

Convex and concave functions



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Convex functions

Introduction: Convex functions

- ▶ In this section we define the **convex functions** of n real variables.
- ▶ The definition compares the value of the function at a **convex combination** of two points with the convex combination of the values at those two points.
- ▶ Convex functions are the basic building blocks of **convex optimization**: with them we will build convex sets and convex optimization problems.

Definition 1: convex function

*A function $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is **convex** in its domain D if:*

$\forall \mathbf{p}, \mathbf{w} \in D$ and $\forall \lambda \in [0, 1]$ we have

$$f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \leq \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})$$

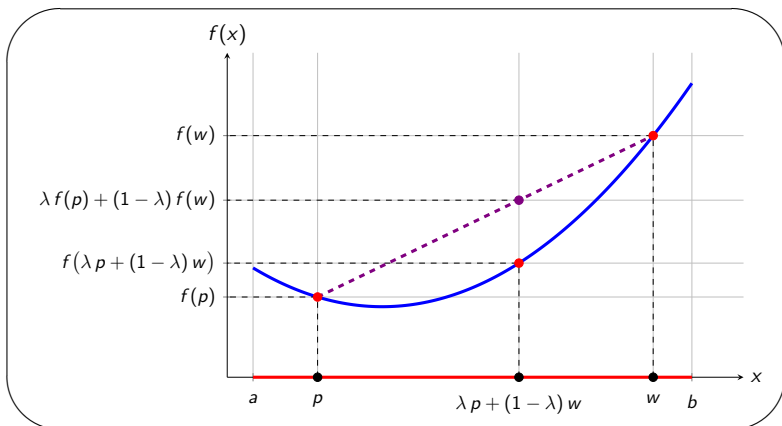
and the domain D of the function f is a convex set.

Convex function: interpretation

- ▶ A function is convex in its domain if, for any two points in its convex domain, the value at any convex combination of the two points is **smaller than or equal to** the convex combination of the values at those two points.
- ▶ Let us now see what this means for a function of one variable.

Graph of a convex function

If $D = [a, b] \subseteq \mathbb{R}$ (an interval which is by nature a convex set), the graph of a convex function is given by the following picture:



Reading the picture

- ▶ For every pair of points $p, w \in [a, b]$, the violet segment joining $(p, f(p))$ and $(w, f(w))$ represents all the convex combinations of these two points of the graph.
- ▶ For a convex function, the violet segment lies **above** the graph of the function (blue) between p and w .
- ▶ At the point $\lambda p + (1 - \lambda) w$ the value of the function, $f(\lambda p + (1 - \lambda) w)$, is smaller than or equal to the value on the segment, $\lambda f(p) + (1 - \lambda) f(w)$.

Summary

Convex function:

- ▶ $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ with D a convex set
- ▶ $f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \leq \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})$ for all $\mathbf{p}, \mathbf{w} \in D$ and $\lambda \in [0, 1]$

Graph (one variable):

- ▶ The segment joining two points of the graph lies above the graph

Concave functions



Introduction: Concave functions

- ▶ In this section we define the **concave functions**, the counterpart of the convex functions.
- ▶ The definition is the same as for convex functions, with the inequality **reversed**.
- ▶ Concave functions appear naturally when we maximize: they play for maximization the role that convex functions play for minimization.

Concave function

Definition 2: concave function

A function $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is **concave** in its domain D if:

$\forall \mathbf{p}, \mathbf{w} \in D$ and $\forall \lambda \in [0, 1]$ we have

$$f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \geq \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})$$

and the domain D of the function f is a convex set.

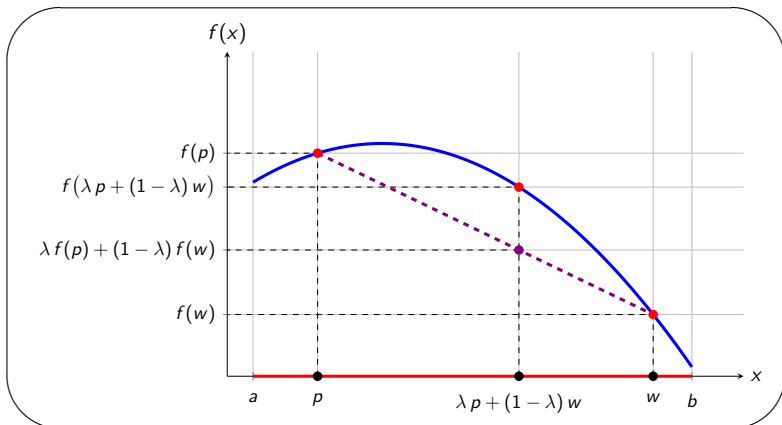
Concave function: interpretation

- ▶ A function is concave in its domain if, for any two points in its convex domain, the value at any convex combination of the two points is **greater than or equal to** the convex combination of the values at those two points.
- ▶ Let us now see what this means for a function of one variable.



Graph of a concave function

If $D = [a, b] \subseteq \mathbb{R}$ (an interval which is by nature a convex set), the graph of a concave function is given by the following picture:



Reading the picture

- ▶ For every pair of points $p, w \in [a, b]$, the violet segment joining $(p, f(p))$ and $(w, f(w))$ represents all the convex combinations of these two points of the graph.
- ▶ For a concave function, the violet segment lies **below** the graph of the function (blue) between p and w .
- ▶ At the point $\lambda p + (1 - \lambda) w$ the value of the function, $f(\lambda p + (1 - \lambda) w)$, is greater than or equal to the value on the segment, $\lambda f(p) + (1 - \lambda) f(w)$.

Summary

Concave function:

- ▶ $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ with D a convex set
- ▶ $f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \geq \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})$ for all $\mathbf{p}, \mathbf{w} \in D$ and $\lambda \in [0, 1]$

Graph (one variable):

- ▶ The segment joining two points of the graph lies below the graph

Examples of convex and concave functions



Introduction: Examples of convex and concave functions

- ▶ In this section we prove that some functions of one and two variables are convex or concave.
- ▶ We always use the **definition**: this is the method we will use in all the exercises.
- ▶ We see quadratic functions, a linear function, the absolute value and the minimum of two linear functions.

How to prove convexity

- ▶ In this chapter we also consider nonlinear functions.
- ▶ The convexity of a function is always proved by checking the definition: for arbitrary points $\mathbf{p}, \mathbf{w}$ of the domain and an arbitrary $\lambda \in [0, 1]$, we verify the inequality of the definition.
- ▶ For the quadratic functions of the first examples, we use the following inequality.

