

Convex and concave optimization problems



Outline

Convex and concave optimization problems

Local optima of convex and concave problems

Examples of convex and concave optimization problems



Convex and concave optimization problems



Introduction: Convex and concave optimization problems

- ▶ In this section we define **convex** and **concave optimization problems**: problems with a convex feasible region and an objective function that is convex or concave over it.
- ▶ These problems are important because, as we will see, their local optima are also optimal solutions.
- ▶ We use the convex sets and the convex and concave functions of the previous chapters, and the optimization problems of the chapter «Optimization problems».

Convex and concave optimization problems

Definition 1: convex and concave optimization problems

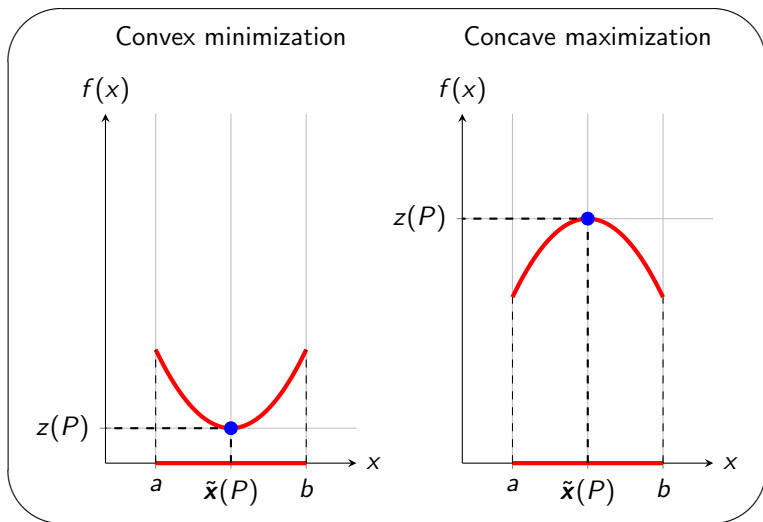
*Given an optimization problem with a **convex feasible region**,*

- ▶ *if the objective function is convex over the feasible region, then the problem is called a **convex optimization problem**;*
- ▶ *if the objective function is concave over the feasible region, then the problem is called a **concave optimization problem**.*

The figures of this chapter

- ▶ In the figures of this chapter, as in the chapter «Optimization problems»:
 - ▶ the graph of the objective function over the feasible region is red;
 - ▶ the feasible region is the red part of the x-axis;
 - ▶ the optimal solutions are blue.
- ▶ In the next slide: a convex minimization problem and a concave maximization problem, with the feasible region $[a, b] \subseteq \mathbb{R}$.

Convex minimization and concave maximization



The feasible region is convex

- ▶ The definitions of convex and concave function of the chapter «Convex and concave functions» require a convex domain: if the objective function is convex or concave over the feasible region, then the feasible region is a convex set.
- ▶ The feasible region is often described by a finite number of constraints:

$$F = \{ \mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \leq 0, i \in \{1, 2, \dots, m\} \}$$

Feasible regions defined by convex functions

- ▶ If every function g_i is convex, then F is a convex set, by the property of the sets defined by convex functions of the chapter «Convex and concave functions».
- ▶ For instance, the interval

$$[a, b] = \{ x \in \mathbb{R} : a - x \leq 0, x - b \leq 0 \}$$

is defined by two linear, hence convex, functions.

Summary

Convex and concave optimization problems:

- ▶ The feasible region is a **convex set**.
- ▶ The objective function is **convex** (convex problem) or **concave** (concave problem) over the feasible region.

Convex feasible regions:

- ▶ A feasible region $F = \{ \mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \leq 0, i \in \{1, 2, \dots, m\} \}$ with all functions g_i convex is a convex set.
- ▶ Example: the interval $[a, b]$, defined by two linear functions.

Local optima of convex and concave problems



Introduction: Local optima of convex and concave problems

- ▶ In this section we prove the main property of convex minimization problems: every **point of local minimum** is a **point of minimum**.
- ▶ The same property holds for the points of local maximum of concave maximization problems.
- ▶ This property is very useful: to find an optimal solution, it is enough to find a point of **local** optimum, which is checked only in a neighborhood of the point.



