



SAPIENZA
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Convex Optimization

Volume 2 — Optimization problems

Optimization problems with their optimal solutions, local and global optima, convex and concave problems, where every local optimum is global. With worked examples.

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Optimization problems

5.1 Optimization problems

Definition 5.1: optimization problem in minimization and maximization form

Given a set $F \subseteq \mathbb{R}^n$ and a function $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, with $F \subseteq D$, the associated **optimization problem** P consists of:

- in the **minimization form**, finding a point $\tilde{\mathbf{x}}(P) \in F$ such that

$$f(\tilde{\mathbf{x}}(P)) \leq f(\mathbf{x}), \quad \forall \mathbf{x} \in F;$$

- in the **maximization form**, finding a point $\tilde{\mathbf{x}}(P) \in F$ such that

$$f(\tilde{\mathbf{x}}(P)) \geq f(\mathbf{x}), \quad \forall \mathbf{x} \in F.$$

5.1.1 Feasible region and objective function

- The set F is called the **feasible region** of the optimization problem; its points $\mathbf{x} \in F$ are called the **feasible solutions** of the optimization problem.
- The function f is the **objective function** of the optimization problem; it assigns an objective value to each feasible solution.

Definition 5.2: optimal solution and optimal value

- A point $\tilde{\mathbf{x}}(P) \in F$ is called an **optimal solution** of the optimization problem.

1. In the minimization form,

$$\tilde{\mathbf{x}}(P) \in \arg \min \{ f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F \}.$$

2. In the maximization form,

$$\tilde{\mathbf{x}}(P) \in \arg \max \{ f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F \}.$$

- The value $f(\tilde{\mathbf{x}}(P))$ is called the **optimal value** of the optimization problem, and it is denoted by $z(P)$.

1. In the minimization form,

$$z(P) = \underbrace{\min\{f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F\}}_{=f(\tilde{\mathbf{x}}(P))}.$$

2. In the maximization form,

$$z(P) = \underbrace{\max\{f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F\}}_{=f(\tilde{\mathbf{x}}(P))}.$$

- An optimization problem may admit multiple (possibly infinitely many) optimal solutions, or none at all. In such cases, the operators $\arg \min$ and $\arg \max$ return the full set of optimal solutions when they exist, and return the empty set otherwise.
- With a slight abuse of notation:

1. In the **minimization form**, we write the optimization problem as

$$\min\{f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F\}.$$

2. In the **maximization form**, we write the optimization problem as

$$\max\{f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F\}.$$

Although we use the symbols “min” and “max”, the goal of an optimization problem is to **find an optimal solution** $\tilde{\mathbf{x}}(P)$, whenever it exists, rather than merely determining the optimal value $z(P)$.

- In the figures of this chapter, the graph of the objective function over the feasible region is red, the feasible region is the red part of the x -axis and the optimal solutions are blue. An empty circle marks an endpoint that does not belong to the feasible region.

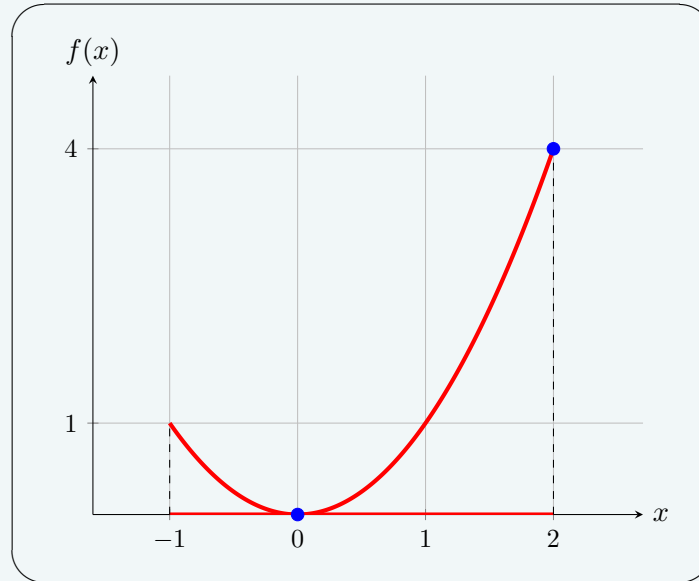
Example 5.1: an optimization problem with a quadratic objective function

Consider the following objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, f : x \mapsto x^2 \quad \text{and} \quad F = \{x \in \mathbb{R} : -1 \leq x \leq 2\} = [-1, 2].$$

The associated optimization problems in minimization and maximization forms are:

$$\min \left\{ x^2 \text{ subject to } -1 \leq x \leq 2 \right\} \quad \text{and} \quad \max \left\{ x^2 \text{ subject to } -1 \leq x \leq 2 \right\}$$



- For the minimization form, since $f(x) = x^2 \geq 0 = f(0)$ for every feasible x , and $x^2 = 0$ only for $x = 0$, the unique optimal solution and the optimal value are:

$$\tilde{\mathbf{x}}(P) = (0) \quad \text{and} \quad z(P) = 0.$$

- For the maximization form, for every feasible x we have $|x| \leq 2$ and then $f(x) = |x|^2 \leq 4 = f(2)$, with $x^2 = 4$ only for $x = -2$ and $x = 2$; since $-2 \notin F$, the unique optimal solution and the optimal value are:

$$\tilde{\mathbf{x}}(P) = (2) \quad \text{and} \quad z(P) = 4.$$

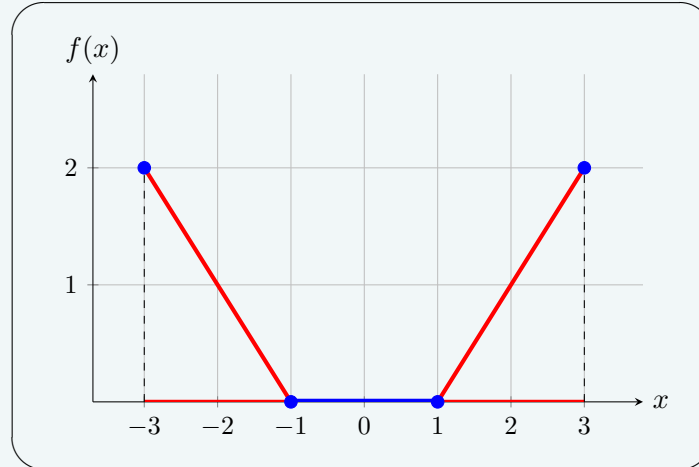
The optimal solutions are found directly from the definition: we compare the value of the objective function at a candidate point with its value at every feasible solution.

Example 5.2: an optimization problem with infinitely many optimal solutions

Consider the following objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, f : x \mapsto \max\{0, |x| - 1\} \quad \text{and} \quad F = [-3, 3].$$

The objective function is $f(x) = 0$ for $-1 \leq x \leq 1$ and $f(x) = |x| - 1$ otherwise: its graph is made of three segments.



- For the minimization form, $f(x) \geq 0$ for every feasible x , and $f(x) = 0$ exactly for $-1 \leq x \leq 1$. The problem has **infinitely many** optimal solutions (the blue segment) and the optimal value is:

$$\arg \min \{ \max\{0, |x| - 1\} \text{ subject to } -3 \leq x \leq 3 \} = [-1, 1] \quad \text{and} \quad z(P) = 0.$$

- For the maximization form, for every feasible x we have $|x| \leq 3$ and then $f(x) \leq \max\{0, 3 - 1\} = 2$, with $f(x) = 2$ only for $x = -3$ and $x = 3$. The problem has **two** optimal solutions and the optimal value is:

$$\arg \max \{ \max\{0, |x| - 1\} \text{ subject to } -3 \leq x \leq 3 \} = \{-3, 3\} \quad \text{and} \quad z(P) = 2.$$

- An optimal solution of an optimization problem may not exist for one of the following three reasons:
 1. If the objective function decreases indefinitely in a minimization problem or increases indefinitely in a maximization problem, the optimization problem is said to be **unbounded**. In these cases, $z(P) = -\infty$ and $z(P) = +\infty$, respectively.
 2. If the feasible region is empty, meaning that no solutions exist, the optimization problem is said to be **infeasible**. In these cases, $z(P)$ is not defined.
 3. If the infimum (or supremum) of the objective function exists but is not attained within the feasible region, the optimization problem is said to have an **unattainable infimum (or supremum)**, as in Examples 5.5 and 5.6. In these cases, $z(P)$ is not defined.

- In the **minimization form**, an optimal solution is a **point of minimum**, and the corresponding optimal value is the **minimum** of the objective function over the feasible region.