An inequality for squares

Observation 1

For every $a, b \in \mathbb{R}$ and every $\lambda \in [0, 1]$, we have

$$(\lambda a + (1 - \lambda)b)^2 \leq \lambda a^2 + (1 - \lambda)b^2.$$

Proof

For every $a, b \in \mathbb{R}$ and every $\lambda \in [0, 1]$, we have:

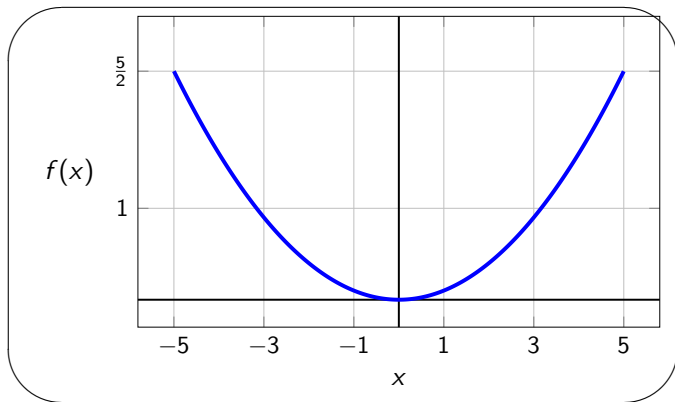
$$\begin{aligned}(\lambda a + (1 - \lambda)b)^2 &= \lambda^2 a^2 + 2\lambda(1 - \lambda)ab + (1 - \lambda)^2 b^2 \\&= \lambda a^2 + (1 - \lambda)b^2 \\&\quad - \lambda(1 - \lambda)a^2 + 2\lambda(1 - \lambda)ab - \lambda(1 - \lambda)b^2 \\&= \lambda a^2 + (1 - \lambda)b^2 - \lambda(1 - \lambda)(a^2 - 2ab + b^2) \\&= \lambda a^2 + (1 - \lambda)b^2 - \underbrace{\lambda(1 - \lambda)(a - b)^2}_{\geq 0} \\&\leq \lambda a^2 + (1 - \lambda)b^2 \\&\quad (\text{since } \lambda \geq 0, 1 - \lambda \geq 0 \text{ and } (a - b)^2 \geq 0)\end{aligned}$$



Example: a convex quadratic function (1/3)

Consider the following function:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto \frac{x^2}{10}$$



Example: a convex quadratic function (2/3)

For every pair of points $p, w \in \mathbb{R}$ and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned}f(\lambda p + (1 - \lambda)w) &= \frac{1}{10}(\lambda p + (1 - \lambda)w)^2 \\&\leq \frac{1}{10}(\lambda p^2 + (1 - \lambda)w^2) \\&\quad \text{(by the inequality for squares)} \\&= \lambda \frac{p^2}{10} + (1 - \lambda) \frac{w^2}{10} \\&= \lambda f(p) + (1 - \lambda)f(w)\end{aligned}$$

Example: a convex quadratic function (3/3)

Therefore,

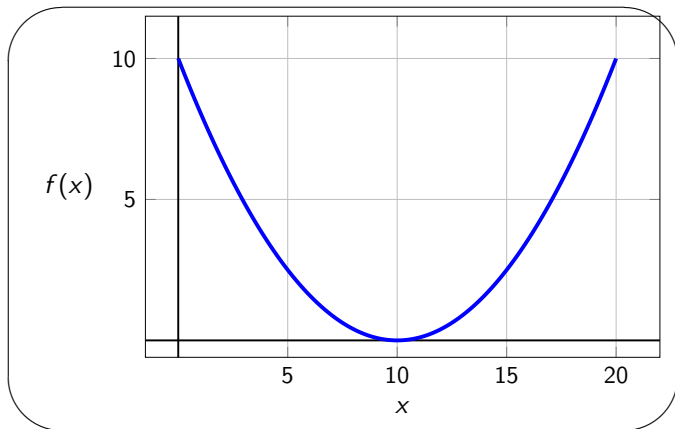
$$f(\lambda p + (1 - \lambda)w) \leq \lambda f(p) + (1 - \lambda)f(w)$$

By nature, the domain $\mathbb{R}$ is a convex set. Accordingly, the function is convex in its domain.

Example: a translated convex quadratic function (1/3)

Consider the following function:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto \frac{1}{10}(x - 10)^2$$



Example: a translated convex quadratic function (2/3)

For every pair of points $p, w \in \mathbb{R}$ and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned}f(\lambda p + (1 - \lambda)w) &= \frac{1}{10}(\lambda p + (1 - \lambda)w - 10)^2 \\&= \frac{1}{10}(\lambda(p - 10) + (1 - \lambda)(w - 10))^2 \\&\quad (\text{since } \lambda + (1 - \lambda) = 1) \\&\leq \frac{1}{10}(\lambda(p - 10)^2 + (1 - \lambda)(w - 10)^2) \\&\quad (\text{by the inequality for squares}) \\&= \lambda \frac{(p - 10)^2}{10} + (1 - \lambda) \frac{(w - 10)^2}{10} \\&= \lambda f(p) + (1 - \lambda)f(w)\end{aligned}$$

Example: a translated convex quadratic function (3/3)

Therefore,

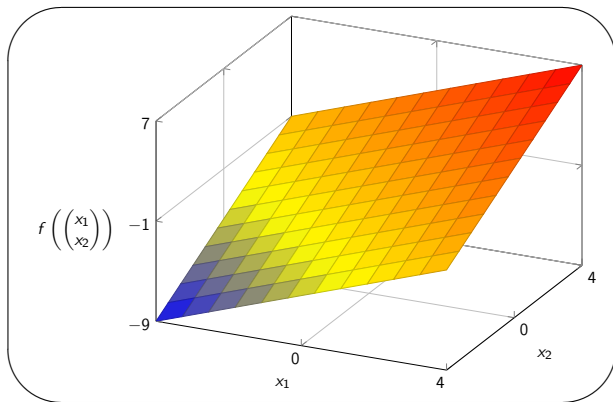
$$f(\lambda p + (1 - \lambda)w) \leq \lambda f(p) + (1 - \lambda)f(w)$$

By nature, the domain $\mathbb{R}$ is a convex set. Accordingly, the function is convex in its domain.

