geq 1 \text{ and } \boldsymbol{w}_j \leq \boldsymbol{k}. \\ \qquad \left. p_j + v(j-1, \boldsymbol{k} - \boldsymbol{w}_j) \right\} & \end{cases}$$

The optimal value is $v(n, \boldsymbol{c})$. The tables s store $s[j][\boldsymbol{k}] = 1$ if item j is selected; an optimal solution is reconstructed from $j = n$ and $\boldsymbol{k} = \boldsymbol{c}$ with $x_j \leftarrow s[j][\boldsymbol{k}]$, $\boldsymbol{k} \leftarrow \boldsymbol{k} - x_j \boldsymbol{w}_j$.

Exercise 7.1 — Exercise 1: Multidimensional Knapsack Problem

Consider the following MDKP instance, with $m = 2$ dimensions, capacities $c_1 = 3$ and $c_2 = 3$, and 3 items with profits and weights:

	$j = 1$	$j = 2$	$j = 3$
p_j	9	8	6
w_{1j}	2	1	2
w_{2j}	2	2	1

Questions

1. Execute the bottom-up DP algorithm: for each item j, write the matrices $v[j][k_1][k_2]$ and $s[j][k_1][k_2]$, and show the computation of the cell $\boldsymbol{k} = \boldsymbol{c}$.

2. Using the tables s, reconstruct an optimal solution (x_1, x_2, x_3) and report the optimal value.

3. How many cells does the table of the DP algorithm have, including the row $j = 0$? How many would it have with a third dimension of capacity $c_3 = 3$?

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. The row $j = 0$ is $v[0][\boldsymbol{k}] = 0$ for every $\boldsymbol{k}$. In each matrix there is one row for each k_1 and one column for each k_2; in case of ties item j is not selected.

   Item $j = 1$, with $p_1 = 9$ and $\boldsymbol{w}_1 = (2, 2)$: $v[1][(3, 3)] = \max\{v[0][(3, 3)], p_1 + v[0][(1, 1)]\} = \max\{0, 9+0\} = 9$; $\boldsymbol{w}_1 = (2, 2) \not\leq (0, 0)$, so $v[1][(0, 0)] = v[0][(0, 0)] = 0$.

$v[1][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$
$k_1 = 0$	0	0	0	0
$k_1 = 1$	0	0	0	0
$k_1 = 2$	0	0	9	9
$k_1 = 3$	0	0	9	9

$s[1][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$
$k_1 = 0$	0	0	0	0
$k_1 = 1$	0	0	0	0
$k_1 = 2$	0	0	1	1
$k_1 = 3$	0	0	1	1

   Item $j = 2$, with $p_2 = 8$ and $\boldsymbol{w}_2 = (1, 2)$: $v[2][(3, 3)] = \max\{v[1][(3, 3)], p_2 + v[1][(2, 1)]\} = \max\{9, 8 + 0\} = 9$; $v[2][(1, 2)] = \max\{v[1][(1, 2)], p_2 + v[1][(0, 0)]\} = \max\{0, 8 + 0\} = 8$.

$v[2][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$
$k_1 = 0$	0	0	0	0
$k_1 = 1$	0	0	8	8
$k_1 = 2$	0	0	9	9
$k_1 = 3$	0	0	9	9

$s[2][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$
$k_1 = 0$	0	0	0	0
$k_1 = 1$	0	0	1	1
$k_1 = 2$	0	0	0	0
$k_1 = 3$	0	0	0	0

   Item $j = 3$, with $p_3 = 6$ and $\boldsymbol{w}_3 = (2, 1)$: $v[3][(3, 3)] = \max\{v[2][(3, 3)], p_3 + v[2][(1, 2)]\} = \max\{9, 6 + 8\} = 14$.

$v[3][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	0	0
$k_1=1$	0	0	8	8
$k_1=2$	0	6	9	9
$k_1=3$	0	6	9	14

$s[3][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	0	0
$k_1=1$	0	0	0	0
$k_1=2$	0	1	0	0
$k_1=3$	0	1	0	1

2. Starting from $j=3$ and $\boldsymbol{k}=(3,3)$:

Iteration	j	$\boldsymbol{k}$	$s[j][\boldsymbol{k}]$	$x[1]$	$x[2]$	$x[3]$	Action
–	–	–	–	0	0	0	$\boldsymbol{k}=(3,3)$
1	3	$(3,3)$	1	0	0	1	$x[3]\leftarrow 1$, $\boldsymbol{k}\leftarrow(3,3)-(2,1)=(1,2)$
2	2	$(1,2)$	1	0	1	1	$x[2]\leftarrow 1$, $\boldsymbol{k}\leftarrow(1,2)-(1,2)=(0,0)$
3	1	$(0,0)$	0	0	1	1	$x[1]\leftarrow 0$, $\boldsymbol{k}=(0,0)$

   The vector $(0,1,1)$ is an optimal solution: its weights are $1+2=3\le c_1$ and $2+1=3\le c_2$, and the optimal value is $v[3][(3,3)]=8+6=14$.

3. The table has $(n+1)(c_1+1)(c_2+1)=4\cdot 4\cdot 4=64$ cells. A third dimension of capacity $c_3=3$ multiplies this number by $c_3+1=4$, giving 256 cells: every new dimension multiplies the number of states.

Exercise 7.2 — Exercise 2: Multidimensional Knapsack Problem

Consider the following MDKP instance, with $m=2$ dimensions, capacities $c_1=4$ and $c_2=3$, and 3 items with profits and weights:

	$j=1$	$j=2$	$j=3$
p_j	4	7	5
w_{1j}	3	2	1
w_{2j}	1	2	2

Questions

1. Execute the bottom-up DP algorithm: for each item j, write the matrices $v[j][k_1][k_2]$ and $s[j][k_1][k_2]$, and show the computation of the cell $\boldsymbol{k}=\boldsymbol{c}$.
2. Using the tables s, reconstruct an optimal solution (x_1,x_2,x_3) and report the optimal value.
3. How many cells does the table of the DP algorithm have, including the row $j=0$? How many would it have with a third dimension of capacity $c_3=3$?

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. The row $j = 0$ is $v[0][\boldsymbol{k}] = 0$ for every $\boldsymbol{k}$. In each matrix there is one row for each k_1 and one column for each k_2; in case of ties item j is not selected.

   Item $j = 1$, with $p_1 = 4$ and $\boldsymbol{w}_1 = (3, 1)$: $v[1][(4,3)] = \max\{v[0][(4,3)], p_1 + v[0][(1,2)]\} = \max\{0, 4+0\} = 4$; $v[1][(3,1)] = \max\{v[0][(3,1)], p_1 + v[0][(0,0)]\} = \max\{0, 4+0\} = 4$.

$v[1][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	0	0
$k_1=1$	0	0	0	0
$k_1=2$	0	0	0	0
$k_1=3$	0	4	4	4
$k_1=4$	0	4	4	4

$s[1][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	0	0
$k_1=1$	0	0	0	0
$k_1=2$	0	0	0	0
$k_1=3$	0	1	1	1
$k_1=4$	0	1	1	1

   Item $j = 2$, with $p_2 = 7$ and $\boldsymbol{w}_2 = (2, 2)$: $v[2][(4,3)] = \max\{v[1][(4,3)], p_2 + v[1][(2,1)]\} = \max\{4, 7+0\} = 7$; $\boldsymbol{w}_2 = (2,2) \not\leq (3,1)$, so $v[2][(3,1)] = v[1][(3,1)] = 4$.

$v[2][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	0	0
$k_1=1$	0	0	0	0
$k_1=2$	0	0	7	7
$k_1=3$	0	4	7	7
$k_1=4$	0	4	7	7

$s[2][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	0	0
$k_1=1$	0	0	0	0
$k_1=2$	0	0	1	1
$k_1=3$	0	0	1	1
$k_1=4$	0	0	1	1

   Item $j = 3$, with $p_3 = 5$ and $\boldsymbol{w}_3 = (1, 2)$: $v[3][(4,3)] = \max\{v[2][(4,3)], p_3 + v[2][(3,1)]\} = \max\{7, 5+4\} = 9$.

$v[3][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	0	0
$k_1=1$	0	0	5	5
$k_1=2$	0	0	7	7
$k_1=3$	0	4	7	7
$k_1=4$	0	4	7	9

$s[3][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	0	0
$k_1=1$	0	0	1	1
$k_1=2$	0	0	0	0
$k_1=3$	0	0	0	0
$k_1=4$	0	0	0	1

2. Starting from $j=3$ and $\boldsymbol{k}=(4,3)$:

Iteration	j	$\boldsymbol{k}$	$s[j][\boldsymbol{k}]$	$x[1]$	$x[2]$	$x[3]$	Action
–	–	–	–	0	0	0	$\boldsymbol{k}=(4,3)$
1	3	$(4,3)$	1	0	0	1	$x[3]\leftarrow 1$, $\boldsymbol{k}\leftarrow(4,3)-(1,2)=(3,1)$
2	2	$(3,1)$	0	0	0	1	$x[2]\leftarrow 0$, $\boldsymbol{k}=(3,1)$
3	1	$(3,1)$	1	1	0	1	$x[1]\leftarrow 1$, $\boldsymbol{k}\leftarrow(3,1)-(3,1)=(0,0)$

   The vector $(1,0,1)$ is an optimal solution: its weights are $3+1=4\le c_1$ and $1+2=3\le c_2$, and the optimal value is $v[3][(4,3)]=4+5=9$.

3. The table has $(n+1)(c_1+1)(c_2+1)=4\cdot5\cdot4=80$ cells. A third dimension of capacity $c_3=3$ multiplies this number by $c_3+1=4$, giving 320 cells: every new dimension multiplies the number of states.

Exercise 7.3 — Exercise 3: Multidimensional Knapsack Problem

Consider the following MDKP instance, with $m=2$ dimensions, capacities $c_1=3$ and $c_2=4$, and 3 items with profits and weights:

	$j=1$	$j=2$	$j=3$
p_j	6	6	9
w_{1j}	2	1	2
w_{2j}	1	2	3

Questions

1. Execute the bottom-up DP algorithm: for each item j, write the matrices $v[j][k_1][k_2]$ and $s[j][k_1][k_2]$, and show the computation of the cell $\boldsymbol{k}=\boldsymbol{c}$.

2. Using the tables s, reconstruct an optimal solution (x_1,x_2,x_3) and report the optimal value.

3. How many cells does the table of the DP algorithm have, including the row $j=0$? How many would it have with a third dimension of capacity $c_3=3$?

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. The row $j = 0$ is $v[0][\boldsymbol{k}] = 0$ for every $\boldsymbol{k}$. In each matrix there is one row for each k_1 and one column for each k_2; in case of ties item j is not selected.

   Item $j = 1$, with $p_1 = 6$ and $\boldsymbol{w}_1 = (2, 1)$: $v[1][(3, 4)] = \max\{v[0][(3, 4)], p_1 + v[0][(1, 3)]\} = \max\{0, 6 + 0\} = 6$; $v[1][(2, 2)] = \max\{v[0][(2, 2)], p_1 + v[0][(0, 1)]\} = \max\{0, 6 + 0\} = 6$.

$v[1][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$	$k_2 = 4$
$k_1 = 0$	0	0	0	0	0
$k_1 = 1$	0	0	0	0	0
$k_1 = 2$	0	6	6	6	6
$k_1 = 3$	0	6	6	6	6

$s[1][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$	$k_2 = 4$
$k_1 = 0$	0	0	0	0	0
$k_1 = 1$	0	0	0	0	0
$k_1 = 2$	0	1	1	1	1
$k_1 = 3$	0	1	1	1	1

   Item $j = 2$, with $p_2 = 6$ and $\boldsymbol{w}_2 = (1, 2)$: $v[2][(3, 4)] = \max\{v[1][(3, 4)], p_2 + v[1][(2, 2)]\} = \max\{6, 6 + 6\} = 12$.

$v[2][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$	$k_2 = 4$
$k_1 = 0$	0	0	0	0	0
$k_1 = 1$	0	0	6	6	6
$k_1 = 2$	0	6	6	6	6
$k_1 = 3$	0	6	6	12	12

$s[2][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$	$k_2 = 4$
$k_1 = 0$	0	0	0	0	0
$k_1 = 1$	0	0	1	1	1
$k_1 = 2$	0	0	0	0	0
$k_1 = 3$	0	0	0	1	1

   Item $j = 3$, with $p_3 = 9$ and $\boldsymbol{w}_3 = (2, 3)$: $v[3][(3, 4)] = \max\{v[2][(3, 4)], p_3 + v[2][(1, 1)]\} = \max\{12, 9 + 0\} = 12$.

$v[3][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$	$k_2 = 4$
$k_1 = 0$	0	0	0	0	0
$k_1 = 1$	0	0	6	6	6
$k_1 = 2$	0	6	6	9	9
$k_1 = 3$	0	6	6	12	12

$s[3][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$	$k_2=4$
$k_1=0$	0	0	0	0	0
$k_1=1$	0	0	0	0	0
$k_1=2$	0	0	0	1	1
$k_1=3$	0	0	0	0	0

2. Starting from $j=3$ and $\boldsymbol{k}=(3,4)$:

Iteration	j	$\boldsymbol{k}$	$s[j][\boldsymbol{k}]$	$x[1]$	$x[2]$	$x[3]$	Action
–	–	–	–	0	0	0	$\boldsymbol{k}=(3,4)$
1	3	$(3,4)$	0	0	0	0	$x[3]\leftarrow 0$, $\boldsymbol{k}=(3,4)$
2	2	$(3,4)$	1	0	1	0	$x[2]\leftarrow 1$, $\boldsymbol{k}\leftarrow(3,4)-(1,2)=(2,2)$
3	1	$(2,2)$	1	1	1	0	$x[1]\leftarrow 1$, $\boldsymbol{k}\leftarrow(2,2)-(2,1)=(0,1)$

   The vector $(1,1,0)$ is an optimal solution: its weights are $2+1=3\le c_1$ and $1+2=3\le c_2$, and the optimal value is $v[3][(3,4)]=6+6=12$.

3. The table has $(n+1)(c_1+1)(c_2+1)=4\cdot 4\cdot 5=80$ cells. A third dimension of capacity $c_3=3$ multiplies this number by $c_3+1=4$, giving 320 cells: every new dimension multiplies the number of states.

Exercise 7.4 — Exercise 4: Multidimensional Knapsack Problem

Consider the following MDKP instance, with $m=2$ dimensions, capacities $c_1=4$ and $c_2=3$, and 4 items with profits and weights:

	$j=1$	$j=2$	$j=3$	$j=4$
p_j	7	3	7	5
w_{1j}	2	1	1	2
w_{2j}	1	2	1	1

Questions

1. Execute the bottom-up DP algorithm: for each item j, write the matrices $v[j][k_1][k_2]$ and $s[j][k_1][k_2]$, and show the computation of the cell $\boldsymbol{k}=\boldsymbol{c}$.
2. Using the tables s, reconstruct an optimal solution (x_1,x_2,x_3,x_4) and report the optimal value.
3. How many cells does the table of the DP algorithm have, including the row $j=0$? How many would it have with a third dimension of capacity $c_3=3$?

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. The row $j = 0$ is $v[0][\boldsymbol{k}] = 0$ for every $\boldsymbol{k}$. In each matrix there is one row for each k_1 and one column for each k_2; in case of ties item j is not selected.

   Item $j = 1$, with $p_1 = 7$ and $\boldsymbol{w}_1 = (2, 1)$: $v[1][(4,3)] = \max\{v[0][(4,3)],\ p_1 + v[0][(2,2)]\} = \max\{0,\ 7+0\} = 7$; $v[1][(3,2)] = \max\{v[0][(3,2)],\ p_1 + v[0][(1,1)]\} = \max\{0,\ 7+0\} = 7$.

$v[1][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	0	0
$k_1=1$	0	0	0	0
$k_1=2$	0	7	7	7
$k_1=3$	0	7	7	7
$k_1=4$	0	7	7	7

$s[1][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	0	0
$k_1=1$	0	0	0	0
$k_1=2$	0	1	1	1
$k_1=3$	0	1	1	1
$k_1=4$	0	1	1	1

   Item $j = 2$, with $p_2 = 3$ and $\boldsymbol{w}_2 = (1, 2)$: $v[2][(4,3)] = \max\{v[1][(4,3)],\ p_2 + v[1][(3,1)]\} = \max\{7,\ 3+7\} = 10$; $v[2][(3,2)] = \max\{v[1][(3,2)],\ p_2 + v[1][(2,0)]\} = \max\{7,\ 3+0\} = 7$.

$v[2][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	0	0
$k_1=1$	0	0	3	3
$k_1=2$	0	7	7	7
$k_1=3$	0	7	7	10
$k_1=4$	0	7	7	10

$s[2][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	0	0
$k_1=1$	0	0	1	1
$k_1=2$	0	0	0	0
$k_1=3$	0	0	0	1
$k_1=4$	0	0	0	1

   Item $j = 3$, with $p_3 = 7$ and $\boldsymbol{w}_3 = (1, 1)$: $v[3][(4,3)] = \max\{v[2][(4,3)],\ p_3 + v[2][(3,2)]\} = \max\{10,\ 7+7\} = 14$.

$v[3][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	0	0
$k_1=1$	0	7	7	7
$k_1=2$	0	7	7	10
$k_1=3$	0	7	14	14
$k_1=4$	0	7	14	14

$s[3][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	0	0
$k_1=1$	0	1	1	1
$k_1=2$	0	0	0	1
$k_1=3$	0	0	1	1
$k_1=4$	0	0	1	1

Item $j=4$, with $p_4=5$ and $\boldsymbol{w}_4=(2,1)$: $v[4][(4,3)] = \max\{v[3][(4,3)], p_4 + v[3][(2,2)]\} = \max\{14, 5+7\} = 14$.

$v[4][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	0	0
$k_1=1$	0	7	7	7
$k_1=2$	0	7	7	10
$k_1=3$	0	7	14	14
$k_1=4$	0	7	14	14

$s[4][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	0	0
$k_1=1$	0	0	0	0
$k_1=2$	0	0	0	0
$k_1=3$	0	0	0	0
$k_1=4$	0	0	0	0

2. Starting from $j=4$ and $\boldsymbol{k}=(4,3)$:

Iteration	j	$\boldsymbol{k}$	$s[j][\boldsymbol{k}]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	Action
–	–	–	–	0	0	0	0	$\boldsymbol{k}=(4,3)$
1	4	(4,3)	0	0	0	0	0	$x[4]\leftarrow 0$, $\boldsymbol{k}=(4,3)$
2	3	(4,3)	1	0	0	1	0	$x[3]\leftarrow 1$, $\boldsymbol{k}\leftarrow(4,3)-(1,1)=(3,2)$
3	2	(3,2)	0	0	0	1	0	$x[2]\leftarrow 0$, $\boldsymbol{k}=(3,2)$
4	1	(3,2)	1	1	0	1	0	$x[1]\leftarrow 1$, $\boldsymbol{k}\leftarrow(3,2)-(2,1)=(1,1)$

The vector $(1,0,1,0)$ is an optimal solution: its weights are $2+1=3\leq c_1$ and $1+1=2\leq c_2$, and the optimal value is $v[4][(4,3)]=7+7=14$.

3. The table has $(n+1)(c_1+1)(c_2+1)=5\cdot 5\cdot 4=100$ cells. A third dimension of capacity $c_3=3$ multiplies this number by $c_3+1=4$, giving 400 cells: every new dimension multiplies the number of states.

Exercise 7.5 — Exercise 5: Multidimensional Knapsack Problem

Consider the following MDKP instance, with $m=2$ dimensions, capacities $c_1=4$ and $c_2=4$, and 4 items with profits and weights:

	$j=1$	$j=2$	$j=3$	$j=4$
p_j	6	7	2	8
w_{1j}	2	3	1	1
w_{2j}	1	1	3	1

Questions

1. Execute the bottom-up DP algorithm: for each item j, write the matrices $v[j][k_1][k_2]$ and $s[j][k_1][k_2]$, and show the computation of the cell $\boldsymbol{k} = \boldsymbol{c}$.
2. Using the tables s, reconstruct an optimal solution (x_1, x_2, x_3, x_4) and report the optimal value.
3. How many cells does the table of the DP algorithm have, including the row $j = 0$? How many would it have with a third dimension of capacity $c_3 = 3$?

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. The row $j = 0$ is $v[0][\boldsymbol{k}] = 0$ for every $\boldsymbol{k}$. In each matrix there is one row for each k_1 and one column for each k_2; in case of ties item j is not selected.

   Item $j = 1$, with $p_1 = 6$ and $\boldsymbol{w}_1 = (2, 1)$: $v[1][(4,4)] = \max\{v[0][(4,4)],\ p_1 + v[0][(2,3)]\} = \max\{0,\ 6+0\} = 6$; $\boldsymbol{w}_1 = (2,1) \not\leq (0,2)$, so $v[1][(0,2)] = v[0][(0,2)] = 0$.

$v[1][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$	$k_2=4$
$k_1=0$	0	0	0	0	0
$k_1=1$	0	0	0	0	0
$k_1=2$	0	6	6	6	6
$k_1=3$	0	6	6	6	6
$k_1=4$	0	6	6	6	6

$s[1][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$	$k_2=4$
$k_1=0$	0	0	0	0	0
$k_1=1$	0	0	0	0	0
$k_1=2$	0	1	1	1	1
$k_1=3$	0	1	1	1	1
$k_1=4$	0	1	1	1	1

   Item $j = 2$, with $p_2 = 7$ and $\boldsymbol{w}_2 = (3, 1)$: $v[2][(4,4)] = \max\{v[1][(4,4)],\ p_2 + v[1][(1,3)]\} = \max\{6,\ 7+0\} = 7$; $v[2][(3,3)] = \max\{v[1][(3,3)],\ p_2 + v[1][(0,2)]\} = \max\{6,\ 7+0\} = 7$.

$v[2][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$	$k_2=4$
$k_1=0$	0	0	0	0	0
$k_1=1$	0	0	0	0	0
$k_1=2$	0	6	6	6	6
$k_1=3$	0	7	7	7	7
$k_1=4$	0	7	7	7	7

$s[2][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$	$k_2=4$
$k_1=0$	0	0	0	0	0
$k_1=1$	0	0	0	0	0
$k_1=2$	0	0	0	0	0
$k_1=3$	0	1	1	1	1
$k_1=4$	0	1	1	1	1

Item $j=3$, with $p_3=2$ and $\boldsymbol{w}_3=(1,3)$: $v[3][(4,4)]=\max\{v[2][(4,4)],\, p_3+v[2][(3,1)]\}=\max\{7,\, 2+7\}=9$; $v[3][(3,3)]=\max\{v[2][(3,3)],\, p_3+v[2][(2,0)]\}=\max\{7,\, 2+0\}=7$.

$v[3][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$	$k_2=4$
$k_1=0$	0	0	0	0	0
$k_1=1$	0	0	0	2	2
$k_1=2$	0	6	6	6	6
$k_1=3$	0	7	7	7	8
$k_1=4$	0	7	7	7	9

$s[3][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$	$k_2=4$
$k_1=0$	0	0	0	0	0
$k_1=1$	0	0	0	1	1
$k_1=2$	0	0	0	0	0
$k_1=3$	0	0	0	0	1
$k_1=4$	0	0	0	0	1

Item $j=4$, with $p_4=8$ and $\boldsymbol{w}_4=(1,1)$: $v[4][(4,4)]=\max\{v[3][(4,4)],\, p_4+v[3][(3,3)]\}=\max\{9,\, 8+7\}=15$.

$v[4][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$	$k_2=4$
$k_1=0$	0	0	0	0	0
$k_1=1$	0	8	8	8	8
$k_1=2$	0	8	8	8	10
$k_1=3$	0	8	14	14	14
$k_1=4$	0	8	15	15	15

$s[4][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$	$k_2=4$
$k_1=0$	0	0	0	0	0
$k_1=1$	0	1	1	1	1
$k_1=2$	0	1	1	1	1
$k_1=3$	0	1	1	1	1
$k_1=4$	0	1	1	1	1

2. Starting from $j = 4$ and $\boldsymbol{k} = (4, 4)$:

Iteration	j	$\boldsymbol{k}$	$s[j][\boldsymbol{k}]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	Action
–	–	–	–	0	0	0	0	$\boldsymbol{k} = (4, 4)$
1	4	$(4, 4)$	1	0	0	0	1	$x[4] \leftarrow 1$, $\boldsymbol{k} \leftarrow (4, 4) - (1, 1) = (3, 3)$
2	3	$(3, 3)$	0	0	0	0	1	$x[3] \leftarrow 0$, $\boldsymbol{k} = (3, 3)$
3	2	$(3, 3)$	1	0	1	0	1	$x[2] \leftarrow 1$, $\boldsymbol{k} \leftarrow (3, 3) - (3, 1) = (0, 2)$
4	1	$(0, 2)$	0	0	1	0	1	$x[1] \leftarrow 0$, $\boldsymbol{k} = (0, 2)$

   The vector $(0, 1, 0, 1)$ is an optimal solution: its weights are $3 + 1 = 4 \leq c_1$ and $1 + 1 = 2 \leq c_2$, and the optimal value is $v[4][(4, 4)] = 7 + 8 = 15$.

