

Convex sets

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Convex combination



Introduction: Convex combination

- ▶ Before studying convex sets, we need a way to **combine points**: the **convex combination**.
- ▶ A convex combination of two points is a point of the **segment** that joins them; with m points we obtain the points of their **convex hull**.
- ▶ It is the basic tool of the whole course: convex sets, and later convex functions, are defined through convex combinations.



Definition 1: convex combination

Given two points

$$\underbrace{\begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}}_{\mathbf{p}} \in \mathbb{R}^n \quad \text{and} \quad \underbrace{\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}}_{\mathbf{w}} \in \mathbb{R}^n$$

their **convex combination** with a scalar $\lambda \in [0, 1]$ is the point

$$\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}$$

Convex combination: components

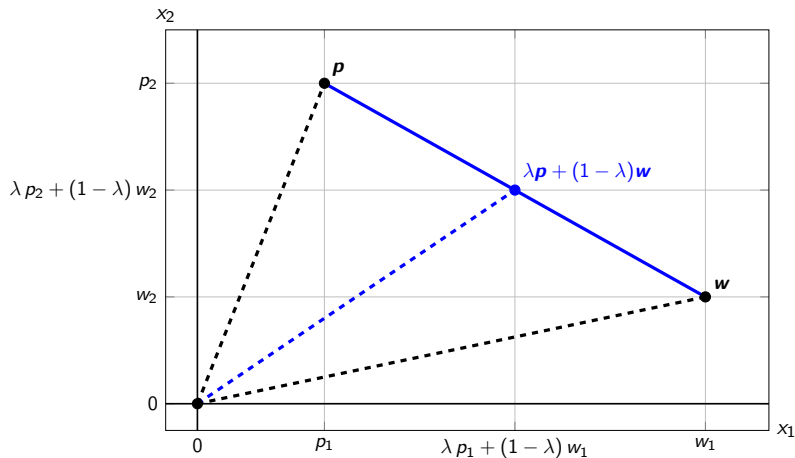
Component by component, the convex combination of $\mathbf{p}$ and $\mathbf{w}$ is the point

$$\underbrace{\begin{pmatrix} \lambda p_1 + (1 - \lambda)w_1 \\ \lambda p_2 + (1 - \lambda)w_2 \\ \vdots \\ \lambda p_n + (1 - \lambda)w_n \end{pmatrix}}_{\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}}$$

Convex combinations and segments

- ▶ The convex combinations for all values $\lambda \in [0, 1]$ are the points on the segment connecting the two points $\mathbf{p}$ and $\mathbf{w}$.
- ▶ For $\lambda = 1$ we obtain $\mathbf{p}$, for $\lambda = 0$ we obtain $\mathbf{w}$, and for $\lambda = \frac{1}{2}$ the midpoint of the segment.
- ▶ In $\mathbb{R}^2$, the convex combinations are the points of the blue segment of the following figure.

The segment between p and w



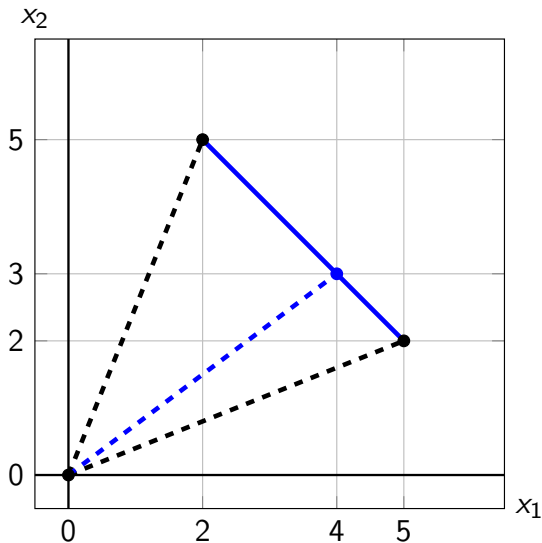
Example (1/2)

Let us consider the points $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ in $\mathbb{R}^2$.