Example: a linear function of two variables (1/3)

Consider the following function:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1 + x_2 - 1$$



Example: a linear function of two variables (2/3)

For every pair of points $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{R}^2$ and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned} f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) &= (\lambda p_1 + (1 - \lambda) w_1) \\ &\quad + (\lambda p_2 + (1 - \lambda) w_2) - 1 \end{aligned}$$

$$\begin{aligned} &= \lambda(p_1 + p_2) + (1 - \lambda)(w_1 + w_2) \\ &\quad - \lambda - (1 - \lambda) \end{aligned}$$

$$(\text{since } \lambda + (1 - \lambda) = 1)$$

$$= \lambda(p_1 + p_2 - 1) + (1 - \lambda)(w_1 + w_2 - 1)$$

$$= \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})$$

Example: a linear function of two variables (3/3)

Therefore,

$$f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \leq \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})$$

(with equality).

- ▶ By nature, the domain $\mathbb{R}^2$ is a convex set. Accordingly, the function is convex in its domain.
- ▶ Since the equality holds, also the inequality with $\geq$ holds:

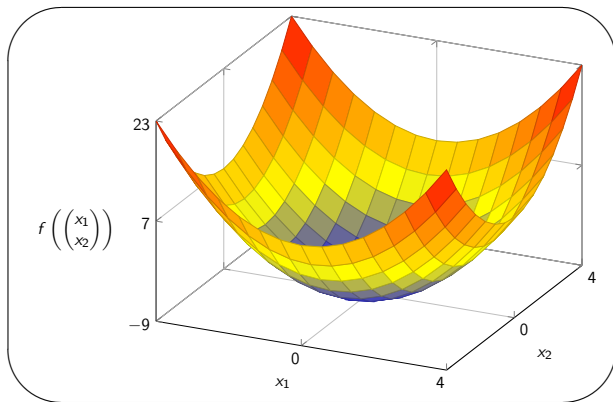
$$f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \geq \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})$$

the function is also **concave** in its domain.

Example: a convex quadratic function of two variables (1/3)

Consider the following function:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1^2 + x_2^2 - 9$$



Example: a convex quadratic function of two variables (2/3)

For every pair of points $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{R}^2$ and every $\lambda \in [0, 1]$, we have:

$$f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) = (\lambda p_1 + (1 - \lambda) w_1)^2 \\ + (\lambda p_2 + (1 - \lambda) w_2)^2 - 9$$

$$\leq \lambda p_1^2 + (1 - \lambda) w_1^2 + \lambda p_2^2 + (1 - \lambda) w_2^2 - 9$$

(by the inequality for squares, twice)

$$= \lambda(p_1^2 + p_2^2 - 9) + (1 - \lambda)(w_1^2 + w_2^2 - 9)$$

(since $\lambda + (1 - \lambda) = 1$)

$$= \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})$$

Example: a convex quadratic function of two variables (3/3)

Therefore,

$$f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \leq \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})$$

By nature, the domain $\mathbb{R}^2$ is a convex set. Accordingly, the function is convex in its domain.

Two functions whose graph is made of two segments

- ▶ The following two examples are functions of one variable whose graph is made of two segments.
- ▶ The first one, the absolute value, is convex.
- ▶ The second one, the minimum of two linear functions, is concave.



Example: the absolute value is a convex function (1/3)

Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = |x|$. For every pair of points $p, w \in \mathbb{R}$ and every $\lambda \in [0, 1]$, we have:

$$f(\lambda p + (1 - \lambda)w) = |\lambda p + (1 - \lambda)w|$$

$$\leq |\lambda p| + |(1 - \lambda)w|$$

(by the triangle inequality)

$$= |\lambda| |p| + |1 - \lambda| |w|$$

(by the absolute homogeneity)

$$= \lambda |p| + (1 - \lambda) |w|$$

(since $\lambda \geq 0$ and $1 - \lambda \geq 0$)

$$= \lambda f(p) + (1 - \lambda)f(w)$$

Example: the absolute value is a convex function (2/3)

Therefore,

$$f(\lambda p + (1 - \lambda)w) \leq \lambda f(p) + (1 - \lambda)f(w)$$

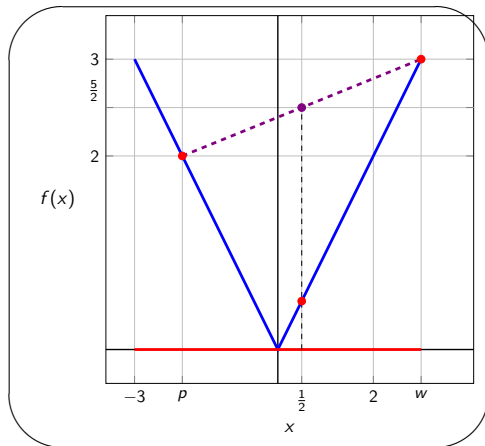
By nature, the domain $\mathbb{R}$ is a convex set. Accordingly, the function is convex in its domain.

For example, with $p = -2$, $w = 3$ and $\lambda = \frac{1}{2}$ we obtain $\lambda p + (1 - \lambda)w = \frac{1}{2}$ and

$$f\left(\frac{1}{2}\right) = \frac{1}{2} \leq \frac{1}{2} f(-2) + \frac{1}{2} f(3) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 3 = \frac{5}{2}.$$

Example: the absolute value is a convex function (3/3)

In the figure, the segment joining $(p, f(p))$ and $(w, f(w))$ (violet) lies above the graph of f (blue).



Example: the minimum of two linear functions (1/4)

Consider the function $f : [-2, 4] \rightarrow \mathbb{R}$,

$f(x) = \min\{x + 2, 4 - \frac{1}{2}x\}$. For every pair of points $p, w \in [-2, 4]$ and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned} f(\lambda p + (1 - \lambda)w) &= \min \left\{ \lambda p + (1 - \lambda)w + 2, \right. \\ &\quad \left. 4 - \frac{1}{2}(\lambda p + (1 - \lambda)w) \right\} \\ &= \min \left\{ \lambda(p + 2) + (1 - \lambda)(w + 2), \right. \\ &\quad \left. \lambda(4 - \frac{1}{2}p) + (1 - \lambda)(4 - \frac{1}{2}w) \right\} \\ &\quad (\text{since } \lambda + (1 - \lambda) = 1) \end{aligned}$$

Example: the minimum of two linear functions (2/4)

$$\geq \min \left\{ \begin{array}{l} \lambda f(p) + (1 - \lambda)f(w), \\ \lambda f(p) + (1 - \lambda)f(w) \end{array} \right\}$$

$$(\text{since } f(p) \leq p + 2, \quad f(p) \leq 4 - \frac{1}{2}p,$$

$$f(w) \leq w + 2, \quad f(w) \leq 4 - \frac{1}{2}w,$$

$$\lambda \geq 0 \text{ and } 1 - \lambda \geq 0)$$

$$= \lambda f(p) + (1 - \lambda)f(w)$$

Example: the minimum of two linear functions (3/4)

Therefore,

$$f(\lambda p + (1 - \lambda)w) \geq \lambda f(p) + (1 - \lambda)f(w)$$

The domain $[-2, 4]$ is an interval, hence a convex set. Accordingly, the function is concave in its domain.