3. The table has $(n + 1)(c_1 + 1)(c_2 + 1) = 5 \cdot 5 \cdot 5 = 125$ cells. A third dimension of capacity $c_3 = 3$ multiplies this number by $c_3 + 1 = 4$, giving 500 cells: every new dimension multiplies the number of states.

Chapter 8

DP for the knapsack problem with setups

Recurrence used in these exercises

The DP algorithm of the chapter «DP for the knapsack problem with setups». The items are numbered class by class, $\mathscr{J}_h = \{\ell_{h-1}+1, \dots, \ell_h\}$, $t_h = |\mathscr{J}_h|$, and $j = \ell_{h-1} + i$ is the i-th item of class h:

$$u(h,i,k) = \begin{cases} -\infty & \text{if } i = 0 \text{ and } k < s_h, \\ v(h-1, k-s_h) - f_h & \text{if } i = 0 \text{ and } k \geq s_h, \\ u(h, i-1, k) & \text{if } i \geq 1 \text{ and } w_j > k, \\ \max\Big\{u(h,i-1,k), & \text{if } i \geq 1 \text{ and } w_j \leq k, \\ \quad p_j + u(h, i-1, k-w_j)\Big\} & \end{cases}$$

$$v(h,k) = \begin{cases} 0 & \text{if } h = 0, \\ \max\big\{v(h-1,k),\ u(h,t_h,k)\big\} & \text{if } h \geq 1. \end{cases}$$

The optimal value is $v(q,c)$. The choice tables store $s[h][i][k] = 1$ if the i-th item of class h is selected, and $o[h][k] = 1$ if class h is open. The reconstruction scans the classes from $h = q$ and $k = c$: an open class is scanned as in the KP, and then k decreases by s_h.

Exercise 8.1 — Exercise 1: Knapsack Problem with Setups

Consider the following KPS instance, with a knapsack of capacity $c = 7$ and 4 items partitioned into $q = 2$ classes: $\mathscr{J}_1 = \{1,2\}$ with $f_1 = 2$ and $s_1 = 1$, $\mathscr{J}_2 = \{3,4\}$ with $f_2 = 2$ and $s_2 = 1$. The revenues and weights are:

	$j=1$	$j=2$	$j=3$	$j=4$
p_j	2	9	3	8
w_j	2	1	2	1

Questions

1. Execute the bottom-up DP algorithm and write the table of all the rows $v[h][\cdot]$ and $u[h][i][\cdot]$, in the order of computation.
2. Write the choice tables s and o, reconstruct an optimal solution (x_1, x_2, x_3, x_4), say which classes are open, and report the optimal value.

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. For each class h the DP algorithm computes the setup row $u[h][0][\cdot]$, the rows of the items of class h, and the row $v[h][\cdot]$. In case of ties the item is not selected and the class stays closed. This is the table after the execution of the DP algorithm:

Row	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$v[0][k]$	0	0	0	0	0	0	0	0
$u[1][0][k]$ (setup)	$-\infty$	-2	-2	-2	-2	-2	-2	-2
$u[1][1][k]$ $(j=1)$	$-\infty$	-2	-2	0	0	0	0	0
$u[1][2][k]$ $(j=2)$	$-\infty$	-2	7	7	9	9	9	9
$v[1][k]$	0	0	7	7	9	9	9	9
$u[2][0][k]$ (setup)	$-\infty$	-2	-2	5	5	7	7	7
$u[2][1][k]$ $(j=3)$	$-\infty$	-2	-2	5	5	8	8	10
$u[2][2][k]$ $(j=4)$	$-\infty$	-2	6	6	13	13	16	16
$v[2][k]$	0	0	7	7	13	13	16	16

   These are the computations of the cells used by the reconstruction, in the order in which it reads them:

   - $v[2][7] = \max\{v[1][7],\ u[2][2][7]\} = \max\{9,\ 16\} = 16$
   - $u[2][2][7] = \max\{u[2][1][7],\ p_4 + u[2][1][6]\} = \max\{10,\ 16\} = 16$
   - $u[2][1][6] = \max\{u[2][0][6],\ p_3 + u[2][0][4]\} = \max\{7,\ 8\} = 8$
   - $u[2][0][4] = v[1][3] - f_2 = 7 - 2 = 5$
   - $v[1][3] = \max\{v[0][3],\ u[1][2][3]\} = \max\{0,\ 7\} = 7$
   - $u[1][2][3] = \max\{u[1][1][3],\ p_2 + u[1][1][2]\} = \max\{0,\ 7\} = 7$
   - $u[1][1][2] = \max\{u[1][0][2],\ p_1 + u[1][0][0]\} = \max\{-2,\ -\infty\} = -2$
   - $u[1][0][2] = v[0][1] - f_1 = 0 - 2 = -2$

2. These are the choice tables computed by the DP algorithm:

Row	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$s[1][1][k]$ $(j=1)$	0	0	0	1	1	1	1	1
$s[1][2][k]$ $(j=2)$	0	0	1	1	1	1	1	1
$o[1][k]$	0	0	1	1	1	1	1	1
$s[2][1][k]$ $(j=3)$	0	0	0	0	0	1	1	1
$s[2][2][k]$ $(j=4)$	0	0	1	1	1	1	1	1
$o[2][k]$	0	0	0	0	1	1	1	1

Starting from $h = 2$ and $k = 7$:

Iteration	Stage	k	Choice	$x[1]$	$x[2]$	$x[3]$	$x[4]$	Action
–	–	–	–	0	0	0	0	$k = 7$
1	$h = 2$	7	$o[2][7] = 1$	0	0	0	0	class 2 open
2	$j = 4$	7	$s[2][2][7] = 1$	0	0	0	1	$k \leftarrow 7 - 1 = 6$
3	$j = 3$	6	$s[2][1][6] = 1$	0	0	1	1	$k \leftarrow 6 - 2 = 4$
–	–	4	–	0	0	1	1	setup: $k \leftarrow 4 - s_2 = 3$
4	$h = 1$	3	$o[1][3] = 1$	0	0	1	1	class 1 open
5	$j = 2$	3	$s[1][2][3] = 1$	0	1	1	1	$k \leftarrow 3 - 1 = 2$
6	$j = 1$	2	$s[1][1][2] = 0$	0	1	1	1	$k = 2$
–	–	2	–	0	1	1	1	setup: $k \leftarrow 2 - s_1 = 1$

The vector $(0, 1, 1, 1)$ is an optimal solution: both classes are open, and the items $\{2, 3, 4\}$ are selected. Its total weight is $1 + 2 + 1 + s_1 + s_2 = 6 \leq c$, and the optimal value is $v[2][7] = 9 + 3 + 8 - f_1 - f_2 = 16$.

Exercise 8.2 — Exercise 2: Knapsack Problem with Setups

Consider the following KPS instance, with a knapsack of capacity $c = 7$ and 4 items partitioned into $q = 2$ classes: $\mathscr{J}_1 = \{1, 2\}$ with $f_1 = 4$ and $s_1 = 2$, $\mathscr{J}_2 = \{3, 4\}$ with $f_2 = 4$ and $s_2 = 1$. The revenues and weights are:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
p_j	7	5	9	2
w_j	3	3	3	3

Questions

1. Execute the bottom-up DP algorithm and write the table of all the rows $v[h][\cdot]$ and $u[h][i][\cdot]$, in the order of computation.
2. Write the choice tables s and o, reconstruct an optimal solution (x_1, x_2, x_3, x_4), say which classes are open, and report the optimal value.

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. For each class h the DP algorithm computes the setup row $u[h][0][\cdot]$, the rows of the items of class h, and the row $v[h][\cdot]$. In case of ties the item is not selected and the class stays closed. This is the table after the execution of the DP algorithm:

Row	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$v[0][k]$	0	0	0	0	0	0	0	0
$u[1][0][k]$ (setup)	$-\infty$	$-\infty$	-4	-4	-4	-4	-4	-4
$u[1][1][k]$ $(j=1)$	$-\infty$	$-\infty$	-4	-4	-4	3	3	3
$u[1][2][k]$ $(j=2)$	$-\infty$	$-\infty$	-4	-4	-4	3	3	3
$v[1][k]$	0	0	0	0	0	3	3	3
$u[2][0][k]$ (setup)	$-\infty$	-4	-4	-4	-4	-4	-1	-1
$u[2][1][k]$ $(j=3)$	$-\infty$	-4	-4	-4	5	5	5	5
$u[2][2][k]$ $(j=4)$	$-\infty$	-4	-4	-4	5	5	5	7
$v[2][k]$	0	0	0	0	5	5	5	7

These are the computations of the cells used by the reconstruction, in the order in which it reads them:

- $v[2][7] = \max\{v[1][7],\, u[2][2][7]\} = \max\{3,\, 7\} = 7$
- $u[2][2][7] = \max\{u[2][1][7],\, p_4 + u[2][1][4]\} = \max\{5,\, 7\} = 7$
- $u[2][1][4] = \max\{u[2][0][4],\, p_3 + u[2][0][1]\} = \max\{-4,\, 5\} = 5$
- $u[2][0][1] = v[1][0] - f_2 = 0 - 4 = -4$
- $v[1][0] = \max\{v[0][0],\, u[1][2][0]\} = \max\{0,\, -\infty\} = 0$

2. These are the choice tables computed by the DP algorithm:

Row	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$s[1][1][k]$ $(j=1)$	0	0	0	0	0	1	1	1
$s[1][2][k]$ $(j=2)$	0	0	0	0	0	0	0	0
$o[1][k]$	0	0	0	0	0	1	1	1
$s[2][1][k]$ $(j=3)$	0	0	0	0	1	1	1	1
$s[2][2][k]$ $(j=4)$	0	0	0	0	0	0	0	1
$o[2][k]$	0	0	0	0	1	1	1	1

Starting from $h=2$ and $k=7$:

Iteration	Stage	k	Choice	$x[1]$	$x[2]$	$x[3]$	$x[4]$	Action
–	–	–	–	0	0	0	0	$k=7$
1	$h=2$	7	$o[2][7]=1$	0	0	0	0	class 2 open
2	$j=4$	7	$s[2][2][7]=1$	0	0	0	1	$k \leftarrow 7-3=4$
3	$j=3$	4	$s[2][1][4]=1$	0	0	1	1	$k \leftarrow 4-3=1$
–	–	1	–	0	0	1	1	setup: $k \leftarrow 1-s_2=0$
4	$h=1$	0	$o[1][0]=0$	0	0	1	1	class 1 closed

The vector $(0,0,1,1)$ is an optimal solution: only class 2 is open, and the items $\{3,4\}$ are selected. Its total weight is $3+3+s_2 = 7 \le c$, and the optimal value is $v[2][7] = 9+2-f_2 = 7$.

Exercise 8.3 — Exercise 3: Knapsack Problem with Setups

Consider the following KPS instance, with a knapsack of capacity $c = 8$ and 5 items partitioned into $q = 2$ classes: $\mathscr{J}_1 = \{1, 2\}$ with $f_1 = 1$ and $s_1 = 1$, $\mathscr{J}_2 = \{3, 4, 5\}$ with $f_2 = 2$ and $s_2 = 2$. The revenues and weights are:

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
p_j	3	9	5	7	9
w_j	4	1	1	3	1

Questions

1. Execute the bottom-up DP algorithm and write the table of all the rows $v[h][\cdot]$ and $u[h][i][\cdot]$, in the order of computation.
2. Write the choice tables s and o, reconstruct an optimal solution $(x_1, x_2, x_3, x_4, x_5)$, say which classes are open, and report the optimal value.

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. For each class h the DP algorithm computes the setup row $u[h][0][\cdot]$, the rows of the items of class h, and the row $v[h][\cdot]$. In case of ties the item is not selected and the class stays closed. This is the table after the execution of the DP algorithm:

Row	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$v[0][k]$	0	0	0	0	0	0	0	0	0
$u[1][0][k]$ (setup)	$-\infty$	-1	-1	-1	-1	-1	-1	-1	-1
$u[1][1][k]$ ($j=1$)	$-\infty$	-1	-1	-1	-1	2	2	2	2
$u[1][2][k]$ ($j=2$)	$-\infty$	-1	8	8	8	8	11	11	11
$v[1][k]$	0	0	8	8	8	8	11	11	11
$u[2][0][k]$ (setup)	$-\infty$	$-\infty$	-2	-2	6	6	6	6	9
$u[2][1][k]$ ($j=3$)	$-\infty$	$-\infty$	-2	3	6	11	11	11	11
$u[2][2][k]$ ($j=4$)	$-\infty$	$-\infty$	-2	3	6	11	11	13	18
$u[2][3][k]$ ($j=5$)	$-\infty$	$-\infty$	-2	7	12	15	20	20	22
$v[2][k]$	0	0	8	8	12	15	20	20	22

   These are the computations of the cells used by the reconstruction, in the order in which it reads them:

   - $v[2][8] = \max\{v[1][8],\ u[2][3][8]\} = \max\{11,\ 22\} = 22$
   - $u[2][3][8] = \max\{u[2][2][8],\ p_5 + u[2][2][7]\} = \max\{18,\ 22\} = 22$
   - $u[2][2][7] = \max\{u[2][1][7],\ p_4 + u[2][1][4]\} = \max\{11,\ 13\} = 13$
   - $u[2][1][4] = \max\{u[2][0][4],\ p_3 + u[2][0][3]\} = \max\{6,\ 3\} = 6$
   - $u[2][0][4] = v[1][2] - f_2 = 8 - 2 = 6$

- $v[1][2] = \max\{v[0][2],\, u[1][2][2]\} = \max\{0,\, 8\} = 8$
- $u[1][2][2] = \max\{u[1][1][2],\, p_2 + u[1][1][1]\} = \max\{-1,\, 8\} = 8$
- $u[1][1][1] = u[1][0][1] = -1$, since $w_1 > 1$
- $u[1][0][1] = v[0][0] - f_1 = 0 - 1 = -1$

2. These are the choice tables computed by the DP algorithm:

Row	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$s[1][1][k]$ $(j=1)$	0	0	0	0	0	1	1	1	1
$s[1][2][k]$ $(j=2)$	0	0	1	1	1	1	1	1	1
$o[1][k]$	0	0	1	1	1	1	1	1	1
$s[2][1][k]$ $(j=3)$	0	0	0	1	0	1	1	1	1
$s[2][2][k]$ $(j=4)$	0	0	0	0	0	0	0	1	1
$s[2][3][k]$ $(j=5)$	0	0	0	1	1	1	1	1	1
$o[2][k]$	0	0	0	0	1	1	1	1	1

Starting from $h = 2$ and $k = 8$:

Iteration	Stage	k	Choice	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	Action
–	–	–	–	0	0	0	0	0	$k = 8$
1	$h=2$	8	$o[2][8]=1$	0	0	0	0	0	class 2 open
2	$j=5$	8	$s[2][3][8]=1$	0	0	0	0	1	$k \leftarrow 8-1=7$
3	$j=4$	7	$s[2][2][7]=1$	0	0	0	1	1	$k \leftarrow 7-3=4$
4	$j=3$	4	$s[2][1][4]=0$	0	0	0	1	1	$k = 4$
–	–	4	–	0	0	0	1	1	setup: $k \leftarrow 4 - s_2 = 2$
5	$h=1$	2	$o[1][2]=1$	0	0	0	1	1	class 1 open
6	$j=2$	2	$s[1][2][2]=1$	0	1	0	1	1	$k \leftarrow 2-1=1$
7	$j=1$	1	$s[1][1][1]=0$	0	1	0	1	1	$k = 1$
–	–	1	–	0	1	0	1	1	setup: $k \leftarrow 1 - s_1 = 0$

The vector $(0, 1, 0, 1, 1)$ is an optimal solution: both classes are open, and the items $\{2, 4, 5\}$ are selected. Its total weight is $1 + 3 + 1 + s_1 + s_2 = 8 \leq c$, and the optimal value is $v[2][8] = 9 + 7 + 9 - f_1 - f_2 = 22$.

Exercise 8.4 — Exercise 4: Knapsack Problem with Setups

Consider the following KPS instance, with a knapsack of capacity $c = 8$ and 5 items partitioned into $q = 2$ classes: $\mathscr{J}_1 = \{1, 2, 3\}$ with $f_1 = 2$ and $s_1 = 1$, $\mathscr{J}_2 = \{4, 5\}$ with $f_2 = 1$ and $s_2 = 1$. The revenues and weights are:

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
p_j	6	9	9	7	5
w_j	1	1	4	3	4

Questions

1. Execute the bottom-up DP algorithm and write the table of all the rows $v[h][\cdot]$ and $u[h][i][\cdot]$, in the order of computation.
2. Write the choice tables s and o, reconstruct an optimal solution $(x_1, x_2, x_3, x_4, x_5)$, say which classes are open, and report the optimal value.

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. For each class h the DP algorithm computes the setup row $u[h][0][\cdot]$, the rows of the items of class h, and the row $v[h][\cdot]$. In case of ties the item is not selected and the class stays closed. This is the table after the execution of the DP algorithm:

Row	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$v[0][k]$	0	0	0	0	0	0	0	0	0
$u[1][0][k]$ (setup)	$-\infty$	-2	-2	-2	-2	-2	-2	-2	-2
$u[1][1][k]$ $(j=1)$	$-\infty$	-2	4	4	4	4	4	4	4
$u[1][2][k]$ $(j=2)$	$-\infty$	-2	7	13	13	13	13	13	13
$u[1][3][k]$ $(j=3)$	$-\infty$	-2	7	13	13	13	16	22	22
$v[1][k]$	0	0	7	13	13	13	16	22	22
$u[2][0][k]$ (setup)	$-\infty$	-1	-1	6	12	12	12	15	21
$u[2][1][k]$ $(j=4)$	$-\infty$	-1	-1	6	12	12	13	19	21
$u[2][2][k]$ $(j=5)$	$-\infty$	-1	-1	6	12	12	13	19	21
$v[2][k]$	0	0	7	13	13	13	16	22	22

   These are the computations of the cells used by the reconstruction, in the order in which it reads them:

   - $v[2][8] = \max\{v[1][8],\ u[2][2][8]\} = \max\{22,\ 21\} = 22$
   - $v[1][8] = \max\{v[0][8],\ u[1][3][8]\} = \max\{0,\ 22\} = 22$
   - $u[1][3][8] = \max\{u[1][2][8],\ p_3 + u[1][2][4]\} = \max\{13,\ 22\} = 22$
   - $u[1][2][4] = \max\{u[1][1][4],\ p_2 + u[1][1][3]\} = \max\{4,\ 13\} = 13$
   - $u[1][1][3] = \max\{u[1][0][3],\ p_1 + u[1][0][2]\} = \max\{-2,\ 4\} = 4$
   - $u[1][0][2] = v[0][1] - f_1 = 0 - 2 = -2$

2. These are the choice tables computed by the DP algorithm:

Row	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$s[1][1][k]$ $(j=1)$	0	0	1	1	1	1	1	1	1
$s[1][2][k]$ $(j=2)$	0	0	1	1	1	1	1	1	1
$s[1][3][k]$ $(j=3)$	0	0	0	0	0	0	1	1	1
$o[1][k]$	0	0	1	1	1	1	1	1	1
$s[2][1][k]$ $(j=4)$	0	0	0	0	0	0	1	1	0
$s[2][2][k]$ $(j=5)$	0	0	0	0	0	0	0	0	0
$o[2][k]$	0	0	0	0	0	0	0	0	0

Starting from $h = 2$ and $k = 8$:

Iteration	Stage	k	Choice	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	Action
–	–	–	–	0	0	0	0	0	$k = 8$
1	$h = 2$	8	$o[2][8] = 0$	0	0	0	0	0	class 2 closed
2	$h = 1$	8	$o[1][8] = 1$	0	0	0	0	0	class 1 open
3	$j = 3$	8	$s[1][3][8] = 1$	0	0	1	0	0	$k \leftarrow 8 - 4 = 4$
4	$j = 2$	4	$s[1][2][4] = 1$	0	1	1	0	0	$k \leftarrow 4 - 1 = 3$
5	$j = 1$	3	$s[1][1][3] = 1$	1	1	1	0	0	$k \leftarrow 3 - 1 = 2$
–	–	2	–	1	1	1	0	0	setup: $k \leftarrow 2 - s_1 = 1$

The vector $(1, 1, 1, 0, 0)$ is an optimal solution: only class 1 is open, and the items $\{1, 2, 3\}$ are selected. Its total weight is $1 + 1 + 4 + s_1 = 7 \leq c$, and the optimal value is $v[2][8] = 6 + 9 + 9 - f_1 = 22$.

Exercise 8.5 — Exercise 5: Knapsack Problem with Setups

Consider the following KPS instance, with a knapsack of capacity $c = 9$ and 5 items partitioned into $q = 2$ classes: $\mathscr{J}_1 = \{1, 2\}$ with $f_1 = 3$ and $s_1 = 2$, $\mathscr{J}_2 = \{3, 4, 5\}$ with $f_2 = 2$ and $s_2 = 2$. The revenues and weights are:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
p_j	6	5	8	6	5
w_j	3	1	1	4	3

Questions

1. Execute the bottom-up DP algorithm and write the table of all the rows $v[h][\cdot]$ and $u[h][i][\cdot]$, in the order of computation.
2. Write the choice tables s and o, reconstruct an optimal solution $(x_1, x_2, x_3, x_4, x_5)$, say which classes are open, and report the optimal value.

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. For each class h the DP algorithm computes the setup row $u[h][0][\cdot]$, the rows of the items of class h, and the row $v[h][\cdot]$. In case of ties the item is not selected and the class stays closed. This is the table after the execution of the DP algorithm:

Row	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$v[0][k]$	0	0	0	0	0	0	0	0	0	0
$u[1][0][k]$ (setup)	$-\infty$	$-\infty$	-3	-3	-3	-3	-3	-3	-3	-3
$u[1][1][k]$ $(j=1)$	$-\infty$	$-\infty$	-3	-3	-3	3	3	3	3	3
$u[1][2][k]$ $(j=2)$	$-\infty$	$-\infty$	-3	2	2	3	8	8	8	8
$v[1][k]$	0	0	0	2	2	3	8	8	8	8
$u[2][0][k]$ (setup)	$-\infty$	$-\infty$	-2	-2	-2	0	0	1	6	6
$u[2][1][k]$ $(j=3)$	$-\infty$	$-\infty$	-2	6	6	6	8	8	9	14
$u[2][2][k]$ $(j=4)$	$-\infty$	$-\infty$	-2	6	6	6	8	12	12	14
$u[2][3][k]$ $(j=5)$	$-\infty$	$-\infty$	-2	6	6	6	11	12	12	14
$v[2][k]$	0	0	0	6	6	6	11	12	12	14

These are the computations of the cells used by the reconstruction, in the order in which it reads them:

- $v[2][9] = \max\{v[1][9],\ u[2][3][9]\} = \max\{8,\ 14\} = 14$
- $u[2][3][9] = \max\{u[2][2][9],\ p_5 + u[2][2][6]\} = \max\{14,\ 13\} = 14$
- $u[2][2][9] = \max\{u[2][1][9],\ p_4 + u[2][1][5]\} = \max\{14,\ 12\} = 14$
- $u[2][1][9] = \max\{u[2][0][9],\ p_3 + u[2][0][8]\} = \max\{6,\ 14\} = 14$
- $u[2][0][8] = v[1][6] - f_2 = 8 - 2 = 6$
- $v[1][6] = \max\{v[0][6],\ u[1][2][6]\} = \max\{0,\ 8\} = 8$
- $u[1][2][6] = \max\{u[1][1][6],\ p_2 + u[1][1][5]\} = \max\{3,\ 8\} = 8$
- $u[1][1][5] = \max\{u[1][0][5],\ p_1 + u[1][0][2]\} = \max\{-3,\ 3\} = 3$
- $u[1][0][2] = v[0][0] - f_1 = 0 - 3 = -3$

2. These are the choice tables computed by the DP algorithm:

Row	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$s[1][1][k]$ $(j=1)$	0	0	0	0	0	1	1	1	1	1
$s[1][2][k]$ $(j=2)$	0	0	0	1	1	0	1	1	1	1
$o[1][k]$	0	0	0	1	1	1	1	1	1	1
$s[2][1][k]$ $(j=3)$	0	0	0	1	1	1	1	1	1	1
$s[2][2][k]$ $(j=4)$	0	0	0	0	0	0	0	1	1	0
$s[2][3][k]$ $(j=5)$	0	0	0	0	0	0	1	0	0	0
$o[2][k]$	0	0	0	1	1	1	1	1	1	1

Starting from $h = 2$ and $k = 9$:

Iteration	Stage	k	Choice	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	Action
–	–	–	–	0	0	0	0	0	$k = 9$
1	$h = 2$	9	$o[2][9] = 1$	0	0	0	0	0	class 2 open
2	$j = 5$	9	$s[2][3][9] = 0$	0	0	0	0	0	$k = 9$
3	$j = 4$	9	$s[2][2][9] = 0$	0	0	0	0	0	$k = 9$
4	$j = 3$	9	$s[2][1][9] = 1$	0	0	1	0	0	$k \leftarrow 9 - 1 = 8$
–	–	8	–	0	0	1	0	0	setup: $k \leftarrow 8 - s_2 = 6$
5	$h = 1$	6	$o[1][6] = 1$	0	0	1	0	0	class 1 open
6	$j = 2$	6	$s[1][2][6] = 1$	0	1	1	0	0	$k \leftarrow 6 - 1 = 5$
7	$j = 1$	5	$s[1][1][5] = 1$	1	1	1	0	0	$k \leftarrow 5 - 3 = 2$
–	–	2	–	1	1	1	0	0	setup: $k \leftarrow 2 - s_1 = 0$

The vector $(1, 1, 1, 0, 0)$ is an optimal solution: both classes are open, and the items $\{1, 2, 3\}$ are selected. Its total weight is $3 + 1 + 1 + s_1 + s_2 = 9 \leq c$, and the optimal value is $v[2][9] = 6 + 5 + 8 - f_1 - f_2 = 14$.