Their convex combination with $\lambda = \frac{1}{3}$ is the point:

$$\begin{pmatrix} \frac{1}{3} \cdot 2 + (1 - \frac{1}{3})5 \\ \frac{1}{3} \cdot 5 + (1 - \frac{1}{3})2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

Example (2/2)



From two points to m points

- ▶ With two points, the convex combination uses the two weights λ and $1 - \lambda$: both are nonnegative and their sum is 1.
- ▶ The same idea works for any number m of points: we use m weights $\lambda_1, \lambda_2, \dots, \lambda_m$, nonnegative and with sum 1.
- ▶ For $m = 2$, with $\lambda_1 = \lambda$ and $\lambda_2 = 1 - \lambda$, we find again the previous definition.



Definition 2: convex combination of m points

Given m points

$$\underbrace{\begin{pmatrix} p_1^{(1)} \\ p_2^{(1)} \\ \vdots \\ p_n^{(1)} \end{pmatrix}}_{\mathbf{p}^{(1)}} \in \mathbb{R}^n \quad \underbrace{\begin{pmatrix} p_1^{(2)} \\ p_2^{(2)} \\ \vdots \\ p_n^{(2)} \end{pmatrix}}_{\mathbf{p}^{(2)}} \in \mathbb{R}^n \quad \dots \quad \underbrace{\begin{pmatrix} p_1^{(m)} \\ p_2^{(m)} \\ \vdots \\ p_n^{(m)} \end{pmatrix}}_{\mathbf{p}^{(m)}} \in \mathbb{R}^n$$

their **convex combination** with scalars $\lambda_1, \lambda_2, \dots, \lambda_m \geq 0$ satisfying $\sum_{k=1}^m \lambda_k = 1$ is the point

$$\sum_{k=1}^m \lambda_k \mathbf{p}^{(k)}$$

Convex combination of m points: components

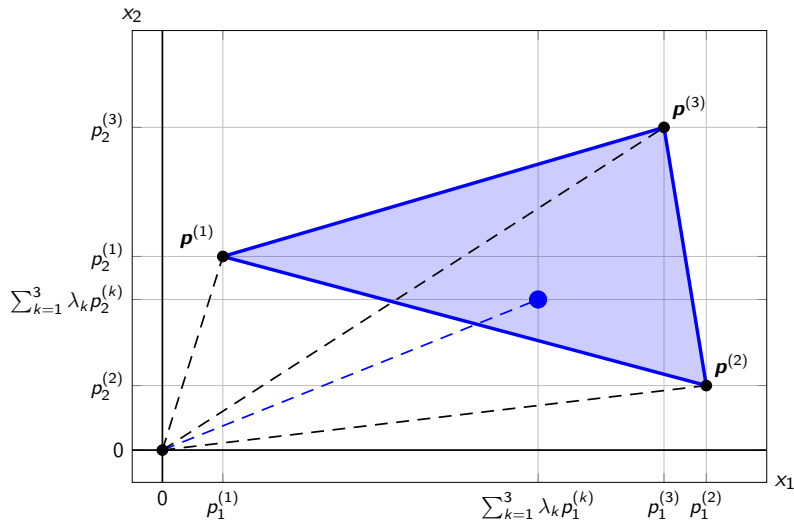
Component by component, the convex combination of the m points is the point

$$\underbrace{\begin{pmatrix} \lambda_1 p_1^{(1)} + \lambda_2 p_1^{(2)} + \cdots + \lambda_m p_1^{(m)} \\ \lambda_1 p_2^{(1)} + \lambda_2 p_2^{(2)} + \cdots + \lambda_m p_2^{(m)} \\ \vdots \\ \lambda_1 p_n^{(1)} + \lambda_2 p_n^{(2)} + \cdots + \lambda_m p_n^{(m)} \end{pmatrix}}_{\sum_{k=1}^m \lambda_k \mathbf{p}^{(k)}}$$

Convex hull

- ▶ The convex combinations for all valid scalars $\lambda_1, \lambda_2, \dots, \lambda_m \geq 0$ are the points in the **convex hull** of the m points.
- ▶ In $\mathbb{R}^2$ with $m = 3$ points, the convex hull is the set of all the points inside (and on the boundary of) the triangle connecting the three points.
- ▶ In the following figure the convex hull is the blue filled region, and the blue point is the convex combination with $\lambda_1 = \frac{1}{3}$, $\lambda_2 = \frac{1}{2}$ and $\lambda_3 = \frac{1}{6}$.