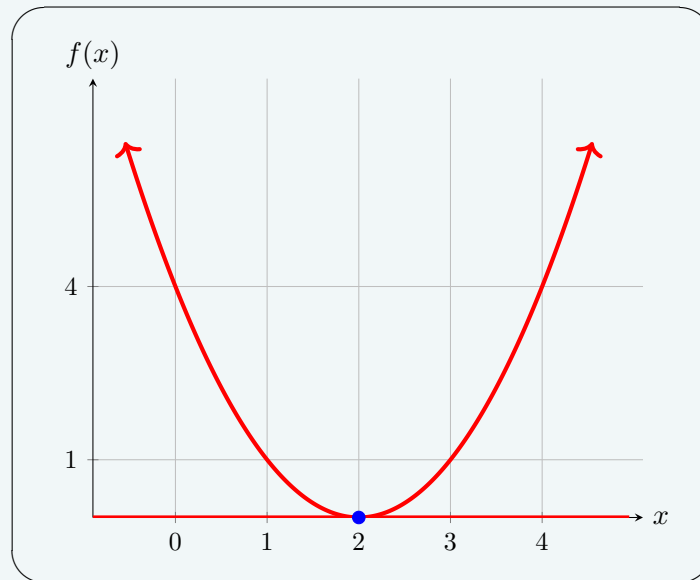
Similarly, in the **maximization form**, an optimal solution is a **point of maximum**, and the corresponding optimal value is the **maximum** of the objective function over the feasible region.

Example 5.3: an unbounded optimization problem

Consider the following objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, f : x \mapsto (x - 2)^2 \quad \text{and} \quad F = \mathbb{R}.$$

The feasible region is the whole real line: the graph continues upward on both sides of the figure.



- For the minimization form, since $f(x) = (x - 2)^2 \geq 0 = f(2)$ for every $x \in \mathbb{R}$, and $(x - 2)^2 = 0$ only for $x = 2$, the unique optimal solution and the optimal value are:

$$\tilde{x}(P) = (2) \quad \text{and} \quad z(P) = 0.$$

- For the maximization form, the objective function increases indefinitely: for every value $M \geq 0$, the feasible solution $x = M + 3$ has

$$f(M + 3) = (M + 1)^2 \geq M + 1 > M.$$

No feasible solution can be optimal and the problem is **unbounded**: $z(P) = +\infty$.

Example 5.4: an infeasible optimization problem

Consider the following objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, f : x \mapsto x^2 \quad \text{and} \quad F = \{x \in \mathbb{R} : x^2 \leq -1\}.$$

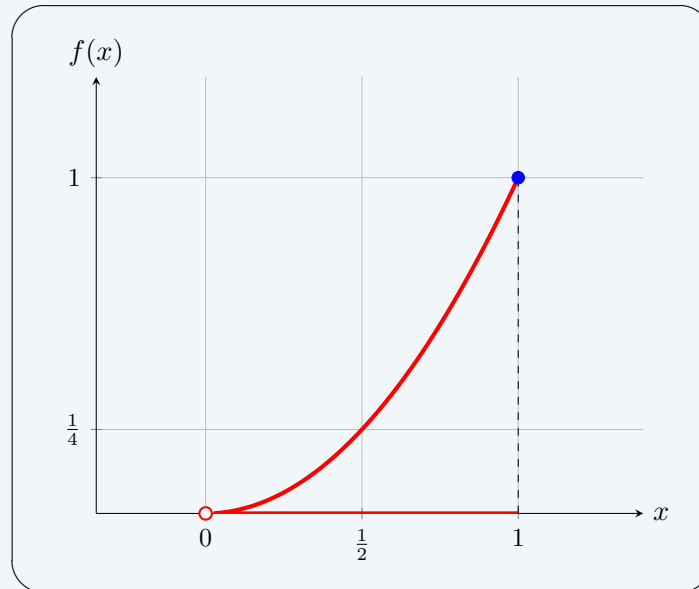
Since $x^2 \geq 0 > -1$ for every $x \in \mathbb{R}$, no point satisfies the constraint and the feasible region is empty: $F = \emptyset$. Both the minimization and the maximization form are **infeasible**: they have no feasible solution, hence no optimal solution, and $z(P)$ is not defined.

Example 5.5: an optimization problem with an unattainable infimum

Consider the following objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, f : x \mapsto x^2 \quad \text{and} \quad F = \{x \in \mathbb{R} : 0 < x \leq 1\} = (0, 1].$$

The point $x = 0$ does not belong to the feasible region (empty circle in the figure).



- For the minimization form, $f(x) = x^2 > 0$ for every feasible x , and the values of f get as close to 0 as we want: the infimum of f over F is 0. However, no feasible solution is optimal: for every $x \in F$, the point $\frac{x}{2}$ is feasible and

$$f\left(\frac{x}{2}\right) = \frac{x^2}{4} < x^2 = f(x).$$

The problem has an **unattainable infimum**: there is no optimal solution and $z(P)$ is not defined.

- For the maximization form, for every feasible x we have $0 < x \leq 1$ and then $f(x) = x^2 \leq 1 = f(1)$, with $x^2 = 1$ only for $x = 1$ in F . The unique optimal solution and the optimal value are:

$$\tilde{x}(P) = (1) \quad \text{and} \quad z(P) = 1.$$

Example 5.6: an unattainable infimum at infinity

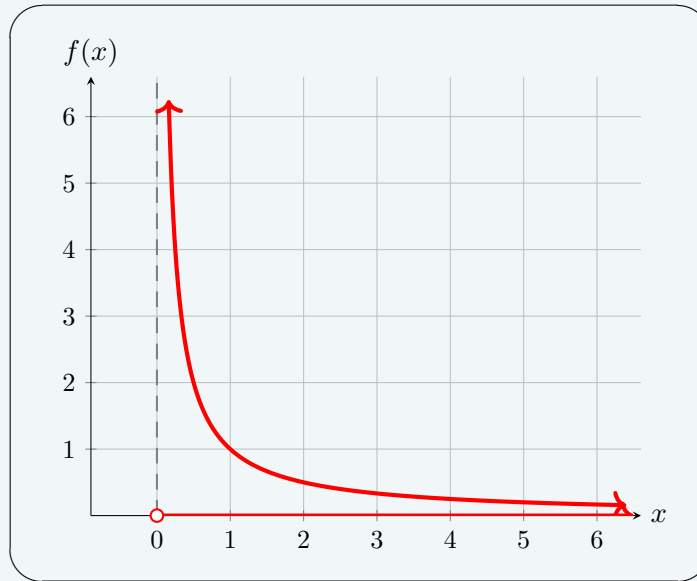
Consider the following objective function and feasible region:

$$f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}, \quad f : x \mapsto \frac{1}{x} \quad \text{and} \quad F = \{x \in \mathbb{R} : x > 0\} = (0, +\infty).$$

The associated optimization problems in minimization and maximization forms are:

$$\min \left\{ \frac{1}{x} \text{ subject to } x > 0 \right\} \quad \text{and} \quad \max \left\{ \frac{1}{x} \text{ subject to } x > 0 \right\}$$

The point $x = 0$ does not belong to the feasible region (empty circle in the figure). The graph continues on the right, approaching the x -axis, and upward, approaching the dashed line $x = 0$.



- For the minimization form, $f(x) = \frac{1}{x} > 0$ for every feasible x , and the values of f get as close to 0 as we want as $x \rightarrow +\infty$: the infimum of f over F is 0. However, no feasible solution is optimal: for every $x \in F$, the point $2x$ is feasible and

$$f(2x) = \frac{1}{2x} < \frac{1}{x} = f(x).$$

The problem has an **unattainable infimum**: there is no optimal solution and

$$\inf \left\{ \frac{1}{x} \text{ subject to } x > 0 \right\} = 0, \quad z(P) \text{ is not defined.}$$

- For the maximization form, the objective function increases indefinitely as $x \rightarrow 0^+$: for every value $M > 0$, the feasible solution $x = \frac{1}{M+1}$ has

$$f\left(\frac{1}{M+1}\right) = M+1 > M.$$

No feasible solution can be optimal and the problem is **unbounded**:

$$\sup \left\{ \frac{1}{x} \text{ subject to } x > 0 \right\} = +\infty \quad \text{and} \quad z(P) = +\infty.$$

5.1.2 Feasible solutions and bounds on the optimal value

- Given

$$\underbrace{\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{pmatrix}}_{\bar{\mathbf{x}}(P)} \in F$$

a **feasible solution** of an optimization problem, we have

1. in **minimization** form, the following **upper bound**:

$$z(P) \leq \underbrace{f(\bar{\mathbf{x}}(P))}_{UB(\bar{\mathbf{x}})}.$$

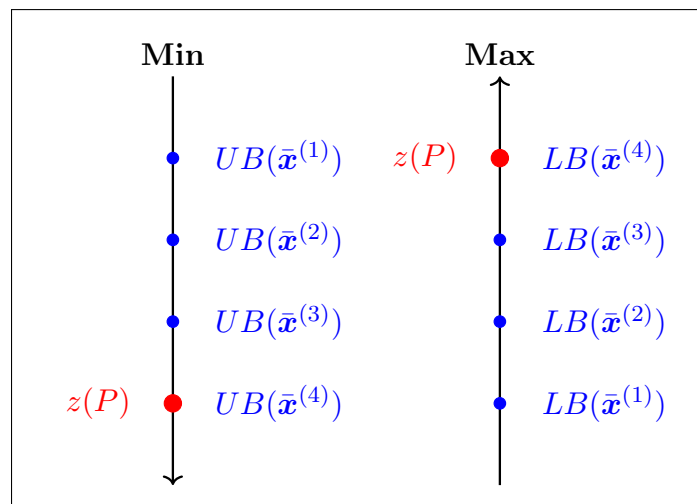
2. in **maximization** form, the following **lower bound**:

$$\underbrace{f(\bar{\mathbf{x}}(P))}_{LB(\bar{\mathbf{x}})} \leq z(P).$$

As an illustrative example, the figure below considers four feasible solutions

$$\bar{\mathbf{x}}^{(1)}, \bar{\mathbf{x}}^{(2)}, \bar{\mathbf{x}}^{(3)}, \bar{\mathbf{x}}^{(4)} \in F$$

and shows how they provide lower bounds in the maximization case and upper bounds in the minimization case for the optimal value $z(P)$ (the fourth one attains the optimal value).