Chapter 9

DP for the multiple knapsack problem

Recurrence used in these exercises

The DP algorithm of the chapter «DP for the multiple knapsack problem», with m knapsacks, the vector $\boldsymbol{k} = (k_1, \ldots, k_m)$ of their residual capacities and the unit vectors $\boldsymbol{e}_i$:

$$v(j, \boldsymbol{k}) = \begin{cases} 0 & \text{if } j = 0, \\ v(j-1, \boldsymbol{k}) & \text{if } j \geq 1 \text{ and } w_j > k_i \text{ for all } i, \\ \max\left\{ v(j-1, \boldsymbol{k}), \right. & \text{if } j \geq 1 \text{ and} \\ \quad \left. p_j + \max\limits_{i:\, w_j \leq k_i} v(j-1, \boldsymbol{k} - w_j \boldsymbol{e}_i) \right\} & w_j \leq k_i \text{ for some } i. \end{cases}$$

The optimal value is $v(n, \boldsymbol{c})$. The tables s store $s[j][\boldsymbol{k}] = i$ if item j is inserted into knapsack i, and 0 if it is left out; an optimal solution is reconstructed from $j = n$ and $\boldsymbol{k} = \boldsymbol{c}$ with $a_j \leftarrow s[j][\boldsymbol{k}]$ and, if $a_j = i \geq 1$, $\boldsymbol{k} \leftarrow \boldsymbol{k} - w_j \boldsymbol{e}_i$.

Exercise 9.1 — Exercise 1: Multiple Knapsack Problem

Consider the following MKP instance, with $m = 2$ knapsacks of capacities $c_1 = 3$ and $c_2 = 2$, and 3 items with profits and weights:

	$j = 1$	$j = 2$	$j = 3$
p_j	8	7	6
w_j	2	3	3

Questions

1. Execute the bottom-up DP algorithm: for each item j, write the matrices $v[j][k_1][k_2]$ and $s[j][k_1][k_2]$, and show the computation of the cell $\boldsymbol{k} = \boldsymbol{c}$.
2. Using the tables s, reconstruct an optimal solution as the vector (a_1, a_2, a_3) of the knapsacks, and report the optimal value.
3. Solve by enumeration the KP with the same items and capacity $c_1 + c_2 = 5$, and compare its optimal value with the optimal value of the MKP.

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. The row $j = 0$ is $v[0][\boldsymbol{k}] = 0$ for every $\boldsymbol{k}$. In each matrix there is one row for each k_1 and one column for each k_2; in s, 0 means that item j is left out, and 1 or 2 the knapsack into which it is inserted. In case of ties the first option is kept: item j left out, then knapsack 1.

   Item $j = 1$, with $p_1 = 8$ and $w_1 = 2$: $v[1][(3,2)] = \max\{v[0][(3,2)],\ p_1 + v[0][(1,2)],\ p_1 + v[0][(3,0)]\} = \max\{0,\ 8 + 0,\ 8 + 0\} = 8$; $v[1][(0,2)] = \max\{v[0][(0,2)],\ p_1 + v[0][(0,0)]\} = \max\{0,\ 8 + 0\} = 8$.

$v[1][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$
$k_1 = 0$	0	0	8
$k_1 = 1$	0	0	8
$k_1 = 2$	8	8	8
$k_1 = 3$	8	8	8

$s[1][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$
$k_1 = 0$	0	0	2
$k_1 = 1$	0	0	2
$k_1 = 2$	1	1	1
$k_1 = 3$	1	1	1

   Item $j = 2$, with $p_2 = 7$ and $w_2 = 3$: $v[2][(3,2)] = \max\{v[1][(3,2)],\ p_2 + v[1][(0,2)]\} = \max\{8,\ 7 + 8\} = 15$.

$v[2][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$
$k_1 = 0$	0	0	8
$k_1 = 1$	0	0	8
$k_1 = 2$	8	8	8
$k_1 = 3$	8	8	15

$s[2][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$
$k_1 = 0$	0	0	0
$k_1 = 1$	0	0	0
$k_1 = 2$	0	0	0
$k_1 = 3$	0	0	1

   Item $j = 3$, with $p_3 = 6$ and $w_3 = 3$: $v[3][(3,2)] = \max\{v[2][(3,2)],\ p_3 + v[2][(0,2)]\} = \max\{15,\ 6 + 8\} = 15$.

$v[3][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$
$k_1 = 0$	0	0	8
$k_1 = 1$	0	0	8
$k_1 = 2$	8	8	8
$k_1 = 3$	8	8	15

$s[3][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$
$k_1=0$	0	0	0
$k_1=1$	0	0	0
$k_1=2$	0	0	0
$k_1=3$	0	0	0

2. Starting from $j=3$ and $\boldsymbol{k}=(3,2)$:

Iteration	j	$\boldsymbol{k}$	$s[j][\boldsymbol{k}]$	$a[1]$	$a[2]$	$a[3]$	Action
–	–	–	–	0	0	0	$\boldsymbol{k}=(3,2)$
1	3	$(3,2)$	0	0	0	0	$a[3]\leftarrow 0$, $\boldsymbol{k}=(3,2)$
2	2	$(3,2)$	1	0	1	0	$a[2]\leftarrow 1$, $\boldsymbol{k}\leftarrow(0,2)$
3	1	$(0,2)$	2	2	1	0	$a[1]\leftarrow 2$, $\boldsymbol{k}\leftarrow(0,0)$

   The vector $(2,1,0)$ is an optimal solution: knapsack 1 contains the items $\{2\}$, of total weight $3\le c_1$, and knapsack 2 the items $\{1\}$, of total weight $2\le c_2$. The optimal value is $v[3][(3,2)]=7+8=15$.

3. The KP with capacity 5 has optimal value 15, equal to the optimal value of the MKP: here the upper bound given by the total capacity is tight.

Exercise 9.2 — Exercise 2: Multiple Knapsack Problem

Consider the following MKP instance, with $m=2$ knapsacks of capacities $c_1=4$ and $c_2=3$, and 3 items with profits and weights:

	$j=1$	$j=2$	$j=3$
p_j	7	3	8
w_j	4	3	3

Questions

1. Execute the bottom-up DP algorithm: for each item j, write the matrices $v[j][k_1][k_2]$ and $s[j][k_1][k_2]$, and show the computation of the cell $\boldsymbol{k}=\boldsymbol{c}$.
2. Using the tables s, reconstruct an optimal solution as the vector (a_1,a_2,a_3) of the knapsacks, and report the optimal value.
3. Solve by enumeration the KP with the same items and capacity $c_1+c_2=7$, and compare its optimal value with the optimal value of the MKP.

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. The row $j=0$ is $v[0][\boldsymbol{k}]=0$ for every $\boldsymbol{k}$. In each matrix there is one row for each k_1 and one column for each k_2; in s, 0 means that item j is left out, and 1 or 2 the

knapsack into which it is inserted. In case of ties the first option is kept: item j left out, then knapsack 1.

Item $j = 1$, with $p_1 = 7$ and $w_1 = 4$: $v[1][(4,3)] = \max\{v[0][(4,3)],\ p_1 + v[0][(0,3)]\} = \max\{0,\ 7+0\} = 7$; $v[1][(4,0)] = \max\{v[0][(4,0)],\ p_1 + v[0][(0,0)]\} = \max\{0,\ 7+0\} = 7$.

$v[1][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$
$k_1 = 0$	0	0	0	0
$k_1 = 1$	0	0	0	0
$k_1 = 2$	0	0	0	0
$k_1 = 3$	0	0	0	0
$k_1 = 4$	7	7	7	7

$s[1][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$
$k_1 = 0$	0	0	0	0
$k_1 = 1$	0	0	0	0
$k_1 = 2$	0	0	0	0
$k_1 = 3$	0	0	0	0
$k_1 = 4$	1	1	1	1

Item $j = 2$, with $p_2 = 3$ and $w_2 = 3$: $v[2][(4,3)] = \max\{v[1][(4,3)],\ p_2 + v[1][(1,3)],\ p_2 + v[1][(4,0)]\} = \max\{7,\ 3+0,\ 3+7\} = 10$; $v[2][(4,0)] = \max\{v[1][(4,0)],\ p_2 + v[1][(1,0)]\} = \max\{7,\ 3+0\} = 7$.

$v[2][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$
$k_1 = 0$	0	0	0	3
$k_1 = 1$	0	0	0	3
$k_1 = 2$	0	0	0	3
$k_1 = 3$	3	3	3	3
$k_1 = 4$	7	7	7	10

$s[2][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$
$k_1 = 0$	0	0	0	2
$k_1 = 1$	0	0	0	2
$k_1 = 2$	0	0	0	2
$k_1 = 3$	1	1	1	1
$k_1 = 4$	0	0	0	2

Item $j = 3$, with $p_3 = 8$ and $w_3 = 3$: $v[3][(4,3)] = \max\{v[2][(4,3)],\ p_3 + v[2][(1,3)],\ p_3 + v[2][(4,0)]\} = \max\{10,\ 8+3,\ 8+7\} = 15$.

$v[3][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$
$k_1 = 0$	0	0	0	8
$k_1 = 1$	0	0	0	8
$k_1 = 2$	0	0	0	8
$k_1 = 3$	8	8	8	11
$k_1 = 4$	8	8	8	15

$s[3][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	0	2
$k_1=1$	0	0	0	2
$k_1=2$	0	0	0	2
$k_1=3$	1	1	1	1
$k_1=4$	1	1	1	2

2. Starting from $j=3$ and $\boldsymbol{k}=(4,3)$:

Iteration	j	$\boldsymbol{k}$	$s[j][\boldsymbol{k}]$	$a[1]$	$a[2]$	$a[3]$	Action
–	–	–	–	0	0	0	$\boldsymbol{k}=(4,3)$
1	3	$(4,3)$	2	0	0	2	$a[3]\leftarrow 2$, $\boldsymbol{k}\leftarrow(4,0)$
2	2	$(4,0)$	0	0	0	2	$a[2]\leftarrow 0$, $\boldsymbol{k}=(4,0)$
3	1	$(4,0)$	1	1	0	2	$a[1]\leftarrow 1$, $\boldsymbol{k}\leftarrow(0,0)$

   The vector $(1,0,2)$ is an optimal solution: knapsack 1 contains the items $\{1\}$, of total weight $4\le c_1$, and knapsack 2 the items $\{3\}$, of total weight $3\le c_2$. The optimal value is $v[3][(4,3)]=7+8=15$.

3. The KP with capacity 7 has optimal value 15, equal to the optimal value of the MKP: here the upper bound given by the total capacity is tight.

Exercise 9.3 — Exercise 3: Multiple Knapsack Problem

Consider the following MKP instance, with $m=2$ knapsacks of capacities $c_1=5$ and $c_2=2$, and 3 items with profits and weights:

	$j=1$	$j=2$	$j=3$
p_j	2	5	9
w_j	2	3	4

Questions

1. Execute the bottom-up DP algorithm: for each item j, write the matrices $v[j][k_1][k_2]$ and $s[j][k_1][k_2]$, and show the computation of the cell $\boldsymbol{k}=\boldsymbol{c}$.
2. Using the tables s, reconstruct an optimal solution as the vector (a_1,a_2,a_3) of the knapsacks, and report the optimal value.
3. Solve by enumeration the KP with the same items and capacity $c_1+c_2=7$, and compare its optimal value with the optimal value of the MKP.

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. The row $j=0$ is $v[0][\boldsymbol{k}]=0$ for every $\boldsymbol{k}$. In each matrix there is one row for each

k_1 and one column for each k_2; in s, 0 means that item j is left out, and 1 or 2 the knapsack into which it is inserted. In case of ties the first option is kept: item j left out, then knapsack 1.

Item $j = 1$, with $p_1 = 2$ and $w_1 = 2$: $v[1][(5,2)] = \max\{v[0][(5,2)], p_1 + v[0][(3,2)], p_1 + v[0][(5,0)]\} = \max\{0, 2 + 0, 2 + 0\} = 2$; $v[1][(1,2)] = \max\{v[0][(1,2)], p_1 + v[0][(1,0)]\} = \max\{0, 2 + 0\} = 2$.

$v[1][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$
$k_1 = 0$	0	0	2
$k_1 = 1$	0	0	2
$k_1 = 2$	2	2	2
$k_1 = 3$	2	2	2
$k_1 = 4$	2	2	2
$k_1 = 5$	2	2	2

$s[1][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$
$k_1 = 0$	0	0	2
$k_1 = 1$	0	0	2
$k_1 = 2$	1	1	1
$k_1 = 3$	1	1	1
$k_1 = 4$	1	1	1
$k_1 = 5$	1	1	1

Item $j = 2$, with $p_2 = 5$ and $w_2 = 3$: $v[2][(5,2)] = \max\{v[1][(5,2)], p_2 + v[1][(2,2)]\} = \max\{2, 5 + 2\} = 7$; $w_2 = 3$ fits in no knapsack, so $v[2][(1,2)] = v[1][(1,2)] = 2$.

$v[2][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$
$k_1 = 0$	0	0	2
$k_1 = 1$	0	0	2
$k_1 = 2$	2	2	2
$k_1 = 3$	5	5	7
$k_1 = 4$	5	5	7
$k_1 = 5$	7	7	7

$s[2][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$
$k_1 = 0$	0	0	0
$k_1 = 1$	0	0	0
$k_1 = 2$	0	0	0
$k_1 = 3$	1	1	1
$k_1 = 4$	1	1	1
$k_1 = 5$	1	1	1

Item $j = 3$, with $p_3 = 9$ and $w_3 = 4$: $v[3][(5,2)] = \max\{v[2][(5,2)], p_3 + v[2][(1,2)]\} = \max\{7, 9 + 2\} = 11$.

$v[3][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$
$k_1=0$	0	0	2
$k_1=1$	0	0	2
$k_1=2$	2	2	2
$k_1=3$	5	5	7
$k_1=4$	9	9	11
$k_1=5$	9	9	11

$s[3][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$
$k_1=0$	0	0	0
$k_1=1$	0	0	0
$k_1=2$	0	0	0
$k_1=3$	0	0	0
$k_1=4$	1	1	1
$k_1=5$	1	1	1

2. Starting from $j=3$ and $\boldsymbol{k}=(5,2)$:

Iteration	j	$\boldsymbol{k}$	$s[j][\boldsymbol{k}]$	$a[1]$	$a[2]$	$a[3]$	Action
–	–	–	–	0	0	0	$\boldsymbol{k}=(5,2)$
1	3	$(5,2)$	1	0	0	1	$a[3]\leftarrow 1$, $\boldsymbol{k}\leftarrow(1,2)$
2	2	$(1,2)$	0	0	0	1	$a[2]\leftarrow 0$, $\boldsymbol{k}=(1,2)$
3	1	$(1,2)$	2	2	0	1	$a[1]\leftarrow 2$, $\boldsymbol{k}\leftarrow(1,0)$

   The vector $(2,0,1)$ is an optimal solution: knapsack 1 contains the items $\{3\}$, of total weight $4\leq c_1$, and knapsack 2 the items $\{1\}$, of total weight $2\leq c_2$. The optimal value is $v[3][(5,2)]=9+2=11$.

3. The KP with capacity 7 has optimal value $14>11$, with the items $\{2,3\}$ of total weight $7\leq 7$. These items cannot be divided between the two knapsacks of capacities 5 and 2: the KP with the total capacity gives only an upper bound on the optimal value of the MKP.

Exercise 9.4 — Exercise 4: Multiple Knapsack Problem

Consider the following MKP instance, with $m=2$ knapsacks of capacities $c_1=4$ and $c_2=3$, and 4 items with profits and weights:

	$j=1$	$j=2$	$j=3$	$j=4$
p_j	5	2	6	6
w_j	2	1	2	4

Questions

1. Execute the bottom-up DP algorithm: for each item j, write the matrices $v[j][k_1][k_2]$ and $s[j][k_1][k_2]$, and show the computation of the cell $\boldsymbol{k}=\boldsymbol{c}$.

2. Using the tables s, reconstruct an optimal solution as the vector (a_1, a_2, a_3, a_4) of the knapsacks, and report the optimal value.

3. Solve by enumeration the KP with the same items and capacity $c_1 + c_2 = 7$, and compare its optimal value with the optimal value of the MKP.

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. The row $j = 0$ is $v[0][\boldsymbol{k}] = 0$ for every $\boldsymbol{k}$. In each matrix there is one row for each k_1 and one column for each k_2; in s, 0 means that item j is left out, and 1 or 2 the knapsack into which it is inserted. In case of ties the first option is kept: item j left out, then knapsack 1.

   Item $j = 1$, with $p_1 = 5$ and $w_1 = 2$: $v[1][(4,3)] = \max\{v[0][(4,3)], p_1 + v[0][(2,3)], p_1 + v[0][(4,1)]\} = \max\{0, 5+0, 5+0\} = 5$; $w_1 = 2$ fits in no knapsack, so $v[1][(0,0)] = v[0][(0,0)] = 0$.

$v[1][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	5	5
$k_1=1$	0	0	5	5
$k_1=2$	5	5	5	5
$k_1=3$	5	5	5	5
$k_1=4$	5	5	5	5

$s[1][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	2	2
$k_1=1$	0	0	2	2
$k_1=2$	1	1	1	1
$k_1=3$	1	1	1	1
$k_1=4$	1	1	1	1

   Item $j = 2$, with $p_2 = 2$ and $w_2 = 1$: $v[2][(4,3)] = \max\{v[1][(4,3)], p_2 + v[1][(3,3)], p_2 + v[1][(4,2)]\} = \max\{5, 2+5, 2+5\} = 7$; $v[2][(0,1)] = \max\{v[1][(0,1)], p_2 + v[1][(0,0)]\} = \max\{0, 2+0\} = 2$.

$v[2][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	2	5	7
$k_1=1$	2	2	7	7
$k_1=2$	5	7	7	7
$k_1=3$	7	7	7	7
$k_1=4$	7	7	7	7

$s[2][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	2	0	2
$k_1=1$	1	1	1	1
$k_1=2$	0	2	1	1
$k_1=3$	1	1	1	1
$k_1=4$	1	1	1	1

Item $j = 3$, with $p_3 = 6$ and $w_3 = 2$: $v[3][(4,3)] = \max\{v[2][(4,3)],\ p_3 + v[2][(2,3)],\ p_3 + v[2][(4,1)]\} = \max\{7,\ 6+7,\ 6+7\} = 13$; $v[3][(0,3)] = \max\{v[2][(0,3)],\ p_3 + v[2][(0,1)]\} = \max\{7,\ 6+2\} = 8$.

$v[3][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	2	6	8
$k_1=1$	2	2	8	8
$k_1=2$	6	8	11	13
$k_1=3$	8	8	13	13
$k_1=4$	11	13	13	13

$s[3][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	2	2
$k_1=1$	0	0	2	2
$k_1=2$	1	1	1	1
$k_1=3$	1	1	1	1
$k_1=4$	1	1	1	1

Item $j = 4$, with $p_4 = 6$ and $w_4 = 4$: $v[4][(4,3)] = \max\{v[3][(4,3)],\ p_4 + v[3][(0,3)]\} = \max\{13,\ 6+8\} = 14$.

$v[4][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	2	6	8
$k_1=1$	2	2	8	8
$k_1=2$	6	8	11	13
$k_1=3$	8	8	13	13
$k_1=4$	11	13	13	14

$s[4][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	0	0
$k_1=1$	0	0	0	0
$k_1=2$	0	0	0	0
$k_1=3$	0	0	0	0
$k_1=4$	0	0	0	1

2. Starting from $j = 4$ and $\boldsymbol{k} = (4,3)$:

Iteration	j	$\boldsymbol{k}$	$s[j][\boldsymbol{k}]$	$a[1]$	$a[2]$	$a[3]$	$a[4]$	Action
–	–	–	–	0	0	0	0	$\boldsymbol{k} = (4,3)$
1	4	$(4,3)$	1	0	0	0	1	$a[4] \leftarrow 1$, $\boldsymbol{k} \leftarrow (0,3)$
2	3	$(0,3)$	2	0	0	2	1	$a[3] \leftarrow 2$, $\boldsymbol{k} \leftarrow (0,1)$
3	2	$(0,1)$	2	0	2	2	1	$a[2] \leftarrow 2$, $\boldsymbol{k} \leftarrow (0,0)$
4	1	$(0,0)$	0	0	2	2	1	$a[1] \leftarrow 0$, $\boldsymbol{k} = (0,0)$

The vector $(0,2,2,1)$ is an optimal solution: knapsack 1 contains the items $\{4\}$, of total weight $4 \leq c_1$, and knapsack 2 the items $\{2,3\}$, of total weight $1+2 = 3 \leq c_2$. The optimal value is $v[4][(4,3)] = 6+2+6 = 14$.

3. The KP with capacity 7 has optimal value 14, equal to the optimal value of the MKP: here the upper bound given by the total capacity is tight.

Exercise 9.5 — Exercise 5: Multiple Knapsack Problem

Consider the following MKP instance, with $m = 2$ knapsacks of capacities $c_1 = 5$ and $c_2 = 3$, and 4 items with profits and weights:

	$j=1$	$j=2$	$j=3$	$j=4$
p_j	5	3	7	2
w_j	4	4	2	2

Questions

1. Execute the bottom-up DP algorithm: for each item j, write the matrices $v[j][k_1][k_2]$ and $s[j][k_1][k_2]$, and show the computation of the cell $\boldsymbol{k} = \boldsymbol{c}$.
2. Using the tables s, reconstruct an optimal solution as the vector (a_1, a_2, a_3, a_4) of the knapsacks, and report the optimal value.
3. Solve by enumeration the KP with the same items and capacity $c_1 + c_2 = 8$, and compare its optimal value with the optimal value of the MKP.

Solution

The recurrence and the conventions are those of the box at the beginning of these exercises.

1. The row $j = 0$ is $v[0][\boldsymbol{k}] = 0$ for every $\boldsymbol{k}$. In each matrix there is one row for each k_1 and one column for each k_2; in s, 0 means that item j is left out, and 1 or 2 the knapsack into which it is inserted. In case of ties the first option is kept: item j left out, then knapsack 1.

   Item $j = 1$, with $p_1 = 5$ and $w_1 = 4$: $v[1][(5,3)] = \max\{v[0][(5,3)],\ p_1 + v[0][(1,3)]\} = \max\{0,\ 5+0\} = 5$; $v[1][(5,1)] = \max\{v[0][(5,1)],\ p_1 + v[0][(1,1)]\} = \max\{0,\ 5+0\} = 5$.

$v[1][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	0	0
$k_1=1$	0	0	0	0
$k_1=2$	0	0	0	0
$k_1=3$	0	0	0	0
$k_1=4$	5	5	5	5
$k_1=5$	5	5	5	5

$s[1][k_1][k_2]$	$k_2=0$	$k_2=1$	$k_2=2$	$k_2=3$
$k_1=0$	0	0	0	0
$k_1=1$	0	0	0	0
$k_1=2$	0	0	0	0
$k_1=3$	0	0	0	0
$k_1=4$	1	1	1	1
$k_1=5$	1	1	1	1

Item $j = 2$, with $p_2 = 3$ and $w_2 = 4$: $v[2][(5,3)] = \max\{v[1][(5,3)],\ p_2 + v[1][(1,3)]\} = \max\{5,\ 3+0\} = 5$; $v[2][(5,1)] = \max\{v[1][(5,1)],\ p_2 + v[1][(1,1)]\} = \max\{5,\ 3+0\} = 5$.