The convex hull of three points



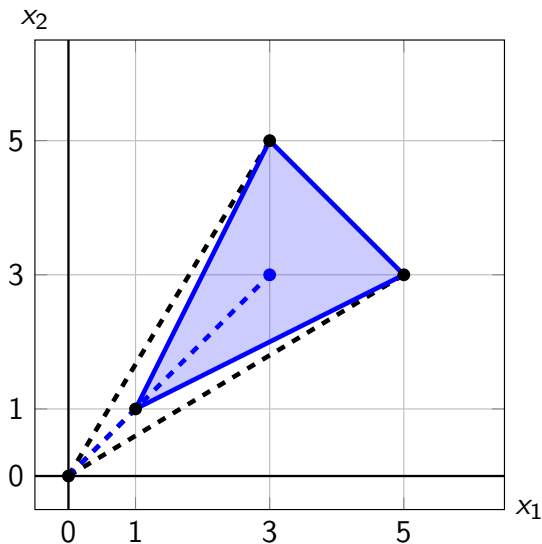
Example (1/2)

Let us consider the three points $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ in $\mathbb{R}^2$.

Their convex combination with $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$ is the point:

$$\begin{pmatrix} \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 5 + \frac{1}{3} \cdot 3 \\ \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 3 + \frac{1}{3} \cdot 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

Example (2/2)



Summary

Convex combination of two points:

- ▶ $\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}$ with $\lambda \in [0, 1]$
- ▶ For all values of λ we obtain the segment that joins $\mathbf{p}$ and $\mathbf{w}$

Convex combination of m points:

- ▶ $\sum_{k=1}^m \lambda_k \mathbf{p}^{(k)}$ with $\lambda_1, \lambda_2, \dots, \lambda_m \geq 0$ and $\sum_{k=1}^m \lambda_k = 1$
- ▶ For all valid scalars we obtain the convex hull of the m points

Convex sets

Introduction: Convex sets

- ▶ In this section we define **convex sets**: the sets that contain the whole segment between any two of their points.
- ▶ We compare convex and non-convex sets in $\mathbb{R}^2$, both with curved boundaries and with straight edges.
- ▶ Convex sets are the domains of convex optimization: the feasible regions of the problems we will study in this course are convex sets.

Definition 3: convex set

A set $A \subseteq \mathbb{R}^n$ is *convex* if

$\forall \mathbf{p}, \mathbf{w} \in A$ and $\forall \lambda \in [0, 1]$ we have $\underbrace{\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}}_{\text{convex combination}} \in A$

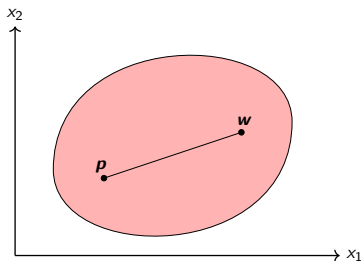
Convex sets in words

- ▶ A subset of the n -dimensional space is convex if for any of its two points, any convex combination of the two points belongs to the subset.
- ▶ In other words, if all points on the segment that joins any two points of the subset belong to the subset.
- ▶ To show that a set is **not** convex, one pair of points $\mathbf{p}, \mathbf{w}$ whose segment leaves the set is enough.