Example 5.7: bounds from feasible solutions

Consider again the objective function $f(x) = x^2$ and the feasible region $F = [-1, 2]$ of the example with a quadratic objective function, where $z(P) = 0$ in minimization form and $z(P) = 4$ in maximization form. The feasible solutions

$$\bar{x}^{(1)} = (-1), \quad \bar{x}^{(2)} = \left(\frac{1}{2}\right), \quad \bar{x}^{(3)} = (1)$$

have objective values 1, $\frac{1}{4}$ and 1. Therefore:

- in minimization form, they give the upper bounds $UB(\bar{x}^{(1)}) = 1$, $UB(\bar{x}^{(2)}) = \frac{1}{4}$ and $UB(\bar{x}^{(3)}) = 1$ on $z(P) = 0$; the best one is $\frac{1}{4}$, the smallest;
- in maximization form, they give the lower bounds $LB(\bar{x}^{(1)}) = 1$, $LB(\bar{x}^{(2)}) = \frac{1}{4}$ and $LB(\bar{x}^{(3)}) = 1$ on $z(P) = 4$; the best one is 1, the largest.

None of these bounds is equal to the optimal value: a bound tells us how far the optimal value can be from the value of a known feasible solution, without computing it.

5.2 Properties of optimization problems

- The following properties hold for every optimization problem.
- In the following, we state properties using min and max only in the case where the minimum and maximum exist (i.e., they are attained on the feasible region); otherwise, the analogous statements hold by replacing min / max with inf / sup.
- For every objective function f and every feasible region F , we have:

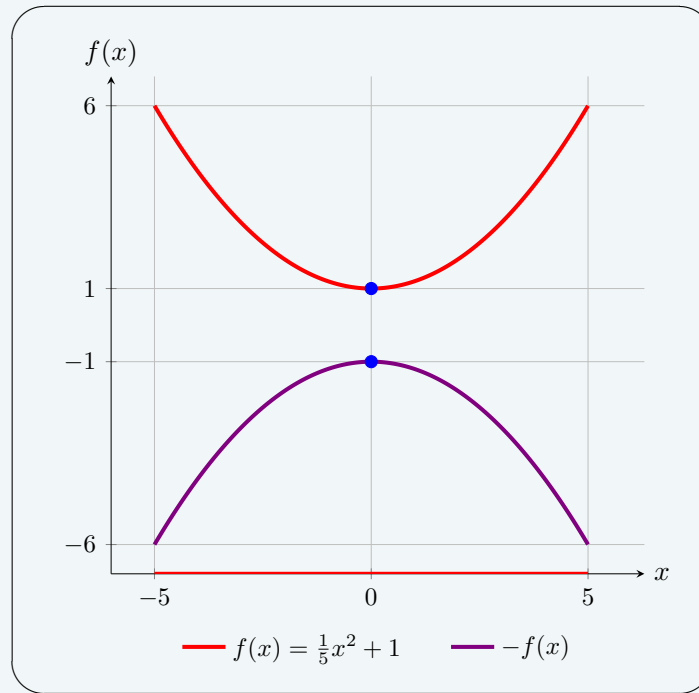
$$\min \left\{ f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F \right\} = - \max \left\{ -f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F \right\} \quad (5.1)$$

- The optimal solutions do not change: a point minimizes f over F if and only if it maximizes $-f$ over F . In the figures of this section, the original objective function is red, the transformed one is violet, the feasible region is the red part of the x -axis and the optimal solutions are blue.

Example 5.8: property: minimum and maximum

$$\begin{aligned}
 & \underbrace{\min \left\{ \frac{1}{5}x^2 + 1 \text{ subject to } -5 \leq x \leq 5 \right\}}_{=1} \\
 &= \underbrace{- \max \left\{ - \left(\frac{1}{5}x^2 + 1 \right) \text{ subject to } -5 \leq x \leq 5 \right\}}_{=-1}
 \end{aligned}$$

Indeed, $\frac{1}{5}x^2 + 1 \geq 1$ and $-\left(\frac{1}{5}x^2 + 1\right) \leq -1$ for every x , with equality only for $x = 0$: both optima are attained at $x = 0$.



- For every objective function, every feasible region, and every $a \in \mathbb{R}$, we have:

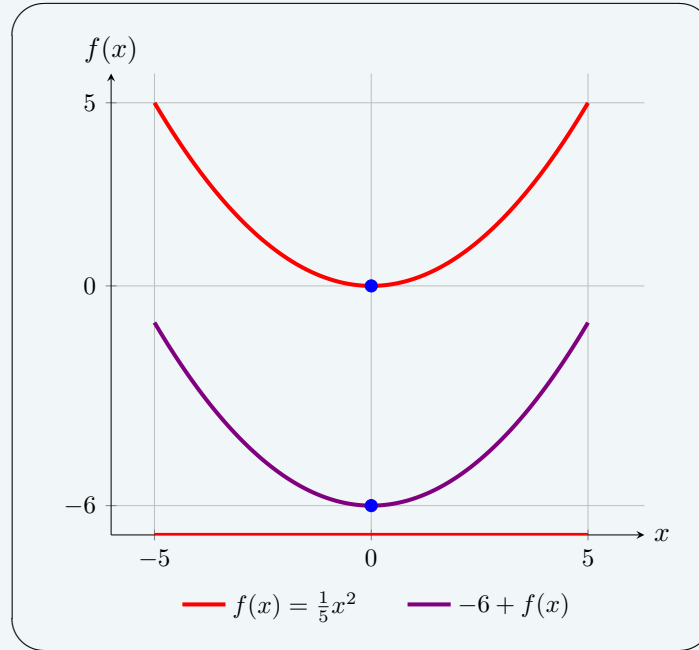
$$\min \left\{ a + f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F \right\} = a + \min \left\{ f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F \right\} \quad (5.2)$$

- Adding a constant moves the graph of the objective function up or down: the optimal solutions do not change.

Example 5.9: property: constant addition

$$\begin{aligned}
 & \min \underbrace{\left\{ \frac{1}{5}x^2 - 6 \text{ subject to } -5 \leq x \leq 5 \right\}}_{=-6} \\
 &= -6 + \min \underbrace{\left\{ \frac{1}{5}x^2 \text{ subject to } -5 \leq x \leq 5 \right\}}_{=0}
 \end{aligned}$$

Both minima are attained at $x = 0$, since $\frac{1}{5}x^2 \geq 0$ for every x , with equality only for $x = 0$.



- For every objective function, every feasible region, and every $a \in \mathbb{R}_{>0}$, we have:

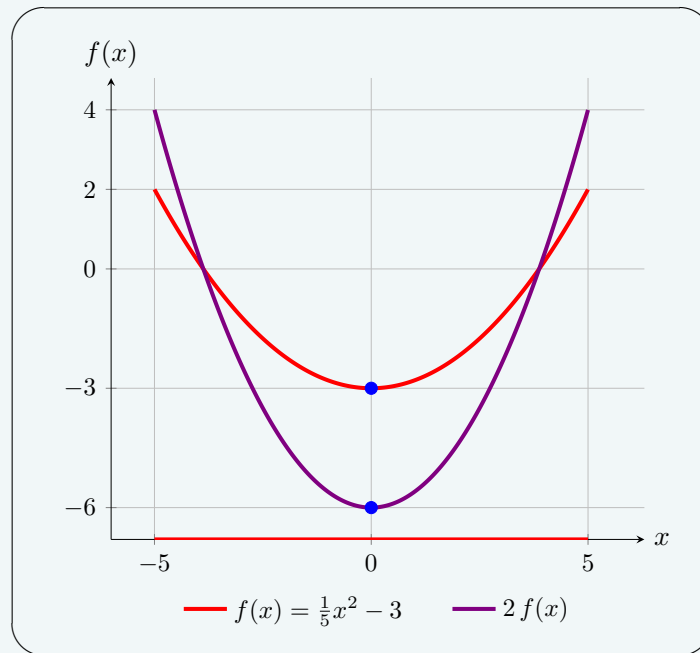
$$\min \left\{ a f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F \right\} = a \min \left\{ f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F \right\} \quad (5.3)$$

- Multiplying by a positive constant stretches (or compresses) the graph vertically: the optimal solutions do not change.

