



SAPIENZA
UNIVERSITÀ DI ROMA

Convex Optimization

Exercises 2 — Optimization problems

Worked exercises on optimization problems, local and global optima, convex and concave problems.

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Online version and downloadable material:
fabiofurini.github.io/convex-optimization/
github.com/fabiofurini/convex-optimization

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Optimization problems

5.1 Problems in one variable

Exercise 5.1 — Optimization problem (E1)

Consider the following objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, f : x \mapsto |x - 1| \quad \text{and} \quad F = \{x \in \mathbb{R} : -1 \leq x \leq 3\} = [-1, 3].$$

Questions

1. Write the associated optimization problems in minimization and maximization forms.
2. Draw the graph of the objective function over the feasible region.
3. Find all optimal solutions and the optimal value for both forms.

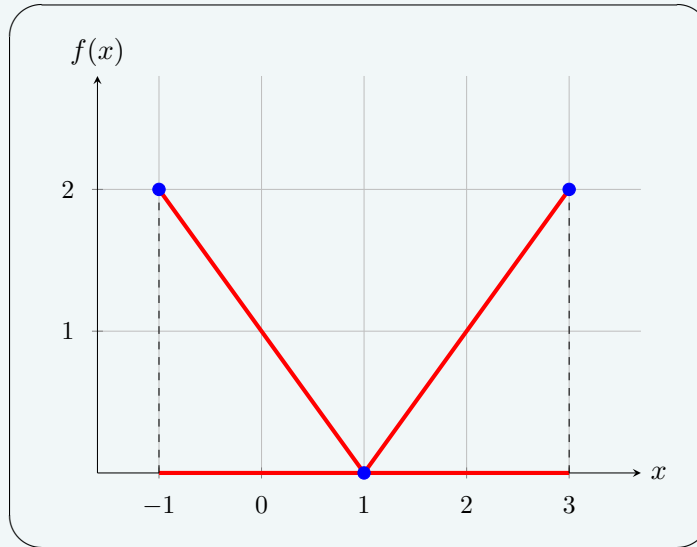
Solution

1. The associated optimization problems in minimization and maximization forms are:

$$\min \left\{ |x - 1| \text{ subject to } -1 \leq x \leq 3 \right\}$$

$$\max \left\{ |x - 1| \text{ subject to } -1 \leq x \leq 3 \right\}$$

2. The graph of the objective function over the feasible region is red; the feasible region is the red part of the x -axis and the optimal solutions are marked in blue. The graph is made of two segments, which meet at the point $(1, 0)$.



3. The feasible region is the interval $F = [-1, 3]$. The objective function is $f(x) = 1 - x$ for $x \leq 1$ and $f(x) = x - 1$ for $x \geq 1$: it is not linear, its graph is made of two segments.

- For the minimization form, since $f(x) = |x - 1| \geq 0 = f(1)$ for every feasible x , and $|x - 1| = 0$ only for $x = 1$, the unique optimal solution and the optimal value are:

$$\tilde{x}(P) = (1) \quad \text{and} \quad z(P) = 0.$$

- For the maximization form, for every feasible x we have $-2 \leq x - 1 \leq 2$ and then $f(x) = |x - 1| \leq 2$, with $|x - 1| = 2$ only for $x = -1$ and $x = 3$. The problem has **two** optimal solutions and the optimal value is:

$$\arg \max \{ |x - 1| \mid \text{subject to } -1 \leq x \leq 3 \} = \{-1, 3\} \quad \text{and} \quad z(P) = 2.$$

Exercise 5.2 — Optimization problem (E2)

Consider the following objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto -x^2 + 4x + 1 \quad \text{and} \quad F = \{x \in \mathbb{R} : 0 \leq x \leq 5\} = [0, 5].$$

Questions

1. Write the associated optimization problems in minimization and maximization forms.
2. Draw the graph of the objective function over the feasible region.
3. Find all optimal solutions and the optimal value for both forms.

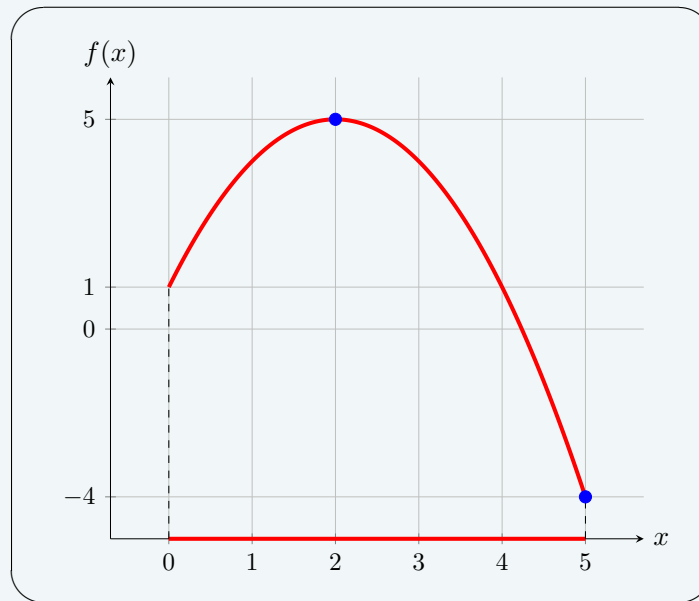
Solution

1. The associated optimization problems in minimization and maximization forms are:

$$\min \left\{ -x^2 + 4x + 1 \text{ subject to } 0 \leq x \leq 5 \right\}$$

$$\max \left\{ -x^2 + 4x + 1 \text{ subject to } 0 \leq x \leq 5 \right\}$$

2. The graph of the objective function over the feasible region is red; the feasible region is the red part of the x -axis and the optimal solutions are marked in blue. Since f takes negative values, the x -axis is drawn at the bottom of the figure; the horizontal grid line at height 0 is the line $f(x) = 0$.



3. Completing the square, the objective function can be written as

$$f(x) = -x^2 + 4x + 1 = 5 - (x^2 - 4x + 4) = 5 - (x - 2)^2.$$

- For the minimization form, for every feasible x we have $-2 \leq x - 2 \leq 3$ and then $(x - 2)^2 \leq 9$, with $(x - 2)^2 = 9$ only for $x = 5$ (the other solution $x = -1$ is not feasible). Therefore $f(x) = 5 - (x - 2)^2 \geq -4 = f(5)$ and the unique optimal solution and the optimal value are:

$$\tilde{x}(P) = (5) \quad \text{and} \quad z(P) = -4.$$

- For the maximization form, since $(x - 2)^2 \geq 0$, we have $f(x) = 5 - (x - 2)^2 \leq 5 = f(2)$ for every feasible x , with equality only for $x = 2$. The unique optimal solution and the optimal value are:

$$\tilde{x}(P) = (2) \quad \text{and} \quad z(P) = 5.$$

The value at the other endpoint of the feasible region, $f(0) = 1$, is neither the minimum nor the maximum.

Exercise 5.3 — Optimization problem (E3)

Consider the following objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, f : x \mapsto |x - 1| + |x + 1| \quad \text{and} \quad F = \{x \in \mathbb{R} : -3 \leq x \leq 2\} = [-3, 2].$$

Questions

1. Write the associated optimization problems in minimization and maximization forms.
2. Draw the graph of the objective function over the feasible region.
3. Find all optimal solutions and the optimal value for both forms.

Solution

1. The associated optimization problems in minimization and maximization forms are:

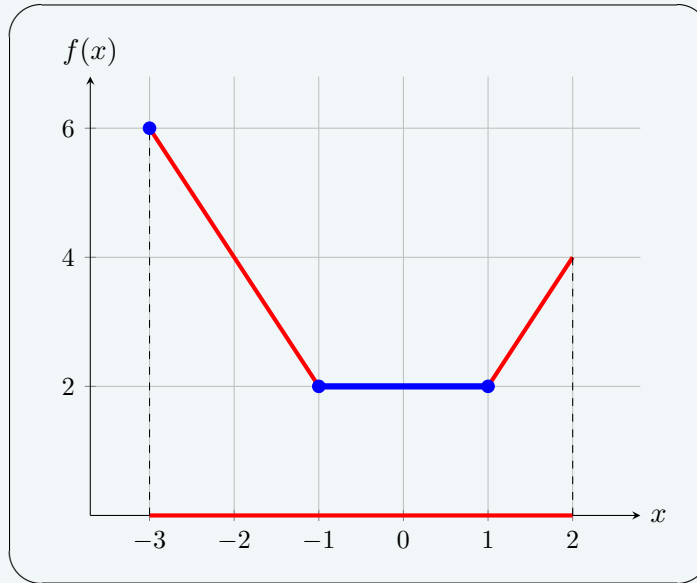
$$\min \left\{ |x - 1| + |x + 1| \text{ subject to } -3 \leq x \leq 2 \right\}$$

$$\max \left\{ |x - 1| + |x + 1| \text{ subject to } -3 \leq x \leq 2 \right\}$$

2. The graph of the objective function over the feasible region is red; the feasible region is the red part of the x -axis and the optimal solutions are marked in blue. Removing the absolute values on each interval, the objective function is

$$f(x) = \begin{cases} -2x & \text{if } x \leq -1, \\ 2 & \text{if } -1 \leq x \leq 1, \\ 2x & \text{if } x \geq 1, \end{cases}$$

and its graph is made of three segments.