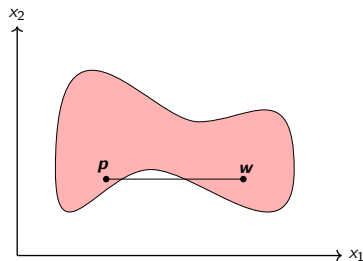


Convex and non-convex sets (1/2)

In $\mathbb{R}^2$ (the 2-dimensional space), an example of a convex set and an example of a non-convex set are given by the following picture, first with curved boundaries:



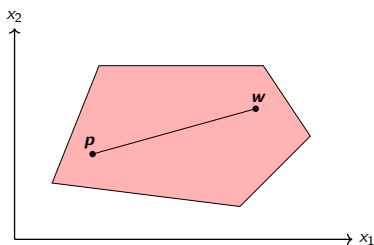
Convex set



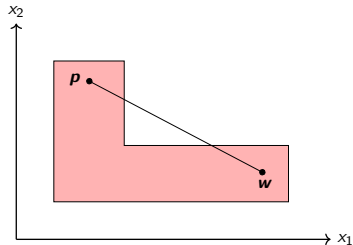
Non-convex set

Convex and non-convex sets (2/2)

The same with straight-edged sets: on the left a convex polygon, on the right a non-convex one.



Convex set



Non-convex set

Summary

Convex set:

- ▶ $A \subseteq \mathbb{R}^n$ is convex if $\lambda \mathbf{p} + (1 - \lambda) \mathbf{w} \in A$ for all $\mathbf{p}, \mathbf{w} \in A$ and all $\lambda \in [0, 1]$
- ▶ Equivalently: the segment that joins any two points of the set belongs to the set

Non-convex set:

- ▶ There are two points of the set whose segment leaves the set

Examples of convex sets



Introduction: Examples of convex sets

- ▶ In this section we prove that three important families of sets are convex: the sets of points at distance at most r from a center $\mathbf{c}$, measured with the norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$.
- ▶ In $\mathbb{R}^2$ they are a diamond, a disk and a square.
- ▶ The three proofs use only the **definition** of convex set: this is the method we will use in the exercises.



How we prove convexity

- ▶ We take any two points $\mathbf{p}, \mathbf{w}$ of the set and any $\lambda \in [0, 1]$.
- ▶ We evaluate the condition of the set at the convex combination $\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}$ and, with a single chain of steps, we show that the condition holds.
- ▶ The steps use two properties of every norm: the **triangle inequality** and the **absolute homogeneity**.



The norm $\|\cdot\|_1$: a diamond

Observation 1

For every vector $\mathbf{c} \in \mathbb{R}^n$ and every scalar $r \in \mathbb{R}_{>0}$, the set

$$\{ \mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{c}\|_1 \leq r \}$$

is convex.

It is the set of points in $\mathbb{R}^n$ whose sum of absolute deviations from $\mathbf{c}$ is at most r . In $\mathbb{R}^2$ this is a diamond (a square rotated by 45°), and in $\mathbb{R}^3$ it is a regular octahedron.

Proof

For every pair of points $\mathbf{p}, \mathbf{w}$ in the set and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned}\|\lambda \mathbf{p} + (1 - \lambda) \mathbf{w} - \mathbf{c}\|_1 &= \|\lambda(\mathbf{p} - \mathbf{c}) + (1 - \lambda)(\mathbf{w} - \mathbf{c})\|_1 \\ &\leq \|\lambda(\mathbf{p} - \mathbf{c})\|_1 + \|(1 - \lambda)(\mathbf{w} - \mathbf{c})\|_1 \\ &\quad \text{(by the triangle inequality)} \\ &= |\lambda| \|\mathbf{p} - \mathbf{c}\|_1 + |1 - \lambda| \|\mathbf{w} - \mathbf{c}\|_1 \\ &\quad \text{(by the absolute homogeneity)} \\ &= \lambda \|\mathbf{p} - \mathbf{c}\|_1 + (1 - \lambda) \|\mathbf{w} - \mathbf{c}\|_1 \\ &\quad \text{(since } \lambda \geq 0, 1 - \lambda \geq 0\text{)} \\ &\leq \lambda r + (1 - \lambda)r = r \\ &\quad \text{(since } \|\mathbf{p} - \mathbf{c}\|_1 \leq r \text{ and } \|\mathbf{w} - \mathbf{c}\|_1 \leq r\text{)}\end{aligned}$$

Therefore, $\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}$ belongs to the set, and the set is convex. □