Example 5.10: property: multiplication by a positive constant

$$\begin{aligned}
 & \min \underbrace{\left\{ 2 \left(\frac{1}{5}x^2 - 3 \right) \text{ subject to } -5 \leq x \leq 5 \right\}}_{=-6} \\
 &= 2 \min \underbrace{\left\{ \frac{1}{5}x^2 - 3 \text{ subject to } -5 \leq x \leq 5 \right\}}_{=-3}
 \end{aligned}$$

Both minima are attained at $x = 0$, since $\frac{1}{5}x^2 - 3 \geq -3$ for every x , with equality only for $x = 0$.



- For every objective function, every feasible region, and every $a \in \mathbb{R}_{<0}$, we have:

$$\min \left\{ a f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F \right\} = a \max \left\{ f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F \right\} \quad (5.4)$$

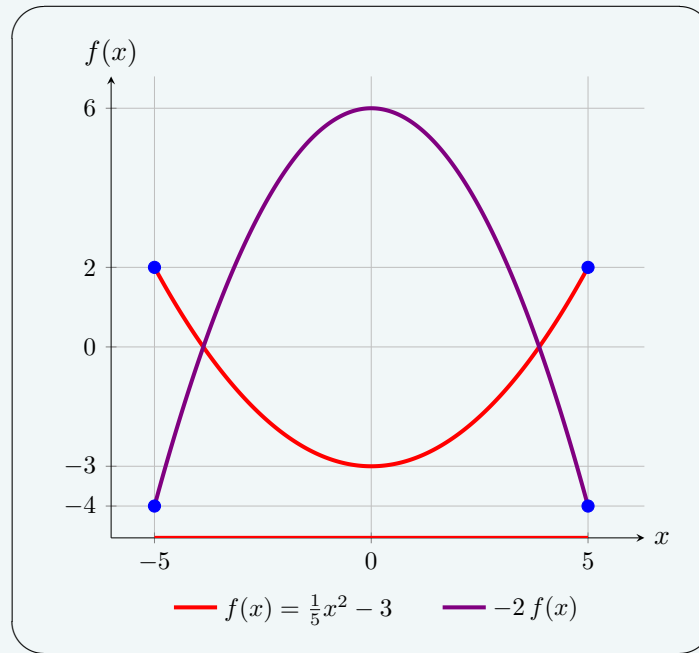
- Multiplying by a negative constant turns the graph upside down: the points of minimum of $a f$ are the points of maximum of f .

Example 5.11: property: multiplication by a negative constant

$$\underbrace{\min \left\{ -2 \left(\frac{1}{5}x^2 - 3 \right) \text{ subject to } -5 \leq x \leq 5 \right\}}_{=-4}$$

$$= -2 \underbrace{\max \left\{ \frac{1}{5}x^2 - 3 \text{ subject to } -5 \leq x \leq 5 \right\}}_{=2}$$

For every feasible x we have $|x| \leq 5$ and then $\frac{1}{5}x^2 - 3 \leq \frac{25}{5} - 3 = 2$, with equality only for $x = -5$ and $x = 5$: both problems have the **two** optimal solutions $x = -5$ and $x = 5$.

**5.3 Local and global optimal solutions****Definition 5.3: point of minimum/maximum and minimum/maximum**

Given a function $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, a point $\tilde{x} \in D$ is

- a **point of minimum** (**global minimum point**) of f in D if

$$f(\tilde{x}) \leq f(x), \quad \forall x \in D$$

and $f(\tilde{x})$ is the **minimum** of f in D ;

- a **point of maximum** (**global maximum point**) of f in D if

$$f(\tilde{x}) \geq f(x), \quad \forall x \in D$$

and $f(\tilde{x})$ is the **maximum** of f in D .

- A point of minimum (maximum) is also called a **minimizer** (**maximizer**) of f in D .
- There can exist multiple points of minimum (maximum), possibly infinitely many, as in Example 5.2.
- The optimal solutions of a minimization (maximization) problem with objective function f and feasible region F are the points of minimum (maximum) of f in F : they are its **global optima**.

Definition 5.4: spherical δ -neighborhood

Given a point $\bar{\mathbf{x}} \in \mathbb{R}^n$ and a value $\delta \in \mathbb{R}_{>0}$, the associated **spherical δ -neighborhood** is:

$$N(\bar{\mathbf{x}}, \delta) = \left\{ \mathbf{x} \in \mathbb{R}^n : \|\bar{\mathbf{x}} - \mathbf{x}\| < \delta \right\}$$

- The value $\|\bar{\mathbf{x}} - \mathbf{x}\|$ is the **Euclidean distance** between a generic point $\mathbf{x} \in \mathbb{R}^n$ and the point $\bar{\mathbf{x}} \in \mathbb{R}^n$, the **center** of the neighborhood:

$$\|\bar{\mathbf{x}} - \mathbf{x}\| = \sqrt{\sum_{j=1}^n (\bar{x}_j - x_j)^2}.$$

The neighborhood $N(\bar{\mathbf{x}}, \delta)$ contains the points whose distance from the center is smaller than the **radius** δ : in $\mathbb{R}$ it is the open interval $(\bar{x} - \delta, \bar{x} + \delta)$, in $\mathbb{R}^2$ it is the open disk of center $\bar{\mathbf{x}}$ and radius δ , without its boundary circle.

Example 5.12: spherical neighborhoods

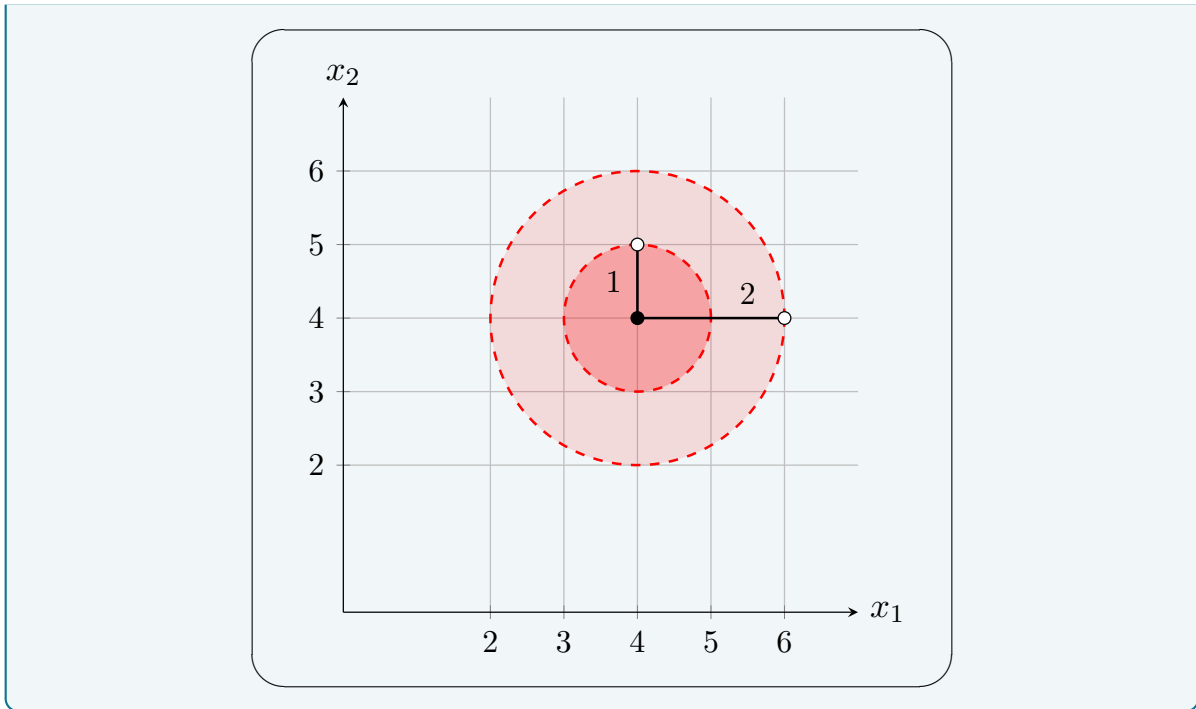
Consider the point $\bar{\mathbf{x}} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \in \mathbb{R}^2$ and the radii $\delta_1 = 1$ and $\delta_2 = 2$. The associated spherical δ -neighborhoods

$$N\left(\begin{pmatrix} 4 \\ 4 \end{pmatrix}, 1\right) = \left\{ \mathbf{x} \in \mathbb{R}^2 : \sqrt{(4 - x_1)^2 + (4 - x_2)^2} < 1 \right\}$$

and

$$N\left(\begin{pmatrix} 4 \\ 4 \end{pmatrix}, 2\right) = \left\{ \mathbf{x} \in \mathbb{R}^2 : \sqrt{(4 - x_1)^2 + (4 - x_2)^2} < 2 \right\}$$

are the red disks in the figure: the darker disk is $N(\bar{\mathbf{x}}, 1)$, which is contained in $N(\bar{\mathbf{x}}, 2)$. The boundary circles are dashed because they do not belong to the neighborhoods: for instance, the points $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 4 \end{pmatrix}$ (empty circles) have distance exactly 1 and 2 from the center, respectively.