3. The feasible region is the interval $F = [-3, 2]$.

- For the minimization form, by the triangle inequality,

$$f(x) = |1 - x| + |x + 1| \geq |(1 - x) + (x + 1)| = 2 \quad \forall x \in \mathbb{R}.$$

For $-1 \leq x \leq 1$ we have $f(x) = (1 - x) + (x + 1) = 2$, while $f(x) = -2x > 2$ for $x < -1$ and $f(x) = 2x > 2$ for $x > 1$. The problem has **infinitely many** optimal solutions (the blue segment) and the optimal value is:

$$\arg \min \{ |x - 1| + |x + 1| \text{ subject to } -3 \leq x \leq 2 \} = [-1, 1] \quad \text{and} \quad z(P) = 2.$$

- For the maximization form, we compare the three pieces: for $-3 \leq x \leq -1$ we have $f(x) = -2x \leq 6$, with equality only for $x = -3$; for $-1 \leq x \leq 1$ we have $f(x) = 2$; for $1 \leq x \leq 2$ we have $f(x) = 2x \leq 4$. The unique optimal solution and the optimal value are:

$$\tilde{x}(P) = (-3) \quad \text{and} \quad z(P) = 6.$$

Exercise 5.4 — Optimization problem (E4)

Consider the following objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto x^2 - 4x + 5 \quad \text{and} \quad F = \{ x \in \mathbb{R} : x \geq 0 \} = [0, +\infty).$$

Questions

1. Write the associated optimization problems in minimization and maximization forms.

2. Draw the graph of the objective function over the feasible region.
3. Find all optimal solutions and the optimal value for both forms, or explain why they do not exist.

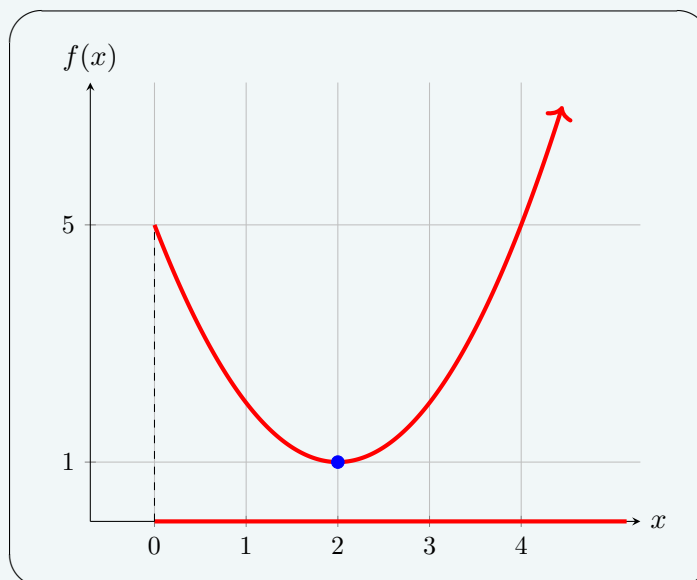
Solution

1. The associated optimization problems in minimization and maximization forms are:

$$\min \left\{ x^2 - 4x + 5 \text{ subject to } x \geq 0 \right\}$$

$$\max \left\{ x^2 - 4x + 5 \text{ subject to } x \geq 0 \right\}$$

2. The graph of the objective function over the feasible region is red; the feasible region is the red part of the x -axis and the optimal solutions are marked in blue. The feasible region is a half-line: the graph continues upward on the right of the figure.



3. Completing the square, the objective function can be written as

$$f(x) = x^2 - 4x + 5 = (x - 2)^2 + 1.$$

- For the minimization form, since $(x - 2)^2 \geq 0$, we have $f(x) \geq 1 = f(2)$ for every feasible x , with equality only for $x = 2$. The unique optimal solution and the optimal value are:

$$\tilde{x}(P) = (2) \quad \text{and} \quad z(P) = 1.$$

- For the maximization form, the objective function increases indefinitely: for every value $M \geq 0$, the point $x = M + 3$ is feasible and

$$f(M + 3) = (M + 1)^2 + 1 > M.$$

Therefore no feasible solution can be optimal: the problem is **unbounded** and

$$z(P) = +\infty.$$

Exercise 5.5 — Optimization problem (E5)

Consider the following objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, f : x \mapsto x^2 + 1 \quad \text{and} \quad F = \{x \in \mathbb{R} : x \geq 3, x^2 \leq 4\}.$$

Questions

1. Write the associated optimization problems in minimization and maximization forms.
2. Draw on the x -axis the set of points that satisfy each constraint.
3. Find all optimal solutions and the optimal value for both forms, or explain why they do not exist.

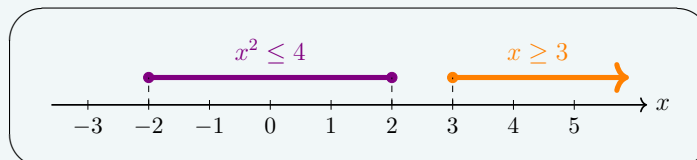
Solution

1. The associated optimization problems in minimization and maximization forms are:

$$\min \left\{ x^2 + 1 \text{ subject to } x \geq 3, x^2 \leq 4 \right\}$$

$$\max \left\{ x^2 + 1 \text{ subject to } x \geq 3, x^2 \leq 4 \right\}$$

2. The constraint $x^2 \leq 4$ is satisfied exactly by the points of the interval $[-2, 2]$ (violet), since $x^2 \leq 4$ if and only if $|x| \leq 2$. The constraint $x \geq 3$ is satisfied by the points of the half-line $[3, +\infty)$ (orange). The two sets have no point in common.



3. Every point with $x \geq 3$ has $x^2 \geq 9 > 4$, so it violates the second constraint: no point satisfies both constraints and the feasible region is empty,

$$F = \{x \in \mathbb{R} : x \geq 3\} \cap [-2, 2] = \emptyset.$$

Both the minimization and the maximization form are **infeasible**: they have no feasible solution, hence no optimal solution, and $z(P)$ is not defined. The objective function plays no role: the same holds for every objective function with this feasible region.

Exercise 5.6 — Optimization problem (E6)

Consider the following objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, f : x \mapsto x^2 - 2x + 2 \quad \text{and} \quad F = \{x \in \mathbb{R} : -1 \leq x < 1\} = [-1, 1).$$

Questions

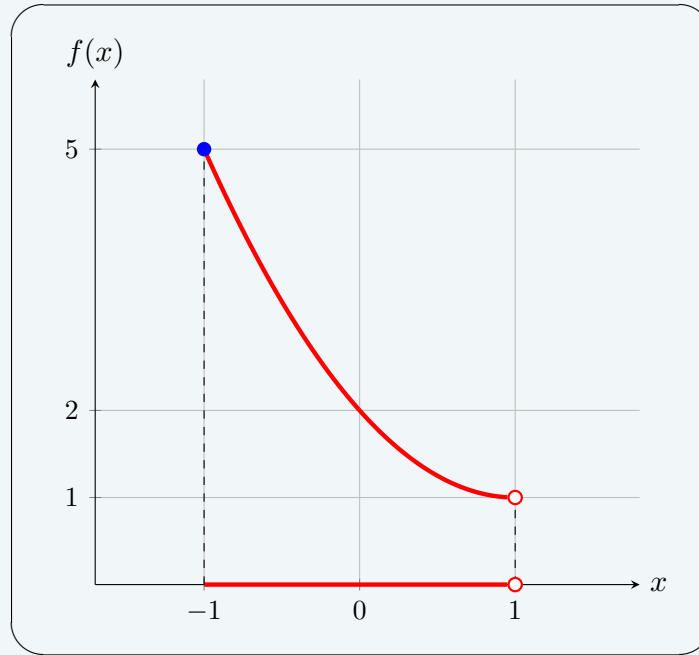
1. Write the associated optimization problems in minimization and maximization forms.
2. Draw the graph of the objective function over the feasible region.
3. Find all optimal solutions and the optimal value for both forms, or explain why they do not exist.

Solution

1. The associated optimization problems in minimization and maximization forms are:

$$\begin{aligned} \min \left\{ x^2 - 2x + 2 \text{ subject to } -1 \leq x < 1 \right\} \\ \max \left\{ x^2 - 2x + 2 \text{ subject to } -1 \leq x < 1 \right\} \end{aligned}$$

2. The graph of the objective function over the feasible region is red; the feasible region is the red part of the x -axis and the optimal solutions are marked in blue. The point $x = 1$ does not belong to the feasible region (empty circles).



3. Completing the square, the objective function can be written as

$$f(x) = x^2 - 2x + 2 = (x - 1)^2 + 1.$$

- For the minimization form, since $x \neq 1$ for every feasible x , we have $f(x) = (x - 1)^2 + 1 > 1$; on the other hand, the values of f get as close to 1 as we want: for example $f(1 - \frac{1}{k}) = 1 + \frac{1}{k^2}$ for every integer $k \geq 1$. The infimum of f over F is 1, but no feasible solution is optimal: for every $x \in F$, the point $\frac{x+1}{2}$ is feasible (it lies between x and 1) and

$$f\left(\frac{x+1}{2}\right) = \left(\frac{x-1}{2}\right)^2 + 1 = \frac{(x-1)^2}{4} + 1 < (x-1)^2 + 1 = f(x).$$

The problem has an **unattainable infimum**: there is no optimal solution and $z(P)$ is not defined.