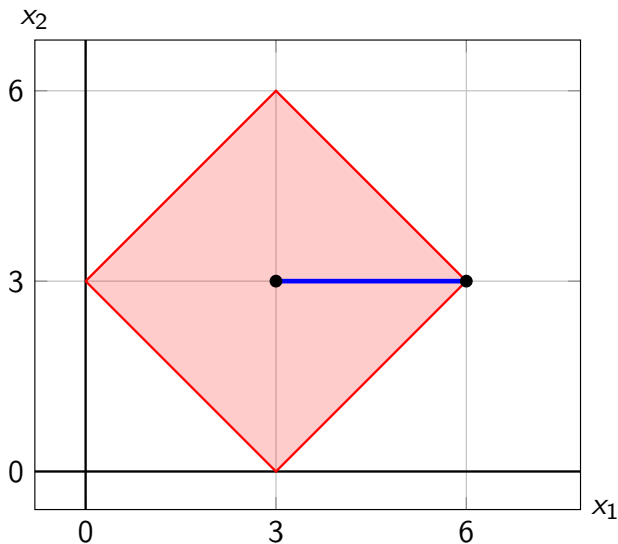
Example (1/2)

For example, in $\mathbb{R}^2$, with $r = 3$ and $\mathbf{c} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$, we have the following convex set:

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : |x_1 - 3| + |x_2 - 3| \leq 3 \right\}$$

The points of this set are all those lying inside (and on) the diamond (rotated square) centered at $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$, with vertices at $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 6 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, shown as the red shaded region in the figure below.

Example (2/2)



The same proof works for the norm $\|\cdot\|_2$:
only the norm changes, the steps are the same.

The norm $\|\cdot\|_2$: a disk

Observation 2

For every vector $\mathbf{c} \in \mathbb{R}^n$ and every scalar $r \in \mathbb{R}_{>0}$, the set

$$\{ \mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{c}\|_2 \leq r \}$$

is convex.

It is the set of points in $\mathbb{R}^n$ contained in the closed ball of radius r centered at $\mathbf{c}$. In $\mathbb{R}^2$ this is a disk, and in $\mathbb{R}^3$ it is a solid ball.

Proof

For every pair of points $\mathbf{p}, \mathbf{w}$ in the set and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned}\|\lambda \mathbf{p} + (1 - \lambda) \mathbf{w} - \mathbf{c}\|_2 &= \|\lambda(\mathbf{p} - \mathbf{c}) + (1 - \lambda)(\mathbf{w} - \mathbf{c})\|_2 \\ &\leq \|\lambda(\mathbf{p} - \mathbf{c})\|_2 + \|(1 - \lambda)(\mathbf{w} - \mathbf{c})\|_2 \\ &\quad \text{(by the triangle inequality)} \\ &= |\lambda| \|\mathbf{p} - \mathbf{c}\|_2 + |1 - \lambda| \|\mathbf{w} - \mathbf{c}\|_2 \\ &\quad \text{(by the absolute homogeneity)} \\ &= \lambda \|\mathbf{p} - \mathbf{c}\|_2 + (1 - \lambda) \|\mathbf{w} - \mathbf{c}\|_2 \\ &\quad \text{(since } \lambda \geq 0, 1 - \lambda \geq 0\text{)} \\ &\leq \lambda r + (1 - \lambda)r = r \\ &\quad \text{(since } \|\mathbf{p} - \mathbf{c}\|_2 \leq r \text{ and } \|\mathbf{w} - \mathbf{c}\|_2 \leq r\text{)}\end{aligned}$$

Therefore, $\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}$ belongs to the set, and the set is convex. □