Every point of minimum is also a point of local minimum.

The following proposition states that, in a convex minimization problem, the converse also holds.

Local minima of convex minimization problems

Proposition 1: local minima of convex minimization problems

In a convex minimization problem, with objective function f convex over the convex feasible region F , if a point $\bar{\mathbf{x}} \in F$ is a point of local minimum of f in F , then it is a point of minimum of f in F , that is, an optimal solution.

The idea of the proof

- ▶ We compare $f(\bar{\mathbf{x}})$ with the value $f(\mathbf{x})$ at an arbitrary point $\mathbf{x} \in F$.
- ▶ Since F is convex, the segment that joins $\bar{\mathbf{x}}$ to $\mathbf{x}$ is contained in F .
- ▶ The points of the segment close enough to $\bar{\mathbf{x}}$ are in the neighborhood of $\bar{\mathbf{x}}$, where $\bar{\mathbf{x}}$ is a point of minimum.
- ▶ The convexity of f along the segment then gives $f(\bar{\mathbf{x}}) \leq f(\mathbf{x})$.



Proof

If $\bar{\mathbf{x}} \in F$ is a point of local minimum, then there exists a spherical neighborhood $N(\bar{\mathbf{x}}, \delta)$ with $\delta > 0$ such that:

$$f(\bar{\mathbf{x}}) \leq f(\mathbf{x}), \quad \forall \mathbf{x} \in N(\bar{\mathbf{x}}, \delta) \cap F$$

For every $\mathbf{x} \in F \setminus \{\bar{\mathbf{x}}\}$, since F is convex, all points of the form

$$\hat{\mathbf{x}} = \lambda \mathbf{x} + (1 - \lambda) \bar{\mathbf{x}}$$

with $\lambda \in (0, 1]$ also belong to F .

Moreover, for λ sufficiently small, we have $\hat{\mathbf{x}} \in N(\bar{\mathbf{x}}, \delta) \cap F$, and therefore:

$$f(\bar{\mathbf{x}}) \leq f(\hat{\mathbf{x}})$$

Proof (continued)

Since f is convex, it also holds that:

$$f(\hat{\mathbf{x}}) \leq \lambda f(\mathbf{x}) + (1 - \lambda) f(\bar{\mathbf{x}})$$

Combining the two inequalities, we obtain:

$$f(\bar{\mathbf{x}}) \leq \lambda f(\mathbf{x}) + (1 - \lambda) f(\bar{\mathbf{x}})$$

$$\Downarrow$$

$$\lambda f(\bar{\mathbf{x}}) \leq \lambda f(\mathbf{x})$$

$$\Downarrow$$

$$f(\bar{\mathbf{x}}) \leq f(\mathbf{x}) \quad (\text{since } \lambda \in (0, 1])$$

Hence,

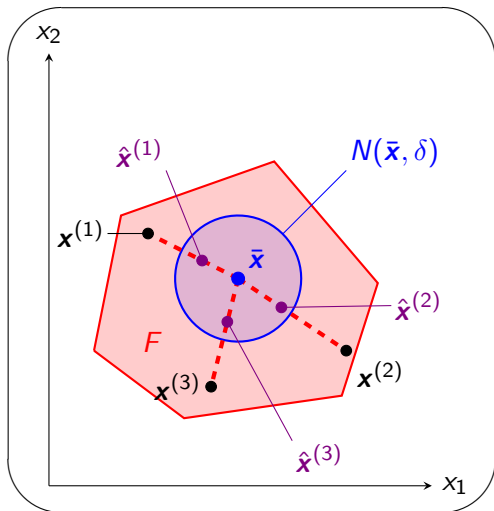
$$f(\bar{\mathbf{x}}) \leq f(\mathbf{x}), \quad \forall \mathbf{x} \in F$$

and $\bar{\mathbf{x}}$ is a point of minimum. □

The idea of the proof in $\mathbb{R}^2$

- ▶ In $\mathbb{R}^2$, the idea of the proof is given by the picture in the next slide, with:
 - ▶ three points $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}$ in F ;
 - ▶ three points $\hat{\mathbf{x}}^{(1)}, \hat{\mathbf{x}}^{(2)}, \hat{\mathbf{x}}^{(3)}$ in $N(\bar{\mathbf{x}}, \delta) \cap F$.
- ▶ Each point $\hat{\mathbf{x}}^{(i)}$ lies on the segment that joins $\bar{\mathbf{x}}$ to the corresponding $\mathbf{x}^{(i)}$.