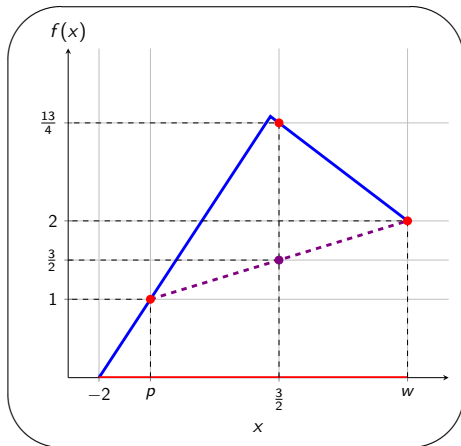
The two lines meet at $x = \frac{4}{3}$, where $f(\frac{4}{3}) = \frac{10}{3}$.

For example, with $p = -1$, $w = 4$ and $\lambda = \frac{1}{2}$ we obtain $\lambda p + (1 - \lambda)w = \frac{3}{2}$ and

$$\begin{aligned} f\left(\frac{3}{2}\right) &= \min\left\{\frac{7}{2}, \frac{13}{4}\right\} = \frac{13}{4} \\ &\geq \frac{1}{2} f(-1) + \frac{1}{2} f(4) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 2 = \frac{3}{2}. \end{aligned}$$

Example: the minimum of two linear functions (4/4)

In the figure, the segment joining $(p, f(p))$ and $(w, f(w))$ (violet) lies below the graph of f (blue).



Summary

Proving convexity:

- ▶ Always by the definition, for arbitrary $\mathbf{p}, \mathbf{w}$ and $\lambda \in [0, 1]$
- ▶ For squares: $(\lambda a + (1 - \lambda)b)^2 \leq \lambda a^2 + (1 - \lambda)b^2$

Examples:

- ▶ $\frac{x^2}{10}, \frac{1}{10}(x - 10)^2, x_1^2 + x_2^2 - 9$ and $|x|$ are convex
- ▶ $x_1 + x_2 - 1$ is both convex and concave (equality holds)
- ▶ $\min\{x + 2, 4 - \frac{1}{2}x\}$ is concave in $[-2, 4]$

Convex functions defined by norms



Introduction: Convex functions defined by norms

- ▶ In this section we introduce the **norms**, which measure the length of a vector, and the norms 1, 2, q and ∞ .
- ▶ We prove that **every norm** is a convex function, and that the squared Euclidean norm is convex.
- ▶ Norms are used to measure **distances**: the distance from a point and the squared distance from a point are convex functions.

Definition 3: norm

A *norm* on $\mathbb{R}^n$ is a function $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}$ such that, for every $\mathbf{p}, \mathbf{w} \in \mathbb{R}^n$ and every $\alpha \in \mathbb{R}$:

- ▶ $\|\mathbf{p}\| \geq 0$, and $\|\mathbf{p}\| = 0$ if and only if $\mathbf{p} = \mathbf{0}$ (positivity);
- ▶ $\|\alpha \mathbf{p}\| = |\alpha| \|\mathbf{p}\|$ (absolute homogeneity);
- ▶ $\|\mathbf{p} + \mathbf{w}\| \leq \|\mathbf{p}\| + \|\mathbf{w}\|$ (triangle inequality).

Definition 4: the norms 1, 2, q and ∞

For every $\mathbf{x} \in \mathbb{R}^n$:

$$\|\mathbf{x}\|_1 = \sum_{j=1}^n |x_j| \quad (\text{norm 1, or Manhattan norm})$$

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{j=1}^n x_j^2} \quad (\text{norm 2, or Euclidean norm})$$

$$\|\mathbf{x}\|_q = \left(\sum_{j=1}^n |x_j|^q \right)^{\frac{1}{q}} \quad (\text{norm } q, \text{ with } q \geq 1)$$

$$\|\mathbf{x}\|_\infty = \max_{j=1,\dots,n} |x_j| \quad (\text{norm } \infty, \text{ or maximum norm})$$

The norms 1 and 2 are the norm q with $q = 1$ and $q = 2$, and

$$\|\mathbf{x}\|_\infty = \lim_{q \rightarrow \infty} \|\mathbf{x}\|_q.$$

Example: the four norms of a vector

For $\mathbf{x} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ we have:

$$\|\mathbf{x}\|_1 = |3| + |-4| = 7, \quad \|\mathbf{x}\|_2 = \sqrt{3^2 + (-4)^2} = 5,$$

$$\|\mathbf{x}\|_3 = (|3|^3 + |-4|^3)^{\frac{1}{3}} = \sqrt[3]{91}, \quad \|\mathbf{x}\|_\infty = \max\{|3|, |-4|\} = 4.$$



We now prove that the norms define convex functions,
starting from the Euclidean norm.

Convexity of the Euclidean norm

Observation 2: convexity of the Euclidean norm

The function

$$f : \mathbb{R}^n \rightarrow \mathbb{R}, \quad f : \mathbf{x} \mapsto \|\mathbf{x}\|_2$$

is convex in its domain.

Proof

For every pair of points $\mathbf{p}, \mathbf{w} \in \mathbb{R}^n$ and every $\lambda \in [0, 1]$, we have:

$$f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) = \|\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}\|_2$$

$$\leq \|\lambda \mathbf{p}\|_2 + \|(1 - \lambda) \mathbf{w}\|_2$$

(by the triangle inequality)

$$= |\lambda| \|\mathbf{p}\|_2 + |1 - \lambda| \|\mathbf{w}\|_2$$

(by the absolute homogeneity)

$$= \lambda \|\mathbf{p}\|_2 + (1 - \lambda) \|\mathbf{w}\|_2$$

(since $\lambda \geq 0$ and $1 - \lambda \geq 0$)

$$= \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})$$

Proof (continued)

Therefore,

$$f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \leq \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})$$

By nature, the domain $\mathbb{R}^n$ is a convex set. Accordingly, the function is convex in its domain.