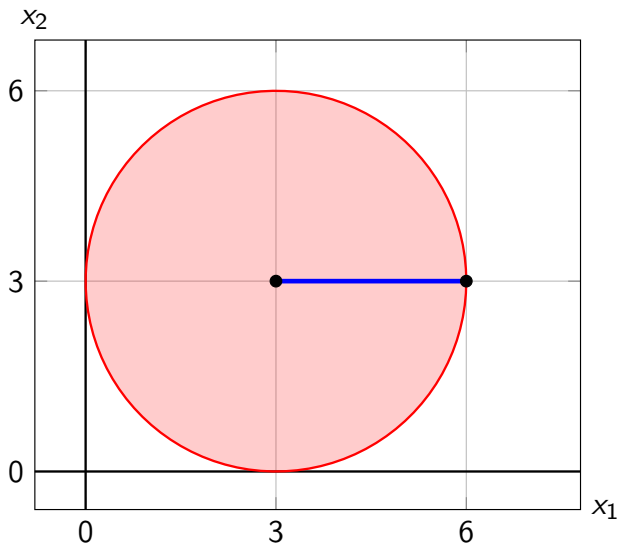
Example (1/2)

For example, in $\mathbb{R}^2$, with $r = 3$ and $\mathbf{c} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$, we have the following convex set:

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : \sqrt{(x_1 - 3)^2 + (x_2 - 3)^2} \leq 3 \right\}$$

The points of this set are all those lying inside (and on the boundary of) the disk of radius 3 centered at $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$, shown as the red shaded region in the figure below.

Example (2/2)



And once more for the norm $\|\cdot\|_\infty$,
with the same steps.

The norm $\|\cdot\|_\infty$: a square

Observation 3

For every vector $\mathbf{c} \in \mathbb{R}^n$ and every scalar $r \in \mathbb{R}_{>0}$, the set

$$\{ \mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{c}\|_\infty \leq r \}$$

is convex.

It is the set of points in $\mathbb{R}^n$ whose maximum absolute deviation from $\mathbf{c}$ is at most r . In $\mathbb{R}^2$ this is an axis-aligned square of side $2r$ centered at $\mathbf{c}$, and in $\mathbb{R}^3$ it is a cube.

Proof

For every pair of points $\mathbf{p}, \mathbf{w}$ in the set and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned}\|\lambda \mathbf{p} + (1 - \lambda) \mathbf{w} - \mathbf{c}\|_{\infty} &= \|\lambda(\mathbf{p} - \mathbf{c}) + (1 - \lambda)(\mathbf{w} - \mathbf{c})\|_{\infty} \\ &\leq \|\lambda(\mathbf{p} - \mathbf{c})\|_{\infty} + \|(1 - \lambda)(\mathbf{w} - \mathbf{c})\|_{\infty} \\ &\quad \text{(by the triangle inequality)} \\ &= |\lambda| \|\mathbf{p} - \mathbf{c}\|_{\infty} + |1 - \lambda| \|\mathbf{w} - \mathbf{c}\|_{\infty} \\ &\quad \text{(by the absolute homogeneity)} \\ &= \lambda \|\mathbf{p} - \mathbf{c}\|_{\infty} + (1 - \lambda) \|\mathbf{w} - \mathbf{c}\|_{\infty} \\ &\quad \text{(since } \lambda \geq 0, 1 - \lambda \geq 0\text{)} \\ &\leq \lambda r + (1 - \lambda)r = r \\ &\quad \text{(since } \|\mathbf{p} - \mathbf{c}\|_{\infty} \leq r \text{ and } \|\mathbf{w} - \mathbf{c}\|_{\infty} \leq r\text{)}\end{aligned}$$

Therefore, $\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}$ belongs to the set, and the set is convex.



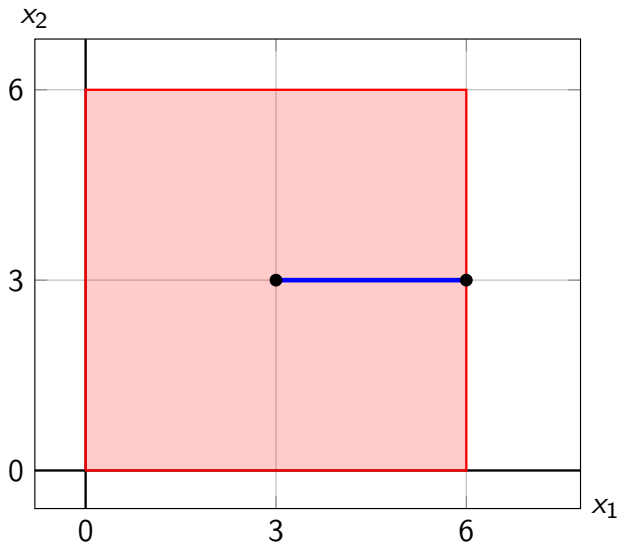
Example (1/2)