The idea of the proof in $\mathbb{R}^2$



Local minimum and minimum

- ▶ In a convex minimization problem we have shown that:

point of local minimum $\implies$ point of minimum

- ▶ The implication also holds in the other direction, because a point of minimum satisfies the defining condition at every point of F , and therefore in particular at every point of a neighborhood.
- ▶ The two notions are equivalent.

For concave maximization problems we obtain the same property, applying the proposition to the convex function $-f$.



Local maxima of concave maximization problems

Corollary 1: local maxima of concave maximization problems

In a concave maximization problem, with objective function f concave over the convex feasible region F , a point $\bar{x} \in F$ is a point of local maximum of f in F if and only if it is a point of maximum of f in F , that is, an optimal solution.

Proof

If f is concave over F , then $-f$ is convex over F , and $\bar{\mathbf{x}}$ is a point of local maximum of f if and only if it is a point of local minimum of $-f$.

By the previous proposition, it is a point of minimum of $-f$ in F , that is, a point of maximum of f in F , since

$$\max\{f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F\} = -\min\{-f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F\}$$



Local optima are optimal solutions

- ▶ In a convex minimization problem, every **point of local minimum** is also a **point of minimum**: every point of local minimum in the feasible region is an optimal solution.
- ▶ In a concave maximization problem, every **point of local maximum** is also a **point of maximum**: every point of local maximum in the feasible region is an optimal solution.



Summary

Convex minimization problems:

- ▶ Point of local minimum $\iff$ point of minimum.
- ▶ Proof: the segment from $\bar{\mathbf{x}}$ to any $\mathbf{x} \in F$ is in F , its points close to $\bar{\mathbf{x}}$ are in the neighborhood, and f is convex along it.

Concave maximization problems:

- ▶ Point of local maximum $\iff$ point of maximum.
- ▶ Proof: $-f$ is convex, and a maximum of f is a minimum of $-f$.

Examples of convex and concave optimization problems

Introduction: Examples of convex and concave optimization problems

- ▶ In this section we solve four problems: two with a quadratic and two with a cubic objective function.
- ▶ The examples show how to recognize a convex (concave) problem and how to use the local optima to find an optimal solution.
- ▶ The cubic examples show that the same objective function can give a convex problem or a concave problem, depending on the feasible region.

How we solve the examples

- ▶ We first check that the feasible region is a convex set and that the objective function is convex (or concave) over it, with the definition of the chapter «Convex and concave functions».
- ▶ Then we find a point of local minimum (or maximum), comparing the value of the objective function at a candidate point with its values at the feasible points of a neighborhood.
- ▶ By the proposition and the corollary of the previous section, this point is an optimal solution.

The candidate point

- ▶ The candidate point is the point where the derivative of the objective function vanishes.
- ▶ The derivative only suggests the candidate: the local optimality is checked directly with the definition of the chapter «Optimization problems».



Example: a convex minimization problem (1/6)

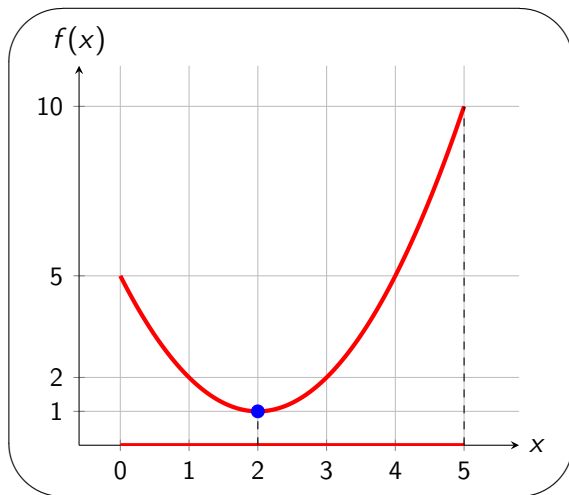
Consider the following objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, f : x \mapsto (x - 2)^2 + 1 \quad \text{and} \quad F = [0, 5].$$

The associated optimization problem in minimization form is:

$$\min \left\{ (x - 2)^2 + 1 \text{ subject to } 0 \leq x \leq 5 \right\}$$

Example: a convex minimization problem (2/6)



Example: a convex minimization problem (3/6)

Feasible region. The feasible region $F = [0, 5]$ is an interval, hence a convex set.