Definition 5.5: point of local minimum/maximum and local minimum/maximum

Given a function $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, a point $\bar{x} \in D$ is

- a **point of local minimum** of f in D if there exists a spherical neighborhood $N(\bar{x}, \delta)$ with $\delta \in \mathbb{R}_{>0}$ such that:

$$f(\bar{x}) \leq f(x), \quad \forall x \in N(\bar{x}, \delta) \cap D$$

and $f(\bar{x})$ is a **local minimum** of f in D ;

- a **point of local maximum** of f in D if there exists a spherical neighborhood $N(\bar{x}, \delta)$ with $\delta \in \mathbb{R}_{>0}$ such that:

$$f(\bar{x}) \geq f(x), \quad \forall x \in N(\bar{x}, \delta) \cap D$$

and $f(\bar{x})$ is a **local maximum** of f in D .

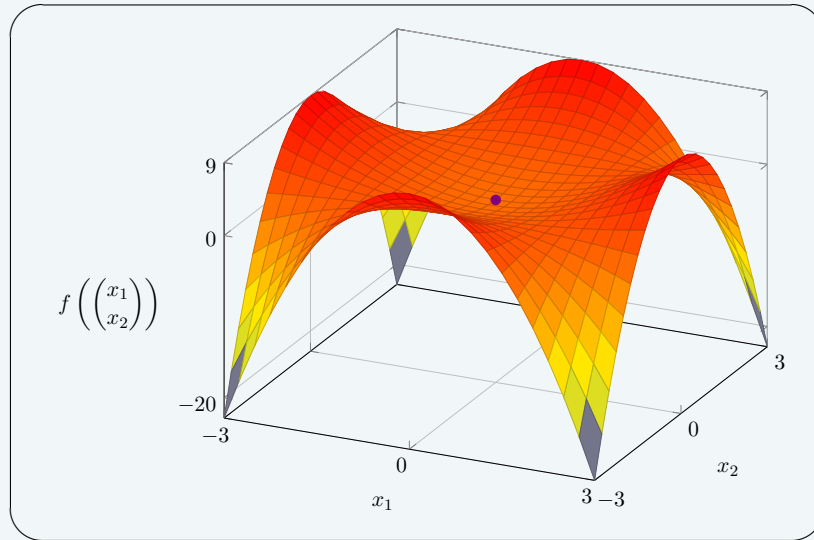
- There can exist multiple points of local minimum (maximum), possibly infinitely many.
- To prove that a point is a point of local minimum (maximum) with the definition, it is enough to exhibit **one** radius δ for which the inequality holds in $N(\bar{x}, \delta) \cap D$; any smaller radius also works.

Example 5.13: points of local minimum

Consider the following function:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1^2 + x_2^2 - \frac{1}{2}x_1^2x_2^2.$$

This is the graph of the function in $[-3, 3] \times [-3, 3]$; the violet point is $(0, 0, f(0, 0)) = (0, 0, 0)$.



To prove that the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a point of local minimum, we use the spherical δ -neighborhood $N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 1\right)$. For every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 1\right)$ we have $x_2^2 \leq x_1^2 + x_2^2 < 1$, hence $1 - \frac{x_2^2}{2} > \frac{1}{2}$, and then:

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) - f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = x_1^2 + x_2^2 - \frac{1}{2}x_1^2x_2^2 = x_1^2\left(1 - \frac{x_2^2}{2}\right) + x_2^2 \geq \frac{1}{2}x_1^2 + x_2^2 \geq 0,$$

with equality only for $x_1 = x_2 = 0$. It follows that

$$f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \leq f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right), \quad \forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 1\right),$$

and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a point of local minimum of f in $\mathbb{R}^2$, with local minimum 0. It is not a point of (global) minimum, since

$$f\left(\begin{pmatrix} 3 \\ 3 \end{pmatrix}\right) = 9 + 9 - \frac{81}{2} = -\frac{45}{2} < 0 = f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right).$$

- Every point of (global) minimum is a point of local minimum (with any δ), and every point of (global) maximum is a point of local maximum. The converse is false: a point of

local minimum may not be a point of minimum, as the previous example and the following one show.

Example 5.14: a local minimum that is not a global minimum

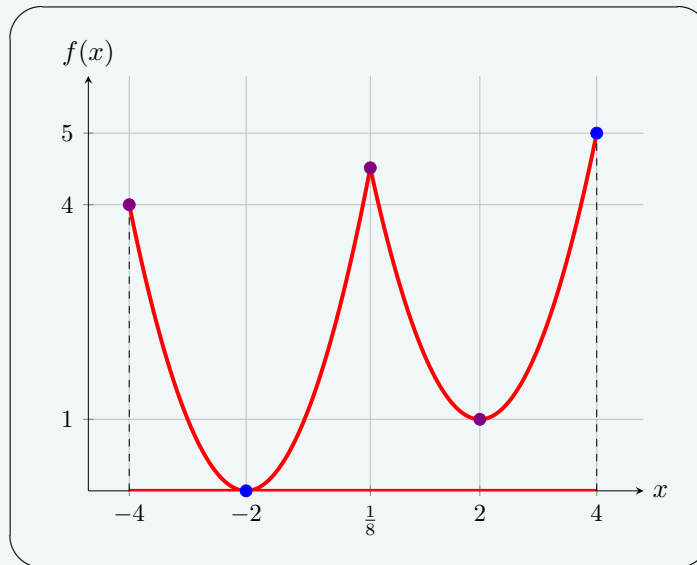
Consider the following function:

$$f : [-4, 4] \rightarrow \mathbb{R}, \quad f : x \mapsto \min \{ (x+2)^2, (x-2)^2 + 1 \}.$$

Since $(x+2)^2 \leq (x-2)^2 + 1$ if and only if $8x \leq 1$, we have

$$f(x) = \begin{cases} (x+2)^2 & \text{if } -4 \leq x \leq \frac{1}{8}, \\ (x-2)^2 + 1 & \text{if } \frac{1}{8} \leq x \leq 4. \end{cases}$$

In the figure, the points of global minimum and maximum are blue and the other points of local minimum and maximum are violet.



- **Global minimum.** Both $(x+2)^2$ and $(x-2)^2 + 1$ are nonnegative, so $f(x) \geq 0 = f(-2)$ for every $x \in [-4, 4]$; since $(x-2)^2 + 1 \geq 1$, $f(x) = 0$ only for $x = -2$. The point $x = -2$ is the unique point of minimum and the minimum is 0.
- **Local minimum that is not global.** In the neighborhood $N((2), 1) = (1, 3)$ we have $x \geq \frac{1}{8}$, and then

$$f(x) = (x-2)^2 + 1 \geq 1 = f(2), \quad \forall x \in N((2), 1) \cap [-4, 4] = (1, 3).$$

The point $x = 2$ is a point of local minimum and 1 is a local minimum; it is not a point of (global) minimum, since $f(-2) = 0 < 1 = f(2)$.

- **Global maximum.** For $-4 \leq x \leq \frac{1}{8}$ we have $-2 \leq x+2 \leq \frac{17}{8}$ and then $f(x) = (x+2)^2 \leq (\frac{17}{8})^2 = \frac{289}{64} < 5$. For $\frac{1}{8} \leq x \leq 4$ we have $-\frac{15}{8} \leq x-2 \leq 2$ and then $f(x) = (x-2)^2 + 1 \leq 4 + 1 = 5$, with equality only for $x = 4$. The point $x = 4$ is the unique point of maximum and the maximum is 5.

- **Local maxima that are not global.** In $N((-4), 1) \cap [-4, 4] = [-4, -3)$ we have $-2 \leq x + 2 < -1$ and then $f(x) = (x + 2)^2 \leq 4 = f(-4)$: the point $x = -4$ is a point of local maximum. In $N((\frac{1}{8}), 1) = (-\frac{7}{8}, \frac{9}{8})$ we have $0 < x + 2 \leq \frac{17}{8}$ when $x \leq \frac{1}{8}$, so that $f(x) = (x + 2)^2 \leq \frac{289}{64}$, and $-\frac{15}{8} \leq x - 2 < 0$ when $x \geq \frac{1}{8}$, so that $f(x) = (x - 2)^2 + 1 \leq \frac{225}{64} + 1 = \frac{289}{64}$; in both cases $f(x) \leq f(\frac{1}{8})$: the point $x = \frac{1}{8}$ is a point of local maximum. Neither of them is a point of (global) maximum, since $f(-4) = 4 < 5$ and $f(\frac{1}{8}) = \frac{289}{64} < 5$.

Consider the optimization problem $\min\{f(x) \text{ subject to } -4 \leq x \leq 4\}$: a method that only looks at the points close to $x = 2$ finds the local minimum 1, but the optimal value is $z(P) = 0$, attained at $\tilde{x}(P) = (-2)$.