For example, in $\mathbb{R}^2$, with $r = 3$ and $\mathbf{c} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$, we have the following convex set:

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : \max \{ |x_1 - 3|, |x_2 - 3| \} \leq 3 \right\}$$

The points of this set are all those lying inside (and on) the axis-aligned square of side 6 centered at $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$, with vertices at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 6 \\ 6 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 6 \end{pmatrix}$, shown as the red shaded region in the figure below.

Example (2/2)



Convex sets defined by a norm:

- ▶ For every $\mathbf{c} \in \mathbb{R}^n$ and $r \in \mathbb{R}_{>0}$, the sets $\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{c}\| \leq r\}$ with the norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are convex
- ▶ In $\mathbb{R}^2$: a diamond, a disk and a square; in $\mathbb{R}^3$: an octahedron, a solid ball and a cube

The proof:

- ▶ From the definition, with the triangle inequality and the absolute homogeneity of the norm

Properties of convex sets



Introduction: Properties of convex sets

- ▶ In this section we prove that the **intersection** of convex sets is a convex set.
- ▶ This property lets us build new convex sets from simpler ones: a set described by several conditions, each one defining a convex set, is convex.
- ▶ The feasible regions of optimization problems are intersections of this kind: one set for each constraint.

Observation 4

For every m convex sets $A_i \subseteq \mathbb{R}^n$ with $i \in \{1, 2, \dots, m\}$, the set

$$\bigcap_{i=1}^m A_i$$

is convex.

It is the set of points in $\mathbb{R}^n$ that belong to all the sets $A_1, A_2, \dots, A_m$ at the same time.

Proof

For every pair of points $\mathbf{p}, \mathbf{w} \in \bigcap_{i=1}^m A_i$ and every $\lambda \in [0, 1]$, we have:

$$\mathbf{p}, \mathbf{w} \in \bigcap_{i=1}^m A_i \implies \mathbf{p} \in A_i \text{ and } \mathbf{w} \in A_i, \quad \forall i \in \{1, 2, \dots, m\}$$

(by definition of intersection)

$$\implies \lambda \mathbf{p} + (1 - \lambda) \mathbf{w} \in A_i, \quad \forall i \in \{1, 2, \dots, m\}$$

(since each A_i is convex)

$$\implies \lambda \mathbf{p} + (1 - \lambda) \mathbf{w} \in \bigcap_{i=1}^m A_i$$

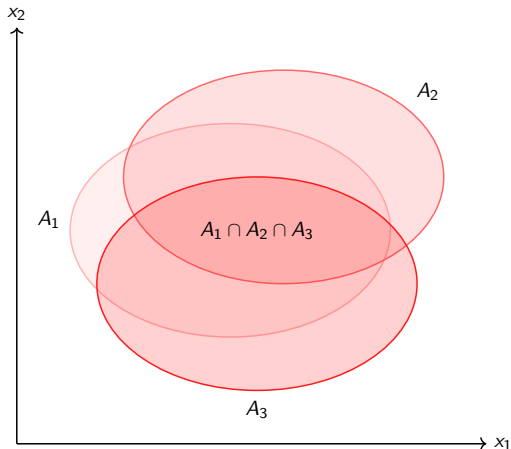
(by definition of intersection)

Therefore, $\lambda \mathbf{p} + (1 - \lambda) \mathbf{w} \in \bigcap_{i=1}^m A_i$, and the set is convex. □

Let us now see some pictures of the intersection of three convex sets A_1 , A_2 and A_3 in $\mathbb{R}^2$.