$v[2][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$
$k_1 = 0$	0	0	0	0
$k_1 = 1$	0	0	0	0
$k_1 = 2$	0	0	0	0
$k_1 = 3$	0	0	0	0
$k_1 = 4$	5	5	5	5
$k_1 = 5$	5	5	5	5

$s[2][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$
$k_1 = 0$	0	0	0	0
$k_1 = 1$	0	0	0	0
$k_1 = 2$	0	0	0	0
$k_1 = 3$	0	0	0	0
$k_1 = 4$	0	0	0	0
$k_1 = 5$	0	0	0	0

Item $j = 3$, with $p_3 = 7$ and $w_3 = 2$: $v[3][(5,3)] = \max\{v[2][(5,3)],\ p_3 + v[2][(3,3)],\ p_3 + v[2][(5,1)]\} = \max\{5,\ 7+0,\ 7+5\} = 12$.

$v[3][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$
$k_1 = 0$	0	0	7	7
$k_1 = 1$	0	0	7	7
$k_1 = 2$	7	7	7	7
$k_1 = 3$	7	7	7	7
$k_1 = 4$	7	7	12	12
$k_1 = 5$	7	7	12	12

$s[3][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$
$k_1 = 0$	0	0	2	2
$k_1 = 1$	0	0	2	2
$k_1 = 2$	1	1	1	1
$k_1 = 3$	1	1	1	1
$k_1 = 4$	1	1	2	2
$k_1 = 5$	1	1	2	2

Item $j = 4$, with $p_4 = 2$ and $w_4 = 2$: $v[4][(5,3)] = \max\{v[3][(5,3)],\ p_4 + v[3][(3,3)],\ p_4 + v[3][(5,1)]\} = \max\{12,\ 2+7,\ 2+7\} = 12$.

$v[4][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$
$k_1 = 0$	0	0	7	7
$k_1 = 1$	0	0	7	7
$k_1 = 2$	7	7	9	9
$k_1 = 3$	7	7	9	9
$k_1 = 4$	9	9	12	12
$k_1 = 5$	9	9	12	12

$s[4][k_1][k_2]$	$k_2 = 0$	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$
$k_1 = 0$	0	0	0	0
$k_1 = 1$	0	0	0	0
$k_1 = 2$	0	0	1	1
$k_1 = 3$	0	0	1	1
$k_1 = 4$	1	1	0	0
$k_1 = 5$	1	1	0	0

2. Starting from $j = 4$ and $\boldsymbol{k} = (5, 3)$:

Iteration	j	$\boldsymbol{k}$	$s[j][\boldsymbol{k}]$	$a[1]$	$a[2]$	$a[3]$	$a[4]$	Action
–	–	–	–	0	0	0	0	$\boldsymbol{k} = (5, 3)$
1	4	$(5, 3)$	0	0	0	0	0	$a[4] \leftarrow 0$, $\boldsymbol{k} = (5, 3)$
2	3	$(5, 3)$	2	0	0	2	0	$a[3] \leftarrow 2$, $\boldsymbol{k} \leftarrow (5, 1)$
3	2	$(5, 1)$	0	0	0	2	0	$a[2] \leftarrow 0$, $\boldsymbol{k} = (5, 1)$
4	1	$(5, 1)$	1	1	0	2	0	$a[1] \leftarrow 1$, $\boldsymbol{k} \leftarrow (1, 1)$

   The vector $(1, 0, 2, 0)$ is an optimal solution: knapsack 1 contains the items $\{1\}$, of total weight $4 \leq c_1$, and knapsack 2 the items $\{3\}$, of total weight $2 \leq c_2$. The optimal value is $v[4][(5, 3)] = 5 + 7 = 12$.

3. The KP with capacity 8 has optimal value $14 > 12$, with the items $\{1, 3, 4\}$ of total weight $8 \leq 8$. These items cannot be divided between the two knapsacks of capacities 5 and 3: the KP with the total capacity gives only an upper bound on the optimal value of the MKP.

Lot sizing problems

Chapter 10

DP for the uncapacitated lot sizing problem

Exercise 10.1

Consider the following ULS instance with $m = 4$ periods, where the demands, the setup costs, the variable production costs, and the holding costs are:

t	1	2	3	4
d_t	30	10	40	10
sc_t	50	80	70	60
vc_t	2	3	1	2
hc_t	1	2	2	1

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each iteration, the values of the matrix c, of the vector v, and of the vector r.

2. Using the vector r, reconstruct an optimal solution $(x_1, x_2, \ldots, x_m)$, and report the optimal value.

Solution

1. Step-by-step execution of the bottom-up DP algorithm

   First, the matrix c is computed. For each step, the table shows the values of j and t,

the current value of u, the cell $c[j][t]$, and the operation performed; the cell $c[j][t]$ is highlighted in blue when assigned.

j	t	u	$c[j][t]$	Action
1	1	2	110	$u \leftarrow vc[1] = 2$, $g \leftarrow sc[1]+vc[1] \cdot d[1] = 50+2 \cdot 30 = 110$, $c[1][1] \leftarrow 110$
1	2	3	140	$u \leftarrow u+hc[1] = 2+1 = 3$, $g \leftarrow g+u \cdot d[2] = 110+3 \cdot 10 = 140$, $c[1][2] \leftarrow 140$
1	3	5	340	$u \leftarrow u+hc[2] = 3+2 = 5$, $g \leftarrow g+u \cdot d[3] = 140+5 \cdot 40 = 340$, $c[1][3] \leftarrow 340$
1	4	7	410	$u \leftarrow u+hc[3] = 5+2 = 7$, $g \leftarrow g+u \cdot d[4] = 340+7 \cdot 10 = 410$, $c[1][4] \leftarrow 410$
2	2	3	110	$u \leftarrow vc[2] = 3$, $g \leftarrow sc[2]+vc[2] \cdot d[2] = 80+3 \cdot 10 = 110$, $c[2][2] \leftarrow 110$
2	3	5	310	$u \leftarrow u+hc[2] = 3+2 = 5$, $g \leftarrow g+u \cdot d[3] = 110+5 \cdot 40 = 310$, $c[2][3] \leftarrow 310$
2	4	7	380	$u \leftarrow u+hc[3] = 5+2 = 7$, $g \leftarrow g+u \cdot d[4] = 310+7 \cdot 10 = 380$, $c[2][4] \leftarrow 380$
3	3	1	110	$u \leftarrow vc[3] = 1$, $g \leftarrow sc[3]+vc[3] \cdot d[3] = 70+1 \cdot 40 = 110$, $c[3][3] \leftarrow 110$
3	4	3	140	$u \leftarrow u+hc[3] = 1+2 = 3$, $g \leftarrow g+u \cdot d[4] = 110+3 \cdot 10 = 140$, $c[3][4] \leftarrow 140$
4	4	2	80	$u \leftarrow vc[4] = 2$, $g \leftarrow sc[4]+vc[4] \cdot d[4] = 60+2 \cdot 10 = 80$, $c[4][4] \leftarrow 80$

This is the matrix c:

$c[j][t]$	$t=1$	$t=2$	$t=3$	$t=4$
$j=1$	110	140	340	410
$j=2$	–	110	310	380
$j=3$	–	–	110	140
$j=4$	–	–	–	80

For each step of the DP algorithm, the table shows the values of t and j, the cells $v[t]$ and $r[t]$, the current value of b, and the current value of a. The cell of b is highlighted in red when updated; the cell of a is highlighted in blue when updated; the cells of $v[t]$ and $r[t]$ are highlighted in blue when assigned.

t	j	$v[t]$	$r[t]$	b	a	Action
–	–	0	–	–	–	$v[0] \leftarrow 0$
	–	–	–	$+\infty$	0	$b \leftarrow +\infty$, $a \leftarrow 0$
1	1	–	–	110	1	$v[0]+c[1][1] = 0+110 = 110 < +\infty$: $b \leftarrow 110$, $a \leftarrow 1$
	–	110	1	110	1	$v[1] \leftarrow b = 110$, $r[1] \leftarrow a = 1$
	–	–	–	$+\infty$	0	$b \leftarrow +\infty$, $a \leftarrow 0$
2	1	–	–	140	1	$v[0]+c[1][2] = 0+140 = 140 < +\infty$: $b \leftarrow 140$, $a \leftarrow 1$
	2	–	–	140	1	$v[1]+c[2][2] = 110+110 = 220 > 140$: no update
	–	140	1	140	1	$v[2] \leftarrow b = 140$, $r[2] \leftarrow a = 1$
	–	–	–	$+\infty$	0	$b \leftarrow +\infty$, $a \leftarrow 0$
3	1	–	–	340	1	$v[0]+c[1][3] = 0+340 = 340 < +\infty$: $b \leftarrow 340$, $a \leftarrow 1$
	2	–	–	340	1	$v[1]+c[2][3] = 110+310 = 420 > 340$: no update
	3	–	–	250	3	$v[2]+c[3][3] = 140+110 = 250 < 340$: $b \leftarrow 250$, $a \leftarrow 3$
	–	250	3	250	3	$v[3] \leftarrow b = 250$, $r[3] \leftarrow a = 3$
	–	–	–	$+\infty$	0	$b \leftarrow +\infty$, $a \leftarrow 0$
4	1	–	–	410	1	$v[0]+c[1][4] = 0+410 = 410 < +\infty$: $b \leftarrow 410$, $a \leftarrow 1$
	2	–	–	410	1	$v[1]+c[2][4] = 110+380 = 490 > 410$: no update
	3	–	–	280	3	$v[2]+c[3][4] = 140+140 = 280 < 410$: $b \leftarrow 280$, $a \leftarrow 3$
	4	–	–	280	3	$v[3]+c[4][4] = 250+80 = 330 > 280$: no update
	–	280	3	280	3	$v[4] \leftarrow b = 280$, $r[4] \leftarrow a = 3$

This is the vector v after the execution of the DP algorithm:

	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$
$v[t]$	0	110	140	250	280

This is the vector r after the execution of the DP algorithm:

	$t=1$	$t=2$	$t=3$	$t=4$
$r[t]$	1	1	3	3

2. We have the following vector r:

	$t=1$	$t=2$	$t=3$	$t=4$
$r[t]$	1	1	3	3

We can reconstruct an optimal solution as follows:

Iteration	t	$r[t]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$t \leftarrow r[t]-1$
–	–	–	0	0	0	0	$t=4$
1	4	3	0	0	50	0	$t=3-1=2$
2	2	1	40	0	50	0	$t=1-1=0$

- $t=4$, $r[4]=3$: the last production run of periods $1,\dots,4$ is in period 3 and covers the demands of periods $3,\dots,4$, so $x[3] \leftarrow d_3+d_4 = 40+10 = 50$
- $t=2$, $r[2]=1$: the last production run of periods $1,\dots,2$ is in period 1 and covers the demands of periods $1,\dots,2$, so $x[1] \leftarrow d_1+d_2 = 30+10 = 40$

Thus, the vector

$$(40,0,50,0)$$

is an optimal solution. The setups are in periods 1, 3, the inventories are $(s_1,\dots,s_4) = (10,0,10,0)$, and the total cost is

$$\underbrace{50+70}_{\text{setup}} + \underbrace{2\cdot 40+3\cdot 0+1\cdot 50+2\cdot 0}_{\text{production}} + \underbrace{1\cdot 10+2\cdot 0+2\cdot 10+1\cdot 0}_{\text{holding}} = 120+130+30 = 280.$$

The optimal value is

$$v[4] = 280.$$

Exercise 10.2

Consider the following ULS instance with $m = 4$ periods, where the demands, the setup costs, the variable production costs, and the holding costs are:

t	1	2	3	4
d_t	30	40	10	20
sc_t	50	80	80	50
vc_t	1	3	2	1
hc_t	1	1	1	2

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each iteration, the values of the matrix c, of the vector v, and of the vector r.

2. Using the vector r, reconstruct an optimal solution $(x_1, x_2, \ldots, x_m)$, and report the optimal value.

Solution

1. Step-by-step execution of the bottom-up DP algorithm

   First, the matrix c is computed. For each step, the table shows the values of j and t, the current value of u, the cell $c[j][t]$, and the operation performed; the cell $c[j][t]$ is highlighted in blue when assigned.

j	t	u	$c[j][t]$	Action
1	1	1	80	$u \leftarrow vc[1] = 1$, $g \leftarrow sc[1]+vc[1] \cdot d[1] = 50+1 \cdot 30 = 80$, $c[1][1] \leftarrow 80$
1	2	2	160	$u \leftarrow u+hc[1] = 1+1 = 2$, $g \leftarrow g+u \cdot d[2] = 80+2 \cdot 40 = 160$, $c[1][2] \leftarrow 160$
1	3	3	190	$u \leftarrow u+hc[2] = 2+1 = 3$, $g \leftarrow g+u \cdot d[3] = 160+3 \cdot 10 = 190$, $c[1][3] \leftarrow 190$
1	4	4	270	$u \leftarrow u+hc[3] = 3+1 = 4$, $g \leftarrow g+u \cdot d[4] = 190+4 \cdot 20 = 270$, $c[1][4] \leftarrow 270$

2	2	3	200	$u \leftarrow vc[2] = 3$, $g \leftarrow sc[2]+vc[2] \cdot d[2] = 80+3 \cdot 40 = 200$, $c[2][2] \leftarrow 200$
2	3	4	240	$u \leftarrow u+hc[2] = 3+1 = 4$, $g \leftarrow g+u \cdot d[3] = 200+4 \cdot 10 = 240$, $c[2][3] \leftarrow 240$
2	4	5	340	$u \leftarrow u+hc[3] = 4+1 = 5$, $g \leftarrow g+u \cdot d[4] = 240+5 \cdot 20 = 340$, $c[2][4] \leftarrow 340$
3	3	2	100	$u \leftarrow vc[3] = 2$, $g \leftarrow sc[3]+vc[3] \cdot d[3] = 80+2 \cdot 10 = 100$, $c[3][3] \leftarrow 100$
3	4	3	160	$u \leftarrow u+hc[3] = 2+1 = 3$, $g \leftarrow g+u \cdot d[4] = 100+3 \cdot 20 = 160$, $c[3][4] \leftarrow 160$
4	4	1	70	$u \leftarrow vc[4] = 1$, $g \leftarrow sc[4]+vc[4] \cdot d[4] = 50+1 \cdot 20 = 70$, $c[4][4] \leftarrow 70$

This is the matrix c:

$c[j][t]$	$t=1$	$t=2$	$t=3$	$t=4$
$j=1$	80	160	190	270
$j=2$	–	200	240	340
$j=3$	–	–	100	160
$j=4$	–	–	–	70

For each step of the DP algorithm, the table shows the values of t and j, the cells $v[t]$ and $r[t]$, the current value of b, and the current value of a. The cell of b is highlighted in red when updated; the cell of a is highlighted in blue when updated; the cells of $v[t]$ and $r[t]$ are highlighted in blue when assigned.

t	j	$v[t]$	$r[t]$	b	a	Action
–	–	0	–	–	–	$v[0] \leftarrow 0$
	–	–	–	$+\infty$	0	$b \leftarrow +\infty,\ a \leftarrow 0$
1	1	–	–	80	1	$v[0]+c[1][1] = 0+80 = 80 < +\infty$: $b \leftarrow 80,\ a \leftarrow 1$
	–	80	1	80	1	$v[1] \leftarrow b = 80,\ r[1] \leftarrow a = 1$
	–	–	–	$+\infty$	0	$b \leftarrow +\infty,\ a \leftarrow 0$
2	1	–	–	160	1	$v[0]+c[1][2] = 0+160 = 160 < +\infty$: $b \leftarrow 160,\ a \leftarrow 1$
	2	–	–	160	1	$v[1]+c[2][2] = 80+200 = 280 > 160$: no update
	–	160	1	160	1	$v[2] \leftarrow b = 160,\ r[2] \leftarrow a = 1$
	–	–	–	$+\infty$	0	$b \leftarrow +\infty,\ a \leftarrow 0$
3	1	–	–	190	1	$v[0]+c[1][3] = 0+190 = 190 < +\infty$: $b \leftarrow 190,\ a \leftarrow 1$
	2	–	–	190	1	$v[1]+c[2][3] = 80+240 = 320 > 190$: no update
	3	–	–	190	1	$v[2]+c[3][3] = 160+100 = 260 > 190$: no update
	–	190	1	190	1	$v[3] \leftarrow b = 190,\ r[3] \leftarrow a = 1$
	–	–	–	$+\infty$	0	$b \leftarrow +\infty,\ a \leftarrow 0$
4	1	–	–	270	1	$v[0]+c[1][4] = 0+270 = 270 < +\infty$: $b \leftarrow 270,\ a \leftarrow 1$
	2	–	–	270	1	$v[1]+c[2][4] = 80+340 = 420 > 270$: no update
	3	–	–	270	1	$v[2]+c[3][4] = 160+160 = 320 > 270$: no update
	4	–	–	260	4	$v[3]+c[4][4] = 190+70 = 260 < 270$: $b \leftarrow 260,\ a \leftarrow 4$
	–	260	4	260	4	$v[4] \leftarrow b = 260,\ r[4] \leftarrow a = 4$

This is the vector v after the execution of the DP algorithm:

	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$
$v[t]$	0	80	160	190	260

This is the vector r after the execution of the DP algorithm:

	$t=1$	$t=2$	$t=3$	$t=4$
$r[t]$	1	1	1	4

2. We have the following vector r:

	$t=1$	$t=2$	$t=3$	$t=4$
$r[t]$	1	1	1	4

We can reconstruct an optimal solution as follows:

Iteration	t	$r[t]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$t \leftarrow r[t]-1$
–	–	–	0	0	0	0	$t=4$
1	4	4	0	0	0	20	$t=4-1=3$
2	3	1	80	0	0	20	$t=1-1=0$

- $t=4$, $r[4]=4$: period 4 produces only its own demand, so $x[4] \leftarrow d_4 = 20$
- $t=3$, $r[3]=1$: the last production run of periods $1,\dots,3$ is in period 1 and covers the demands of periods $1,\dots,3$, so $x[1] \leftarrow d_1+d_2+d_3 = 30+40+10 = 80$

Thus, the vector

$$(80,0,0,20)$$

is an optimal solution. The setups are in periods 1, 4, the inventories are $(s_1,\dots,s_4) = (50,10,0,0)$, and the total cost is

$$\underbrace{50+50}_{\text{setup}} + \underbrace{1\cdot 80 + 3\cdot 0 + 2\cdot 0 + 1\cdot 20}_{\text{production}} + \underbrace{1\cdot 50 + 1\cdot 10 + 1\cdot 0 + 2\cdot 0}_{\text{holding}} = 100+100+60 = 260.$$

The optimal value is

$$v[4] = 260.$$

Exercise 10.3

Consider the following ULS instance with $m = 5$ periods, where the demands, the setup costs, the variable production costs, and the holding costs are:

t	1	2	3	4	5
d_t	10	30	40	30	40
sc_t	70	80	50	80	50
vc_t	1	1	1	1	2
hc_t	1	1	2	1	2

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each iteration, the values of the matrix c, of the vector v, and of the vector r.

2. Using the vector r, reconstruct an optimal solution $(x_1, x_2, \ldots, x_m)$, and report the optimal value.

Solution

1. Step-by-step execution of the bottom-up DP algorithm

   First, the matrix c is computed. For each step, the table shows the values of j and t, the current value of u, the cell $c[j][t]$, and the operation performed; the cell $c[j][t]$ is highlighted in blue when assigned.

j	t	u	$c[j][t]$	Action
1	1	1	80	$u \leftarrow vc[1] = 1$, $g \leftarrow sc[1]+vc[1] \cdot d[1] = 70+1 \cdot 10 = 80$, $c[1][1] \leftarrow 80$
1	2	2	140	$u \leftarrow u+hc[1] = 1+1 = 2$, $g \leftarrow g+u \cdot d[2] = 80+2 \cdot 30 = 140$, $c[1][2] \leftarrow 140$
1	3	3	260	$u \leftarrow u+hc[2] = 2+1 = 3$, $g \leftarrow g+u \cdot d[3] = 140+3 \cdot 40 = 260$, $c[1][3] \leftarrow 260$
1	4	5	410	$u \leftarrow u+hc[3] = 3+2 = 5$, $g \leftarrow g+u \cdot d[4] = 260+5 \cdot 30 = 410$, $c[1][4] \leftarrow 410$

1	5	6	650	$u \leftarrow u+hc[4] = 5+1 = 6$, $g \leftarrow g+u \cdot d[5] = 410+6 \cdot 40 = 650$, $c[1][5] \leftarrow 650$
2	2	1	110	$u \leftarrow vc[2] = 1$, $g \leftarrow sc[2]+vc[2] \cdot d[2] = 80+1 \cdot 30 = 110$, $c[2][2] \leftarrow 110$
2	3	2	190	$u \leftarrow u+hc[2] = 1+1 = 2$, $g \leftarrow g+u \cdot d[3] = 110+2 \cdot 40 = 190$, $c[2][3] \leftarrow 190$
2	4	4	310	$u \leftarrow u+hc[3] = 2+2 = 4$, $g \leftarrow g+u \cdot d[4] = 190+4 \cdot 30 = 310$, $c[2][4] \leftarrow 310$
2	5	5	510	$u \leftarrow u+hc[4] = 4+1 = 5$, $g \leftarrow g+u \cdot d[5] = 310+5 \cdot 40 = 510$, $c[2][5] \leftarrow 510$
3	3	1	90	$u \leftarrow vc[3] = 1$, $g \leftarrow sc[3]+vc[3] \cdot d[3] = 50+1 \cdot 40 = 90$, $c[3][3] \leftarrow 90$
3	4	3	180	$u \leftarrow u+hc[3] = 1+2 = 3$, $g \leftarrow g+u \cdot d[4] = 90+3 \cdot 30 = 180$, $c[3][4] \leftarrow 180$
3	5	4	340	$u \leftarrow u+hc[4] = 3+1 = 4$, $g \leftarrow g+u \cdot d[5] = 180+4 \cdot 40 = 340$, $c[3][5] \leftarrow 340$
4	4	1	110	$u \leftarrow vc[4] = 1$, $g \leftarrow sc[4]+vc[4] \cdot d[4] = 80+1 \cdot 30 = 110$, $c[4][4] \leftarrow 110$
4	5	2	190	$u \leftarrow u+hc[4] = 1+1 = 2$, $g \leftarrow g+u \cdot d[5] = 110+2 \cdot 40 = 190$, $c[4][5] \leftarrow 190$
5	5	2	130	$u \leftarrow vc[5] = 2$, $g \leftarrow sc[5]+vc[5] \cdot d[5] = 50+2 \cdot 40 = 130$, $c[5][5] \leftarrow 130$

This is the matrix c:

$c[j][t]$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
$j=1$	80	140	260	410	650
$j=2$	–	110	190	310	510
$j=3$	–	–	90	180	340
$j=4$	–	–	–	110	190
$j=5$	–	–	–	–	130

For each step of the DP algorithm, the table shows the values of t and j, the cells $v[t]$ and $r[t]$, the current value of b, and the current value of a. The cell of b is highlighted in red when updated; the cell of a is highlighted in blue when updated; the cells of $v[t]$ and $r[t]$ are highlighted in blue when assigned.

t	j	$v[t]$	$r[t]$	b	a	Action
–	–	0	–	–	–	$v[0] \leftarrow 0$
	–	–	–	$+\infty$	0	$b \leftarrow +\infty$, $a \leftarrow 0$
1	1	–	–	80	1	$v[0]+c[1][1] = 0+80 = 80 < +\infty$: $b \leftarrow 80$, $a \leftarrow 1$
	–	80	1	80	1	$v[1] \leftarrow b = 80$, $r[1] \leftarrow a = 1$
	–	–	–	$+\infty$	0	$b \leftarrow +\infty$, $a \leftarrow 0$
2	1	–	–	140	1	$v[0]+c[1][2] = 0+140 = 140 < +\infty$: $b \leftarrow 140$, $a \leftarrow 1$
	2	–	–	140	1	$v[1]+c[2][2] = 80+110 = 190 > 140$: no update
	–	140	1	140	1	$v[2] \leftarrow b = 140$, $r[2] \leftarrow a = 1$
	–	–	–	$+\infty$	0	$b \leftarrow +\infty$, $a \leftarrow 0$
3	1	–	–	260	1	$v[0]+c[1][3] = 0+260 = 260 < +\infty$: $b \leftarrow 260$, $a \leftarrow 1$
	2	–	–	260	1	$v[1]+c[2][3] = 80+190 = 270 > 260$: no update
	3	–	–	230	3	$v[2]+c[3][3] = 140+90 = 230 < 260$: $b \leftarrow 230$, $a \leftarrow 3$
	–	230	3	230	3	$v[3] \leftarrow b = 230$, $r[3] \leftarrow a = 3$
	–	–	–	$+\infty$	0	$b \leftarrow +\infty$, $a \leftarrow 0$
4	1	–	–	410	1	$v[0]+c[1][4] = 0+410 = 410 < +\infty$: $b \leftarrow 410$, $a \leftarrow 1$
	2	–	–	390	2	$v[1]+c[2][4] = 80+310 = 390 < 410$: $b \leftarrow 390$, $a \leftarrow 2$
	3	–	–	320	3	$v[2]+c[3][4] = 140+180 = 320 < 390$: $b \leftarrow 320$, $a \leftarrow 3$
	4	–	–	320	3	$v[3]+c[4][4] = 230+110 = 340 > 320$: no update
	–	320	3	320	3	$v[4] \leftarrow b = 320$, $r[4] \leftarrow a = 3$
	–	–	–	$+\infty$	0	$b \leftarrow +\infty$, $a \leftarrow 0$
5	1	–	–	650	1	$v[0]+c[1][5] = 0+650 = 650 < +\infty$: $b \leftarrow 650$, $a \leftarrow 1$
	2	–	–	590	2	$v[1]+c[2][5] = 80+510 = 590 < 650$: $b \leftarrow 590$, $a \leftarrow 2$
	3	–	–	480	3	$v[2]+c[3][5] = 140+340 = 480 < 590$: $b \leftarrow 480$, $a \leftarrow 3$
	4	–	–	420	4	$v[3]+c[4][5] = 230+190 = 420 < 480$: $b \leftarrow 420$, $a \leftarrow 4$
	5	–	–	420	4	$v[4]+c[5][5] = 320+130 = 450 > 420$: no update
	–	420	4	420	4	$v[5] \leftarrow b = 420$, $r[5] \leftarrow a = 4$