- For the maximization form, for every feasible x we have $-2 \leq x - 1 < 0$ and then $(x - 1)^2 \leq 4$, with equality only for $x = -1$. Therefore $f(x) \leq 5 = f(-1)$ and the unique optimal solution and the optimal value are:

$$\tilde{x}(P) = (-1) \quad \text{and} \quad z(P) = 5.$$

Exercise 5.7 — Optimization problem (E7)

Consider the following objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto x^3 - 3x^2 - 9x + 5 \quad \text{and} \quad F = \{x \in \mathbb{R} : -2 \leq x \leq 5\} = [-2, 5].$$

Questions

1. Write the associated optimization problems in minimization and maximization forms.
2. Draw the graph of the objective function over the feasible region.
3. Find all optimal solutions and the optimal value for both forms.
4. Prove that $x = -2$ is a point of local minimum of f in F , but not a point of (global) minimum.

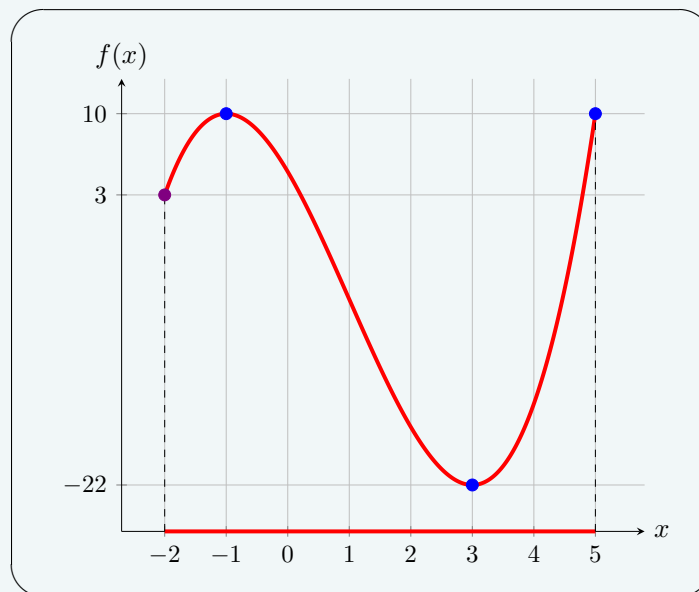
Solution

1. The associated optimization problems in minimization and maximization forms are:

$$\min \left\{ x^3 - 3x^2 - 9x + 5 \text{ subject to } -2 \leq x \leq 5 \right\}$$

$$\max \left\{ x^3 - 3x^2 - 9x + 5 \text{ subject to } -2 \leq x \leq 5 \right\}$$

2. The graph of the objective function over the feasible region is red; the feasible region is the red part of the x -axis and the optimal solutions are marked in blue. The point of local minimum that is not a point of global minimum is violet. Since f takes negative values, the x -axis is drawn at the bottom of the figure.



3. We compare f with the candidate optimal values using the following factorizations, which can be checked by expanding the products:

$$f(x) + 22 = x^3 - 3x^2 - 9x + 27 = (x - 3)^2 (x + 3),$$

$$f(x) - 10 = x^3 - 3x^2 - 9x - 5 = (x + 1)^2 (x - 5).$$

- For the minimization form, for every feasible x we have $x + 3 \geq 1 > 0$ and $(x - 3)^2 \geq 0$, so $f(x) + 22 \geq 0$, that is $f(x) \geq -22 = f(3)$, with equality only for $x = 3$. The unique optimal solution and the optimal value are:

$$\tilde{x}(P) = (3) \quad \text{and} \quad z(P) = -22.$$

- For the maximization form, for every feasible x we have $x - 5 \leq 0$ and $(x + 1)^2 \geq 0$, so $f(x) - 10 \leq 0$, that is $f(x) \leq 10$, with equality only for $x = -1$ and $x = 5$. The problem has **two** optimal solutions and the optimal value is:

$$\arg \max \{ x^3 - 3x^2 - 9x + 5 \text{ subject to } -2 \leq x \leq 5 \} = \{-1, 5\}$$

and

$$z(P) = 10.$$

4. We have $f(-2) = -8 - 12 + 18 + 5 = 3$ and

$$f(x) - 3 = x^3 - 3x^2 - 9x + 2 = (x + 2)(x^2 - 5x + 1).$$

In the spherical δ -neighborhood $N((-2), 1)$ we have $N((-2), 1) \cap [-2, 5] = [-2, -1)$. For every $x \in [-2, -1)$ we have $x + 2 \geq 0$ and, since $x^2 \geq 1$ and $-5x > 5$, $x^2 - 5x + 1 > 7 > 0$. Therefore

$$f(-2) = 3 \leq f(x), \quad \forall x \in N((-2), 1) \cap [-2, 5] = [-2, -1),$$

and $x = -2$ is a point of local minimum of f in F , with local minimum 3. It is not a point of (global) minimum, since $f(3) = -22 < 3 = f(-2)$: an optimization method that only looks at the points close to $x = -2$ would miss the optimal solution of the minimization form.

Exercise 5.8 — Optimization problem (E9)

Consider the following linear objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto x - 1 \quad \text{and} \quad F = \{x \in \mathbb{R} : 0 \leq x < 3\} = [0, 3).$$

Questions

1. Write the associated optimization problems in minimization and maximization forms.
2. Draw the graph of the objective function over the feasible region.
3. Find all optimal solutions and the optimal value for both forms.

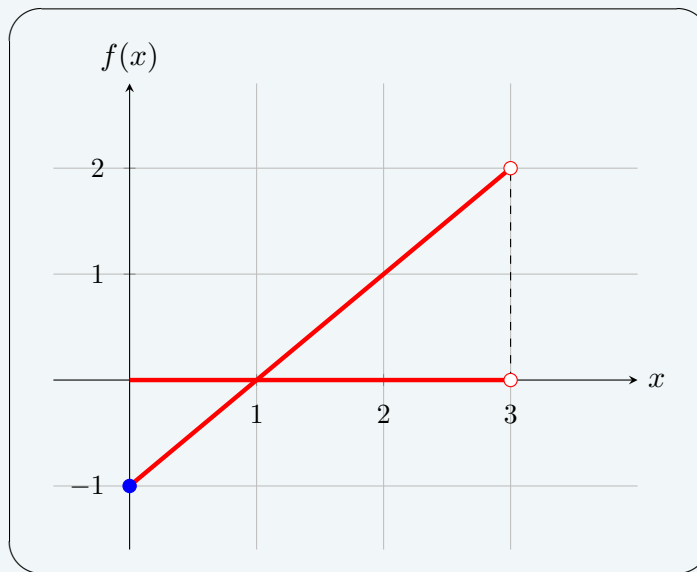
Solution

1. The associated optimization problems in minimization and maximization forms are:

$$\min \left\{ x - 1 \text{ subject to } 0 \leq x < 3 \right\}$$

$$\max \left\{ x - 1 \text{ subject to } 0 \leq x < 3 \right\}$$

2. The graph of the objective function over the feasible region is red; the feasible region is the red part of the x -axis and the optimal solutions are marked in blue. The point $x = 3$ does not belong to the feasible region: the empty circles mark the excluded endpoint of the feasible region and of the graph.



3. The feasible region is the interval $F = [0, 3)$, which contains 0 but not 3, and the linear function f is strictly increasing.

- For the minimization form, since for every feasible x we have $x \geq 0$ and then $f(x) = x - 1 \geq -1 = f(0)$, the unique optimal solution and the optimal value are:

$$\tilde{x}(P) = (0) \quad \text{and} \quad z(P) = -1.$$

- For the maximization form, for every feasible x we have $x < 3$ and then $f(x) = x - 1 < 2$; moreover, for every feasible x the point $\frac{x+3}{2}$ is feasible and

$$f\left(\frac{x+3}{2}\right) = \frac{x+1}{2} > x - 1 = f(x),$$

so no feasible point is optimal, while $f(x)$ gets as close as we want to 2 when x gets close to 3. The problem has an **unattainable supremum**: $\sup\{f(x) : x \in F\} = 2$ is not attained, there is no optimal solution and

$$\arg \max \left\{ x - 1 \text{ subject to } 0 \leq x < 3 \right\} = \emptyset \quad \text{and} \quad z(P) \text{ is not defined.}$$

Exercise 5.9 — Optimization problem (E10)

Consider the following function:

$$f : [-1, 4] \rightarrow \mathbb{R}, \quad f : x \mapsto 2 + 9x - 6x^2 + x^3.$$

Questions

1. Prove, using the definition, that the point (3) is a point of local minimum of f in $[-1, 4]$.
2. Prove, using the definition, that the point (1) is a point of local maximum of f in $[-1, 4]$.
3. Find all the points of (global) minimum and maximum of f in $[-1, 4]$, that is, the optimal solutions of the optimization problems $\min\{f(x) \text{ subject to } -1 \leq x \leq 4\}$ and $\max\{f(x) \text{ subject to } -1 \leq x \leq 4\}$.
4. Draw the graph of the function.

Solution

1. We have $f(3) = 2 + 27 - 54 + 27 = 2$ and

$$f(x) - f(3) = x^3 - 6x^2 + 9x = x(x-3)^2.$$

We use the spherical δ -neighborhood $N((3), \frac{1}{2})$, for which $N((3), \frac{1}{2}) \cap [-1, 4] = (\frac{5}{2}, \frac{7}{2})$. For every x in this interval we have $x > 0$ and $(x-3)^2 \geq 0$, so $f(x) - f(3) \geq 0$. Therefore,

$$f(3) = 2 \leq f(x), \quad \forall x \in N((3), \frac{1}{2}) \cap [-1, 4] = (\frac{5}{2}, \frac{7}{2}),$$

and $x = 3$ is a point of local minimum of f in $[-1, 4]$, with local minimum 2.