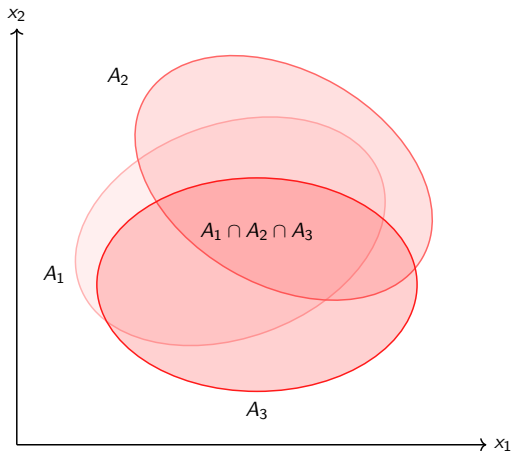
Intersection of convex sets (1/4)

In $\mathbb{R}^2$ (the 2-dimensional space), examples of intersection of three convex sets A_1, A_2 and A_3 are given by the following pictures, first with three ellipses:



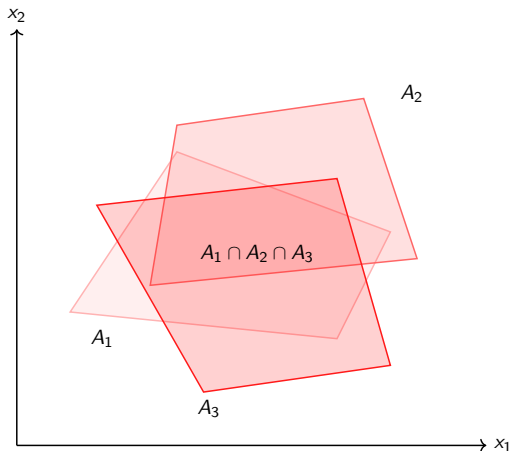
Intersection of convex sets (2/4)

Again three ellipses, two of them rotated:



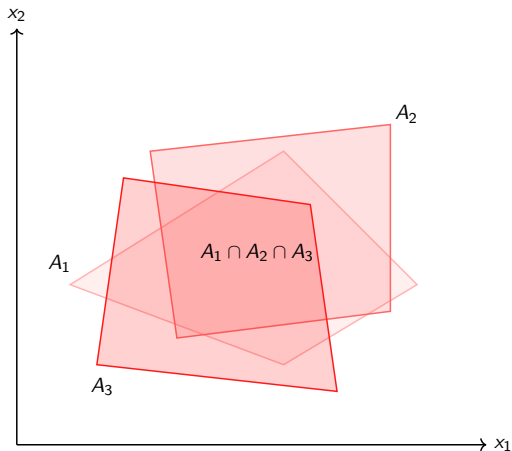
Intersection of convex sets (3/4)

The same with three convex quadrilaterals (straight edges, like the polyhedra of linear programming, which are all convex):



Intersection of convex sets (4/4)

Again three convex quadrilaterals, in a different position:



Summary

Intersection of convex sets:

- ▶ For every m convex sets $A_1, A_2, \dots, A_m \subseteq \mathbb{R}^n$, the set $\bigcap_{i=1}^m A_i$ is convex
- ▶ The proof uses only the definition of intersection and the convexity of each A_i

Why it matters:

- ▶ A set described by several conditions, each defining a convex set, is convex

Key messages

Summary (1/2)

1. Convex combination:

- ▶ Of two points: $\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}$ with $\lambda \in [0, 1]$, the points of the segment
- ▶ Of m points: $\sum_{k=1}^m \lambda_k \mathbf{p}^{(k)}$ with $\lambda_k \geq 0$ and $\sum_{k=1}^m \lambda_k = 1$, the points of the convex hull

2. Convex set:

- ▶ $A \subseteq \mathbb{R}^n$ is convex if it contains every convex combination of any two of its points



Summary (2/2)

3. Examples:

- ▶ The sets $\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{c}\| \leq r\}$ with the norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are convex
- ▶ Convexity is proved from the definition: triangle inequality and absolute homogeneity

4. Properties:

- ▶ The intersection of convex sets is convex