Convex and concave optimization problems

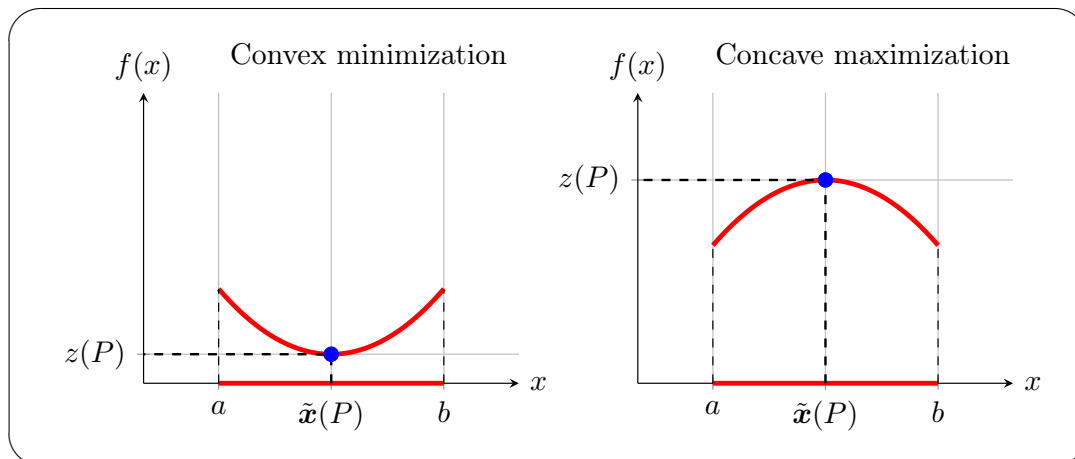
6.1 Convex and concave optimization problems

Definition 6.1: convex and concave optimization problems

Given an optimization problem with a **convex feasible region**,

- if the objective function is convex over the feasible region, then the problem is called a **convex optimization problem**;
- if the objective function is concave over the feasible region, then the problem is called a **concave optimization problem**.

- In the figures of this chapter, as in the chapter «Optimization problems», the graph of the objective function over the feasible region is red, the feasible region is the red part of the x -axis and the optimal solutions are blue. Examples of a convex minimization problem and of a concave maximization problem with the feasible region $[a, b] \subseteq \mathbb{R}$:



- The definitions of convex and concave function of the chapter «Convex and concave functions» require a convex domain: if the objective function is convex or concave over the feasible region, then the feasible region is a convex set.

- The feasible region is often described by a finite number of constraints, $F = \{ \mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \leq 0, i \in \{1, 2, \dots, m\} \}$. If every function g_i is convex, then F is a convex set, by the property of the sets defined by convex functions of the chapter «Convex and concave functions». For instance, the interval $[a, b] = \{ x \in \mathbb{R} : a - x \leq 0, x - b \leq 0 \}$ is defined by two linear, hence convex, functions.

6.2 Local optima of convex and concave problems

- Every point of minimum is also a point of local minimum. The following proposition states that, in a convex minimization problem, the converse also holds.

Proposition 6.2: local minima of convex minimization problems

In a convex minimization problem, with objective function f convex over the convex feasible region F , if a point $\bar{\mathbf{x}} \in F$ is a point of local minimum of f in F , then it is a point of minimum of f in F , that is, an optimal solution.

Proof. If $\bar{\mathbf{x}} \in F$ is a point of local minimum, then there exists a spherical neighborhood $N(\bar{\mathbf{x}}, \delta)$ with $\delta > 0$ such that:

$$f(\bar{\mathbf{x}}) \leq f(\mathbf{x}), \quad \forall \mathbf{x} \in N(\bar{\mathbf{x}}, \delta) \cap F$$

For every $\mathbf{x} \in F \setminus \{\bar{\mathbf{x}}\}$, since F is convex, all points of the form $\hat{\mathbf{x}} = \lambda \mathbf{x} + (1 - \lambda) \bar{\mathbf{x}}$ with $\lambda \in (0, 1]$ also belong to F . Moreover, for λ sufficiently small, we have $\hat{\mathbf{x}} \in N(\bar{\mathbf{x}}, \delta) \cap F$, and therefore:

$$f(\bar{\mathbf{x}}) \leq f(\hat{\mathbf{x}})$$

Since f is convex, it also holds that:

$$f(\hat{\mathbf{x}}) \leq \lambda f(\mathbf{x}) + (1 - \lambda) f(\bar{\mathbf{x}})$$

Combining the two inequalities, we obtain:

$$f(\bar{\mathbf{x}}) \leq \lambda f(\mathbf{x}) + (1 - \lambda) f(\bar{\mathbf{x}})$$

$$\Downarrow$$

$$\lambda f(\bar{\mathbf{x}}) \leq \lambda f(\mathbf{x})$$

$$\Downarrow$$

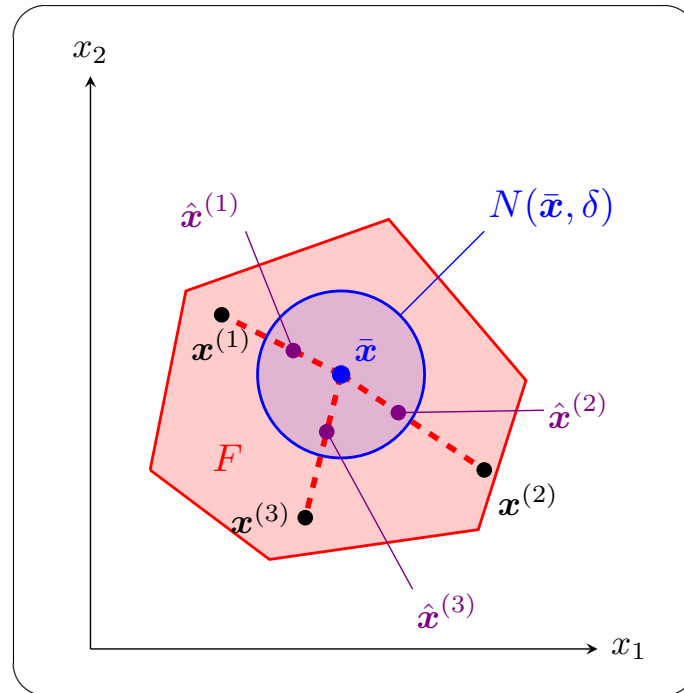
$$f(\bar{\mathbf{x}}) \leq f(\mathbf{x}) \quad (\text{since } \lambda \in (0, 1])$$

Hence,

$$f(\bar{\mathbf{x}}) \leq f(\mathbf{x}), \quad \forall \mathbf{x} \in F$$

and $\bar{\mathbf{x}}$ is a point of minimum. □

- In $\mathbb{R}^2$, the idea of the proof is given by the following picture, with three points $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}$ in F and three points $\hat{\mathbf{x}}^{(1)}, \hat{\mathbf{x}}^{(2)}, \hat{\mathbf{x}}^{(3)}$ in $N(\bar{\mathbf{x}}, \delta) \cap F$, each lying on the segment that joins $\bar{\mathbf{x}}$ to the corresponding $\mathbf{x}^{(i)}$.



- In a convex minimization problem we have shown that:

$$\text{point of local minimum} \implies \text{point of minimum}$$

and the implication also holds in the other direction, because a point of minimum satisfies the defining condition at every point of F , and therefore in particular at every point of a neighborhood. The two notions are equivalent.

Corollary 6.3: local maxima of concave maximization problems

In a concave maximization problem, with objective function f concave over the convex feasible region F , a point $\bar{\mathbf{x}} \in F$ is a point of local maximum of f in F if and only if it is a point of maximum of f in F , that is, an optimal solution.

Proof. If f is concave over F , then $-f$ is convex over F , and $\bar{\mathbf{x}}$ is a point of local maximum of f if and only if it is a point of local minimum of $-f$. By the previous proposition, it is a point of minimum of $-f$ in F , that is, a point of maximum of f in F , since $\max\{f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F\} = -\min\{-f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F\}$. $\square$

- In a convex minimization problem, every **point of local minimum** is also a **point of minimum**: every point of local minimum in the feasible region is an optimal solution.
- In a concave maximization problem, every **point of local maximum** is also a **point of maximum**: every point of local maximum in the feasible region is an optimal solution.

6.3 Examples of convex and concave optimization problems

- In the following examples, we first check that the feasible region is a convex set and that the objective function is convex (or concave) over it, with the definition of the chapter «Convex and concave functions». Then we find a point of local minimum (or maximum), comparing the value of the objective function at a candidate point with its values at the feasible points of a neighborhood. By the proposition and the corollary of the previous section, this point is an optimal solution.
- The candidate point is the point where the derivative of the objective function vanishes. The derivative only suggests the candidate: the local optimality is checked directly with the definition of the chapter «Optimization problems».