Objective function. For every pair of points $p, w \in F$ and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned} f(\lambda p + (1 - \lambda)w) &= (\lambda p + (1 - \lambda)w - 2)^2 + 1 \\ &= (\lambda(p - 2) + (1 - \lambda)(w - 2))^2 + \lambda + (1 - \lambda) \\ &\quad \text{(since } \lambda + (1 - \lambda) = 1\text{)} \\ &\leq \lambda(p - 2)^2 + (1 - \lambda)(w - 2)^2 + \lambda + (1 - \lambda) \\ &\quad \text{(since } (\lambda a + (1 - \lambda)b)^2 \leq \lambda a^2 + (1 - \lambda)b^2, \\ &\quad \text{chapter «Convex and concave functions»)} \end{aligned}$$

Example: a convex minimization problem (4/6)

$$\begin{aligned}f(\lambda p + (1 - \lambda)w) &= \dots \\&\leq \lambda(p - 2)^2 + (1 - \lambda)(w - 2)^2 + \lambda + (1 - \lambda) \\&= \lambda((p - 2)^2 + 1) + (1 - \lambda)((w - 2)^2 + 1) \\&= \lambda f(p) + (1 - \lambda)f(w)\end{aligned}$$

Therefore,

$$f(\lambda p + (1 - \lambda)w) \leq \lambda f(p) + (1 - \lambda)f(w)$$

The objective function is convex over the convex feasible region F , and the problem is a **convex minimization problem**.

Example: a convex minimization problem (5/6)

Point of local minimum. The derivative

$$f'(x) = 2x - 4 = 2(x - 2)$$

vanishes at $x = 2$. In the neighborhood $N((2), 1) = (1, 3)$, all points are feasible and

$$f(x) = (x - 2)^2 + 1 \geq 1 = f(2),$$

$$\forall x \in N((2), 1) \cap F = (1, 3).$$

The point $x = 2$ is a point of local minimum.

Example: a convex minimization problem (6/6)

Optimal solution. By the proposition «local minima of convex minimization problems», the point $x = 2$ is also a point of minimum, and the optimal solution and the optimal value are:

$$\tilde{x}(P) = (2) \quad \text{and} \quad z(P) = f(2) = 1.$$

Example: a concave maximization problem (1/6)

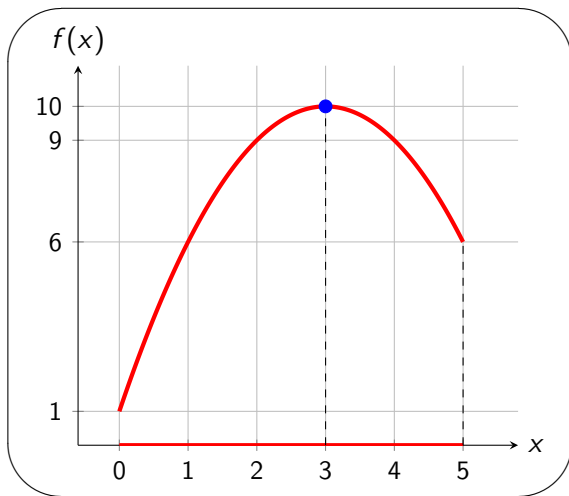
Consider the following objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, f : x \mapsto -(x - 3)^2 + 10 \quad \text{and} \quad F = [0, 5].$$

The associated optimization problem in maximization form is:

$$\max \left\{ -(x - 3)^2 + 10 \text{ subject to } 0 \leq x \leq 5 \right\}$$

Example: a concave maximization problem (2/6)



Example: a concave maximization problem (3/6)

Feasible region. The feasible region $F = [0, 5]$ is an interval, hence a convex set.