This is the vector v after the execution of the DP algorithm:

	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
$v[t]$	0	80	140	230	320	420

This is the vector r after the execution of the DP algorithm:

	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
$r[t]$	1	1	3	3	4

2. We have the following vector r:

	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
$r[t]$	1	1	3	3	4

We can reconstruct an optimal solution as follows:

Iteration	t	$r[t]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	$t \leftarrow r[t]-1$
–	–	–	0	0	0	0	0	$t=5$
1	5	4	0	0	0	70	0	$t=4-1=3$
2	3	3	0	0	40	70	0	$t=3-1=2$
3	2	1	40	0	40	70	0	$t=1-1=0$

- $t=5$, $r[5]=4$: the last production run of periods $1,\dots,5$ is in period 4 and covers the demands of periods $4,\dots,5$, so $x[4] \leftarrow d_4+d_5=30+40=70$
- $t=3$, $r[3]=3$: period 3 produces only its own demand, so $x[3] \leftarrow d_3=40$
- $t=2$, $r[2]=1$: the last production run of periods $1,\dots,2$ is in period 1 and covers the demands of periods $1,\dots,2$, so $x[1] \leftarrow d_1+d_2=10+30=40$

Thus, the vector

$$(40,0,40,70,0)$$

is an optimal solution. The setups are in periods 1, 3, 4, the inventories are $(s_1,\dots,s_5)=(30,0,0,40,0)$, and the total cost is

$$\underbrace{70+50+80}_{\text{setup}}+\underbrace{1\cdot 40+1\cdot 0+1\cdot 40+1\cdot 70+2\cdot 0}_{\text{production}}$$
$$+\underbrace{1\cdot 30+1\cdot 0+2\cdot 0+1\cdot 40+2\cdot 0}_{\text{holding}}=200+150+70=420.$$

The optimal value is

$$v[5]=420.$$

Chapter 11

DP for the capacitated lot sizing problem

Exercise 11.1

Consider the following CLS instance with $m = 3$ periods, where the demands, the setup costs, the variable production costs, the holding costs, and the capacities are:

t	1	2	3
d_t	1	2	3
sc_t	10	5	8
vc_t	1	1	2
hc_t	1	1	2
cap_t	2	4	4

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and r.

2. Using the matrix r, reconstruct an optimal solution $(x_1, x_2, \ldots, x_m)$, and report the optimal value.

Solution

1. Step-by-step execution of the bottom-up DP algorithm

For each step of the DP algorithm, the table shows the values of t and k, the cells $v[t][k]$ and $r[t][k]$, and the operation performed. The cells are computed for increasing values of t and, for each fixed value of t, for increasing values of $k \in \{0, 1, \ldots, D_{t+1,m}\}$. We recall that $f_t(0) = 0$ and $f_t(x) = sc_t + vc_t\, x$ for $x \geq 1$.

t	k	$v[t][k]$	$r[t][k]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $r[0][0] \leftarrow 0$
0	1	$+\infty$	0	$v[0][1] \leftarrow +\infty$, $r[0][1] \leftarrow 0$
0	2	$+\infty$	0	$v[0][2] \leftarrow +\infty$, $r[0][2] \leftarrow 0$
0	3	$+\infty$	0	$v[0][3] \leftarrow +\infty$, $r[0][3] \leftarrow 0$
0	4	$+\infty$	0	$v[0][4] \leftarrow +\infty$, $r[0][4] \leftarrow 0$
0	5	$+\infty$	0	$v[0][5] \leftarrow +\infty$, $r[0][5] \leftarrow 0$
0	6	$+\infty$	0	$v[0][6] \leftarrow +\infty$, $r[0][6] \leftarrow 0$
1	0	11	1	$\min\{v[0][1]+f_1(0), v[0][0]+f_1(1)\}+hc_1 \cdot 0 = \min\{+\infty, 0+11\}+1 \cdot 0 = 11$; first minimum at $x = 1$
1	1	13	2	$\min\{v[0][2]+f_1(0), v[0][1]+f_1(1), v[0][0]+f_1(2)\}+hc_1 \cdot 1 = \min\{+\infty, +\infty, 0+12\}+1 \cdot 1 = 13$; first minimum at $x = 2$
1	2	$+\infty$	0	the candidates use the cells $v[0][1], \ldots, v[0][3]$, which are all $+\infty$: $v[1][2] \leftarrow +\infty$
1	3	$+\infty$	0	the candidates use the cells $v[0][2], \ldots, v[0][4]$, which are all $+\infty$: $v[1][3] \leftarrow +\infty$
1	4	$+\infty$	0	the candidates use the cells $v[0][3], \ldots, v[0][5]$, which are all $+\infty$: $v[1][4] \leftarrow +\infty$
1	5	$+\infty$	0	the candidates use the cells $v[0][4], \ldots, v[0][6]$, which are all $+\infty$: $v[1][5] \leftarrow +\infty$
2	0	18	2	$\min\{v[1][2]+f_2(0), v[1][1]+f_2(1), v[1][0]+f_2(2)\}+hc_2 \cdot 0 = \min\{+\infty, 13+6, 11+7\}+1 \cdot 0 = 18$; first minimum at $x = 2$
2	1	20	3	$\min\{v[1][3]+f_2(0), v[1][2]+f_2(1), v[1][1]+f_2(2), v[1][0]+f_2(3)\}+hc_2 \cdot 1 = \min\{+\infty, +\infty, 13+7, 11+8\}+1 \cdot 1 = 20$; first minimum at $x = 3$
2	2	22	4	$\min\{v[1][4]+f_2(0), v[1][3]+f_2(1), v[1][2]+f_2(2), v[1][1]+f_2(3), v[1][0]+f_2(4)\}+hc_2 \cdot 2 = \min\{+\infty, +\infty, +\infty, 13+8, 11+9\}+1 \cdot 2 = 22$; first minimum at $x = 4$
2	3	25	4	$\min\{v[1][5]+f_2(0), v[1][4]+f_2(1), v[1][3]+f_2(2), v[1][2]+f_2(3), v[1][1]+f_2(4)\}+hc_2 \cdot 3 = \min\{+\infty, +\infty, +\infty, +\infty, 13+9\}+1 \cdot 3 = 25$; first minimum at $x = 4$
3	0	25	0	$\min\{v[2][3]+f_3(0), v[2][2]+f_3(1), v[2][1]+f_3(2), v[2][0]+f_3(3)\}+hc_3 \cdot 0 = \min\{25+0, 22+10, 20+12, 18+14\}+2 \cdot 0 = 25$; first minimum at $x = 0$

These are the matrices v and r obtained during the execution of the DP algorithm (a dot marks a pair (t, k) with $k > D_{t+1,m}$, which is not a state):

$v[t][k]$	0	1	2	3	4	5	6
$t = 0$	0	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
$t = 1$	11	13	$+\infty$	$+\infty$	$+\infty$	$+\infty$	·
$t = 2$	18	20	22	25	·	·	·
$t = 3$	25	·	·	·	·	·	·

$r[t][k]$	0	1	2	3	4	5	6
$t = 0$	0	0	0	0	0	0	0
$t = 1$	1	2	0	0	0	0	·
$t = 2$	2	3	4	4	·	·	·
$t = 3$	0	·	·	·	·	·	·

The optimal value of the CLS instance is $v[3][0] = 25$.

2. Using the matrix r, we can reconstruct an optimal solution as follows:

Iteration	t	k	$r[t][k]$	$x[1]$	$x[2]$	$x[3]$	$k \leftarrow k + d[t] - x[t]$
–	–	–	–	0	0	0	$t = 3,\ k = 0$
1	3	0	0	0	0	0	$k = 0 + 3 - 0 = 3$
2	2	3	4	0	4	0	$k = 3 + 2 - 4 = 1$
3	1	1	2	2	4	0	$k = 1 + 1 - 2 = 0$

Thus, the vector

$$(2, 4, 0)$$

is an optimal solution. The setups are in periods 1, 2, the inventories are $(s_1, \ldots, s_3) = (1, 3, 0)$, and the total cost is

$$\underbrace{10 + 5}_{\text{setup}} + \underbrace{1 \cdot 2 + 1 \cdot 4 + 2 \cdot 0}_{\text{production}} + \underbrace{1 \cdot 1 + 1 \cdot 3 + 2 \cdot 0}_{\text{holding}} = 15 + 6 + 4 = 25.$$

The optimal value is

$$v[3][0] = 25.$$

Exercise 11.2

Consider the following CLS instance with $m = 4$ periods, where the demands, the setup costs, the variable production costs, the holding costs, and the capacities are:

t	1	2	3	4
d_t	3	1	1	2
sc_t	5	6	10	10
vc_t	2	1	1	1
hc_t	1	1	2	2
cap_t	4	3	3	2

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and r.

2. Using the matrix r, reconstruct an optimal solution $(x_1, x_2, \ldots, x_m)$, and report the optimal value.

Solution

1. Step-by-step execution of the bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the values of t and k, the cells $v[t][k]$ and $r[t][k]$, and the operation performed. The cells are computed for increasing values of t and, for each fixed value of t, for increasing values of $k \in \{0, 1, \ldots, D_{t+1,m}\}$. We recall that $f_t(0) = 0$ and $f_t(x) = sc_t + vc_t\, x$ for $x \geq 1$.

t	k	$v[t][k]$	$r[t][k]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $r[0][0] \leftarrow 0$
0	1	$+\infty$	0	$v[0][1] \leftarrow +\infty$, $r[0][1] \leftarrow 0$
0	2	$+\infty$	0	$v[0][2] \leftarrow +\infty$, $r[0][2] \leftarrow 0$
0	3	$+\infty$	0	$v[0][3] \leftarrow +\infty$, $r[0][3] \leftarrow 0$
0	4	$+\infty$	0	$v[0][4] \leftarrow +\infty$, $r[0][4] \leftarrow 0$
0	5	$+\infty$	0	$v[0][5] \leftarrow +\infty$, $r[0][5] \leftarrow 0$

0	6	$+\infty$	0	$v[0][6] \leftarrow +\infty$, $r[0][6] \leftarrow 0$
0	7	$+\infty$	0	$v[0][7] \leftarrow +\infty$, $r[0][7] \leftarrow 0$
1	0	11	3	$\min\{v[0][3]+f_1(0), v[0][2]+f_1(1), v[0][1]+f_1(2), v[0][0]+f_1(3)\}+hc_1 \cdot 0 = \min\{+\infty, +\infty, +\infty, 0+11\}+1 \cdot 0 = 11$; first minimum at $x = 3$
1	1	14	4	$\min\{v[0][4]+f_1(0), v[0][3]+f_1(1), v[0][2]+f_1(2), v[0][1]+f_1(3), v[0][0]+f_1(4)\}+hc_1 \cdot 1 = \min\{+\infty, +\infty, +\infty, +\infty, 0+13\}+1 \cdot 1 = 14$; first minimum at $x = 4$
1	2	$+\infty$	0	the candidates use the cells $v[0][1], \ldots, v[0][5]$, which are all $+\infty$: $v[1][2] \leftarrow +\infty$
1	3	$+\infty$	0	the candidates use the cells $v[0][2], \ldots, v[0][6]$, which are all $+\infty$: $v[1][3] \leftarrow +\infty$
1	4	$+\infty$	0	the candidates use the cells $v[0][3], \ldots, v[0][7]$, which are all $+\infty$: $v[1][4] \leftarrow +\infty$
2	0	14	0	$\min\{v[1][1]+f_2(0), v[1][0]+f_2(1)\}+hc_2 \cdot 0 = \min\{14+0, 11+7\}+1 \cdot 0 = 14$; first minimum at $x = 0$
2	1	20	2	$\min\{v[1][2]+f_2(0), v[1][1]+f_2(1), v[1][0]+f_2(2)\}+hc_2 \cdot 1 = \min\{+\infty, 14+7, 11+8\}+1 \cdot 1 = 20$; first minimum at $x = 2$
2	2	22	3	$\min\{v[1][3]+f_2(0), v[1][2]+f_2(1), v[1][1]+f_2(2), v[1][0]+f_2(3)\}+hc_2 \cdot 2 = \min\{+\infty, +\infty, 14+8, 11+9\}+1 \cdot 2 = 22$; first minimum at $x = 3$
2	3	26	3	$\min\{v[1][4]+f_2(0), v[1][3]+f_2(1), v[1][2]+f_2(2), v[1][1]+f_2(3)\}+hc_2 \cdot 3 = \min\{+\infty, +\infty, +\infty, 14+9\}+1 \cdot 3 = 26$; first minimum at $x = 3$
3	0	20	0	$\min\{v[2][1]+f_3(0), v[2][0]+f_3(1)\}+hc_3 \cdot 0 = \min\{20+0, 14+11\}+2 \cdot 0 = 20$; first minimum at $x = 0$
3	1	24	0	$\min\{v[2][2]+f_3(0), v[2][1]+f_3(1), v[2][0]+f_3(2)\}+hc_3 \cdot 1 = \min\{22+0, 20+11, 14+12\}+2 \cdot 1 = 24$; first minimum at $x = 0$
3	2	30	0	$\min\{v[2][3]+f_3(0), v[2][2]+f_3(1), v[2][1]+f_3(2), v[2][0]+f_3(3)\}+hc_3 \cdot 2 = \min\{26+0, 22+11, 20+12, 14+13\}+2 \cdot 2 = 30$; first minimum at $x = 0$
4	0	30	0	$\min\{v[3][2]+f_4(0), v[3][1]+f_4(1), v[3][0]+f_4(2)\}+hc_4 \cdot 0 = \min\{30+0, 24+11, 20+12\}+2 \cdot 0 = 30$; first minimum at $x = 0$

These are the matrices v and r obtained during the execution of the DP algorithm (a dot marks a pair (t,k) with $k > D_{t+1,m}$, which is not a state):

$v[t][k]$	0	1	2	3	4	5	6	7
$t=0$	0	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
$t=1$	11	14	$+\infty$	$+\infty$	$+\infty$	·	·	·
$t=2$	14	20	22	26	·	·	·	·
$t=3$	20	24	30	·	·	·	·	·
$t=4$	30	·	·	·	·	·	·	·

$r[t][k]$	0	1	2	3	4	5	6	7
$t=0$	0	0	0	0	0	0	0	0
$t=1$	3	4	0	0	0	·	·	·
$t=2$	0	2	3	3	·	·	·	·
$t=3$	0	0	0	·	·	·	·	·
$t=4$	0	·	·	·	·	·	·	·

The optimal value of the CLS instance is $v[4][0] = 30$.

2. Using the matrix r, we can reconstruct an optimal solution as follows:

Iteration	t	k	$r[t][k]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$k \leftarrow k + d[t] - x[t]$
–	–	–	–	0	0	0	0	$t = 4,\ k = 0$
1	4	0	0	0	0	0	0	$k = 0 + 2 - 0 = 2$
2	3	2	0	0	0	0	0	$k = 2 + 1 - 0 = 3$
3	2	3	3	0	3	0	0	$k = 3 + 1 - 3 = 1$
4	1	1	4	4	3	0	0	$k = 1 + 3 - 4 = 0$

Thus, the vector

$$(4, 3, 0, 0)$$

is an optimal solution. The setups are in periods 1, 2, the inventories are $(s_1, \ldots, s_4) = (1, 3, 2, 0)$, and the total cost is

$$\underbrace{5+6}_{\text{setup}} + \underbrace{2\cdot 4 + 1\cdot 3 + 1\cdot 0 + 1\cdot 0}_{\text{production}}$$
$$+ \underbrace{1\cdot 1 + 1\cdot 3 + 2\cdot 2 + 2\cdot 0}_{\text{holding}} = 11 + 11 + 8 = 30.$$

The optimal value is

$$v[4][0] = 30.$$

Miscellaneous problems

DP for the coin change problem

Exercise 12.1

Consider the following CCP instance, where the amount of change to return is $c = 6$, and the denominations are

$$w = (1, 4).$$

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each iteration, the values of the array v and the array s.

2. Using the array s, reconstruct an optimal solution in integer notation (x_1, x_2), and report the optimal value.

Solution

1. Step-by-step execution of the bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the amount of change k, the cells $v[k]$ and $s[k]$, and the operation performed; $s[k]$ is the index j of the denomination of the coin used. In case of ties the smallest index is kept, because m is updated only when the value is strictly smaller.

k	$v[k]$	$s[k]$	Action
0	0	0	$v[0] \leftarrow 0$, $s[0] \leftarrow 0$

1	1	1	$\min\{1 + v[0]\} = \min\{1\} = 1$
2	2	1	$\min\{1 + v[1]\} = \min\{2\} = 2$
3	3	1	$\min\{1 + v[2]\} = \min\{3\} = 3$
4	1	2	$\min\{1 + v[3], 1 + v[0]\} = \min\{4, 1\} = 1$
5	2	1	$\min\{1 + v[4], 1 + v[1]\} = \min\{2, 2\} = 2$
6	3	1	$\min\{1 + v[5], 1 + v[2]\} = \min\{3, 3\} = 3$

This is the array v after the execution of the DP algorithm:

k	0	1	2	3	4	5	6
$v[k]$	0	1	2	3	1	2	3

This is the array s after the execution of the DP algorithm:

k	0	1	2	3	4	5	6
$s[k]$	0	1	1	1	2	1	1

2. We have the following array s:

k	0	1	2	3	4	5	6
$s[k]$	0	1	1	1	2	1	1

We can reconstruct an optimal solution by starting from $k = 6$ as follows:

Iteration	k	$s[k]$	$x[1]$	$x[2]$	$k \leftarrow k - w_{s[k]}$
–	–	–	0	0	$k = 6$
1	6	1	1	0	$k = 6 - 1 = 5$
2	5	1	2	0	$k = 5 - 1 = 4$
3	4	2	2	1	$k = 4 - 4 = 0$

Thus, the vector

$$(2, 1)$$

is an optimal solution: its total value is $2 \cdot 1 + 1 \cdot 4 = 6 = c$. The optimal value is

$$v[6] = 3.$$

Exercise 12.2

Consider the following CCP instance, where the amount of change to return is $c = 11$, and

the denominations are

$$w = (3, 5).$$

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each iteration, the values of the array v and the array s.

2. Using the array s, reconstruct an optimal solution in integer notation (x_1, x_2), and report the optimal value.

Solution

1. Step-by-step execution of the bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the amount of change k, the cells $v[k]$ and $s[k]$, and the operation performed; $s[k]$ is the index j of the denomination of the coin used. In case of ties the smallest index is kept, because m is updated only when the value is strictly smaller. The value $+\infty$ denotes an amount of change that cannot be returned, for which $s[k] = 0$.

k	$v[k]$	$s[k]$	Action
0	0	0	$v[0] \leftarrow 0$, $s[0] \leftarrow 0$
1	$+\infty$	0	no denomination $w_j \leq 1$: $v[1] \leftarrow +\infty$
2	$+\infty$	0	no denomination $w_j \leq 2$: $v[2] \leftarrow +\infty$
3	1	1	$\min\{1 + v[0]\} = \min\{1\} = 1$
4	$+\infty$	0	$\min\{1 + v[1]\} = \min\{+\infty\} = +\infty$
5	1	2	$\min\{1 + v[2], 1 + v[0]\} = \min\{+\infty, 1\} = 1$
6	2	1	$\min\{1 + v[3], 1 + v[1]\} = \min\{2, +\infty\} = 2$
7	$+\infty$	0	$\min\{1 + v[4], 1 + v[2]\} = \min\{+\infty, +\infty\} = +\infty$
8	2	1	$\min\{1 + v[5], 1 + v[3]\} = \min\{2, 2\} = 2$
9	3	1	$\min\{1 + v[6], 1 + v[4]\} = \min\{3, +\infty\} = 3$
10	2	2	$\min\{1 + v[7], 1 + v[5]\} = \min\{+\infty, 2\} = 2$
11	3	1	$\min\{1 + v[8], 1 + v[6]\} = \min\{3, 3\} = 3$

This is the array v after the execution of the DP algorithm:

k	0	1	2	3	4	5	6	7	8	9	10	11
$v[k]$	0	$+\infty$	$+\infty$	1	$+\infty$	1	2	$+\infty$	2	3	2	3

This is the array s after the execution of the DP algorithm:

k	0	1	2	3	4	5	6	7	8	9	10	11
$s[k]$	0	0	0	1	0	2	1	0	1	1	2	1

2. We have the following array s:

k	0	1	2	3	4	5	6	7	8	9	10	11
$s[k]$	0	0	0	1	0	2	1	0	1	1	2	1

We can reconstruct an optimal solution by starting from $k = 11$ as follows:

Iteration	k	$s[k]$	$x[1]$	$x[2]$	$k \leftarrow k - w_{s[k]}$
–	–	–	0	0	$k = 11$
1	11	1	1	0	$k = 11 - 3 = 8$
2	8	1	2	0	$k = 8 - 3 = 5$
3	5	2	2	1	$k = 5 - 5 = 0$

Thus, the vector

$$(2, 1)$$

is an optimal solution: its total value is $2 \cdot 3 + 1 \cdot 5 = 11 = c$. The optimal value is

$$v[11] = 3.$$

Exercise 12.3

Consider the following CCP instance, where the amount of change to return is $c = 7$, and

the denominations are

$$w = (2, 4).$$

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each iteration, the values of the array v and the array s.

2. Using the array s, reconstruct an optimal solution in integer notation (x_1, x_2), or explain why the instance is infeasible, and report the optimal value.

Solution

1. Step-by-step execution of the bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the amount of change k, the cells $v[k]$ and $s[k]$, and the operation performed; $s[k]$ is the index j of the denomination of the coin used. In case of ties the smallest index is kept, because m is updated only when the value is strictly smaller. The value $+\infty$ denotes an amount of change that cannot be returned, for which $s[k] = 0$.

k	$v[k]$	$s[k]$	Action
0	0	0	$v[0] \leftarrow 0$, $s[0] \leftarrow 0$
1	$+\infty$	0	no denomination $w_j \leq 1$: $v[1] \leftarrow +\infty$
2	1	1	$\min\{1 + v[0]\} = \min\{1\} = 1$
3	$+\infty$	0	$\min\{1 + v[1]\} = \min\{+\infty\} = +\infty$
4	1	2	$\min\{1 + v[2], 1 + v[0]\} = \min\{2, 1\} = 1$
5	$+\infty$	0	$\min\{1 + v[3], 1 + v[1]\} = \min\{+\infty, +\infty\} = +\infty$
6	2	1	$\min\{1 + v[4], 1 + v[2]\} = \min\{2, 2\} = 2$
7	$+\infty$	0	$\min\{1 + v[5], 1 + v[3]\} = \min\{+\infty, +\infty\} = +\infty$

This is the array v after the execution of the DP algorithm:

k	0	1	2	3	4	5	6	7
$v[k]$	0	$+\infty$	1	$+\infty$	1	$+\infty$	2	$+\infty$

This is the array s after the execution of the DP algorithm:

k	0	1	2	3	4	5	6	7
$s[k]$	0	0	1	0	2	0	1	0

2. Since $v[7] = +\infty$, the instance is **infeasible**: no combination of coins of denominations $w = (2, 4)$ returns an amount of change exactly equal to $c = 7$. Indeed, every denomination is even, so every amount of change that can be returned is even, while $c = 7$ is odd.