- ▶ The proof uses only the triangle inequality and the absolute homogeneity, which hold for **every** norm.
- ▶ Hence the same proof shows that every norm is a convex function.

Convexity of every norm

Observation 3: convexity of every norm

For every norm $\|\cdot\|$ on $\mathbb{R}^n$ (in particular $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_q$ with $q \geq 1$, and $\|\cdot\|_\infty$), the function

$$f : \mathbb{R}^n \rightarrow \mathbb{R}, \quad f : \mathbf{x} \mapsto \|\mathbf{x}\|$$

is convex in its domain.

Proof

For every pair of points $\mathbf{p}, \mathbf{w} \in \mathbb{R}^n$ and every $\lambda \in [0, 1]$, we have:

$$f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) = \|\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}\|$$

$$\leq \|\lambda \mathbf{p}\| + \|(1 - \lambda) \mathbf{w}\|$$

(by the triangle inequality)

$$= |\lambda| \|\mathbf{p}\| + |1 - \lambda| \|\mathbf{w}\|$$

(by the absolute homogeneity)

$$= \lambda \|\mathbf{p}\| + (1 - \lambda) \|\mathbf{w}\|$$

(since $\lambda \geq 0$ and $1 - \lambda \geq 0$)

$$= \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})$$

Proof (continued)

Therefore,

$$f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \leq \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})$$

By nature, the domain $\mathbb{R}^n$ is a convex set. Accordingly, the function is convex in its domain.



Corollary 1

For every pair of points $\mathbf{p}, \mathbf{w} \in \mathbb{R}^n$ and every $\lambda \in [0, 1]$, we have

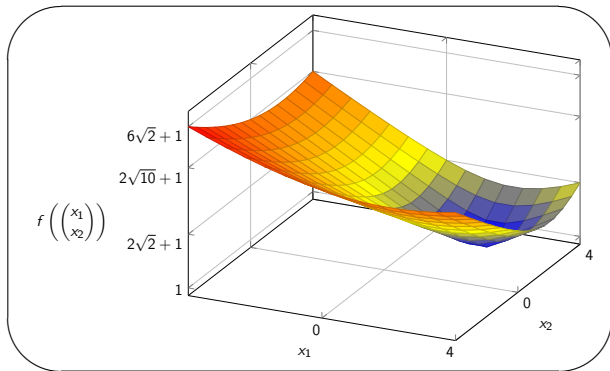
$$\|\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}\|_2 \leq \lambda \|\mathbf{p}\|_2 + (1 - \lambda) \|\mathbf{w}\|_2$$

- We use this inequality to prove that the distance from a point is a convex function.

Example: the distance from a point (1/3)

Let us consider the vector $\mathbf{c} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and the function:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\|_2 + 1$$



Example: the distance from a point (2/3)

For every pair of points $\mathbf{p}, \mathbf{w} \in \mathbb{R}^2$ and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned} f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) &= \|\lambda \mathbf{p} + (1 - \lambda) \mathbf{w} - \mathbf{c}\|_2 + 1 \\ &= \|\lambda(\mathbf{p} - \mathbf{c}) + (1 - \lambda)(\mathbf{w} - \mathbf{c})\|_2 + 1 \\ &\quad (\text{since } \lambda + (1 - \lambda) = 1) \\ &\leq \lambda \|\mathbf{p} - \mathbf{c}\|_2 + (1 - \lambda) \|\mathbf{w} - \mathbf{c}\|_2 + 1 \\ &\quad (\text{by the inequality of the corollary}) \\ &= \lambda(\|\mathbf{p} - \mathbf{c}\|_2 + 1) + (1 - \lambda)(\|\mathbf{w} - \mathbf{c}\|_2 + 1) \\ &= \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w}) \end{aligned}$$

Example: the distance from a point (3/3)

Therefore,

$$f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \leq \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})$$

By nature, the domain $\mathbb{R}^2$ is a convex set. Accordingly, the function is convex in its domain.

Convexity of the squared Euclidean norm

Observation 4: convexity of the squared Euclidean norm

The function

$$f : \mathbb{R}^n \rightarrow \mathbb{R}, \quad f : \mathbf{x} \mapsto \|\mathbf{x}\|_2^2$$

is convex in its domain.

Proof

For every pair of points $\mathbf{p}, \mathbf{w} \in \mathbb{R}^n$ and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned} f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) &= \|\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}\|_2^2 \\ &= (\lambda \mathbf{p} + (1 - \lambda) \mathbf{w})' (\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \\ &= \lambda^2 \|\mathbf{p}\|_2^2 + (1 - \lambda)^2 \|\mathbf{w}\|_2^2 \\ &\quad + 2\lambda(1 - \lambda) (\mathbf{p}' \mathbf{w}) \\ &\leq \lambda^2 \|\mathbf{p}\|_2^2 + (1 - \lambda)^2 \|\mathbf{w}\|_2^2 \\ &\quad + 2\lambda(1 - \lambda) \|\mathbf{p}\|_2 \|\mathbf{w}\|_2 \\ &\quad \text{(by the Cauchy-Schwarz inequality)} \end{aligned}$$

Proof (continued)

$$\begin{aligned} &\leq \lambda^2 \|\mathbf{p}\|_2^2 + (1 - \lambda)^2 \|\mathbf{w}\|_2^2 \\ &\quad + \lambda(1 - \lambda) \|\mathbf{p}\|_2^2 + \lambda(1 - \lambda) \|\mathbf{w}\|_2^2 \\ &\quad \text{(since } 2ab \leq a^2 + b^2 \text{ for all } a, b \in \mathbb{R}, \\ &\quad \text{with } a = \|\mathbf{p}\|_2 \text{ and } b = \|\mathbf{w}\|_2) \\ &= (\lambda^2 + \lambda(1 - \lambda)) \|\mathbf{p}\|_2^2 \\ &\quad + ((1 - \lambda)^2 + \lambda(1 - \lambda)) \|\mathbf{w}\|_2^2 \\ &= \lambda \|\mathbf{p}\|_2^2 + (1 - \lambda) \|\mathbf{w}\|_2^2 \\ &= \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w}) \end{aligned}$$

Proof (continued)

Therefore,

$$f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \leq \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})$$

By nature, the domain $\mathbb{R}^n$ is a convex set. Accordingly, the function is convex in its domain.



- ▶ The next proof is more direct and does not require the use of the Cauchy–Schwarz inequality and the inequality $2ab \leq a^2 + b^2$ for every $a, b \in \mathbb{R}$.