Objective function. For every pair of points $p, w \in F$ and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned} & f(\lambda p + (1 - \lambda)w) \\ &= -(\lambda(p - 3) + (1 - \lambda)(w - 3))^2 + \lambda 10 + (1 - \lambda) 10 \\ &\quad (\text{since } \lambda + (1 - \lambda) = 1) \\ &\geq -\lambda(p - 3)^2 - (1 - \lambda)(w - 3)^2 + \lambda 10 + (1 - \lambda) 10 \\ &\quad (\text{since } (\lambda a + (1 - \lambda)b)^2 \leq \lambda a^2 + (1 - \lambda)b^2, \\ &\quad \text{and } -1 \text{ reverses the inequality}) \end{aligned}$$

Example: a concave maximization problem (4/6)

$$\begin{aligned} & f(\lambda p + (1 - \lambda)w) \\ &= \dots \\ &\geq -\lambda(p - 3)^2 - (1 - \lambda)(w - 3)^2 + \lambda 10 + (1 - \lambda) 10 \\ &= \lambda(- (p - 3)^2 + 10) + (1 - \lambda)(- (w - 3)^2 + 10) \\ &= \lambda f(p) + (1 - \lambda)f(w) \end{aligned}$$

Therefore,

$$f(\lambda p + (1 - \lambda)w) \geq \lambda f(p) + (1 - \lambda)f(w)$$

The objective function is concave over the convex feasible region F , and the problem is a **concave maximization problem**.

Example: a concave maximization problem (5/6)

Point of local maximum. The derivative

$$f'(x) = -2x + 6 = -2(x - 3)$$

vanishes at $x = 3$. In the neighborhood $N((3), 1) = (2, 4)$, all points are feasible and

$$f(x) = -(x - 3)^2 + 10 \leq 10 = f(3),$$

$$\forall x \in N((3), 1) \cap F = (2, 4).$$

The point $x = 3$ is a point of local maximum.

Example: a concave maximization problem (6/6)

Optimal solution. By the corollary «local maxima of concave maximization problems», the point $x = 3$ is also a point of maximum, and the optimal solution and the optimal value are:

$$\tilde{x}(P) = (3) \quad \text{and} \quad z(P) = f(3) = 10.$$

In the last two examples, the objective function is the same cubic polynomial, with two different feasible regions.

A cubic objective function (1/2)

- ▶ The objective function of the last two examples is

$$\begin{aligned} f : \mathbb{R} &\rightarrow \mathbb{R}, \quad f : x \mapsto 2 + 9x - 6x^2 + x^3 \\ &= (x - 2)^3 - 3(x - 2) + 4 \end{aligned}$$

- ▶ It is neither convex nor concave in $\mathbb{R}$:

- ▶ with $p = 0$, $w = 2$ and $\lambda = \frac{1}{2}$ we have

$$f(1) = 6 > 3 = \frac{1}{2} f(0) + \frac{1}{2} f(2)$$

- ▶ with $p = 2$, $w = 4$ and $\lambda = \frac{1}{2}$ we have

$$f(3) = 2 < 5 = \frac{1}{2} f(2) + \frac{1}{2} f(4)$$

A cubic objective function (2/2)

- ▶ It is convex over $[2, 4]$ and concave over $[0, 2]$.
- ▶ The definition of convex (concave) optimization problem only concerns the values of the objective function over the feasible region.
- ▶ To prove it, we use the inequality of the following observation.



An inequality for cubes

Observation 1

For every $\lambda \in [0, 1]$ and every $a, b \in \mathbb{R}$, we have

$$\begin{aligned}\lambda a^3 + (1 - \lambda)b^3 - (\lambda a + (1 - \lambda)b)^3 \\ = \lambda(1 - \lambda)(a - b)^2((1 + \lambda)a + (2 - \lambda)b)\end{aligned}$$

and, therefore,

$$(\lambda a + (1 - \lambda)b)^3 \leq \lambda a^3 + (1 - \lambda)b^3 \quad \text{if } a, b \geq 0,$$

$$(\lambda a + (1 - \lambda)b)^3 \geq \lambda a^3 + (1 - \lambda)b^3 \quad \text{if } a, b \leq 0.$$