2. We have $f(1) = 2 + 9 - 6 + 1 = 6$ and

$$f(x) - f(1) = x^3 - 6x^2 + 9x - 4 = (x-4)(x-1)^2.$$

We use the spherical δ -neighborhood $N((1), \frac{1}{2})$, for which $N((1), \frac{1}{2}) \cap [-1, 4] = (\frac{1}{2}, \frac{3}{2})$. For every x in this interval we have $x-4 < 0$ and $(x-1)^2 \geq 0$, so $f(x) - f(1) \leq 0$. Therefore,

$$f(1) = 6 \geq f(x), \quad \forall x \in N((1), \frac{1}{2}) \cap [-1, 4] = (\frac{1}{2}, \frac{3}{2}),$$

and $x = 1$ is a point of local maximum of f in $[-1, 4]$, with local maximum 6.

3. We study the minimum and the maximum separately.

- **Minimum.** We have $f(-1) = 2 - 9 - 6 - 1 = -14$ and

$$f(x) + 14 = x^3 - 6x^2 + 9x + 16 = (x+1)(x^2 - 7x + 16) = (x+1) \left(\left(x - \frac{7}{2}\right)^2 + \frac{15}{4} \right).$$

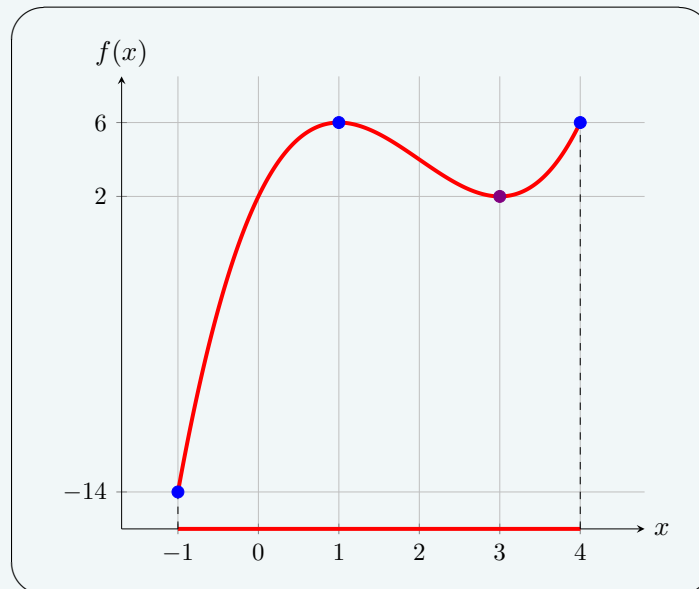
For every $x \in [-1, 4]$ we have $x + 1 \geq 0$, and the second factor is positive, so $f(x) \geq -14 = f(-1)$, with equality only for $x = -1$. The unique point of minimum is $x = -1$ and the minimum is -14 . The point of local minimum $x = 3$ is not a point of (global) minimum, since $f(-1) = -14 < 2 = f(3)$.

- **Maximum.** The factorization of the second question holds for every x : for every $x \in [-1, 4]$ we have $x - 4 \leq 0$, so $f(x) - 6 = (x - 4)(x - 1)^2 \leq 0$, with equality only for $x = 1$ and $x = 4$. The function has **two** points of maximum and the maximum is 6:

$$\arg \max \{ f(x) \text{ subject to } -1 \leq x \leq 4 \} = \{1, 4\}.$$

The point of local maximum $x = 1$ is also a point of (global) maximum.

4. The graph of the function is red; the domain is the red part of the x -axis, the points of (global) minimum and maximum are blue and the points of local minimum or maximum that are not global are violet. Since f takes negative values, the x -axis is drawn at the bottom of the figure.



Exercise 5.10 — Optimization problem (E11)

Consider the following function:

$$f : [0, 4] \rightarrow \mathbb{R}, \quad f : x \mapsto 5 + 54x - 2x^3.$$

Questions

1. Prove, using the definition, that the point (3) is a point of local maximum of f in $[0, 4]$.
2. Prove, using the definition, that the point (4) is a point of local minimum of f in $[0, 4]$.
3. Find all the points of (global) minimum and maximum of f in $[0, 4]$, that is, the optimal solutions of the optimization problems $\min\{f(x) \text{ subject to } 0 \leq x \leq 4\}$ and $\max\{f(x) \text{ subject to } 0 \leq x \leq 4\}$.
4. Draw the graph of the function.

Solution

1. We have $f(3) = 5 + 162 - 54 = 113$ and

$$f(3) - f(x) = 2x^3 - 54x + 108 = 2(x - 3)^2(x + 6).$$

We use the spherical δ -neighborhood $N((3), 1)$, for which $N((3), 1) \cap [0, 4] = (2, 4)$. For every x in this interval we have $x + 6 > 0$ and $(x - 3)^2 \geq 0$, so $f(3) - f(x) \geq 0$. Therefore,

$$f(3) = 113 \geq f(x), \quad \forall x \in N((3), 1) \cap [0, 4] = (2, 4),$$

and $x = 3$ is a point of local maximum of f in $[0, 4]$, with local maximum 113.

2. We have $f(4) = 5 + 216 - 128 = 93$ and

$$f(x) - f(4) = -2x^3 + 54x - 88 = -2(x - 4)(x^2 + 4x - 11).$$

We use the spherical δ -neighborhood $N((4), \frac{1}{2})$, for which $N((4), \frac{1}{2}) \cap [0, 4] = (\frac{7}{2}, 4]$. For every x in this interval we have $x - 4 \leq 0$ and, since $x > \frac{7}{2}$,

$$x^2 + 4x - 11 > \frac{49}{4} + 14 - 11 = \frac{61}{4} > 0,$$

so $f(x) - f(4) = -2(x - 4)(x^2 + 4x - 11) \geq 0$. Therefore,

$$f(4) = 93 \leq f(x), \quad \forall x \in N((4), \frac{1}{2}) \cap [0, 4] = (\frac{7}{2}, 4],$$

and $x = 4$ is a point of local minimum of f in $[0, 4]$, with local minimum 93.

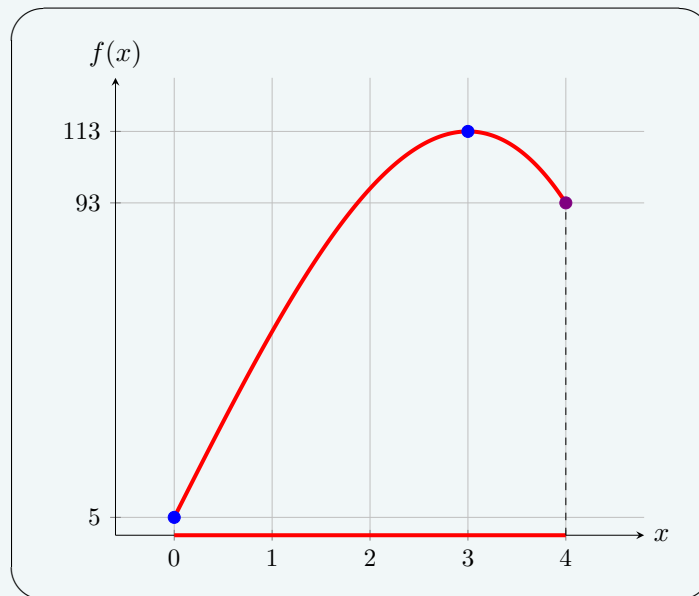
3. We study the minimum and the maximum separately.

- **Maximum.** The factorization of the first question holds for every x : for every $x \in [0, 4]$ we have $x + 6 > 0$, so $f(3) - f(x) = 2(x - 3)^2(x + 6) \geq 0$, with equality only for $x = 3$. The unique point of maximum is $x = 3$ and the maximum is 113: the point of local maximum is also a point of (global) maximum.
- **Minimum.** We have $f(0) = 5$ and

$$f(x) - 5 = 54x - 2x^3 = 2x(27 - x^2).$$

For every $x \in [0, 4]$ we have $x \geq 0$ and $x^2 \leq 16 < 27$, so $f(x) \geq 5 = f(0)$, with equality only for $x = 0$. The unique point of minimum is $x = 0$ and the minimum is 5. The point of local minimum $x = 4$ is not a point of (global) minimum, since $f(0) = 5 < 93 = f(4)$.

4. The graph of the function is red; the domain is the red part of the x -axis, the points of (global) minimum and maximum are blue and the points of local minimum or maximum that are not global are violet.



Exercise 5.11 — Optimization problem (E12)

Consider the following function:

$$f : [-2, 3] \rightarrow \mathbb{R}, \quad f : x \mapsto 1 - 12x - 3x^2 + 2x^3.$$

Questions

1. Prove, using the definition, that the point (-1) is a point of local maximum of f in $[-2, 3]$.
2. Prove, using the definition, that the point (2) is a point of local minimum of f in $[-2, 3]$.
3. Find all the points of (global) minimum and maximum of f in $[-2, 3]$, that is, the optimal solutions of the optimization problems $\min\{f(x) \text{ subject to } -2 \leq x \leq 3\}$ and $\max\{f(x) \text{ subject to } -2 \leq x \leq 3\}$.
4. Draw the graph of the function.