Alternative proof

For every pair of points $\mathbf{p}, \mathbf{w} \in \mathbb{R}^n$ and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned} f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) &= (\lambda \mathbf{p} + (1 - \lambda) \mathbf{w})' (\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \\ &= \lambda^2 \|\mathbf{p}\|_2^2 + (1 - \lambda)^2 \|\mathbf{w}\|_2^2 \\ &\quad + 2\lambda(1 - \lambda) (\mathbf{p}' \mathbf{w}) \\ &= \lambda^2 \|\mathbf{p}\|_2^2 + (1 - \lambda)^2 \|\mathbf{w}\|_2^2 \\ &\quad + \lambda(1 - \lambda) (\|\mathbf{p}\|_2^2 + \|\mathbf{w}\|_2^2 - \|\mathbf{p} - \mathbf{w}\|_2^2) \\ &\quad (\text{since } \|\mathbf{p} - \mathbf{w}\|_2^2 = \|\mathbf{p}\|_2^2 + \|\mathbf{w}\|_2^2 - 2\mathbf{p}' \mathbf{w}) \end{aligned}$$

Alternative proof (continued)

$$\begin{aligned} &= (\lambda^2 + \lambda(1 - \lambda)) \|\mathbf{p}\|_2^2 \\ &\quad + ((1 - \lambda)^2 + \lambda(1 - \lambda)) \|\mathbf{w}\|_2^2 \\ &\quad - \lambda(1 - \lambda) \|\mathbf{p} - \mathbf{w}\|_2^2 \\ &= \lambda \|\mathbf{p}\|_2^2 + (1 - \lambda) \|\mathbf{w}\|_2^2 \\ &\quad - \lambda(1 - \lambda) \|\mathbf{p} - \mathbf{w}\|_2^2 \\ &\leq \lambda \|\mathbf{p}\|_2^2 + (1 - \lambda) \|\mathbf{w}\|_2^2 \\ &\quad (\text{since } \lambda(1 - \lambda) \|\mathbf{p} - \mathbf{w}\|_2^2 \geq 0) \\ &= \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w}) \end{aligned}$$

Alternative proof (continued)

Therefore,

$$f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \leq \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})$$

By nature, the domain $\mathbb{R}^n$ is a convex set. Accordingly, the function is convex in its domain.



Corollary 2

For every pair of points $\mathbf{p}, \mathbf{w} \in \mathbb{R}^n$ and every $\lambda \in [0, 1]$, we have

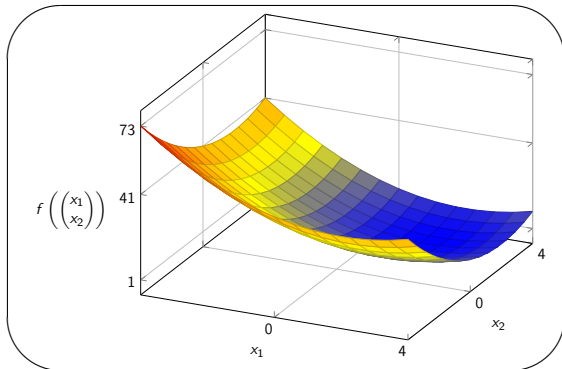
$$\|\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}\|_2^2 \leq \lambda \|\mathbf{p}\|_2^2 + (1 - \lambda) \|\mathbf{w}\|_2^2$$

- We use this inequality to prove that the squared distance from a point is a convex function.

Example: the squared distance from a point (1/3)

Let us consider the vector $\mathbf{c} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and the function:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\|_2^2 + 1$$



Example: the squared distance from a point (2/3)

For every pair of points $\mathbf{p}, \mathbf{w} \in \mathbb{R}^2$ and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned} f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) &= \|\lambda \mathbf{p} + (1 - \lambda) \mathbf{w} - \mathbf{c}\|_2^2 + 1 \\ &= \|\lambda(\mathbf{p} - \mathbf{c}) + (1 - \lambda)(\mathbf{w} - \mathbf{c})\|_2^2 + 1 \\ &\quad (\text{since } \lambda + (1 - \lambda) = 1) \\ &\leq \lambda \|\mathbf{p} - \mathbf{c}\|_2^2 + (1 - \lambda) \|\mathbf{w} - \mathbf{c}\|_2^2 + 1 \\ &\quad (\text{by the inequality of the corollary}) \\ &= \lambda(\|\mathbf{p} - \mathbf{c}\|_2^2 + 1) + (1 - \lambda)(\|\mathbf{w} - \mathbf{c}\|_2^2 + 1) \\ &= \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w}) \end{aligned}$$

Example: the squared distance from a point (3/3)

Therefore,

$$f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \leq \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})$$

By nature, the domain $\mathbb{R}^2$ is a convex set. Accordingly, the function is convex in its domain.

Summary

Norms:

- ▶ Positivity, absolute homogeneity and triangle inequality
- ▶ $\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$, $\|\mathbf{x}\|_q$ ($q \geq 1$) and $\|\mathbf{x}\|_\infty$

Convexity:

- ▶ Every norm is a convex function (triangle inequality + homogeneity)
- ▶ The squared Euclidean norm $\|\mathbf{x}\|_2^2$ is convex
- ▶ The distance and the squared distance from a point $\mathbf{c}$ are convex

Properties of convex and concave functions



Introduction: Properties of convex and concave functions

- ▶ In this section we see how to **build** new convex functions from known ones: the opposite, the sum with nonnegative coefficients and the maximum.
- ▶ These properties let us prove convexity without going back to the definition every time.
- ▶ Finally, we see that the points satisfying constraints $f_i(\mathbf{x}) \leq 0$ with convex f_i form a **convex set**: the basis of convex optimization problems.

Convex and concave: the opposite function

Observation 5

A function $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is convex in D if and only if the function $-f$ is concave in D .

Proof

Let f be convex in D . For every pair of points $\mathbf{p}, \mathbf{w} \in D$ and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned}(-f)(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) &= -f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \\ &\geq -(\lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})) \\ &\quad (\text{since } f \text{ is convex,} \\ &\quad \text{and } -1 \text{ reverses the inequality}) \\ &= \lambda (-f)(\mathbf{p}) + (1 - \lambda) (-f)(\mathbf{w})\end{aligned}$$

Therefore,

$$(-f)(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \geq \lambda (-f)(\mathbf{p}) + (1 - \lambda) (-f)(\mathbf{w})$$

Proof (continued)

- ▶ The domain D is a convex set, since f is convex in D . Accordingly, the function $-f$ is concave in its domain.
- ▶ The converse is proved in the same way, starting from the inequality of the concave function $-f$ and multiplying it by -1 .