Proof

Let $\mu = 1 - \lambda$, so that $1 - \lambda^2 = (1 - \lambda)(1 + \lambda) = \mu(1 + \lambda)$ and $1 - \mu^2 = \lambda(1 + \mu)$. Expanding the cube, we have:

$$\begin{aligned} & \lambda a^3 + \mu b^3 - (\lambda a + \mu b)^3 \\ &= \lambda(1 - \lambda^2) a^3 + \mu(1 - \mu^2) b^3 - 3\lambda^2 \mu a^2 b - 3\lambda \mu^2 a b^2 \\ &= \lambda \mu ((1 + \lambda) a^3 + (1 + \mu) b^3 - 3\lambda a^2 b - 3\mu a b^2) \\ &= \lambda \mu (a^2 - 2ab + b^2) ((1 + \lambda) a + (1 + \mu) b) \\ & \quad (\text{since } (1 + \mu) - 2(1 + \lambda) = -3\lambda \\ & \quad \text{and } (1 + \lambda) - 2(1 + \mu) = -3\mu) \\ &= \lambda(1 - \lambda)(a - b)^2((1 + \lambda) a + (2 - \lambda) b) \end{aligned}$$



Proof (continued)

We have obtained

$$\begin{aligned}\lambda a^3 + (1 - \lambda)b^3 - (\lambda a + (1 - \lambda)b)^3 \\ = \lambda(1 - \lambda)(a - b)^2((1 + \lambda)a + (2 - \lambda)b)\end{aligned}$$

Since $\lambda(1 - \lambda)(a - b)^2 \geq 0$, $1 + \lambda > 0$ and $2 - \lambda > 0$, the right-hand side is:

- ▶ nonnegative if $a, b \geq 0$;
- ▶ nonpositive if $a, b \leq 0$.



Example: convex minimization with a cubic function (1/7)

Consider the following objective function and feasible region:

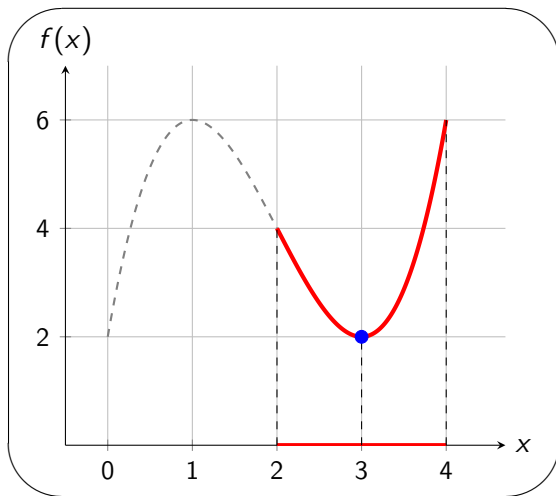
$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto 2 + 9x - 6x^2 + x^3 \quad \text{and} \quad F = [2, 4].$$

The associated optimization problem in minimization form is:

$$\min \left\{ 2 + 9x - 6x^2 + x^3 \quad \text{subject to} \quad 2 \leq x \leq 4 \right\}$$

In the figure, the graph of the objective function outside the feasible region is dashed.

Example: convex minimization with a cubic function (2/7)



Example: convex minimization with a cubic function (3/7)

Feasible region. The feasible region $F = [2, 4]$ is an interval, hence a convex set.

Objective function. For every pair of points $p, w \in F$ and every $\lambda \in [0, 1]$, with $a = p - 2 \geq 0$ and $b = w - 2 \geq 0$, we have:

$$\begin{aligned} & f(\lambda p + (1 - \lambda)w) \\ &= (\lambda p + (1 - \lambda)w - 2)^3 - 3(\lambda p + (1 - \lambda)w - 2) + 4 \\ & \quad (\text{since } f(x) = (x - 2)^3 - 3(x - 2) + 4) \\ &= (\lambda a + (1 - \lambda)b)^3 - 3(\lambda a + (1 - \lambda)b) + \lambda 4 + (1 - \lambda) 4 \\ & \quad (\text{since } \lambda p + (1 - \lambda)w - 2 = \lambda a + (1 - \lambda)b, \\ & \quad \text{and } \lambda + (1 - \lambda) = 1) \end{aligned}$$