   Since $s[7] = 0$, there is no solution to reconstruct. The optimal value is

   $$v[7] = +\infty.$$

Exercise 12.4

Consider the following CCP instance, where the amount of change to return is $c = 12$, and the denominations are

$$w = (3, 7).$$

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each iteration, the values of the array v and the array s.

2. Using the array s, reconstruct an optimal solution in integer notation (x_1, x_2), and report the optimal value.

Solution

1. Step-by-step execution of the bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the amount of change k, the cells $v[k]$ and $s[k]$, and the operation performed; $s[k]$ is the index j of the denomination of the coin used. In case of ties the smallest index is kept, because m is updated only when the value is strictly smaller. The value $+\infty$ denotes an amount of change that cannot be returned, for which $s[k] = 0$.

k	$v[k]$	$s[k]$	Action
0	0	0	$v[0] \leftarrow 0$, $s[0] \leftarrow 0$
1	$+\infty$	0	no denomination $w_j \leq 1$: $v[1] \leftarrow +\infty$
2	$+\infty$	0	no denomination $w_j \leq 2$: $v[2] \leftarrow +\infty$
3	1	1	$\min\{1 + v[0]\} = \min\{1\} = 1$
4	$+\infty$	0	$\min\{1 + v[1]\} = \min\{+\infty\} = +\infty$
5	$+\infty$	0	$\min\{1 + v[2]\} = \min\{+\infty\} = +\infty$
6	2	1	$\min\{1 + v[3]\} = \min\{2\} = 2$
7	1	2	$\min\{1 + v[4], 1 + v[0]\} = \min\{+\infty, 1\} = 1$
8	$+\infty$	0	$\min\{1 + v[5], 1 + v[1]\} = \min\{+\infty, +\infty\} = +\infty$
9	3	1	$\min\{1 + v[6], 1 + v[2]\} = \min\{3, +\infty\} = 3$
10	2	1	$\min\{1 + v[7], 1 + v[3]\} = \min\{2, 2\} = 2$
11	$+\infty$	0	$\min\{1 + v[8], 1 + v[4]\} = \min\{+\infty, +\infty\} = +\infty$
12	4	1	$\min\{1 + v[9], 1 + v[5]\} = \min\{4, +\infty\} = 4$

This is the array v after the execution of the DP algorithm:

k	0	1	2	3	4	5	6	7	8	9	10	11	12
$v[k]$	0	$+\infty$	$+\infty$	1	$+\infty$	$+\infty$	2	1	$+\infty$	3	2	$+\infty$	4

This is the array s after the execution of the DP algorithm:

k	0	1	2	3	4	5	6	7	8	9	10	11	12
$s[k]$	0	0	0	1	0	0	1	2	0	1	1	0	1

2. We have the following array s:

k	0	1	2	3	4	5	6	7	8	9	10	11	12
$s[k]$	0	0	0	1	0	0	1	2	0	1	1	0	1

We can reconstruct an optimal solution by starting from $k = 12$ as follows:

Iteration	k	$s[k]$	$x[1]$	$x[2]$	$k \leftarrow k - w_{s[k]}$
–	–	–	0	0	$k = 12$
1	12	1	1	0	$k = 12 - 3 = 9$
2	9	1	2	0	$k = 9 - 3 = 6$
3	6	1	3	0	$k = 6 - 3 = 3$
4	3	1	4	0	$k = 3 - 3 = 0$

Thus, the vector

$$(4, 0)$$

is an optimal solution: its total value is $4 \cdot 3 = 12 = c$. The optimal value is

$$v[12] = 4.$$

Exercise 12.5

Consider the following CCP instance, where the amount of change to return is $c = 9$, and

the denominations are

$$w = (2, 3, 7).$$

Questions

1. Show the execution of the bottom-up DP algorithm by reporting, for each iteration, the values of the array v and the array s.

2. Using the array s, reconstruct an optimal solution in integer notation (x_1, x_2, x_3), and report the optimal value.

Solution

1. Step-by-step execution of the bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the amount of change k, the cells $v[k]$ and $s[k]$, and the operation performed; $s[k]$ is the index j of the denomination of the coin used. In case of ties the smallest index is kept, because m is updated only when the value is strictly smaller. The value $+\infty$ denotes an amount of change that cannot be returned, for which $s[k] = 0$.

k	$v[k]$	$s[k]$	Action
0	0	0	$v[0] \leftarrow 0$, $s[0] \leftarrow 0$
1	$+\infty$	0	no denomination $w_j \leq 1$: $v[1] \leftarrow +\infty$
2	1	1	$\min\{1 + v[0]\} = \min\{1\} = 1$
3	1	2	$\min\{1 + v[1], 1 + v[0]\} = \min\{+\infty, 1\} = 1$
4	2	1	$\min\{1 + v[2], 1 + v[1]\} = \min\{2, +\infty\} = 2$
5	2	1	$\min\{1 + v[3], 1 + v[2]\} = \min\{2, 2\} = 2$
6	2	2	$\min\{1 + v[4], 1 + v[3]\} = \min\{3, 2\} = 2$
7	1	3	$\min\{1 + v[5], 1 + v[4], 1 + v[0]\} = \min\{3, 3, 1\} = 1$
8	3	1	$\min\{1 + v[6], 1 + v[5], 1 + v[1]\} = \min\{3, 3, +\infty\} = 3$
9	2	1	$\min\{1 + v[7], 1 + v[6], 1 + v[2]\} = \min\{2, 3, 2\} = 2$

This is the array v after the execution of the DP algorithm:

k	0	1	2	3	4	5	6	7	8	9
$v[k]$	0	$+\infty$	1	1	2	2	2	1	3	2

This is the array s after the execution of the DP algorithm:

k	0	1	2	3	4	5	6	7	8	9
$s[k]$	0	0	1	2	1	1	2	3	1	1

2. We have the following array s:

k	0	1	2	3	4	5	6	7	8	9
$s[k]$	0	0	1	2	1	1	2	3	1	1

We can reconstruct an optimal solution by starting from $k = 9$ as follows:

Iteration	k	$s[k]$	$x[1]$	$x[2]$	$x[3]$	$k \leftarrow k - w_{s[k]}$
–	–	–	0	0	0	$k = 9$
1	9	1	1	0	0	$k = 9 - 2 = 7$
2	7	3	1	0	1	$k = 7 - 7 = 0$

Thus, the vector

$$(1, 0, 1)$$

is an optimal solution: its total value is $1 \cdot 2 + 1 \cdot 7 = 9 = c$. The optimal value is

$$v[9] = 2.$$

Chapter 13

DP for the rod cutting problem

Exercise 13.1

Consider the following RCP instance, where the rod has length $n = 6$, and the profits associated with the possible piece lengths are:

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
p_j	2	5	7	8	10	13

Questions

1. Show the execution of the first bottom-up DP algorithm by reporting, for each iteration, the values of the vector v and the values of the vector s.

2. Using the vector s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

3. Show the execution of the second bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and s.

4. Using the matrix s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

Solution

1. Step-by-step execution of the first bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the values of ℓ and j, the cells $v[\ell]$ and $s[\ell]$, the current value of m, and the current value of t. The cell of m is highlighted in red when updated; the cell of t is highlighted in blue when updated; the cells of $v[\ell]$ and $s[\ell]$ are highlighted in blue when assigned.

ℓ	j	$v[\ell]$	$s[\ell]$	m	t	Action
–	–	0	–	–	–	$v[0] \leftarrow 0$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
1	1	–	–	2	1	$p[1] + v[0] = 2 + 0 = 2 > 0$: $m \leftarrow 2,\ t \leftarrow 1$
	–	2	1	2	1	$v[1] \leftarrow m = 2,\ s[1] \leftarrow t = 1$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
2	1	–	–	4	1	$p[1] + v[1] = 2 + 2 = 4 > 0$: $m \leftarrow 4,\ t \leftarrow 1$
	2	–	–	5	2	$p[2] + v[0] = 5 + 0 = 5 > 4$: $m \leftarrow 5,\ t \leftarrow 2$
	–	5	2	5	2	$v[2] \leftarrow m = 5,\ s[2] \leftarrow t = 2$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
3	1	–	–	7	1	$p[1] + v[2] = 2 + 5 = 7 > 0$: $m \leftarrow 7,\ t \leftarrow 1$
	2	–	–	7	1	$p[2] + v[1] = 5 + 2 = 7 = m$: no update
	3	–	–	7	1	$p[3] + v[0] = 7 + 0 = 7 = m$: no update
	–	7	1	7	1	$v[3] \leftarrow m = 7,\ s[3] \leftarrow t = 1$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
4	1	–	–	9	1	$p[1] + v[3] = 2 + 7 = 9 > 0$: $m \leftarrow 9,\ t \leftarrow 1$
	2	–	–	10	2	$p[2] + v[2] = 5 + 5 = 10 > 9$: $m \leftarrow 10,\ t \leftarrow 2$
	3	–	–	10	2	$p[3] + v[1] = 7 + 2 = 9 < 10$: no update
	4	–	–	10	2	$p[4] + v[0] = 8 + 0 = 8 < 10$: no update
	–	10	2	10	2	$v[4] \leftarrow m = 10,\ s[4] \leftarrow t = 2$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
5	1	–	–	12	1	$p[1] + v[4] = 2 + 10 = 12 > 0$: $m \leftarrow 12,\ t \leftarrow 1$
	2	–	–	12	1	$p[2] + v[3] = 5 + 7 = 12 = m$: no update
	3	–	–	12	1	$p[3] + v[2] = 7 + 5 = 12 = m$: no update
	4	–	–	12	1	$p[4] + v[1] = 8 + 2 = 10 < 12$: no update
	5	–	–	12	1	$p[5] + v[0] = 10 + 0 = 10 < 12$: no update
	–	12	1	12	1	$v[5] \leftarrow m = 12,\ s[5] \leftarrow t = 1$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
6	1	–	–	14	1	$p[1] + v[5] = 2 + 12 = 14 > 0$: $m \leftarrow 14,\ t \leftarrow 1$
	2	–	–	15	2	$p[2] + v[4] = 5 + 10 = 15 > 14$: $m \leftarrow 15,\ t \leftarrow 2$
	3	–	–	15	2	$p[3] + v[3] = 7 + 7 = 14 < 15$: no update
	4	–	–	15	2	$p[4] + v[2] = 8 + 5 = 13 < 15$: no update
	5	–	–	15	2	$p[5] + v[1] = 10 + 2 = 12 < 15$: no update
	6	–	–	15	2	$p[6] + v[0] = 13 + 0 = 13 < 15$: no update
	–	15	2	15	2	$v[6] \leftarrow m = 15,\ s[6] \leftarrow t = 2$

This is the vector v after the execution of the DP algorithm:

	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$	$\ell=5$	$\ell=6$
$v[\ell]$	0	2	5	7	10	12	15

This is the vector s after the execution of the DP algorithm:

	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$	$\ell=5$	$\ell=6$
$s[\ell]$	1	2	1	2	1	2

2. We have the following vector s:

	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$	$\ell=5$	$\ell=6$
$s[\ell]$	1	2	1	2	1	2

We can reconstruct an optimal solution for a rod of length $n=6$ as follows:

Iteration	j	$s[j]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	$x[6]$	Update of j
-	-	-	0	0	0	0	0	0	$j=6$
1	6	2	0	1	0	0	0	0	$j=6-2=4$
2	4	2	0	2	0	0	0	0	$j=4-2=2$
3	2	2	0	3	0	0	0	0	$j=2-2=0$

Thus, the vector

$$(0,3,0,0,0,0)$$

is an optimal solution. The optimal value is

$$v[6]=15.$$

3. Step-by-step execution of the second bottom-up DP algorithm

For each step of the DP algorithm, the table shows the values of j and ℓ, the cells $v[j][\ell]$ and $s[j][\ell]$, and the operation performed. The cells are computed for increasing values of j and, for each fixed value of j, for increasing values of ℓ.

j	ℓ	$v[j][\ell]$	$s[j][\ell]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $s[0][0] \leftarrow 0$
0	1	$-\infty$	0	$v[0][1] \leftarrow -\infty$, $s[0][1] \leftarrow 0$
0	2	$-\infty$	0	$v[0][2] \leftarrow -\infty$, $s[0][2] \leftarrow 0$
0	3	$-\infty$	0	$v[0][3] \leftarrow -\infty$, $s[0][3] \leftarrow 0$
0	4	$-\infty$	0	$v[0][4] \leftarrow -\infty$, $s[0][4] \leftarrow 0$
0	5	$-\infty$	0	$v[0][5] \leftarrow -\infty$, $s[0][5] \leftarrow 0$
0	6	$-\infty$	0	$v[0][6] \leftarrow -\infty$, $s[0][6] \leftarrow 0$
1	0	0	0	$j > \ell$: $v[1][0] \leftarrow v[0][0] = 0$, $s[1][0] \leftarrow 0$
1	1	2	1	$p[1] + v[1][0] = 2 + 0 > v[0][1] = -\infty$: $v[1][1] \leftarrow 2$, $s[1][1] \leftarrow 1$
1	2	4	1	$p[1] + v[1][1] = 2 + 2 > v[0][2] = -\infty$: $v[1][2] \leftarrow 4$, $s[1][2] \leftarrow 1$
1	3	6	1	$p[1] + v[1][2] = 2 + 4 > v[0][3] = -\infty$: $v[1][3] \leftarrow 6$, $s[1][3] \leftarrow 1$
1	4	8	1	$p[1] + v[1][3] = 2 + 6 > v[0][4] = -\infty$: $v[1][4] \leftarrow 8$, $s[1][4] \leftarrow 1$
1	5	10	1	$p[1] + v[1][4] = 2 + 8 > v[0][5] = -\infty$: $v[1][5] \leftarrow 10$, $s[1][5] \leftarrow 1$
1	6	12	1	$p[1] + v[1][5] = 2 + 10 > v[0][6] = -\infty$: $v[1][6] \leftarrow 12$, $s[1][6] \leftarrow 1$
2	0	0	0	$j > \ell$: $v[2][0] \leftarrow v[1][0] = 0$, $s[2][0] \leftarrow 0$
2	1	2	0	$j > \ell$: $v[2][1] \leftarrow v[1][1] = 2$, $s[2][1] \leftarrow 0$
2	2	5	1	$p[2] + v[2][0] = 5 + 0 > v[1][2] = 4$: $v[2][2] \leftarrow 5$, $s[2][2] \leftarrow 1$
2	3	7	1	$p[2] + v[2][1] = 5 + 2 > v[1][3] = 6$: $v[2][3] \leftarrow 7$, $s[2][3] \leftarrow 1$
2	4	10	1	$p[2] + v[2][2] = 5 + 5 > v[1][4] = 8$: $v[2][4] \leftarrow 10$, $s[2][4] \leftarrow 1$
2	5	12	1	$p[2] + v[2][3] = 5 + 7 > v[1][5] = 10$: $v[2][5] \leftarrow 12$, $s[2][5] \leftarrow 1$
2	6	15	1	$p[2] + v[2][4] = 5 + 10 > v[1][6] = 12$: $v[2][6] \leftarrow 15$, $s[2][6] \leftarrow 1$
3	0	0	0	$j > \ell$: $v[3][0] \leftarrow v[2][0] = 0$, $s[3][0] \leftarrow 0$
3	1	2	0	$j > \ell$: $v[3][1] \leftarrow v[2][1] = 2$, $s[3][1] \leftarrow 0$
3	2	5	0	$j > \ell$: $v[3][2] \leftarrow v[2][2] = 5$, $s[3][2] \leftarrow 0$
3	3	7	0	$p[3] + v[3][0] = 7 + 0 \leq v[2][3] = 7$: $v[3][3] \leftarrow v[2][3] = 7$, $s[3][3] \leftarrow 0$
3	4	10	0	$p[3] + v[3][1] = 7 + 2 \leq v[2][4] = 10$: $v[3][4] \leftarrow v[2][4] = 10$, $s[3][4] \leftarrow 0$
3	5	12	0	$p[3] + v[3][2] = 7 + 5 \leq v[2][5] = 12$: $v[3][5] \leftarrow v[2][5] = 12$, $s[3][5] \leftarrow 0$
3	6	15	0	$p[3] + v[3][3] = 7 + 7 \leq v[2][6] = 15$: $v[3][6] \leftarrow v[2][6] = 15$, $s[3][6] \leftarrow 0$
4	0	0	0	$j > \ell$: $v[4][0] \leftarrow v[3][0] = 0$, $s[4][0] \leftarrow 0$
4	1	2	0	$j > \ell$: $v[4][1] \leftarrow v[3][1] = 2$, $s[4][1] \leftarrow 0$
4	2	5	0	$j > \ell$: $v[4][2] \leftarrow v[3][2] = 5$, $s[4][2] \leftarrow 0$
4	3	7	0	$j > \ell$: $v[4][3] \leftarrow v[3][3] = 7$, $s[4][3] \leftarrow 0$
4	4	10	0	$p[4] + v[4][0] = 8 + 0 \leq v[3][4] = 10$: $v[4][4] \leftarrow v[3][4] = 10$, $s[4][4] \leftarrow 0$
4	5	12	0	$p[4] + v[4][1] = 8 + 2 \leq v[3][5] = 12$: $v[4][5] \leftarrow v[3][5] = 12$, $s[4][5] \leftarrow 0$
4	6	15	0	$p[4] + v[4][2] = 8 + 5 \leq v[3][6] = 15$: $v[4][6] \leftarrow v[3][6] = 15$, $s[4][6] \leftarrow 0$
5	0	0	0	$j > \ell$: $v[5][0] \leftarrow v[4][0] = 0$, $s[5][0] \leftarrow 0$
5	1	2	0	$j > \ell$: $v[5][1] \leftarrow v[4][1] = 2$, $s[5][1] \leftarrow 0$
5	2	5	0	$j > \ell$: $v[5][2] \leftarrow v[4][2] = 5$, $s[5][2] \leftarrow 0$
5	3	7	0	$j > \ell$: $v[5][3] \leftarrow v[4][3] = 7$, $s[5][3] \leftarrow 0$
5	4	10	0	$j > \ell$: $v[5][4] \leftarrow v[4][4] = 10$, $s[5][4] \leftarrow 0$
5	5	12	0	$p[5] + v[5][0] = 10 + 0 \leq v[4][5] = 12$: $v[5][5] \leftarrow v[4][5] = 12$, $s[5][5] \leftarrow 0$
5	6	15	0	$p[5] + v[5][1] = 10 + 2 \leq v[4][6] = 15$: $v[5][6] \leftarrow v[4][6] = 15$, $s[5][6] \leftarrow 0$
6	0	0	0	$j > \ell$: $v[6][0] \leftarrow v[5][0] = 0$, $s[6][0] \leftarrow 0$
6	1	2	0	$j > \ell$: $v[6][1] \leftarrow v[5][1] = 2$, $s[6][1] \leftarrow 0$
6	2	5	0	$j > \ell$: $v[6][2] \leftarrow v[5][2] = 5$, $s[6][2] \leftarrow 0$
6	3	7	0	$j > \ell$: $v[6][3] \leftarrow v[5][3] = 7$, $s[6][3] \leftarrow 0$
6	4	10	0	$j > \ell$: $v[6][4] \leftarrow v[5][4] = 10$, $s[6][4] \leftarrow 0$

6	5	12	0	$j > \ell$: $v[6][5] \leftarrow v[5][5] = 12$, $s[6][5] \leftarrow 0$
6	6	15	0	$p[6] + v[6][0] = 13 + 0 \leq v[5][6] = 15$: $v[6][6] \leftarrow v[5][6] = 15$, $s[6][6] \leftarrow 0$

These are the matrices v and s obtained during the execution of the DP algorithm:

$v[j][\ell]$	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$	$\ell=5$	$\ell=6$
$j=0$	0	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$j=1$	0	2	4	6	8	10	12
$j=2$	0	2	5	7	10	12	15
$j=3$	0	2	5	7	10	12	15
$j=4$	0	2	5	7	10	12	15
$j=5$	0	2	5	7	10	12	15
$j=6$	0	2	5	7	10	12	15

$s[j][\ell]$	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$	$\ell=5$	$\ell=6$
$j=0$	0	0	0	0	0	0	0
$j=1$	0	1	1	1	1	1	1
$j=2$	0	0	1	1	1	1	1
$j=3$	0	0	0	0	0	0	0
$j=4$	0	0	0	0	0	0	0
$j=5$	0	0	0	0	0	0	0
$j=6$	0	0	0	0	0	0	0

The optimal value of the RCP instance is $v[6][6] = 15$.

4. Using the matrix s, we can reconstruct an optimal solution as follows:

Iteration	j	ℓ	$s[j][\ell]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	$x[6]$	Action
1	6	6	0	0	0	0	0	0	0	$j \leftarrow 5$
2	5	6	0	0	0	0	0	0	0	$j \leftarrow 4$
3	4	6	0	0	0	0	0	0	0	$j \leftarrow 3$
4	3	6	0	0	0	0	0	0	0	$j \leftarrow 2$
5	2	6	1	0	1	0	0	0	0	$\ell \leftarrow 4$
6	2	4	1	0	2	0	0	0	0	$\ell \leftarrow 2$
7	2	2	1	0	3	0	0	0	0	$\ell \leftarrow 0$

Thus, the vector

$$(0, 3, 0, 0, 0, 0)$$

is an optimal solution. The optimal value is

$$v[6][6] = 15.$$

Exercise 13.2

Consider the following RCP instance, where the rod has length $n = 5$, and the profits associated with the possible piece lengths are:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
p_j	1	4	6	8	9

Questions

1. Show the execution of the first bottom-up DP algorithm by reporting, for each iteration, the values of the vector v and the values of the vector s.

2. Using the vector s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

3. Show the execution of the second bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and s.

4. Using the matrix s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

Solution

1. Step-by-step execution of the first bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the values of ℓ and j, the cells

$v[\ell]$ and $s[\ell]$, the current value of m, and the current value of t. The cell of m is highlighted in red when updated; the cell of t is highlighted in blue when updated; the cells of $v[\ell]$ and $s[\ell]$ are highlighted in blue when assigned.