Sum with nonnegative coefficients

Observation 6

Let $f, g : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ be convex in D and let $\alpha, \beta \geq 0$. Then the function $\alpha f + \beta g$ is convex in D . The same holds for concave functions.

Proof

For every pair of points $\mathbf{p}, \mathbf{w} \in D$ and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned}(\alpha f + \beta g)(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) &= \alpha f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \\ &\quad + \beta g(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \\ &\leq \alpha(\lambda f(\mathbf{p}) + (1 - \lambda)f(\mathbf{w})) \\ &\quad + \beta(\lambda g(\mathbf{p}) + (1 - \lambda)g(\mathbf{w})) \\ &\quad \text{(since } f \text{ and } g \text{ are convex,} \\ &\quad \text{and } \alpha \geq 0, \beta \geq 0) \\ &= \lambda(\alpha f(\mathbf{p}) + \beta g(\mathbf{p})) \\ &\quad + (1 - \lambda)(\alpha f(\mathbf{w}) + \beta g(\mathbf{w})) \\ &= \lambda(\alpha f + \beta g)(\mathbf{p}) \\ &\quad + (1 - \lambda)(\alpha f + \beta g)(\mathbf{w})\end{aligned}$$

Proof (continued)

Therefore,

$$\begin{aligned} &(\alpha f + \beta g)(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \\ &\leq \lambda (\alpha f + \beta g)(\mathbf{p}) + (1 - \lambda) (\alpha f + \beta g)(\mathbf{w}) \end{aligned}$$

The domain D is a convex set, since f is convex in D . Accordingly, the function $\alpha f + \beta g$ is convex in its domain. For concave functions the same argument holds with $\geq$.



- ▶ The point-wise maximum of convex functions is itself a convex function: this allows us to combine multiple convex functions into a single one while preserving convexity.

Observation 7

For every m functions $f_i : D_i \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ convex in their domains, with $i \in \{1, 2, \dots, m\}$, the function

$$f : \bigcap_{i=1}^m D_i \subseteq \mathbb{R}^n \rightarrow \mathbb{R}, \quad \mathbf{x} \mapsto \max \{ f_i(\mathbf{x}) : i \in \{1, 2, \dots, m\} \}$$

is convex in its domain. Similarly, the minimum of m concave functions is concave in its domain.

Proof

For every pair of points $\mathbf{p}, \mathbf{w} \in \bigcap_{i=1}^m D_i$ and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned} f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) &= \max \{ f_i(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) : i \in \{1, 2, \dots, m\} \} \\ &= f_k(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \\ &\quad \text{(for an index } k \in \{1, 2, \dots, m\} \\ &\quad \text{attaining the maximum)} \\ &\leq \lambda f_k(\mathbf{p}) + (1 - \lambda) f_k(\mathbf{w}) \\ &\quad \text{(since } f_k \text{ is convex)} \end{aligned}$$

Proof (continued)

$$\begin{aligned} &\leq \lambda \max \{f_i(\mathbf{p}) : i \in \{1, 2, \dots, m\}\} \\ &\quad + (1 - \lambda) \max \{f_i(\mathbf{w}) : i \in \{1, 2, \dots, m\}\} \\ &\quad \text{(since } \lambda \geq 0 \text{ and } 1 - \lambda \geq 0) \\ &= \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w}) \end{aligned}$$

Proof (continued)

Therefore,

$$f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \leq \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})$$

The domain $\bigcap_{i=1}^m D_i$ is a convex set, since it is the intersection of convex sets. Accordingly, the function f is convex in its domain.

The statement for the minimum of concave functions follows by applying this one to $-f_1, \dots, -f_m$.



The two examples with two segments, again

- ▶ The absolute value $|x| = \max\{x, -x\}$ is the maximum of the two functions x and $-x$, which are convex (and concave), since the definition holds with equality:

$$-(\lambda p + (1 - \lambda)w) = \lambda(-p) + (1 - \lambda)(-w)$$

- ▶ The function $\min\{x + 2, 4 - \frac{1}{2}x\}$ is the minimum of the two functions $x + 2$ and $4 - \frac{1}{2}x$, which are concave for the same reason:

$$(\lambda p + (1 - \lambda)w) + 2 = \lambda(p + 2) + (1 - \lambda)(w + 2)$$

$$4 - \frac{1}{2}(\lambda p + (1 - \lambda)w) = \lambda(4 - \frac{1}{2}p) + (1 - \lambda)(4 - \frac{1}{2}w)$$

- ▶ By the previous observation, we find again that the two functions are convex and concave, respectively.

Example: maximum of convex functions (1/2)

Consider the following two convex functions (seen in the examples):

$$f_1 : \mathbb{R} \rightarrow \mathbb{R}, \quad f_1 : x \mapsto \frac{1}{10}(x - 10)^2$$

and

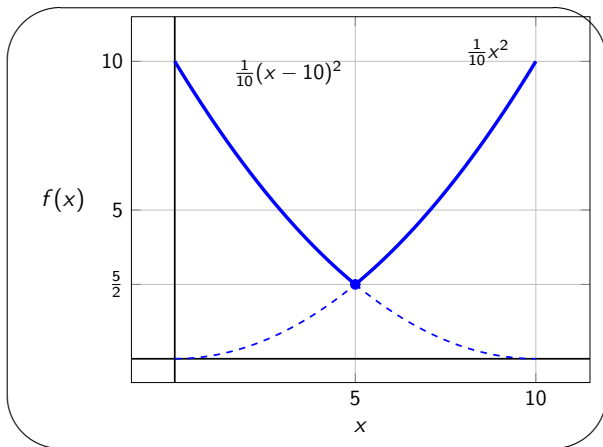
$$f_2 : \mathbb{R} \rightarrow \mathbb{R}, \quad f_2 : x \mapsto \frac{1}{10} x^2$$

According to the previous observation, the following function is convex in its domain:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto \max \left\{ \frac{1}{10}(x - 10)^2, \frac{1}{10} x^2 \right\}$$

Example: maximum of convex functions (2/2)

In the figure, the graph of f is the solid blue line; the dashed parts of the two parabolas lie below it.



We now use convex functions to build convex sets.