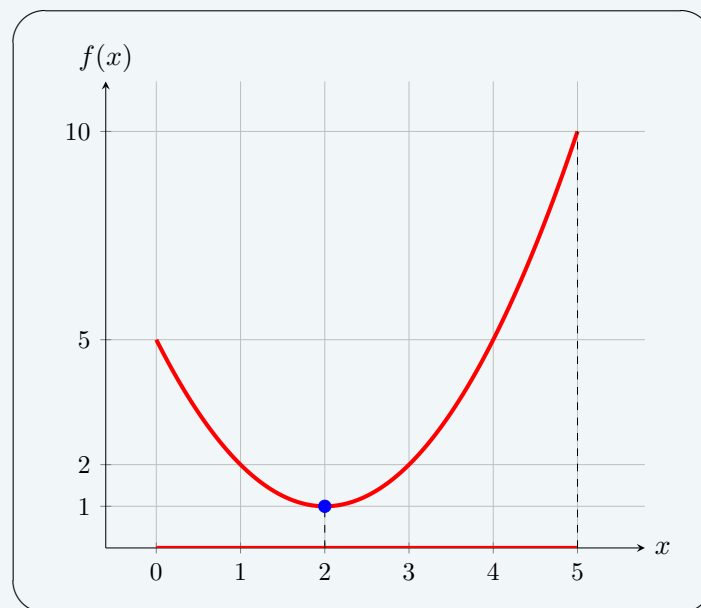
Example 6.1: a convex minimization problem

Consider the following objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, f : x \mapsto (x - 2)^2 + 1 \quad \text{and} \quad F = [0, 5].$$

The associated optimization problem in minimization form is:

$$\min \left\{ (x - 2)^2 + 1 \text{ subject to } 0 \leq x \leq 5 \right\}$$



- **Feasible region.** The feasible region $F = [0, 5]$ is an interval, hence a convex set.
- **Objective function.** For every pair of points $p, w \in F$ and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned}
 f(\lambda p + (1 - \lambda)w) &= (\lambda p + (1 - \lambda)w - 2)^2 + 1 \\
 &= (\lambda(p - 2) + (1 - \lambda)(w - 2))^2 + \lambda + (1 - \lambda) \\
 &\quad (\text{since } \lambda + (1 - \lambda) = 1) \\
 &\leq \lambda(p - 2)^2 + (1 - \lambda)(w - 2)^2 + \lambda + (1 - \lambda) \\
 &\quad (\text{since } (\lambda a + (1 - \lambda)b)^2 \leq \lambda a^2 + (1 - \lambda)b^2, \\
 &\quad \text{chapter «Convex and concave functions»}) \\
 &= \lambda((p - 2)^2 + 1) + (1 - \lambda)((w - 2)^2 + 1) \\
 &= \lambda f(p) + (1 - \lambda)f(w)
 \end{aligned}$$

Therefore,

$$f(\lambda p + (1 - \lambda)w) \leq \lambda f(p) + (1 - \lambda)f(w)$$

The objective function is convex over the convex feasible region F , and the problem is a **convex minimization problem**.

- **Point of local minimum.** The derivative $f'(x) = 2x - 4 = 2(x - 2)$ vanishes at $x = 2$. In the neighborhood $N((2), 1) = (1, 3)$, all points are feasible and

$$f(x) = (x - 2)^2 + 1 \geq 1 = f(2), \quad \forall x \in N((2), 1) \cap F = (1, 3).$$

The point $x = 2$ is a point of local minimum.

- **Optimal solution.** By the proposition «local minima of convex minimization problems», the point $x = 2$ is also a point of minimum, and the optimal solution and the optimal value are:

$$\tilde{x}(P) = (2) \quad \text{and} \quad z(P) = f(2) = 1.$$

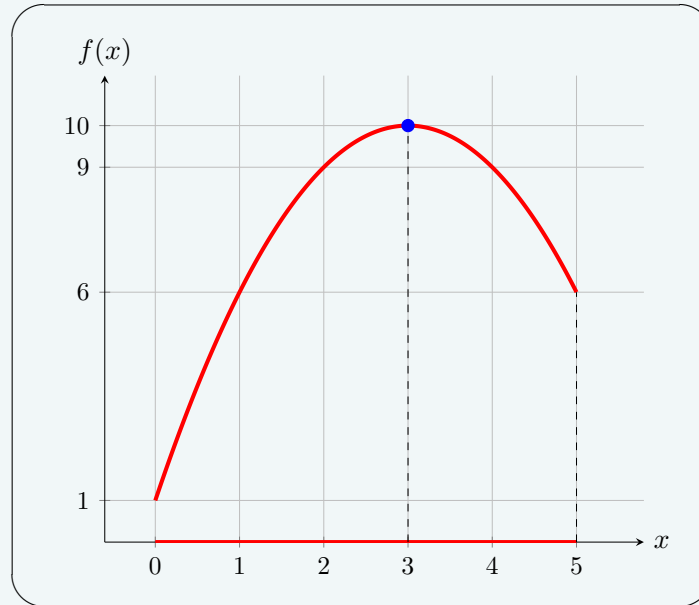
Example 6.2: a concave maximization problem

Consider the following objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto -(x - 3)^2 + 10 \quad \text{and} \quad F = [0, 5].$$

The associated optimization problem in maximization form is:

$$\max \left\{ -(x - 3)^2 + 10 \quad \text{subject to} \quad 0 \leq x \leq 5 \right\}$$



- **Feasible region.** The feasible region $F = [0, 5]$ is an interval, hence a convex set.
- **Objective function.** For every pair of points $p, w \in F$ and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned}
 f(\lambda p + (1 - \lambda)w) &= -(\lambda(p - 3) + (1 - \lambda)(w - 3))^2 + \lambda 10 + (1 - \lambda) 10 \\
 &\quad (\text{since } \lambda + (1 - \lambda) = 1) \\
 &\geq -\lambda(p - 3)^2 - (1 - \lambda)(w - 3)^2 + \lambda 10 + (1 - \lambda) 10 \\
 &\quad (\text{since } (\lambda a + (1 - \lambda)b)^2 \leq \lambda a^2 + (1 - \lambda)b^2, \\
 &\quad \text{and } -1 \text{ reverses the inequality}) \\
 &= \lambda(-(p - 3)^2 + 10) + (1 - \lambda)(-(w - 3)^2 + 10) \\
 &= \lambda f(p) + (1 - \lambda)f(w)
 \end{aligned}$$

Therefore,

$$f(\lambda p + (1 - \lambda)w) \geq \lambda f(p) + (1 - \lambda)f(w)$$

The objective function is concave over the convex feasible region F , and the problem is a **concave maximization problem**.

- **Point of local maximum.** The derivative $f'(x) = -2x + 6 = -2(x - 3)$ vanishes at $x = 3$. In the neighborhood $N((3), 1) = (2, 4)$, all points are feasible and

$$f(x) = -(x - 3)^2 + 10 \leq 10 = f(3), \quad \forall x \in N((3), 1) \cap F = (2, 4).$$

The point $x = 3$ is a point of local maximum.

- **Optimal solution.** By the corollary «local maxima of concave maximization problems», the point $x = 3$ is also a point of maximum, and the optimal solution and the optimal value are:

$$\tilde{x}(P) = (3) \quad \text{and} \quad z(P) = f(3) = 10.$$

- In the last two examples, the objective function is the same cubic polynomial

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto 2 + 9x - 6x^2 + x^3 = (x - 2)^3 - 3(x - 2) + 4,$$

which is neither convex nor concave in $\mathbb{R}$: with $p = 0$, $w = 2$ and $\lambda = \frac{1}{2}$ we have $f(1) = 6 > 3 = \frac{1}{2}f(0) + \frac{1}{2}f(2)$, and with $p = 2$, $w = 4$ and $\lambda = \frac{1}{2}$ we have $f(3) = 2 < 5 = \frac{1}{2}f(2) + \frac{1}{2}f(4)$. It is convex over $[2, 4]$ and concave over $[0, 2]$: the definition of convex (concave) optimization problem only concerns the values of the objective function over the feasible region. To prove it, we use the following inequality.

Observation 6.4

For every $\lambda \in [0, 1]$ and every $a, b \in \mathbb{R}$, we have

$$\lambda a^3 + (1 - \lambda)b^3 - (\lambda a + (1 - \lambda)b)^3 = \lambda(1 - \lambda)(a - b)^2((1 + \lambda)a + (2 - \lambda)b)$$

and, therefore,

$$\begin{aligned} (\lambda a + (1 - \lambda)b)^3 &\leq \lambda a^3 + (1 - \lambda)b^3 && \text{if } a, b \geq 0, \\ (\lambda a + (1 - \lambda)b)^3 &\geq \lambda a^3 + (1 - \lambda)b^3 && \text{if } a, b \leq 0. \end{aligned}$$

Proof. Let $\mu = 1 - \lambda$, so that $1 - \lambda^2 = (1 - \lambda)(1 + \lambda) = \mu(1 + \lambda)$ and $1 - \mu^2 = \lambda(1 + \mu)$. Expanding the cube, we have:

$$\begin{aligned} &\lambda a^3 + \mu b^3 - (\lambda a + \mu b)^3 \\ &= \lambda(1 - \lambda^2)a^3 + \mu(1 - \mu^2)b^3 - 3\lambda^2\mu a^2b - 3\lambda\mu^2 ab^2 \\ &= \lambda\mu((1 + \lambda)a^3 + (1 + \mu)b^3 - 3\lambda a^2b - 3\mu ab^2) \\ &= \lambda\mu(a^2 - 2ab + b^2)((1 + \lambda)a + (1 + \mu)b) \\ &\quad (\text{since } (1 + \mu) - 2(1 + \lambda) = -3\lambda \text{ and } (1 + \lambda) - 2(1 + \mu) = -3\mu) \\ &= \lambda(1 - \lambda)(a - b)^2((1 + \lambda)a + (2 - \lambda)b) \end{aligned}$$