ℓ	j	$v[\ell]$	$s[\ell]$	m	t	Action
–	–	0	–	–	–	$v[0] \leftarrow 0$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
1	1	–	–	1	1	$p[1] + v[0] = 1 + 0 = 1 > 0$: $m \leftarrow 1,\ t \leftarrow 1$
	–	1	1	1	1	$v[1] \leftarrow m = 1,\ s[1] \leftarrow t = 1$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
2	1	–	–	2	1	$p[1] + v[1] = 1 + 1 = 2 > 0$: $m \leftarrow 2,\ t \leftarrow 1$
	2	–	–	4	2	$p[2] + v[0] = 4 + 0 = 4 > 2$: $m \leftarrow 4,\ t \leftarrow 2$
	–	4	2	4	2	$v[2] \leftarrow m = 4,\ s[2] \leftarrow t = 2$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
3	1	–	–	5	1	$p[1] + v[2] = 1 + 4 = 5 > 0$: $m \leftarrow 5,\ t \leftarrow 1$
	2	–	–	5	1	$p[2] + v[1] = 4 + 1 = 5 = m$: no update
	3	–	–	6	3	$p[3] + v[0] = 6 + 0 = 6 > 5$: $m \leftarrow 6,\ t \leftarrow 3$
	–	6	3	6	3	$v[3] \leftarrow m = 6,\ s[3] \leftarrow t = 3$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
4	1	–	–	7	1	$p[1] + v[3] = 1 + 6 = 7 > 0$: $m \leftarrow 7,\ t \leftarrow 1$
	2	–	–	8	2	$p[2] + v[2] = 4 + 4 = 8 > 7$: $m \leftarrow 8,\ t \leftarrow 2$
	3	–	–	8	2	$p[3] + v[1] = 6 + 1 = 7 < 8$: no update
	4	–	–	8	2	$p[4] + v[0] = 8 + 0 = 8 = m$: no update
	–	8	2	8	2	$v[4] \leftarrow m = 8,\ s[4] \leftarrow t = 2$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
5	1	–	–	9	1	$p[1] + v[4] = 1 + 8 = 9 > 0$: $m \leftarrow 9,\ t \leftarrow 1$
	2	–	–	10	2	$p[2] + v[3] = 4 + 6 = 10 > 9$: $m \leftarrow 10,\ t \leftarrow 2$
	3	–	–	10	2	$p[3] + v[2] = 6 + 4 = 10 = m$: no update
	4	–	–	10	2	$p[4] + v[1] = 8 + 1 = 9 < 10$: no update
	5	–	–	10	2	$p[5] + v[0] = 9 + 0 = 9 < 10$: no update
	–	10	2	10	2	$v[5] \leftarrow m = 10,\ s[5] \leftarrow t = 2$

This is the vector v after the execution of the DP algorithm:

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
$v[\ell]$	0	1	4	6	8	10

This is the vector s after the execution of the DP algorithm:

	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
$s[\ell]$	1	2	3	2	2

2. We have the following vector s:

	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
$s[\ell]$	1	2	3	2	2

We can reconstruct an optimal solution for a rod of length $n = 5$ as follows:

Iteration	j	$s[j]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	Update of j
-	-	-	0	0	0	0	0	$j = 5$
1	5	2	0	1	0	0	0	$j = 5 - 2 = 3$
2	3	3	0	1	1	0	0	$j = 3 - 3 = 0$

Thus, the vector

$$(0, 1, 1, 0, 0)$$

is an optimal solution. The optimal value is

$$v[5] = 10.$$

3. Step-by-step execution of the second bottom-up DP algorithm

For each step of the DP algorithm, the table shows the values of j and ℓ, the cells $v[j][\ell]$ and $s[j][\ell]$, and the operation performed. The cells are computed for increasing values of j and, for each fixed value of j, for increasing values of ℓ.

j	ℓ	$v[j][\ell]$	$s[j][\ell]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $s[0][0] \leftarrow 0$
0	1	$-\infty$	0	$v[0][1] \leftarrow -\infty$, $s[0][1] \leftarrow 0$
0	2	$-\infty$	0	$v[0][2] \leftarrow -\infty$, $s[0][2] \leftarrow 0$
0	3	$-\infty$	0	$v[0][3] \leftarrow -\infty$, $s[0][3] \leftarrow 0$
0	4	$-\infty$	0	$v[0][4] \leftarrow -\infty$, $s[0][4] \leftarrow 0$
0	5	$-\infty$	0	$v[0][5] \leftarrow -\infty$, $s[0][5] \leftarrow 0$
1	0	0	0	$j > \ell$: $v[1][0] \leftarrow v[0][0] = 0$, $s[1][0] \leftarrow 0$
1	1	1	1	$p[1] + v[1][0] = 1 + 0 > v[0][1] = -\infty$: $v[1][1] \leftarrow 1$, $s[1][1] \leftarrow 1$
1	2	2	1	$p[1] + v[1][1] = 1 + 1 > v[0][2] = -\infty$: $v[1][2] \leftarrow 2$, $s[1][2] \leftarrow 1$
1	3	3	1	$p[1] + v[1][2] = 1 + 2 > v[0][3] = -\infty$: $v[1][3] \leftarrow 3$, $s[1][3] \leftarrow 1$
1	4	4	1	$p[1] + v[1][3] = 1 + 3 > v[0][4] = -\infty$: $v[1][4] \leftarrow 4$, $s[1][4] \leftarrow 1$
1	5	5	1	$p[1] + v[1][4] = 1 + 4 > v[0][5] = -\infty$: $v[1][5] \leftarrow 5$, $s[1][5] \leftarrow 1$
2	0	0	0	$j > \ell$: $v[2][0] \leftarrow v[1][0] = 0$, $s[2][0] \leftarrow 0$
2	1	1	0	$j > \ell$: $v[2][1] \leftarrow v[1][1] = 1$, $s[2][1] \leftarrow 0$
2	2	4	1	$p[2] + v[2][0] = 4 + 0 > v[1][2] = 2$: $v[2][2] \leftarrow 4$, $s[2][2] \leftarrow 1$
2	3	5	1	$p[2] + v[2][1] = 4 + 1 > v[1][3] = 3$: $v[2][3] \leftarrow 5$, $s[2][3] \leftarrow 1$
2	4	8	1	$p[2] + v[2][2] = 4 + 4 > v[1][4] = 4$: $v[2][4] \leftarrow 8$, $s[2][4] \leftarrow 1$
2	5	9	1	$p[2] + v[2][3] = 4 + 5 > v[1][5] = 5$: $v[2][5] \leftarrow 9$, $s[2][5] \leftarrow 1$
3	0	0	0	$j > \ell$: $v[3][0] \leftarrow v[2][0] = 0$, $s[3][0] \leftarrow 0$
3	1	1	0	$j > \ell$: $v[3][1] \leftarrow v[2][1] = 1$, $s[3][1] \leftarrow 0$
3	2	4	0	$j > \ell$: $v[3][2] \leftarrow v[2][2] = 4$, $s[3][2] \leftarrow 0$
3	3	6	1	$p[3] + v[3][0] = 6 + 0 > v[2][3] = 5$: $v[3][3] \leftarrow 6$, $s[3][3] \leftarrow 1$
3	4	8	0	$p[3] + v[3][1] = 6 + 1 \leq v[2][4] = 8$: $v[3][4] \leftarrow v[2][4] = 8$, $s[3][4] \leftarrow 0$
3	5	10	1	$p[3] + v[3][2] = 6 + 4 > v[2][5] = 9$: $v[3][5] \leftarrow 10$, $s[3][5] \leftarrow 1$
4	0	0	0	$j > \ell$: $v[4][0] \leftarrow v[3][0] = 0$, $s[4][0] \leftarrow 0$
4	1	1	0	$j > \ell$: $v[4][1] \leftarrow v[3][1] = 1$, $s[4][1] \leftarrow 0$
4	2	4	0	$j > \ell$: $v[4][2] \leftarrow v[3][2] = 4$, $s[4][2] \leftarrow 0$
4	3	6	0	$j > \ell$: $v[4][3] \leftarrow v[3][3] = 6$, $s[4][3] \leftarrow 0$
4	4	8	0	$p[4] + v[4][0] = 8 + 0 \leq v[3][4] = 8$: $v[4][4] \leftarrow v[3][4] = 8$, $s[4][4] \leftarrow 0$
4	5	10	0	$p[4] + v[4][1] = 8 + 1 \leq v[3][5] = 10$: $v[4][5] \leftarrow v[3][5] = 10$, $s[4][5] \leftarrow 0$
5	0	0	0	$j > \ell$: $v[5][0] \leftarrow v[4][0] = 0$, $s[5][0] \leftarrow 0$
5	1	1	0	$j > \ell$: $v[5][1] \leftarrow v[4][1] = 1$, $s[5][1] \leftarrow 0$
5	2	4	0	$j > \ell$: $v[5][2] \leftarrow v[4][2] = 4$, $s[5][2] \leftarrow 0$
5	3	6	0	$j > \ell$: $v[5][3] \leftarrow v[4][3] = 6$, $s[5][3] \leftarrow 0$
5	4	8	0	$j > \ell$: $v[5][4] \leftarrow v[4][4] = 8$, $s[5][4] \leftarrow 0$
5	5	10	0	$p[5] + v[5][0] = 9 + 0 \leq v[4][5] = 10$: $v[5][5] \leftarrow v[4][5] = 10$, $s[5][5] \leftarrow 0$

These are the matrices v and s obtained during the execution of the DP algorithm:

$v[j][\ell]$	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
$j = 0$	0	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$j = 1$	0	1	2	3	4	5
$j = 2$	0	1	4	5	8	9
$j = 3$	0	1	4	6	8	10
$j = 4$	0	1	4	6	8	10
$j = 5$	0	1	4	6	8	10

$s[j][\ell]$	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
$j = 0$	0	0	0	0	0	0
$j = 1$	0	1	1	1	1	1
$j = 2$	0	0	1	1	1	1
$j = 3$	0	0	0	1	0	1
$j = 4$	0	0	0	0	0	0
$j = 5$	0	0	0	0	0	0

The optimal value of the RCP instance is $v[5][5] = 10$.

4. Using the matrix s, we can reconstruct an optimal solution as follows:

Iteration	j	ℓ	$s[j][\ell]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	Action
1	5	5	0	0	0	0	0	0	$j \leftarrow 4$
2	4	5	0	0	0	0	0	0	$j \leftarrow 3$
3	3	5	1	0	0	1	0	0	$\ell \leftarrow 2$
4	3	2	0	0	0	1	0	0	$j \leftarrow 2$
5	2	2	1	0	1	1	0	0	$\ell \leftarrow 0$

Thus, the vector

$$(0, 1, 1, 0, 0)$$

is an optimal solution. The optimal value is

$$v[5][5] = 10.$$

Exercise 13.3

Consider the following RCP instance, where the rod has length $n = 5$, and the profits associated with the possible piece lengths are:

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
p_j	3	5	8	9	10

Questions

1. Show the execution of the first bottom-up DP algorithm by reporting, for each iteration, the values of the vector v and the values of the vector s.

2. Using the vector s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

3. Show the execution of the second bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and s.

4. Using the matrix s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

Solution

1. Step-by-step execution of the first bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the values of ℓ and j, the cells $v[\ell]$ and $s[\ell]$, the current value of m, and the current value of t. The cell of m is highlighted in red when updated; the cell of t is highlighted in blue when updated; the cells of $v[\ell]$ and $s[\ell]$ are highlighted in blue when assigned.

ℓ	j	$v[\ell]$	$s[\ell]$	m	t	Action
–	–	0	–	–	–	$v[0] \leftarrow 0$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$

1	1	–	–	3	1	$p[1]+v[0]=3+0=3>0$: $m\leftarrow 3$, $t\leftarrow 1$
	–	3	1	3	1	$v[1]\leftarrow m=3$, $s[1]\leftarrow t=1$
	–	–	–	0	0	$m\leftarrow 0$, $t\leftarrow 0$
2	1	–	–	6	1	$p[1]+v[1]=3+3=6>0$: $m\leftarrow 6$, $t\leftarrow 1$
	2	–	–	6	1	$p[2]+v[0]=5+0=5<6$: no update
	–	6	1	6	1	$v[2]\leftarrow m=6$, $s[2]\leftarrow t=1$
	–	–	–	0	0	$m\leftarrow 0$, $t\leftarrow 0$
3	1	–	–	9	1	$p[1]+v[2]=3+6=9>0$: $m\leftarrow 9$, $t\leftarrow 1$
	2	–	–	9	1	$p[2]+v[1]=5+3=8<9$: no update
	3	–	–	9	1	$p[3]+v[0]=8+0=8<9$: no update
	–	9	1	9	1	$v[3]\leftarrow m=9$, $s[3]\leftarrow t=1$
	–	–	–	0	0	$m\leftarrow 0$, $t\leftarrow 0$
4	1	–	–	12	1	$p[1]+v[3]=3+9=12>0$: $m\leftarrow 12$, $t\leftarrow 1$
	2	–	–	12	1	$p[2]+v[2]=5+6=11<12$: no update
	3	–	–	12	1	$p[3]+v[1]=8+3=11<12$: no update
	4	–	–	12	1	$p[4]+v[0]=9+0=9<12$: no update
	–	12	1	12	1	$v[4]\leftarrow m=12$, $s[4]\leftarrow t=1$
	–	–	–	0	0	$m\leftarrow 0$, $t\leftarrow 0$
5	1	–	–	15	1	$p[1]+v[4]=3+12=15>0$: $m\leftarrow 15$, $t\leftarrow 1$
	2	–	–	15	1	$p[2]+v[3]=5+9=14<15$: no update
	3	–	–	15	1	$p[3]+v[2]=8+6=14<15$: no update
	4	–	–	15	1	$p[4]+v[1]=9+3=12<15$: no update
	5	–	–	15	1	$p[5]+v[0]=10+0=10<15$: no update
	–	15	1	15	1	$v[5]\leftarrow m=15$, $s[5]\leftarrow t=1$

This is the vector v after the execution of the DP algorithm:

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
$v[\ell]$	0	3	6	9	12	15

This is the vector s after the execution of the DP algorithm:

	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
$s[\ell]$	1	1	1	1	1

2. We have the following vector s:

	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
$s[\ell]$	1	1	1	1	1

We can reconstruct an optimal solution for a rod of length $n = 5$ as follows:

Iteration	j	$s[j]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	Update of j
-	-	-	0	0	0	0	0	$j = 5$
1	5	1	1	0	0	0	0	$j = 5 - 1 = 4$
2	4	1	2	0	0	0	0	$j = 4 - 1 = 3$
3	3	1	3	0	0	0	0	$j = 3 - 1 = 2$
4	2	1	4	0	0	0	0	$j = 2 - 1 = 1$
5	1	1	5	0	0	0	0	$j = 1 - 1 = 0$

Thus, the vector

$$(5, 0, 0, 0, 0)$$

is an optimal solution. The optimal value is

$$v[5] = 15.$$

3. Step-by-step execution of the second bottom-up DP algorithm

For each step of the DP algorithm, the table shows the values of j and ℓ, the cells $v[j][\ell]$ and $s[j][\ell]$, and the operation performed. The cells are computed for increasing values of j and, for each fixed value of j, for increasing values of ℓ.

j	ℓ	$v[j][\ell]$	$s[j][\ell]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $s[0][0] \leftarrow 0$
0	1	$-\infty$	0	$v[0][1] \leftarrow -\infty$, $s[0][1] \leftarrow 0$
0	2	$-\infty$	0	$v[0][2] \leftarrow -\infty$, $s[0][2] \leftarrow 0$
0	3	$-\infty$	0	$v[0][3] \leftarrow -\infty$, $s[0][3] \leftarrow 0$
0	4	$-\infty$	0	$v[0][4] \leftarrow -\infty$, $s[0][4] \leftarrow 0$
0	5	$-\infty$	0	$v[0][5] \leftarrow -\infty$, $s[0][5] \leftarrow 0$
1	0	0	0	$j > \ell$: $v[1][0] \leftarrow v[0][0] = 0$, $s[1][0] \leftarrow 0$
1	1	3	1	$p[1] + v[1][0] = 3 + 0 > v[0][1] = -\infty$: $v[1][1] \leftarrow 3$, $s[1][1] \leftarrow 1$
1	2	6	1	$p[1] + v[1][1] = 3 + 3 > v[0][2] = -\infty$: $v[1][2] \leftarrow 6$, $s[1][2] \leftarrow 1$
1	3	9	1	$p[1] + v[1][2] = 3 + 6 > v[0][3] = -\infty$: $v[1][3] \leftarrow 9$, $s[1][3] \leftarrow 1$
1	4	12	1	$p[1] + v[1][3] = 3 + 9 > v[0][4] = -\infty$: $v[1][4] \leftarrow 12$, $s[1][4] \leftarrow 1$
1	5	15	1	$p[1] + v[1][4] = 3 + 12 > v[0][5] = -\infty$: $v[1][5] \leftarrow 15$, $s[1][5] \leftarrow 1$
2	0	0	0	$j > \ell$: $v[2][0] \leftarrow v[1][0] = 0$, $s[2][0] \leftarrow 0$
2	1	3	0	$j > \ell$: $v[2][1] \leftarrow v[1][1] = 3$, $s[2][1] \leftarrow 0$
2	2	6	0	$p[2] + v[2][0] = 5 + 0 \leq v[1][2] = 6$: $v[2][2] \leftarrow v[1][2] = 6$, $s[2][2] \leftarrow 0$
2	3	9	0	$p[2] + v[2][1] = 5 + 3 \leq v[1][3] = 9$: $v[2][3] \leftarrow v[1][3] = 9$, $s[2][3] \leftarrow 0$
2	4	12	0	$p[2] + v[2][2] = 5 + 6 \leq v[1][4] = 12$: $v[2][4] \leftarrow v[1][4] = 12$, $s[2][4] \leftarrow 0$
2	5	15	0	$p[2] + v[2][3] = 5 + 9 \leq v[1][5] = 15$: $v[2][5] \leftarrow v[1][5] = 15$, $s[2][5] \leftarrow 0$
3	0	0	0	$j > \ell$: $v[3][0] \leftarrow v[2][0] = 0$, $s[3][0] \leftarrow 0$
3	1	3	0	$j > \ell$: $v[3][1] \leftarrow v[2][1] = 3$, $s[3][1] \leftarrow 0$
3	2	6	0	$j > \ell$: $v[3][2] \leftarrow v[2][2] = 6$, $s[3][2] \leftarrow 0$
3	3	9	0	$p[3] + v[3][0] = 8 + 0 \leq v[2][3] = 9$: $v[3][3] \leftarrow v[2][3] = 9$, $s[3][3] \leftarrow 0$
3	4	12	0	$p[3] + v[3][1] = 8 + 3 \leq v[2][4] = 12$: $v[3][4] \leftarrow v[2][4] = 12$, $s[3][4] \leftarrow 0$
3	5	15	0	$p[3] + v[3][2] = 8 + 6 \leq v[2][5] = 15$: $v[3][5] \leftarrow v[2][5] = 15$, $s[3][5] \leftarrow 0$
4	0	0	0	$j > \ell$: $v[4][0] \leftarrow v[3][0] = 0$, $s[4][0] \leftarrow 0$
4	1	3	0	$j > \ell$: $v[4][1] \leftarrow v[3][1] = 3$, $s[4][1] \leftarrow 0$
4	2	6	0	$j > \ell$: $v[4][2] \leftarrow v[3][2] = 6$, $s[4][2] \leftarrow 0$
4	3	9	0	$j > \ell$: $v[4][3] \leftarrow v[3][3] = 9$, $s[4][3] \leftarrow 0$
4	4	12	0	$p[4] + v[4][0] = 9 + 0 \leq v[3][4] = 12$: $v[4][4] \leftarrow v[3][4] = 12$, $s[4][4] \leftarrow 0$
4	5	15	0	$p[4] + v[4][1] = 9 + 3 \leq v[3][5] = 15$: $v[4][5] \leftarrow v[3][5] = 15$, $s[4][5] \leftarrow 0$
5	0	0	0	$j > \ell$: $v[5][0] \leftarrow v[4][0] = 0$, $s[5][0] \leftarrow 0$
5	1	3	0	$j > \ell$: $v[5][1] \leftarrow v[4][1] = 3$, $s[5][1] \leftarrow 0$
5	2	6	0	$j > \ell$: $v[5][2] \leftarrow v[4][2] = 6$, $s[5][2] \leftarrow 0$
5	3	9	0	$j > \ell$: $v[5][3] \leftarrow v[4][3] = 9$, $s[5][3] \leftarrow 0$
5	4	12	0	$j > \ell$: $v[5][4] \leftarrow v[4][4] = 12$, $s[5][4] \leftarrow 0$
5	5	15	0	$p[5] + v[5][0] = 10 + 0 \leq v[4][5] = 15$: $v[5][5] \leftarrow v[4][5] = 15$, $s[5][5] \leftarrow 0$

These are the matrices v and s obtained during the execution of the DP algorithm:

$v[j][\ell]$	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
$j = 0$	0	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$j = 1$	0	3	6	9	12	15
$j = 2$	0	3	6	9	12	15
$j = 3$	0	3	6	9	12	15
$j = 4$	0	3	6	9	12	15
$j = 5$	0	3	6	9	12	15

$s[j][\ell]$	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
$j = 0$	0	0	0	0	0	0
$j = 1$	0	1	1	1	1	1
$j = 2$	0	0	0	0	0	0
$j = 3$	0	0	0	0	0	0
$j = 4$	0	0	0	0	0	0
$j = 5$	0	0	0	0	0	0

The optimal value of the RCP instance is $v[5][5] = 15$.

4. Using the matrix s, we can reconstruct an optimal solution as follows:

Iteration	j	ℓ	$s[j][\ell]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	Action
1	5	5	0	0	0	0	0	0	$j \leftarrow 4$
2	4	5	0	0	0	0	0	0	$j \leftarrow 3$
3	3	5	0	0	0	0	0	0	$j \leftarrow 2$
4	2	5	0	0	0	0	0	0	$j \leftarrow 1$
5	1	5	1	1	0	0	0	0	$\ell \leftarrow 4$
6	1	4	1	2	0	0	0	0	$\ell \leftarrow 3$
7	1	3	1	3	0	0	0	0	$\ell \leftarrow 2$
8	1	2	1	4	0	0	0	0	$\ell \leftarrow 1$
9	1	1	1	5	0	0	0	0	$\ell \leftarrow 0$

Thus, the vector

$$(5, 0, 0, 0, 0)$$

is an optimal solution. The optimal value is

$$v[5][5] = 15.$$

Exercise 13.4

Consider the following RCP instance, where the rod has length $n = 6$, and the profits associated with the possible piece lengths are:

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
p_j	2	6	7	11	13	17

Questions

1. Show the execution of the first bottom-up DP algorithm by reporting, for each iteration, the values of the vector v and the values of the vector s.

2. Using the vector s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

3. Show the execution of the second bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and s.

4. Using the matrix s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

Solution

1. Step-by-step execution of the first bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the values of ℓ and j, the cells

$v[\ell]$ and $s[\ell]$, the current value of m, and the current value of t. The cell of m is highlighted in red when updated; the cell of t is highlighted in blue when updated; the cells of $v[\ell]$ and $s[\ell]$ are highlighted in blue when assigned.

ℓ	j	$v[\ell]$	$s[\ell]$	m	t	Action
–	–	0	–	–	–	$v[0] \leftarrow 0$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
1	1	–	–	2	1	$p[1] + v[0] = 2 + 0 = 2 > 0$: $m \leftarrow 2,\ t \leftarrow 1$
	–	2	1	2	1	$v[1] \leftarrow m = 2,\ s[1] \leftarrow t = 1$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
2	1	–	–	4	1	$p[1] + v[1] = 2 + 2 = 4 > 0$: $m \leftarrow 4,\ t \leftarrow 1$
	2	–	–	6	2	$p[2] + v[0] = 6 + 0 = 6 > 4$: $m \leftarrow 6,\ t \leftarrow 2$
	–	6	2	6	2	$v[2] \leftarrow m = 6,\ s[2] \leftarrow t = 2$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
3	1	–	–	8	1	$p[1] + v[2] = 2 + 6 = 8 > 0$: $m \leftarrow 8,\ t \leftarrow 1$
	2	–	–	8	1	$p[2] + v[1] = 6 + 2 = 8 = m$: no update
	3	–	–	8	1	$p[3] + v[0] = 7 + 0 = 7 < 8$: no update
	–	8	1	8	1	$v[3] \leftarrow m = 8,\ s[3] \leftarrow t = 1$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
4	1	–	–	10	1	$p[1] + v[3] = 2 + 8 = 10 > 0$: $m \leftarrow 10,\ t \leftarrow 1$
	2	–	–	12	2	$p[2] + v[2] = 6 + 6 = 12 > 10$: $m \leftarrow 12,\ t \leftarrow 2$
	3	–	–	12	2	$p[3] + v[1] = 7 + 2 = 9 < 12$: no update
	4	–	–	12	2	$p[4] + v[0] = 11 + 0 = 11 < 12$: no update
	–	12	2	12	2	$v[4] \leftarrow m = 12,\ s[4] \leftarrow t = 2$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
5	1	–	–	14	1	$p[1] + v[4] = 2 + 12 = 14 > 0$: $m \leftarrow 14,\ t \leftarrow 1$
	2	–	–	14	1	$p[2] + v[3] = 6 + 8 = 14 = m$: no update
	3	–	–	14	1	$p[3] + v[2] = 7 + 6 = 13 < 14$: no update
	4	–	–	14	1	$p[4] + v[1] = 11 + 2 = 13 < 14$: no update
	5	–	–	14	1	$p[5] + v[0] = 13 + 0 = 13 < 14$: no update
	–	14	1	14	1	$v[5] \leftarrow m = 14,\ s[5] \leftarrow t = 1$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
6	1	–	–	16	1	$p[1] + v[5] = 2 + 14 = 16 > 0$: $m \leftarrow 16,\ t \leftarrow 1$
	2	–	–	18	2	$p[2] + v[4] = 6 + 12 = 18 > 16$: $m \leftarrow 18,\ t \leftarrow 2$
	3	–	–	18	2	$p[3] + v[3] = 7 + 8 = 15 < 18$: no update
	4	–	–	18	2	$p[4] + v[2] = 11 + 6 = 17 < 18$: no update
	5	–	–	18	2	$p[5] + v[1] = 13 + 2 = 15 < 18$: no update
	6	–	–	18	2	$p[6] + v[0] = 17 + 0 = 17 < 18$: no update
	–	18	2	18	2	$v[6] \leftarrow m = 18,\ s[6] \leftarrow t = 2$

This is the vector v after the execution of the DP algorithm:

	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$	$\ell=5$	$\ell=6$
$v[\ell]$	0	2	6	8	12	14	18

This is the vector s after the execution of the DP algorithm:

	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$	$\ell=5$	$\ell=6$
$s[\ell]$	1	2	1	2	1	2

2. We have the following vector s:

	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$	$\ell=5$	$\ell=6$
$s[\ell]$	1	2	1	2	1	2

We can reconstruct an optimal solution for a rod of length $n=6$ as follows:

Iteration	j	$s[j]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	$x[6]$	Update of j
-	-	-	0	0	0	0	0	0	$j=6$
1	6	2	0	1	0	0	0	0	$j=6-2=4$
2	4	2	0	2	0	0	0	0	$j=4-2=2$
3	2	2	0	3	0	0	0	0	$j=2-2=0$

Thus, the vector

$$(0,3,0,0,0,0)$$

is an optimal solution. The optimal value is

$$v[6]=18.$$

3. Step-by-step execution of the second bottom-up DP algorithm

For each step of the DP algorithm, the table shows the values of j and ℓ, the cells $v[j][\ell]$ and $s[j][\ell]$, and the operation performed. The cells are computed for increasing values of j and, for each fixed value of j, for increasing values of ℓ.