Example: convex minimization with a cubic function (4/7)

$$f(\lambda p + (1 - \lambda)w)$$

$$= \dots$$

$$= (\lambda a + (1 - \lambda)b)^3 - 3(\lambda a + (1 - \lambda)b) + \lambda 4 + (1 - \lambda) 4$$

$$\leq \lambda a^3 + (1 - \lambda)b^3 - 3(\lambda a + (1 - \lambda)b) + \lambda 4 + (1 - \lambda) 4$$

(by the previous observation, since $a \geq 0$ and $b \geq 0$)

$$= \lambda(a^3 - 3a + 4) + (1 - \lambda)(b^3 - 3b + 4)$$

$$= \lambda f(p) + (1 - \lambda)f(w)$$

(since $f(p) = a^3 - 3a + 4$ and $f(w) = b^3 - 3b + 4$)

Example: convex minimization with a cubic function (5/7)

Therefore,

$$f(\lambda p + (1 - \lambda)w) \leq \lambda f(p) + (1 - \lambda)f(w)$$

The objective function is convex over the convex feasible region F , and the problem is a **convex minimization problem**.

Example: convex minimization with a cubic function (6/7)

Point of local minimum. The derivative

$$f'(x) = 9 - 12x + 3x^2 = 3(x - 1)(x - 3)$$

vanishes at $x = 1$ and at $x = 3$, and only $x = 3$ is feasible.

For every $x \in \mathbb{R}$ we have $f(x) - 2 = x(x - 3)^2$. In the neighborhood $N((3), 1) = (2, 4)$, all points are feasible and positive, and then

$$f(x) = 2 + x(x - 3)^2 \geq 2 = f(3),$$

$$\forall x \in N((3), 1) \cap F = (2, 4).$$

The point $x = 3$ is a point of local minimum.

Example: convex minimization with a cubic function (7/7)

Optimal solution. By the proposition «local minima of convex minimization problems», the point $x = 3$ is also a point of minimum, and the optimal solution and the optimal value are:

$$\tilde{x}(P) = (3) \quad \text{and} \quad z(P) = f(3) = 2.$$

Example: concave maximization with a cubic function (1/7)

Consider the same objective function with a different feasible region:

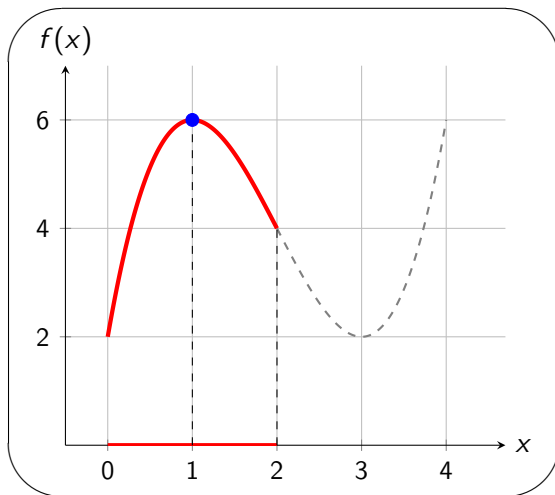
$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto 2 + 9x - 6x^2 + x^3 \quad \text{and} \quad F = [0, 2].$$

The associated optimization problem in maximization form is:

$$\max \left\{ 2 + 9x - 6x^2 + x^3 \quad \text{subject to} \quad 0 \leq x \leq 2 \right\}$$

In the figure, the graph of the objective function outside the feasible region is dashed.

Example: concave maximization with a cubic function (2/7)



Example: concave maximization with a cubic function (3/7)

Feasible region. The feasible region $F = [0, 2]$ is an interval, hence a convex set.