Solution

1. We have $f(-1) = 1 + 12 - 3 - 2 = 8$ and

$$f(x) - f(-1) = 2x^3 - 3x^2 - 12x - 7 = (x + 1)^2 (2x - 7).$$

We use the spherical δ -neighborhood $N((-1), 1)$, for which $N((-1), 1) \cap [-2, 3] = (-2, 0)$. For every x in this interval we have $2x - 7 < 0$ and $(x + 1)^2 \geq 0$, so $f(x) - f(-1) \leq 0$. Therefore,

$$f(-1) = 8 \geq f(x), \quad \forall x \in N((-1), 1) \cap [-2, 3] = (-2, 0),$$

and $x = -1$ is a point of local maximum of f in $[-2, 3]$, with local maximum 8.

2. We have $f(2) = 1 - 24 - 12 + 16 = -19$ and

$$f(x) - f(2) = 2x^3 - 3x^2 - 12x + 20 = (x - 2)^2 (2x + 5).$$

We use the spherical δ -neighborhood $N((2), 1)$, for which $N((2), 1) \cap [-2, 3] = (1, 3)$. For every x in this interval we have $2x + 5 > 0$ and $(x - 2)^2 \geq 0$, so $f(x) - f(2) \geq 0$. Therefore,

$$f(2) = -19 \leq f(x), \quad \forall x \in N((2), 1) \cap [-2, 3] = (1, 3),$$

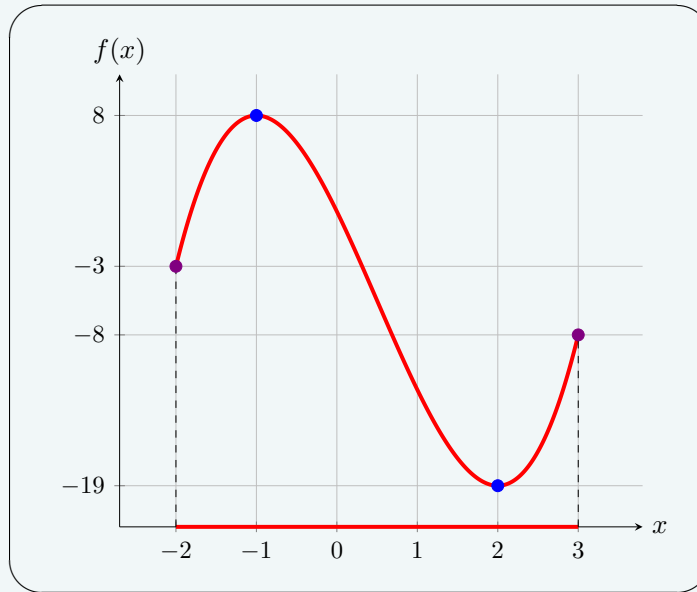
and $x = 2$ is a point of local minimum of f in $[-2, 3]$, with local minimum -19 .

3. The two factorizations hold for every x , and every $x \in [-2, 3]$ satisfies $2x - 7 \leq -1 < 0$ and $2x + 5 \geq 1 > 0$. Then, for every $x \in [-2, 3]$,

$$f(x) - 8 = (x + 1)^2 (2x - 7) \leq 0 \quad \text{and} \quad f(x) + 19 = (x - 2)^2 (2x + 5) \geq 0,$$

with equality only for $x = -1$ and only for $x = 2$, respectively. The unique point of maximum is $x = -1$, with maximum 8, and the unique point of minimum is $x = 2$, with minimum -19 : both points of the first two questions are global optima.

4. The graph of the function is red; the domain is the red part of the x -axis, the points of (global) minimum and maximum are blue. Since f takes negative values, the x -axis is drawn at the bottom of the figure. At the endpoints we have $f(-2) = -3$ and $f(3) = -8$: since f is increasing near both endpoints, $x = -2$ is a point of local minimum and $x = 3$ is a point of local maximum, which are not global, and they are violet.



Exercise 5.12 — Optimization problem (E13)

Consider the following function:

$$f : [-2, 3] \rightarrow \mathbb{R}, \quad f : x \mapsto 3 - 2x^2 + x^4.$$

Questions

1. Prove, using the definition, that the point (-1) is a point of local minimum of f in $[-2, 3]$.
2. Prove, using the definition, that the point (0) is a point of local maximum of f in $[-2, 3]$.
3. Prove, using the definition, that the point (1) is a point of local minimum of f in $[-2, 3]$.
4. Find all the points of (global) minimum and maximum of f in $[-2, 3]$, that is, the optimal solutions of the optimization problems $\min\{f(x) \text{ subject to } -2 \leq x \leq 3\}$ and $\max\{f(x) \text{ subject to } -2 \leq x \leq 3\}$.
5. Draw the graph of the function in the interval $[-2, 2]$.

Solution

1. We have $f(-1) = 3 - 2 + 1 = 2$ and

$$f(x) - f(-1) = x^4 - 2x^2 + 1 = (x^2 - 1)^2 \geq 0, \quad \forall x \in \mathbb{R}.$$

In particular, for the spherical δ -neighborhood $N((-1), \frac{1}{2})$ we have

$$f(-1) = 2 \leq f(x), \quad \forall x \in N((-1), \frac{1}{2}) \cap [-2, 3] = \left(-\frac{3}{2}, -\frac{1}{2}\right),$$

and $x = -1$ is a point of local minimum of f in $[-2, 3]$, with local minimum 2.

2. We have $f(0) = 3$ and

$$f(x) - f(0) = x^4 - 2x^2 = x^2(x^2 - 2).$$

We use the spherical δ -neighborhood $N((0), \frac{1}{2})$, for which $N((0), \frac{1}{2}) \cap [-2, 3] = (-\frac{1}{2}, \frac{1}{2})$. For every x in this interval we have $x^2 < \frac{1}{4} < 2$, so $f(x) - f(0) \leq 0$. Therefore,

$$f(0) = 3 \geq f(x), \quad \forall x \in N((0), \frac{1}{2}) \cap [-2, 3] = \left(-\frac{1}{2}, \frac{1}{2}\right),$$

and $x = 0$ is a point of local maximum of f in $[-2, 3]$, with local maximum 3.

3. We have $f(1) = 2$ and, as in the first question, $f(x) - f(1) = (x^2 - 1)^2 \geq 0$ for every $x \in \mathbb{R}$. In particular,

$$f(1) = 2 \leq f(x), \quad \forall x \in N((1), \frac{1}{2}) \cap [-2, 3] = \left(\frac{1}{2}, \frac{3}{2}\right),$$

and $x = 1$ is a point of local minimum of f in $[-2, 3]$, with local minimum 2.

4. We study the minimum and the maximum separately.

- **Minimum.** For every $x \in [-2, 3]$ we have $f(x) - 2 = (x^2 - 1)^2 \geq 0$, with equality only for $x = -1$ and $x = 1$. The function has **two** points of minimum and the minimum is 2:

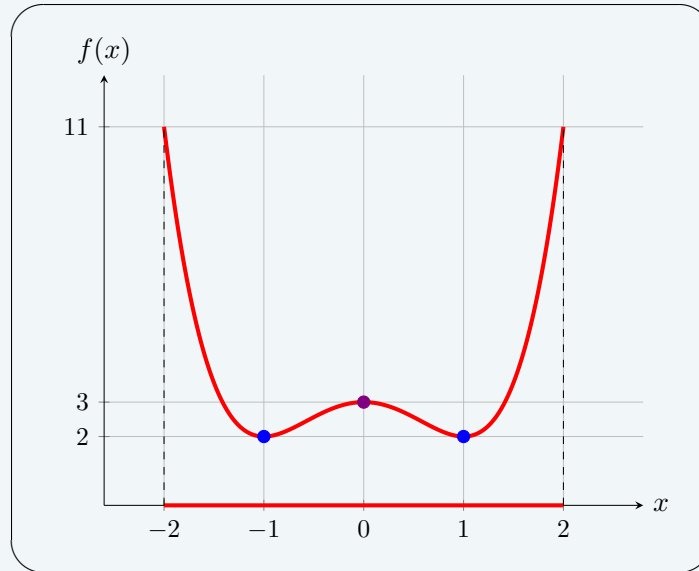
$$\arg \min \{ f(x) \text{ subject to } -2 \leq x \leq 3 \} = \{-1, 1\}.$$

- **Maximum.** We have $f(3) = 3 - 18 + 81 = 66$ and

$$66 - f(x) = 63 + 2x^2 - x^4 = (9 - x^2)(x^2 + 7).$$

For every $x \in [-2, 3]$ we have $x^2 \leq 9$, so $f(x) \leq 66 = f(3)$, with equality only for $x = 3$ (the value $x = -3$ is not in the domain). The unique point of maximum is $x = 3$ and the maximum is 66. The point of local maximum $x = 0$ is not a point of (global) maximum, since $f(0) = 3 < 66 = f(3)$.

5. The graph of the function in the interval $[-2, 2]$ is red and the interval $[-2, 2]$, which is a part of the domain, is the red part of the x -axis; the points of (global) minimum are blue and the point of local maximum that is not global is violet. The point of maximum $x = 3$, with $f(3) = 66$, is outside the figure.