j	ℓ	$v[j][\ell]$	$s[j][\ell]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $s[0][0] \leftarrow 0$
0	1	$-\infty$	0	$v[0][1] \leftarrow -\infty$, $s[0][1] \leftarrow 0$
0	2	$-\infty$	0	$v[0][2] \leftarrow -\infty$, $s[0][2] \leftarrow 0$
0	3	$-\infty$	0	$v[0][3] \leftarrow -\infty$, $s[0][3] \leftarrow 0$
0	4	$-\infty$	0	$v[0][4] \leftarrow -\infty$, $s[0][4] \leftarrow 0$
0	5	$-\infty$	0	$v[0][5] \leftarrow -\infty$, $s[0][5] \leftarrow 0$
0	6	$-\infty$	0	$v[0][6] \leftarrow -\infty$, $s[0][6] \leftarrow 0$
1	0	0	0	$j > \ell$: $v[1][0] \leftarrow v[0][0] = 0$, $s[1][0] \leftarrow 0$
1	1	2	1	$p[1] + v[1][0] = 2 + 0 > v[0][1] = -\infty$: $v[1][1] \leftarrow 2$, $s[1][1] \leftarrow 1$
1	2	4	1	$p[1] + v[1][1] = 2 + 2 > v[0][2] = -\infty$: $v[1][2] \leftarrow 4$, $s[1][2] \leftarrow 1$
1	3	6	1	$p[1] + v[1][2] = 2 + 4 > v[0][3] = -\infty$: $v[1][3] \leftarrow 6$, $s[1][3] \leftarrow 1$
1	4	8	1	$p[1] + v[1][3] = 2 + 6 > v[0][4] = -\infty$: $v[1][4] \leftarrow 8$, $s[1][4] \leftarrow 1$
1	5	10	1	$p[1] + v[1][4] = 2 + 8 > v[0][5] = -\infty$: $v[1][5] \leftarrow 10$, $s[1][5] \leftarrow 1$
1	6	12	1	$p[1] + v[1][5] = 2 + 10 > v[0][6] = -\infty$: $v[1][6] \leftarrow 12$, $s[1][6] \leftarrow 1$
2	0	0	0	$j > \ell$: $v[2][0] \leftarrow v[1][0] = 0$, $s[2][0] \leftarrow 0$
2	1	2	0	$j > \ell$: $v[2][1] \leftarrow v[1][1] = 2$, $s[2][1] \leftarrow 0$
2	2	6	1	$p[2] + v[2][0] = 6 + 0 > v[1][2] = 4$: $v[2][2] \leftarrow 6$, $s[2][2] \leftarrow 1$
2	3	8	1	$p[2] + v[2][1] = 6 + 2 > v[1][3] = 6$: $v[2][3] \leftarrow 8$, $s[2][3] \leftarrow 1$
2	4	12	1	$p[2] + v[2][2] = 6 + 6 > v[1][4] = 8$: $v[2][4] \leftarrow 12$, $s[2][4] \leftarrow 1$
2	5	14	1	$p[2] + v[2][3] = 6 + 8 > v[1][5] = 10$: $v[2][5] \leftarrow 14$, $s[2][5] \leftarrow 1$
2	6	18	1	$p[2] + v[2][4] = 6 + 12 > v[1][6] = 12$: $v[2][6] \leftarrow 18$, $s[2][6] \leftarrow 1$
3	0	0	0	$j > \ell$: $v[3][0] \leftarrow v[2][0] = 0$, $s[3][0] \leftarrow 0$
3	1	2	0	$j > \ell$: $v[3][1] \leftarrow v[2][1] = 2$, $s[3][1] \leftarrow 0$
3	2	6	0	$j > \ell$: $v[3][2] \leftarrow v[2][2] = 6$, $s[3][2] \leftarrow 0$
3	3	8	0	$p[3] + v[3][0] = 7 + 0 \leq v[2][3] = 8$: $v[3][3] \leftarrow v[2][3] = 8$, $s[3][3] \leftarrow 0$
3	4	12	0	$p[3] + v[3][1] = 7 + 2 \leq v[2][4] = 12$: $v[3][4] \leftarrow v[2][4] = 12$, $s[3][4] \leftarrow 0$
3	5	14	0	$p[3] + v[3][2] = 7 + 6 \leq v[2][5] = 14$: $v[3][5] \leftarrow v[2][5] = 14$, $s[3][5] \leftarrow 0$
3	6	18	0	$p[3] + v[3][3] = 7 + 8 \leq v[2][6] = 18$: $v[3][6] \leftarrow v[2][6] = 18$, $s[3][6] \leftarrow 0$
4	0	0	0	$j > \ell$: $v[4][0] \leftarrow v[3][0] = 0$, $s[4][0] \leftarrow 0$
4	1	2	0	$j > \ell$: $v[4][1] \leftarrow v[3][1] = 2$, $s[4][1] \leftarrow 0$
4	2	6	0	$j > \ell$: $v[4][2] \leftarrow v[3][2] = 6$, $s[4][2] \leftarrow 0$
4	3	8	0	$j > \ell$: $v[4][3] \leftarrow v[3][3] = 8$, $s[4][3] \leftarrow 0$
4	4	12	0	$p[4] + v[4][0] = 11 + 0 \leq v[3][4] = 12$: $v[4][4] \leftarrow v[3][4] = 12$, $s[4][4] \leftarrow 0$
4	5	14	0	$p[4] + v[4][1] = 11 + 2 \leq v[3][5] = 14$: $v[4][5] \leftarrow v[3][5] = 14$, $s[4][5] \leftarrow 0$
4	6	18	0	$p[4] + v[4][2] = 11 + 6 \leq v[3][6] = 18$: $v[4][6] \leftarrow v[3][6] = 18$, $s[4][6] \leftarrow 0$
5	0	0	0	$j > \ell$: $v[5][0] \leftarrow v[4][0] = 0$, $s[5][0] \leftarrow 0$
5	1	2	0	$j > \ell$: $v[5][1] \leftarrow v[4][1] = 2$, $s[5][1] \leftarrow 0$
5	2	6	0	$j > \ell$: $v[5][2] \leftarrow v[4][2] = 6$, $s[5][2] \leftarrow 0$
5	3	8	0	$j > \ell$: $v[5][3] \leftarrow v[4][3] = 8$, $s[5][3] \leftarrow 0$
5	4	12	0	$j > \ell$: $v[5][4] \leftarrow v[4][4] = 12$, $s[5][4] \leftarrow 0$
5	5	14	0	$p[5] + v[5][0] = 13 + 0 \leq v[4][5] = 14$: $v[5][5] \leftarrow v[4][5] = 14$, $s[5][5] \leftarrow 0$
5	6	18	0	$p[5] + v[5][1] = 13 + 2 \leq v[4][6] = 18$: $v[5][6] \leftarrow v[4][6] = 18$, $s[5][6] \leftarrow 0$
6	0	0	0	$j > \ell$: $v[6][0] \leftarrow v[5][0] = 0$, $s[6][0] \leftarrow 0$
6	1	2	0	$j > \ell$: $v[6][1] \leftarrow v[5][1] = 2$, $s[6][1] \leftarrow 0$
6	2	6	0	$j > \ell$: $v[6][2] \leftarrow v[5][2] = 6$, $s[6][2] \leftarrow 0$
6	3	8	0	$j > \ell$: $v[6][3] \leftarrow v[5][3] = 8$, $s[6][3] \leftarrow 0$
6	4	12	0	$j > \ell$: $v[6][4] \leftarrow v[5][4] = 12$, $s[6][4] \leftarrow 0$

6	5	14	0	$j > \ell$: $v[6][5] \leftarrow v[5][5] = 14$, $s[6][5] \leftarrow 0$
6	6	18	0	$p[6]+v[6][0] = 17+0 \leq v[5][6] = 18$: $v[6][6] \leftarrow v[5][6] = 18$, $s[6][6] \leftarrow 0$

These are the matrices v and s obtained during the execution of the DP algorithm:

$v[j][\ell]$	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$	$\ell=5$	$\ell=6$
$j=0$	0	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$j=1$	0	2	4	6	8	10	12
$j=2$	0	2	6	8	12	14	18
$j=3$	0	2	6	8	12	14	18
$j=4$	0	2	6	8	12	14	18
$j=5$	0	2	6	8	12	14	18
$j=6$	0	2	6	8	12	14	18

$s[j][\ell]$	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$	$\ell=5$	$\ell=6$
$j=0$	0	0	0	0	0	0	0
$j=1$	0	1	1	1	1	1	1
$j=2$	0	0	1	1	1	1	1
$j=3$	0	0	0	0	0	0	0
$j=4$	0	0	0	0	0	0	0
$j=5$	0	0	0	0	0	0	0
$j=6$	0	0	0	0	0	0	0

The optimal value of the RCP instance is $v[6][6] = 18$.

4. Using the matrix s, we can reconstruct an optimal solution as follows:

Iteration	j	ℓ	$s[j][\ell]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	$x[6]$	Action
1	6	6	0	0	0	0	0	0	0	$j \leftarrow 5$
2	5	6	0	0	0	0	0	0	0	$j \leftarrow 4$
3	4	6	0	0	0	0	0	0	0	$j \leftarrow 3$
4	3	6	0	0	0	0	0	0	0	$j \leftarrow 2$
5	2	6	1	0	1	0	0	0	0	$\ell \leftarrow 4$
6	2	4	1	0	2	0	0	0	0	$\ell \leftarrow 2$
7	2	2	1	0	3	0	0	0	0	$\ell \leftarrow 0$

Thus, the vector

$$(0, 3, 0, 0, 0, 0)$$

is an optimal solution. The optimal value is

$$v[6][6] = 18.$$

Exercise 13.5

Consider the following RCP instance, where the rod has length $n = 6$, and the profits associated with the possible piece lengths are:

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
p_j	1	5	8	9	12	14

Questions

1. Show the execution of the first bottom-up DP algorithm by reporting, for each iteration, the values of the vector v and the values of the vector s.

2. Using the vector s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

3. Show the execution of the second bottom-up DP algorithm by reporting, for each iteration, the values of the matrices v and s.

4. Using the matrix s, reconstruct an optimal solution in integer notation $(x_1, x_2, \ldots, x_n)$, and report the optimal value.

Solution

1. Step-by-step execution of the first bottom-up DP algorithm

   For each step of the DP algorithm, the table shows the values of ℓ and j, the cells

$v[\ell]$ and $s[\ell]$, the current value of m, and the current value of t. The cell of m is highlighted in red when updated; the cell of t is highlighted in blue when updated; the cells of $v[\ell]$ and $s[\ell]$ are highlighted in blue when assigned.

ℓ	j	$v[\ell]$	$s[\ell]$	m	t	Action
–	–	0	–	–	–	$v[0] \leftarrow 0$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
1	1	–	–	1	1	$p[1] + v[0] = 1 + 0 = 1 > 0$: $m \leftarrow 1,\ t \leftarrow 1$
	–	1	1	1	1	$v[1] \leftarrow m = 1,\ s[1] \leftarrow t = 1$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
2	1	–	–	2	1	$p[1] + v[1] = 1 + 1 = 2 > 0$: $m \leftarrow 2,\ t \leftarrow 1$
	2	–	–	5	2	$p[2] + v[0] = 5 + 0 = 5 > 2$: $m \leftarrow 5,\ t \leftarrow 2$
	–	5	2	5	2	$v[2] \leftarrow m = 5,\ s[2] \leftarrow t = 2$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
3	1	–	–	6	1	$p[1] + v[2] = 1 + 5 = 6 > 0$: $m \leftarrow 6,\ t \leftarrow 1$
	2	–	–	6	1	$p[2] + v[1] = 5 + 1 = 6 = m$: no update
	3	–	–	8	3	$p[3] + v[0] = 8 + 0 = 8 > 6$: $m \leftarrow 8,\ t \leftarrow 3$
	–	8	3	8	3	$v[3] \leftarrow m = 8,\ s[3] \leftarrow t = 3$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
4	1	–	–	9	1	$p[1] + v[3] = 1 + 8 = 9 > 0$: $m \leftarrow 9,\ t \leftarrow 1$
	2	–	–	10	2	$p[2] + v[2] = 5 + 5 = 10 > 9$: $m \leftarrow 10,\ t \leftarrow 2$
	3	–	–	10	2	$p[3] + v[1] = 8 + 1 = 9 < 10$: no update
	4	–	–	10	2	$p[4] + v[0] = 9 + 0 = 9 < 10$: no update
	–	10	2	10	2	$v[4] \leftarrow m = 10,\ s[4] \leftarrow t = 2$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
5	1	–	–	11	1	$p[1] + v[4] = 1 + 10 = 11 > 0$: $m \leftarrow 11,\ t \leftarrow 1$
	2	–	–	13	2	$p[2] + v[3] = 5 + 8 = 13 > 11$: $m \leftarrow 13,\ t \leftarrow 2$
	3	–	–	13	2	$p[3] + v[2] = 8 + 5 = 13 = m$: no update
	4	–	–	13	2	$p[4] + v[1] = 9 + 1 = 10 < 13$: no update
	5	–	–	13	2	$p[5] + v[0] = 12 + 0 = 12 < 13$: no update
	–	13	2	13	2	$v[5] \leftarrow m = 13,\ s[5] \leftarrow t = 2$
	–	–	–	0	0	$m \leftarrow 0,\ t \leftarrow 0$
6	1	–	–	14	1	$p[1] + v[5] = 1 + 13 = 14 > 0$: $m \leftarrow 14,\ t \leftarrow 1$
	2	–	–	15	2	$p[2] + v[4] = 5 + 10 = 15 > 14$: $m \leftarrow 15,\ t \leftarrow 2$
	3	–	–	16	3	$p[3] + v[3] = 8 + 8 = 16 > 15$: $m \leftarrow 16,\ t \leftarrow 3$
	4	–	–	16	3	$p[4] + v[2] = 9 + 5 = 14 < 16$: no update
	5	–	–	16	3	$p[5] + v[1] = 12 + 1 = 13 < 16$: no update
	6	–	–	16	3	$p[6] + v[0] = 14 + 0 = 14 < 16$: no update
	–	16	3	16	3	$v[6] \leftarrow m = 16,\ s[6] \leftarrow t = 3$

This is the vector v after the execution of the DP algorithm:

	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$	$\ell=5$	$\ell=6$
$v[\ell]$	0	1	5	8	10	13	16

This is the vector s after the execution of the DP algorithm:

	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$	$\ell=5$	$\ell=6$
$s[\ell]$	1	2	3	2	2	3

2. We have the following vector s:

	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$	$\ell=5$	$\ell=6$
$s[\ell]$	1	2	3	2	2	3

We can reconstruct an optimal solution for a rod of length $n=6$ as follows:

Iteration	j	$s[j]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	$x[6]$	Update of j
-	-	-	0	0	0	0	0	0	$j=6$
1	6	3	0	0	1	0	0	0	$j=6-3=3$
2	3	3	0	0	2	0	0	0	$j=3-3=0$

Thus, the vector

$$(0,0,2,0,0,0)$$

is an optimal solution. The optimal value is

$$v[6]=16.$$

3. Step-by-step execution of the second bottom-up DP algorithm

For each step of the DP algorithm, the table shows the values of j and ℓ, the cells $v[j][\ell]$ and $s[j][\ell]$, and the operation performed. The cells are computed for increasing values of j and, for each fixed value of j, for increasing values of ℓ.

j	ℓ	$v[j][\ell]$	$s[j][\ell]$	Action
0	0	0	0	$v[0][0] \leftarrow 0$, $s[0][0] \leftarrow 0$
0	1	$-\infty$	0	$v[0][1] \leftarrow -\infty$, $s[0][1] \leftarrow 0$
0	2	$-\infty$	0	$v[0][2] \leftarrow -\infty$, $s[0][2] \leftarrow 0$
0	3	$-\infty$	0	$v[0][3] \leftarrow -\infty$, $s[0][3] \leftarrow 0$
0	4	$-\infty$	0	$v[0][4] \leftarrow -\infty$, $s[0][4] \leftarrow 0$
0	5	$-\infty$	0	$v[0][5] \leftarrow -\infty$, $s[0][5] \leftarrow 0$
0	6	$-\infty$	0	$v[0][6] \leftarrow -\infty$, $s[0][6] \leftarrow 0$
1	0	0	0	$j > \ell$: $v[1][0] \leftarrow v[0][0] = 0$, $s[1][0] \leftarrow 0$
1	1	1	1	$p[1] + v[1][0] = 1 + 0 > v[0][1] = -\infty$: $v[1][1] \leftarrow 1$, $s[1][1] \leftarrow 1$
1	2	2	1	$p[1] + v[1][1] = 1 + 1 > v[0][2] = -\infty$: $v[1][2] \leftarrow 2$, $s[1][2] \leftarrow 1$
1	3	3	1	$p[1] + v[1][2] = 1 + 2 > v[0][3] = -\infty$: $v[1][3] \leftarrow 3$, $s[1][3] \leftarrow 1$
1	4	4	1	$p[1] + v[1][3] = 1 + 3 > v[0][4] = -\infty$: $v[1][4] \leftarrow 4$, $s[1][4] \leftarrow 1$
1	5	5	1	$p[1] + v[1][4] = 1 + 4 > v[0][5] = -\infty$: $v[1][5] \leftarrow 5$, $s[1][5] \leftarrow 1$
1	6	6	1	$p[1] + v[1][5] = 1 + 5 > v[0][6] = -\infty$: $v[1][6] \leftarrow 6$, $s[1][6] \leftarrow 1$
2	0	0	0	$j > \ell$: $v[2][0] \leftarrow v[1][0] = 0$, $s[2][0] \leftarrow 0$
2	1	1	0	$j > \ell$: $v[2][1] \leftarrow v[1][1] = 1$, $s[2][1] \leftarrow 0$
2	2	5	1	$p[2] + v[2][0] = 5 + 0 > v[1][2] = 2$: $v[2][2] \leftarrow 5$, $s[2][2] \leftarrow 1$
2	3	6	1	$p[2] + v[2][1] = 5 + 1 > v[1][3] = 3$: $v[2][3] \leftarrow 6$, $s[2][3] \leftarrow 1$
2	4	10	1	$p[2] + v[2][2] = 5 + 5 > v[1][4] = 4$: $v[2][4] \leftarrow 10$, $s[2][4] \leftarrow 1$
2	5	11	1	$p[2] + v[2][3] = 5 + 6 > v[1][5] = 5$: $v[2][5] \leftarrow 11$, $s[2][5] \leftarrow 1$
2	6	15	1	$p[2] + v[2][4] = 5 + 10 > v[1][6] = 6$: $v[2][6] \leftarrow 15$, $s[2][6] \leftarrow 1$
3	0	0	0	$j > \ell$: $v[3][0] \leftarrow v[2][0] = 0$, $s[3][0] \leftarrow 0$
3	1	1	0	$j > \ell$: $v[3][1] \leftarrow v[2][1] = 1$, $s[3][1] \leftarrow 0$
3	2	5	0	$j > \ell$: $v[3][2] \leftarrow v[2][2] = 5$, $s[3][2] \leftarrow 0$
3	3	8	1	$p[3] + v[3][0] = 8 + 0 > v[2][3] = 6$: $v[3][3] \leftarrow 8$, $s[3][3] \leftarrow 1$
3	4	10	0	$p[3] + v[3][1] = 8 + 1 \leq v[2][4] = 10$: $v[3][4] \leftarrow v[2][4] = 10$, $s[3][4] \leftarrow 0$
3	5	13	1	$p[3] + v[3][2] = 8 + 5 > v[2][5] = 11$: $v[3][5] \leftarrow 13$, $s[3][5] \leftarrow 1$
3	6	16	1	$p[3] + v[3][3] = 8 + 8 > v[2][6] = 15$: $v[3][6] \leftarrow 16$, $s[3][6] \leftarrow 1$
4	0	0	0	$j > \ell$: $v[4][0] \leftarrow v[3][0] = 0$, $s[4][0] \leftarrow 0$
4	1	1	0	$j > \ell$: $v[4][1] \leftarrow v[3][1] = 1$, $s[4][1] \leftarrow 0$
4	2	5	0	$j > \ell$: $v[4][2] \leftarrow v[3][2] = 5$, $s[4][2] \leftarrow 0$
4	3	8	0	$j > \ell$: $v[4][3] \leftarrow v[3][3] = 8$, $s[4][3] \leftarrow 0$
4	4	10	0	$p[4] + v[4][0] = 9 + 0 \leq v[3][4] = 10$: $v[4][4] \leftarrow v[3][4] = 10$, $s[4][4] \leftarrow 0$
4	5	13	0	$p[4] + v[4][1] = 9 + 1 \leq v[3][5] = 13$: $v[4][5] \leftarrow v[3][5] = 13$, $s[4][5] \leftarrow 0$
4	6	16	0	$p[4] + v[4][2] = 9 + 5 \leq v[3][6] = 16$: $v[4][6] \leftarrow v[3][6] = 16$, $s[4][6] \leftarrow 0$
5	0	0	0	$j > \ell$: $v[5][0] \leftarrow v[4][0] = 0$, $s[5][0] \leftarrow 0$
5	1	1	0	$j > \ell$: $v[5][1] \leftarrow v[4][1] = 1$, $s[5][1] \leftarrow 0$
5	2	5	0	$j > \ell$: $v[5][2] \leftarrow v[4][2] = 5$, $s[5][2] \leftarrow 0$
5	3	8	0	$j > \ell$: $v[5][3] \leftarrow v[4][3] = 8$, $s[5][3] \leftarrow 0$
5	4	10	0	$j > \ell$: $v[5][4] \leftarrow v[4][4] = 10$, $s[5][4] \leftarrow 0$
5	5	13	0	$p[5] + v[5][0] = 12 + 0 \leq v[4][5] = 13$: $v[5][5] \leftarrow v[4][5] = 13$, $s[5][5] \leftarrow 0$
5	6	16	0	$p[5] + v[5][1] = 12 + 1 \leq v[4][6] = 16$: $v[5][6] \leftarrow v[4][6] = 16$, $s[5][6] \leftarrow 0$
6	0	0	0	$j > \ell$: $v[6][0] \leftarrow v[5][0] = 0$, $s[6][0] \leftarrow 0$
6	1	1	0	$j > \ell$: $v[6][1] \leftarrow v[5][1] = 1$, $s[6][1] \leftarrow 0$
6	2	5	0	$j > \ell$: $v[6][2] \leftarrow v[5][2] = 5$, $s[6][2] \leftarrow 0$
6	3	8	0	$j > \ell$: $v[6][3] \leftarrow v[5][3] = 8$, $s[6][3] \leftarrow 0$
6	4	10	0	$j > \ell$: $v[6][4] \leftarrow v[5][4] = 10$, $s[6][4] \leftarrow 0$

6	5	13	0	$j > \ell$: $v[6][5] \leftarrow v[5][5] = 13$, $s[6][5] \leftarrow 0$
6	6	16	0	$p[6]+v[6][0] = 14+0 \leq v[5][6] = 16$: $v[6][6] \leftarrow v[5][6] = 16$, $s[6][6] \leftarrow 0$

These are the matrices v and s obtained during the execution of the DP algorithm:

$v[j][\ell]$	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$	$\ell=5$	$\ell=6$
$j=0$	0	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$j=1$	0	1	2	3	4	5	6
$j=2$	0	1	5	6	10	11	15
$j=3$	0	1	5	8	10	13	16
$j=4$	0	1	5	8	10	13	16
$j=5$	0	1	5	8	10	13	16
$j=6$	0	1	5	8	10	13	16

$s[j][\ell]$	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$	$\ell=5$	$\ell=6$
$j=0$	0	0	0	0	0	0	0
$j=1$	0	1	1	1	1	1	1
$j=2$	0	0	1	1	1	1	1
$j=3$	0	0	0	1	0	1	1
$j=4$	0	0	0	0	0	0	0
$j=5$	0	0	0	0	0	0	0
$j=6$	0	0	0	0	0	0	0

The optimal value of the RCP instance is $v[6][6] = 16$.

4. Using the matrix s, we can reconstruct an optimal solution as follows:

Iteration	j	ℓ	$s[j][\ell]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	$x[6]$	Action
1	6	6	0	0	0	0	0	0	0	$j \leftarrow 5$
2	5	6	0	0	0	0	0	0	0	$j \leftarrow 4$
3	4	6	0	0	0	0	0	0	0	$j \leftarrow 3$
4	3	6	1	0	0	1	0	0	0	$\ell \leftarrow 3$
5	3	3	1	0	0	2	0	0	0	$\ell \leftarrow 0$

Thus, the vector

$$(0, 0, 2, 0, 0, 0)$$

is an optimal solution. The optimal value is

$$v[6][6] = 16.$$