Objective function. For every pair of points $p, w \in F$ and every $\lambda \in [0, 1]$, with $a = p - 2 \leq 0$ and $b = w - 2 \leq 0$, we have:

$$\begin{aligned} & f(\lambda p + (1 - \lambda)w) \\ &= (\lambda p + (1 - \lambda)w - 2)^3 - 3(\lambda p + (1 - \lambda)w - 2) + 4 \\ & \quad (\text{since } f(x) = (x - 2)^3 - 3(x - 2) + 4) \\ &= (\lambda a + (1 - \lambda)b)^3 - 3(\lambda a + (1 - \lambda)b) + \lambda 4 + (1 - \lambda) 4 \\ & \quad (\text{since } \lambda p + (1 - \lambda)w - 2 = \lambda a + (1 - \lambda)b, \\ & \quad \text{and } \lambda + (1 - \lambda) = 1) \end{aligned}$$

Example: concave maximization with a cubic function (4/7)

$$f(\lambda p + (1 - \lambda)w)$$

$$= \dots$$

$$= (\lambda a + (1 - \lambda)b)^3 - 3(\lambda a + (1 - \lambda)b) + \lambda 4 + (1 - \lambda) 4$$

$$\geq \lambda a^3 + (1 - \lambda)b^3 - 3(\lambda a + (1 - \lambda)b) + \lambda 4 + (1 - \lambda) 4$$

(by the previous observation, since $a \leq 0$ and $b \leq 0$)

$$= \lambda(a^3 - 3a + 4) + (1 - \lambda)(b^3 - 3b + 4)$$

$$= \lambda f(p) + (1 - \lambda)f(w)$$

(since $f(p) = a^3 - 3a + 4$ and $f(w) = b^3 - 3b + 4$)

Example: concave maximization with a cubic function (5/7)

Therefore,

$$f(\lambda p + (1 - \lambda)w) \geq \lambda f(p) + (1 - \lambda)f(w)$$

The objective function is concave over the convex feasible region F , and the problem is a **concave maximization problem**.

Example: concave maximization with a cubic function (6/7)

Point of local maximum. The derivative

$$f'(x) = 3(x-1)(x-3)$$

vanishes at $x = 1$ and at $x = 3$, and only $x = 1$ is feasible.

For every $x \in \mathbb{R}$ we have $6 - f(x) = (x-1)^2(4-x)$. In the neighborhood $N((1), 1) = (0, 2)$, all points are feasible and smaller than 4, and then

$$f(x) = 6 - (x-1)^2(4-x) \leq 6 = f(1),$$

$$\forall x \in N((1), 1) \cap F = (0, 2).$$

The point $x = 1$ is a point of local maximum.

Example: concave maximization with a cubic function (7/7)

Optimal solution. By the corollary «local maxima of concave maximization problems», the point $x = 1$ is also a point of maximum, and the optimal solution and the optimal value are:

$$\tilde{x}(P) = (1) \quad \text{and} \quad z(P) = f(1) = 6.$$

Solving a convex (concave) problem:

- ▶ Check that the feasible region is convex and that the objective function is convex (concave) over it, with the definition.
- ▶ Find a candidate point where the derivative vanishes, and check with the definition that it is a point of local minimum (maximum).
- ▶ The point of local optimum is an optimal solution.

The feasible region matters:

- ▶ $2 + 9x - 6x^2 + x^3$ is convex over $[2, 4]$ and concave over $[0, 2]$, but neither convex nor concave in $\mathbb{R}$.

Summary (1/2)

Convex and concave optimization problems:

- ▶ Convex feasible region, objective function convex (convex problem) or concave (concave problem) over it.
- ▶ Feasible regions defined by convex functions $g_i(\mathbf{x}) \leq 0$ are convex sets.

Local optima:

- ▶ Convex minimization: point of local minimum $\iff$ point of minimum.
- ▶ Concave maximization: point of local maximum $\iff$ point of maximum.

Summary (2/2)

Examples:

- ▶ Two quadratic problems: $(x - 2)^2 + 1$ minimized and $-(x - 3)^2 + 10$ maximized over $[0, 5]$.
- ▶ Two cubic problems: $2 + 9x - 6x^2 + x^3$ minimized over $[2, 4]$ (convex) and maximized over $[0, 2]$ (concave).

Method:

- ▶ Convexity (concavity) with the definition, a point of local optimum with the definition, and then the optimal solution by the proposition or the corollary.