Exercise 5.13 — Optimization problem (E14)

Consider the following function:

$$f : [0, 5] \rightarrow \mathbb{R}, \quad f : x \mapsto x^3 - 9x^2 + 24x - 10.$$

Questions

1. Prove, using the definition, that the point (2) is a point of local maximum of f in $[0, 5]$.
2. Prove, using the definition, that the point (4) is a point of local minimum of f in $[0, 5]$.
3. Find all the points of (global) minimum and maximum of f in $[0, 5]$, that is, the optimal solutions of the optimization problems $\min\{f(x) \text{ subject to } 0 \leq x \leq 5\}$ and $\max\{f(x) \text{ subject to } 0 \leq x \leq 5\}$.
4. Draw the graph of the function.

Solution

1. We have $f(2) = 8 - 36 + 48 - 10 = 10$ and

$$f(x) - f(2) = x^3 - 9x^2 + 24x - 20 = (x - 2)^2 (x - 5).$$

We use the spherical δ -neighborhood $N((2), 1)$, for which $N((2), 1) \cap [0, 5] = (1, 3)$. For every x in this interval we have $x - 5 < 0$ and $(x - 2)^2 \geq 0$, so $f(x) - f(2) \leq 0$. Therefore,

$$f(2) = 10 \geq f(x), \quad \forall x \in N((2), 1) \cap [0, 5] = (1, 3),$$

and $x = 2$ is a point of local maximum of f in $[0, 5]$, with local maximum 10.

2. We have $f(4) = 64 - 144 + 96 - 10 = 6$ and

$$f(x) - f(4) = x^3 - 9x^2 + 24x - 16 = (x - 4)^2 (x - 1).$$

We use the spherical δ -neighborhood $N((4), 1)$, for which $N((4), 1) \cap [0, 5] = (3, 5)$. For every x in this interval we have $x - 1 > 0$ and $(x - 4)^2 \geq 0$, so $f(x) - f(4) \geq 0$. Therefore,

$$f(4) = 6 \leq f(x), \quad \forall x \in N((4), 1) \cap [0, 5] = (3, 5),$$

and $x = 4$ is a point of local minimum of f in $[0, 5]$, with local minimum 6.

3. We study the minimum and the maximum separately.

- **Maximum.** The factorization of the first question holds for every x : for every $x \in [0, 5]$ we have $x - 5 \leq 0$, so $f(x) - 10 = (x - 2)^2 (x - 5) \leq 0$, with equality only for $x = 2$ and $x = 5$. The function has **two** points of maximum and the maximum is 10:

$$\arg \max \{ f(x) \text{ subject to } 0 \leq x \leq 5 \} = \{2, 5\}.$$

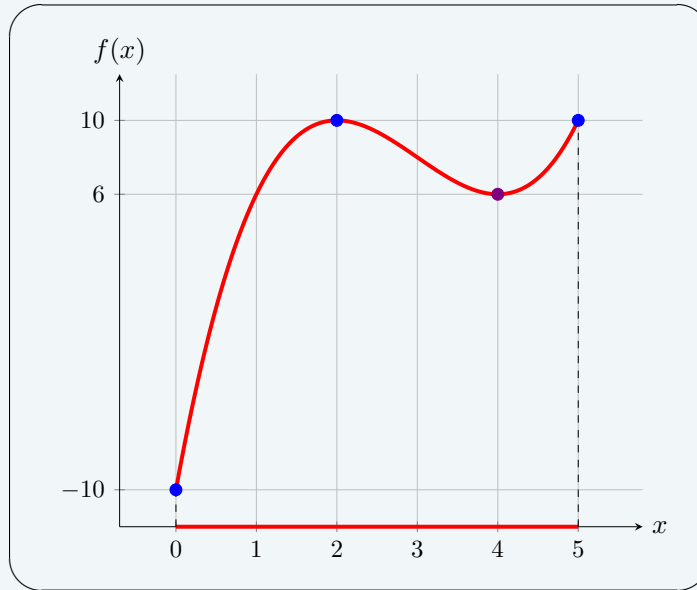
The point of local maximum $x = 2$ is also a point of (global) maximum.

- **Minimum.** We have $f(0) = -10$ and

$$f(x) + 10 = x^3 - 9x^2 + 24x = x(x^2 - 9x + 24) = x \left(\left(x - \frac{9}{2}\right)^2 + \frac{15}{4} \right).$$

For every $x \in [0, 5]$ we have $x \geq 0$, and the second factor is positive, so $f(x) \geq -10 = f(0)$, with equality only for $x = 0$. The unique point of minimum is $x = 0$ and the minimum is -10 . The point of local minimum $x = 4$ is not a point of (global) minimum, since $f(0) = -10 < 6 = f(4)$.

4. The graph of the function is red; the domain is the red part of the x -axis, the points of (global) minimum and maximum are blue and the points of local minimum or maximum that are not global are violet. Since f takes negative values, the x -axis is drawn at the bottom of the figure.



Exercise 5.14 — Optimization problem (E15)

Consider the following function:

$$f : [-1, 3] \rightarrow \mathbb{R}, \quad f : x \mapsto 2x^3 - 3x^2 - 12x + 5.$$

Questions

1. Prove, using the definition, that the point (-1) is a point of local maximum of f in $[-1, 3]$.
2. Prove, using the definition, that the point (2) is a point of local minimum of f in $[-1, 3]$.
3. Find all the points of (global) minimum and maximum of f in $[-1, 3]$, that is, the optimal solutions of the optimization problems $\min\{f(x) \text{ subject to } -1 \leq x \leq 3\}$ and $\max\{f(x) \text{ subject to } -1 \leq x \leq 3\}$.
4. Draw the graph of the function.

Solution

1. We have $f(-1) = -2 - 3 + 12 + 5 = 12$ and

$$f(x) - f(-1) = 2x^3 - 3x^2 - 12x - 7 = (x+1)^2(2x-7).$$

We use the spherical δ -neighborhood $N((-1), \frac{1}{2}) = (-\frac{3}{2}, -\frac{1}{2})$. The point $x = -1$ is the left endpoint of the domain, so only the points on its right are considered: $N((-1), \frac{1}{2}) \cap [-1, 3] = [-1, -\frac{1}{2})$. For every x in this interval we have $2x - 7 < 0$ and

$(x+1)^2 \geq 0$, so $f(x) - f(-1) \leq 0$. Therefore,

$$f(-1) = 12 \geq f(x), \quad \forall x \in N\left((-1), \frac{1}{2}\right) \cap [-1, 3] = \left[-1, -\frac{1}{2}\right),$$

and $x = -1$ is a point of local maximum of f in $[-1, 3]$, with local maximum 12.

2. We have $f(2) = 16 - 12 - 24 + 5 = -15$ and

$$f(x) - f(2) = 2x^3 - 3x^2 - 12x + 20 = (x-2)^2(2x+5).$$

We use the spherical δ -neighborhood $N\left((2), \frac{1}{2}\right)$, for which $N\left((2), \frac{1}{2}\right) \cap [-1, 3] = \left(\frac{3}{2}, \frac{5}{2}\right)$. For every x in this interval we have $2x+5 > 0$ and $(x-2)^2 \geq 0$, so $f(x) - f(2) \geq 0$. Therefore,

$$f(2) = -15 \leq f(x), \quad \forall x \in N\left((2), \frac{1}{2}\right) \cap [-1, 3] = \left(\frac{3}{2}, \frac{5}{2}\right),$$

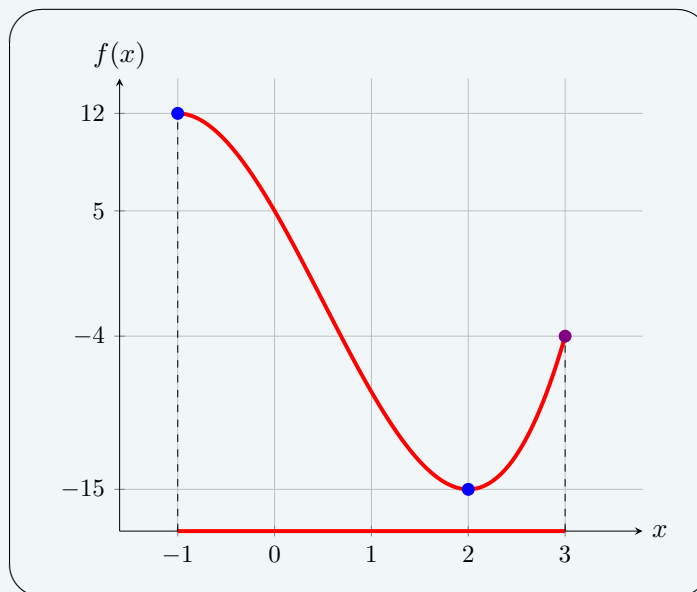
and $x = 2$ is a point of local minimum of f in $[-1, 3]$, with local minimum -15 .

3. The two factorizations hold for every x , and every $x \in [-1, 3]$ satisfies $2x-7 \leq -1 < 0$ and $2x+5 \geq 3 > 0$. Then, for every $x \in [-1, 3]$,

$$f(x) - 12 = (x+1)^2(2x-7) \leq 0 \quad \text{and} \quad f(x) + 15 = (x-2)^2(2x+5) \geq 0,$$

with equality only for $x = -1$ and only for $x = 2$, respectively. The unique point of maximum is $x = -1$, with maximum 12, and the unique point of minimum is $x = 2$, with minimum -15 : both points of the first two questions are global optima.

4. The graph of the function is red; the domain is the red part of the x -axis, the points of (global) minimum and maximum are blue. Since f takes negative values, the x -axis is drawn at the bottom of the figure. At the right endpoint we have $f(3) = -4$: since f is increasing near $x = 3$, it is a point of local maximum which is not global, and it is violet.