Since $\lambda(1 - \lambda)(a - b)^2 \geq 0$, $1 + \lambda > 0$ and $2 - \lambda > 0$, the right-hand side is nonnegative if $a, b \geq 0$, and nonpositive if $a, b \leq 0$. $\square$

Example 6.3: a convex minimization problem with a cubic objective function

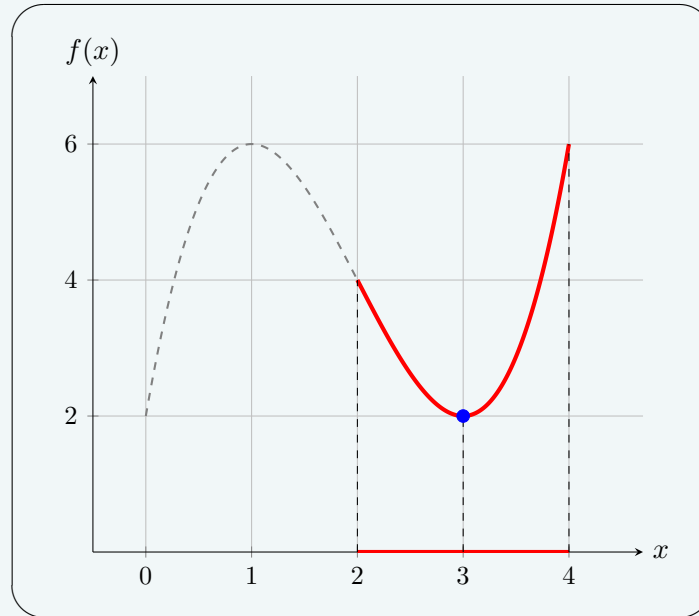
Consider the following objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto 2 + 9x - 6x^2 + x^3 \quad \text{and} \quad F = [2, 4].$$

The associated optimization problem in minimization form is:

$$\min \left\{ 2 + 9x - 6x^2 + x^3 \quad \text{subject to} \quad 2 \leq x \leq 4 \right\}$$

In the figure, the graph of the objective function outside the feasible region is dashed.



- **Feasible region.** The feasible region $F = [2, 4]$ is an interval, hence a convex set.
- **Objective function.** For every pair of points $p, w \in F$ and every $\lambda \in [0, 1]$, with $a = p - 2 \geq 0$ and $b = w - 2 \geq 0$, we have:

$$\begin{aligned}
 f(\lambda p + (1 - \lambda)w) &= (\lambda p + (1 - \lambda)w - 2)^3 - 3(\lambda p + (1 - \lambda)w - 2) + 4 \\
 &\quad (\text{since } f(x) = (x - 2)^3 - 3(x - 2) + 4) \\
 &= (\lambda a + (1 - \lambda)b)^3 - 3(\lambda a + (1 - \lambda)b) + \lambda 4 + (1 - \lambda) 4 \\
 &\quad (\text{since } \lambda p + (1 - \lambda)w - 2 = \lambda a + (1 - \lambda)b, \\
 &\quad \text{and } \lambda + (1 - \lambda) = 1) \\
 &\leq \lambda a^3 + (1 - \lambda)b^3 - 3(\lambda a + (1 - \lambda)b) + \lambda 4 + (1 - \lambda) 4 \\
 &\quad (\text{by the previous observation, since } a \geq 0 \text{ and } b \geq 0) \\
 &= \lambda(a^3 - 3a + 4) + (1 - \lambda)(b^3 - 3b + 4) \\
 &= \lambda f(p) + (1 - \lambda)f(w) \\
 &\quad (\text{since } f(p) = a^3 - 3a + 4 \text{ and } f(w) = b^3 - 3b + 4)
 \end{aligned}$$

Therefore,

$$f(\lambda p + (1 - \lambda)w) \leq \lambda f(p) + (1 - \lambda)f(w)$$

The objective function is convex over the convex feasible region F , and the problem is a **convex minimization problem**.

- **Point of local minimum.** The derivative $f'(x) = 9 - 12x + 3x^2 = 3(x - 1)(x - 3)$ vanishes at $x = 1$ and at $x = 3$, and only $x = 3$ is feasible. For every $x \in \mathbb{R}$ we have

$f(x) - 2 = x(x - 3)^2$. In the neighborhood $N((3), 1) = (2, 4)$, all points are feasible and positive, and then

$$f(x) = 2 + x(x - 3)^2 \geq 2 = f(3), \quad \forall x \in N((3), 1) \cap F = (2, 4).$$

The point $x = 3$ is a point of local minimum.

- **Optimal solution.** By the proposition «local minima of convex minimization problems», the point $x = 3$ is also a point of minimum, and the optimal solution and the optimal value are:

$$\tilde{x}(P) = (3) \quad \text{and} \quad z(P) = f(3) = 2.$$

Example 6.4: a concave maximization problem with a cubic objective function

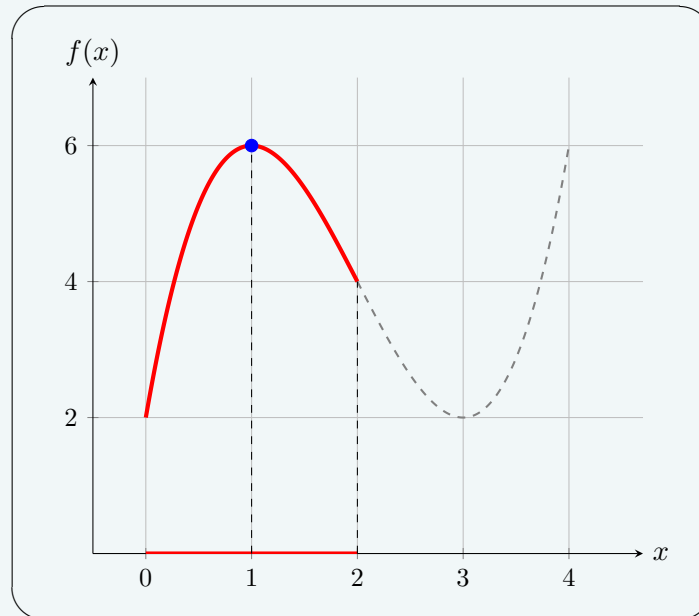
Consider the same objective function with a different feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto 2 + 9x - 6x^2 + x^3 \quad \text{and} \quad F = [0, 2].$$

The associated optimization problem in maximization form is:

$$\max \left\{ 2 + 9x - 6x^2 + x^3 \quad \text{subject to} \quad 0 \leq x \leq 2 \right\}$$

In the figure, the graph of the objective function outside the feasible region is dashed.



- **Feasible region.** The feasible region $F = [0, 2]$ is an interval, hence a convex set.
- **Objective function.** For every pair of points $p, w \in F$ and every $\lambda \in [0, 1]$, with

$a = p - 2 \leq 0$ and $b = w - 2 \leq 0$, we have:

$$\begin{aligned}
 f(\lambda p + (1 - \lambda)w) &= (\lambda p + (1 - \lambda)w - 2)^3 - 3(\lambda p + (1 - \lambda)w - 2) + 4 \\
 &\quad (\text{since } f(x) = (x - 2)^3 - 3(x - 2) + 4) \\
 &= (\lambda a + (1 - \lambda)b)^3 - 3(\lambda a + (1 - \lambda)b) + \lambda 4 + (1 - \lambda) 4 \\
 &\quad (\text{since } \lambda p + (1 - \lambda)w - 2 = \lambda a + (1 - \lambda)b, \\
 &\quad \text{and } \lambda + (1 - \lambda) = 1) \\
 &\geq \lambda a^3 + (1 - \lambda)b^3 - 3(\lambda a + (1 - \lambda)b) + \lambda 4 + (1 - \lambda) 4 \\
 &\quad (\text{by the previous observation, since } a \leq 0 \text{ and } b \leq 0) \\
 &= \lambda(a^3 - 3a + 4) + (1 - \lambda)(b^3 - 3b + 4) \\
 &= \lambda f(p) + (1 - \lambda)f(w) \\
 &\quad (\text{since } f(p) = a^3 - 3a + 4 \text{ and } f(w) = b^3 - 3b + 4)
 \end{aligned}$$

Therefore,

$$f(\lambda p + (1 - \lambda)w) \geq \lambda f(p) + (1 - \lambda)f(w)$$

The objective function is concave over the convex feasible region F , and the problem is a **concave maximization problem**.

- **Point of local maximum.** The derivative $f'(x) = 3(x - 1)(x - 3)$ vanishes at $x = 1$ and at $x = 3$, and only $x = 1$ is feasible. For every $x \in \mathbb{R}$ we have $6 - f(x) = (x - 1)^2(4 - x)$. In the neighborhood $N((1), 1) = (0, 2)$, all points are feasible and smaller than 4, and then

$$f(x) = 6 - (x - 1)^2(4 - x) \leq 6 = f(1), \quad \forall x \in N((1), 1) \cap F = (0, 2).$$

The point $x = 1$ is a point of local maximum.

- **Optimal solution.** By the corollary «local maxima of concave maximization problems», the point $x = 1$ is also a point of maximum, and the optimal solution and the optimal value are:

$$\tilde{x}(P) = (1) \quad \text{and} \quad z(P) = f(1) = 6.$$