Observation 8

For every m functions $f_i : D_i \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ convex in their domains, with $i \in \{1, 2, \dots, m\}$, the set

$$\left\{ \mathbf{x} \in \bigcap_{i=1}^m D_i : f_i(\mathbf{x}) \leq 0, \forall i \in \{1, 2, \dots, m\} \right\}$$

is convex.

Proof

For every pair of points $\mathbf{p}$ and $\mathbf{w}$ in the set and every $\lambda \in [0, 1]$, the point $\lambda\mathbf{p} + (1 - \lambda)\mathbf{w}$ belongs to $\bigcap_{i=1}^m D_i$, since $\mathbf{p}, \mathbf{w} \in D_i$ and each domain D_i is convex, and for every $i \in \{1, 2, \dots, m\}$ we have:

$$f_i(\lambda\mathbf{p} + (1 - \lambda)\mathbf{w}) \leq \lambda f_i(\mathbf{p}) + (1 - \lambda)f_i(\mathbf{w})$$

(since f_i is convex)

$$\leq \lambda 0 + (1 - \lambda) 0$$

(since $f_i(\mathbf{p}) \leq 0$, $f_i(\mathbf{w}) \leq 0$,

$\lambda \geq 0$ and $1 - \lambda \geq 0$)

$$= 0$$

Proof (continued)

Therefore,

$$f_i(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \leq 0, \quad \forall i \in \{1, 2, \dots, m\}$$

Accordingly, $\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}$ belongs to the set and the set is convex.



Example: convex set from convex functions (1/2)

Consider the following two convex functions (seen in the examples):

$$f_1 : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f_1 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1^2 + x_2^2 - 9$$

and

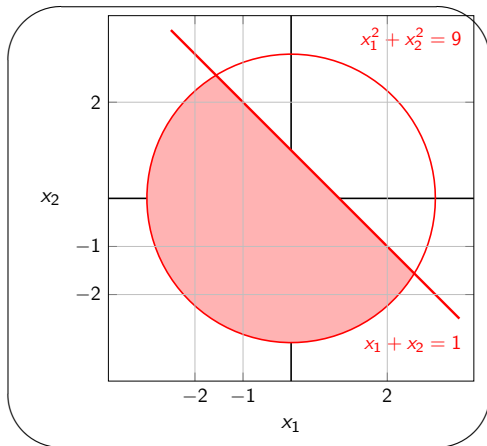
$$f_2 : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f_2 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1 + x_2 - 1$$

According to the previous observation, the following is a convex set:

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1^2 + x_2^2 - 9 \leq 0 \quad \text{and} \quad x_1 + x_2 - 1 \leq 0 \right\}$$

Example: convex set from convex functions (2/2)

It is the disk of radius 3 centered at the origin (red circle) cut by the half-plane $x_1 + x_2 \leq 1$ (red line), shaded in red.



Example: another convex set from convex functions (1/3)

Consider the following two convex functions:

$$f_1 : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f_1 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto |x_1| + |x_2| - 2$$

and

$$f_2 : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f_2 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1 + x_2 - 1.$$

- ▶ The function f_1 is convex because $|x_1| = \max\{x_1, -x_1\}$ and $|x_2| = \max\{x_2, -x_2\}$ are maxima of convex functions (as $|x| = \max\{x, -x\}$ above), and the sum of convex functions is convex.
- ▶ The function f_2 is convex by the example of the linear function of two variables.

Example: another convex set from convex functions (2/3)

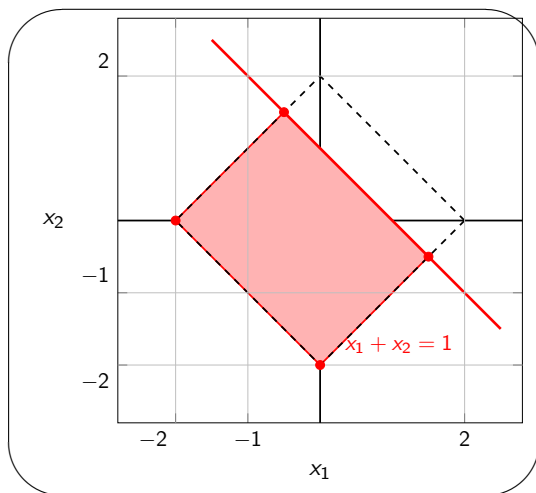
Also in this case, according to the same observation, the following is a convex set:

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : |x_1| + |x_2| - 2 \leq 0 \text{ and } x_1 + x_2 - 1 \leq 0 \right\}.$$

It is the square with vertices $(\pm 2, 0)$ and $(0, \pm 2)$ (dashed) cut by the half-plane $x_1 + x_2 \leq 1$ (red line): the quadrilateral with vertices $(-2, 0)$, $(0, -2)$, $(\frac{3}{2}, -\frac{1}{2})$ and $(-\frac{1}{2}, \frac{3}{2})$ shaded in red.



Example: another convex set from convex functions (3/3)



Convex sets defined by constraints

- ▶ The points that satisfy a finite number of constraints $f_i(\mathbf{x}) \leq 0$ with convex functions f_i form a convex set.
- ▶ For instance, every half-space $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}'\mathbf{x} \leq b\}$, with $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$, is a convex set of this kind, since the linear function $\mathbf{a}'\mathbf{x} - b$ is convex.
- ▶ So is every intersection of finitely many half-spaces, as the polyhedra of linear programming.

Summary

Building convex functions:

- ▶ f is convex if and only if $-f$ is concave
- ▶ $\alpha f + \beta g$ with $\alpha, \beta \geq 0$ is convex if f and g are convex
- ▶ The maximum of convex functions is convex; the minimum of concave functions is concave

Convex sets:

- ▶ $\{\mathbf{x} : f_i(\mathbf{x}) \leq 0, i = 1, \dots, m\}$ is convex if every f_i is convex
- ▶ Half-spaces and polyhedra are convex sets of this kind

Key messages

Summary (1/2)

1. Definitions:

- ▶ Convex: $f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \leq \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})$, with convex domain
- ▶ Concave: the same with $\geq$

2. How to prove convexity:

- ▶ Check the definition: start from $f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w})$ and arrive at $\lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})$, justifying every step

Summary (2/2)

3. Convex functions to remember:

- ▶ Linear functions (convex and concave), $|x|$, convex quadratic functions
- ▶ Every norm, the squared Euclidean norm, the (squared) distance from a point

4. Properties:

- ▶ Opposite, sums with nonnegative coefficients, maximum
- ▶ Constraints $f_i(\mathbf{x}) \leq 0$ with convex f_i define a convex set