5.2 Problems in two variables

Exercise 5.15 — Optimization problem (E8)

Consider the following objective function of two variables and feasible region:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto (x_1 - x_2)^2$$

and

$$F = \{ \mathbf{x} \in \mathbb{R}^2 : 0 \leq x_1 \leq 2, 0 \leq x_2 \leq 2 \} = [0, 2] \times [0, 2].$$

Questions

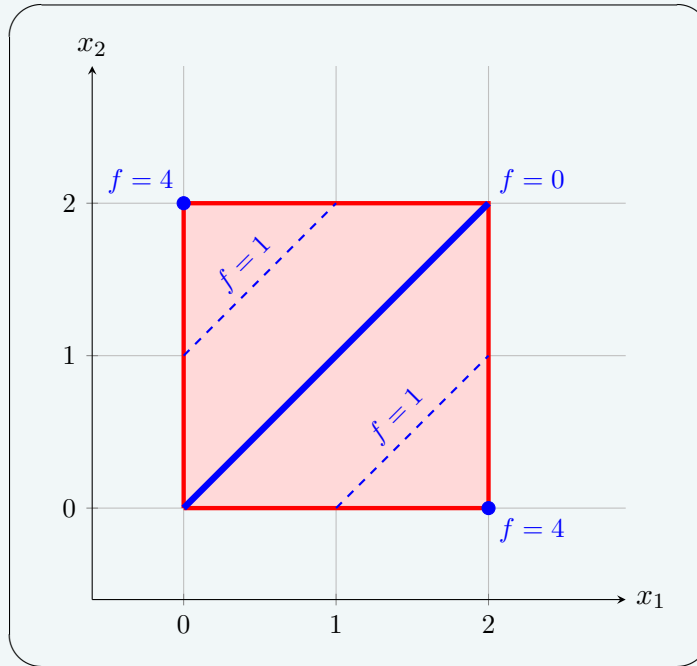
1. Write the associated optimization problems in minimization and maximization forms.
2. Draw the feasible region and the level curves of the objective function with values 0, 1 and 4.
3. Find all optimal solutions and the optimal value for both forms.

Solution

1. The associated optimization problems in minimization and maximization forms are:

$$\begin{aligned} \min & \left\{ (x_1 - x_2)^2 \text{ subject to } 0 \leq x_1 \leq 2, 0 \leq x_2 \leq 2 \right\} \\ \max & \left\{ (x_1 - x_2)^2 \text{ subject to } 0 \leq x_1 \leq 2, 0 \leq x_2 \leq 2 \right\} \end{aligned}$$

2. The feasible region is the red square. Since $f(\mathbf{x}) = c$ if and only if $x_1 - x_2 = \sqrt{c}$ or $x_1 - x_2 = -\sqrt{c}$, the level curves are straight lines parallel to the line $x_1 = x_2$: the level curve with value 0 is the line $x_1 = x_2$, the one with value 1 is made of the two lines $x_1 - x_2 = \pm 1$ (dashed), and the one with value 4 is made of the two lines $x_1 - x_2 = \pm 2$, which touch the square only at its corners (2, 0) and (0, 2). The optimal solutions are blue.



3. The feasible region is the square $F = [0, 2] \times [0, 2]$.

- For the minimization form, $f(\mathbf{x}) = (x_1 - x_2)^2 \geq 0$ for every feasible $\mathbf{x}$, with equality exactly when $x_1 = x_2$. The problem has **infinitely many** optimal solutions (the blue diagonal of the square) and the optimal value is:

$$\arg \min \{ (x_1 - x_2)^2 \text{ subject to } \mathbf{x} \in F \} = \left\{ \begin{pmatrix} t \\ t \end{pmatrix} : 0 \leq t \leq 2 \right\} \quad \text{and} \quad z(P) = 0.$$

- For the maximization form, for every feasible $\mathbf{x}$ we have $0 \leq x_1 \leq 2$ and $-2 \leq -x_2 \leq 0$, so $-2 \leq x_1 - x_2 \leq 2$ and $f(\mathbf{x}) \leq 4$. The equality $x_1 - x_2 = 2$ holds only for $x_1 = 2$ and $x_2 = 0$, and $x_1 - x_2 = -2$ only for $x_1 = 0$ and $x_2 = 2$. The problem has **two** optimal solutions and the optimal value is:

$$\arg \max \{ (x_1 - x_2)^2 \text{ subject to } \mathbf{x} \in F \} = \left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\} \quad \text{and} \quad z(P) = 4.$$

Exercise 5.16 — Optimization problem (E16)

Consider the following function:

$$f : D \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1 + x_2, \quad \text{with} \quad D = [-2, 2] \times [-2, 2].$$

Questions

1. Prove, using the definition, that the point $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$ is the unique point of minimum of f in D .
2. Prove, using the definition, that the point $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is the unique point of maximum of f in D .
3. Draw the graph of the function in the domain D .

Solution

1. For every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D$ we have $x_1 \geq -2$ and $x_2 \geq -2$. Therefore,

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = x_1 + x_2 \geq -2 + (-2) = -4 = f\left(\begin{pmatrix} -2 \\ -2 \end{pmatrix}\right),$$

with equality only for $x_1 = x_2 = -2$. Accordingly,

$$f\left(\begin{pmatrix} -2 \\ -2 \end{pmatrix}\right) \leq f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right), \quad \forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D,$$

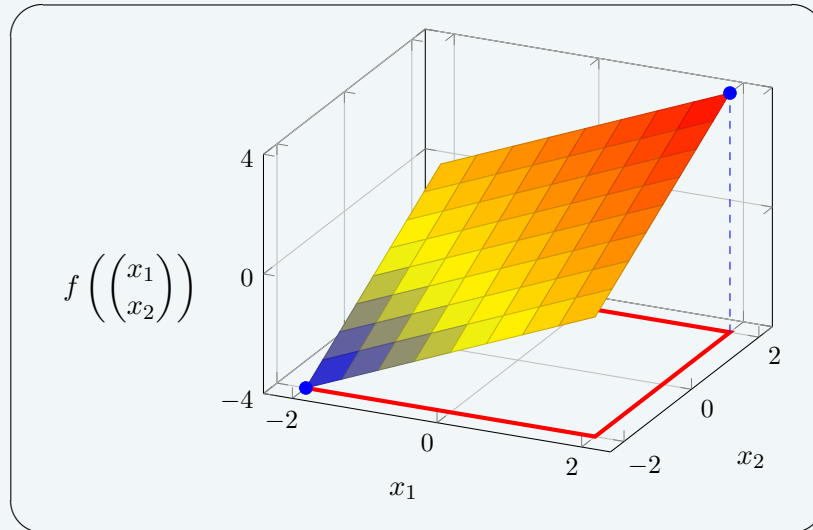
and $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$ is the unique point of minimum of f in D , with minimum -4 .

2. For every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D$ we have $x_1 \leq 2$ and $x_2 \leq 2$. Therefore,

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = x_1 + x_2 \leq 2 + 2 = 4 = f\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right),$$

with equality only for $x_1 = x_2 = 2$, and $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is the unique point of maximum of f in D , with maximum 4 .

3. The graph of the function is the surface over the domain D , which is the red square drawn on the bottom plane; the points of the graph corresponding to the point of minimum and to the point of maximum are blue, with a dashed segment that joins them to the bottom plane.



Exercise 5.17 — Optimization problem (E17)

Consider the following function:

$$f : D \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto (x_1 - 2)^2 + (x_2 - 2)^2 + 1, \quad \text{with} \quad D = [-4, 4] \times [-4, 4].$$

Questions

1. Prove, using the definition, that the point $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is a point of local minimum of f in D .
2. Prove that $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is also the unique point of (global) minimum of f in D , and find all the points of maximum of f in D .
3. Draw the graph of the function in the domain D .

Solution

1. We have $f\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) = 1$. We use the spherical δ -neighborhood $N\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}, 1\right)$: for every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in it,

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) - f\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) = (x_1 - 2)^2 + (x_2 - 2)^2 \geq 0.$$

Therefore,

$$f\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) \leq f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right), \quad \forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in N\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}, 1\right) \cap D,$$

and $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is a point of local minimum of f in D , with local minimum 1.

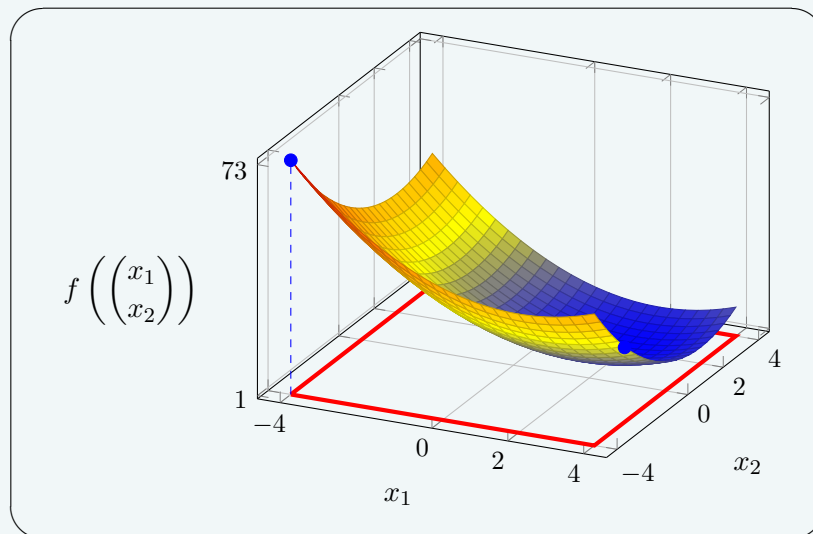
2. We study the minimum and the maximum separately.

- **Minimum.** The inequality $f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) - f\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) = (x_1 - 2)^2 + (x_2 - 2)^2 \geq 0$ holds for every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D$, not only in the neighborhood, with equality only for $x_1 = x_2 = 2$. The point $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is the unique point of minimum of f in D and the minimum is 1.
- **Maximum.** For every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D$ we have $-6 \leq x_1 - 2 \leq 2$ and $-6 \leq x_2 - 2 \leq 2$, so $(x_1 - 2)^2 \leq 36$ and $(x_2 - 2)^2 \leq 36$, with equality only for $x_1 = -4$ and $x_2 = -4$, respectively. Therefore,

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \leq 36 + 36 + 1 = 73 = f\left(\begin{pmatrix} -4 \\ -4 \end{pmatrix}\right),$$

and $\begin{pmatrix} -4 \\ -4 \end{pmatrix}$ is the unique point of maximum of f in D , with maximum 73.

3. The graph of the function is the surface over the domain D , which is the red square drawn on the bottom plane; the points of the graph corresponding to the point of minimum and to the point of maximum are blue, with a dashed segment that joins them to the bottom plane.



Exercise 5.18 — Optimization problem (E18)

Consider the following function:

$$f : D \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1^2 + x_2^2 + 2 - \frac{1}{10}(x_1^4 + x_2^4), \quad \text{with} \quad D = [-4, 4] \times [-4, 4].$$

Questions

1. Prove, using the definition, that the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a point of local minimum of f in D .
2. Prove that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not a point of (global) minimum of f in D , and find all the points of minimum of f in D , that is, the optimal solutions of the optimization problem $\min\{f(\mathbf{x}) \text{ subject to } \mathbf{x} \in D\}$.
3. Draw the graph of the function in the domain D .

Solution

1. We have $f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = 2$. We use the spherical δ -neighborhood $N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 2\right)$: for every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in it we have $x_1^2 \leq x_1^2 + x_2^2 < 4$ and $x_2^2 < 4$, hence

$$1 - \frac{x_1^2}{10} > 1 - \frac{4}{10} = \frac{3}{5} > 0 \quad \text{and} \quad 1 - \frac{x_2^2}{10} > \frac{3}{5} > 0,$$

and then

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) - f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = x_1^2 \left(1 - \frac{x_1^2}{10}\right) + x_2^2 \left(1 - \frac{x_2^2}{10}\right) \geq 0.$$

Therefore,

$$f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \leq f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right), \quad \forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 2\right) \cap D,$$

and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a point of local minimum of f in D , with local minimum 2.

2. We write f as a sum of two functions of one variable:

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = 2 + g(x_1) + g(x_2), \quad \text{with} \quad g(t) = t^2 - \frac{t^4}{10}.$$

We have $g(4) = 16 - \frac{256}{10} = -\frac{48}{5}$ and

$$g(t) + \frac{48}{5} = \frac{-t^4 + 10t^2 + 96}{10} = \frac{(16 - t^2)(t^2 + 6)}{10}.$$

For every $t \in [-4, 4]$ we have $t^2 \leq 16$, so $g(t) \geq -\frac{48}{5}$, with equality only for $t = -4$ and $t = 4$. Therefore, for every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D$,

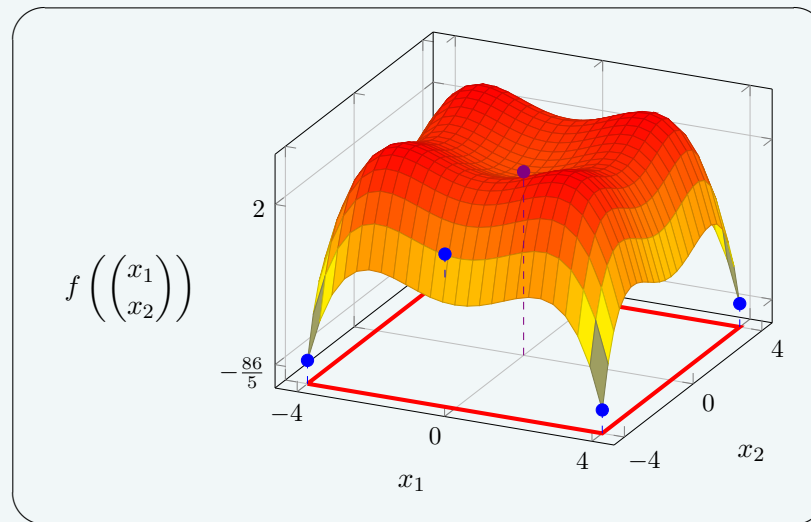
$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \geq 2 - \frac{48}{5} - \frac{48}{5} = -\frac{86}{5},$$

with equality only when both x_1 and x_2 are equal to -4 or 4 . The function has **four** points of minimum, the corners of D , and the minimum is $-\frac{86}{5}$:

$$\arg \min \left\{ f(\mathbf{x}) \text{ subject to } \mathbf{x} \in D \right\} = \left\{ \begin{pmatrix} -4 \\ -4 \end{pmatrix}, \begin{pmatrix} -4 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ -4 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix} \right\}.$$

The point of local minimum $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not a point of (global) minimum, since $-\frac{86}{5} < 2$.

3. The graph of the function is the surface over the domain D , which is the red square drawn on the bottom plane; the points of the graph corresponding to the four points of minimum are blue and the point corresponding to the point of local minimum that is not global is violet, with a dashed segment that joins them to the bottom plane.



Exercise 5.19 — Optimization problem (E19)

Consider the following function:

$$f : D \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1^2 + x_2^2 - \frac{1}{2}x_1^2x_2^2, \quad \text{with} \quad D = [-3, 3] \times [-3, 3].$$

Questions

1. Prove, using the definition, that the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a point of local minimum of f in D .
2. Prove that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not a point of (global) minimum of f in D , and find all the points of minimum of f in D , that is, the optimal solutions of the optimization problem $\min\{f(\mathbf{x}) \text{ subject to } \mathbf{x} \in D\}$.
3. Draw the graph of the function in the domain D .

Solution

1. We have $f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = 0$. We use the spherical δ -neighborhood $N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 1\right)$: for every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in it we have $x_2^2 \leq x_1^2 + x_2^2 < 1$, hence $1 - \frac{x_2^2}{2} > \frac{1}{2}$, and then

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) - f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = x_1^2 \left(1 - \frac{x_2^2}{2}\right) + x_2^2 \geq \frac{1}{2}x_1^2 + x_2^2 \geq 0.$$

Therefore,

$$f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \leq f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right), \quad \forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 1\right) \cap D,$$

and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a point of local minimum of f in D , with local minimum 0.

2. We have $f\left(\begin{pmatrix} 3 \\ 3 \end{pmatrix}\right) = 9 + 9 - \frac{81}{2} = -\frac{45}{2} < 0$, so $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not a point of (global) minimum. To find the points of minimum, we write

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = 2 - \frac{1}{2}uv, \quad \text{with} \quad u = x_1^2 - 2 \quad \text{and} \quad v = x_2^2 - 2.$$

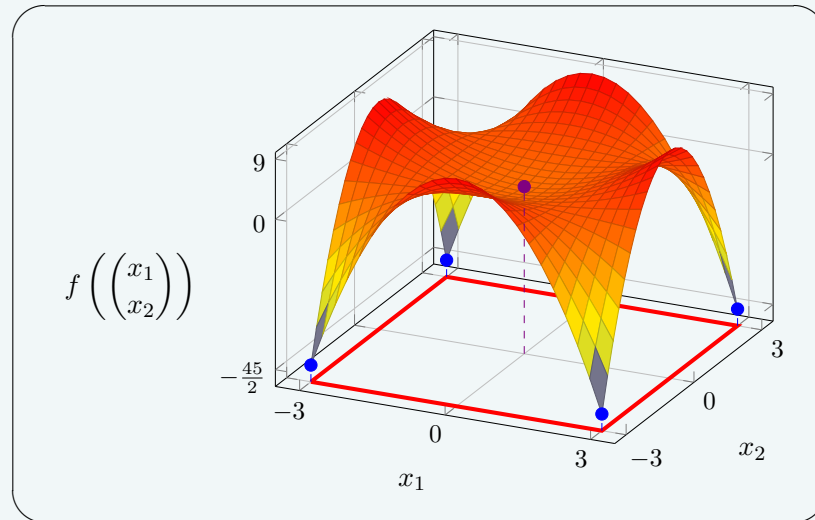
For every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D$ we have $0 \leq x_1^2 \leq 9$ and $0 \leq x_2^2 \leq 9$, so $-2 \leq u \leq 7$ and $-2 \leq v \leq 7$. Then:

- if $u \geq 0$ and $v \geq 0$, we have $uv \leq 7 \cdot 7 = 49$, with equality only for $u = v = 7$;
- if u and v have opposite signs, we have $uv \leq 0$;
- if $u < 0$ and $v < 0$, we have $uv = |u||v| \leq 2 \cdot 2 = 4$.

In all cases $uv \leq 49$, so $f(\mathbf{x}) \geq 2 - \frac{49}{2} = -\frac{45}{2}$, with equality only for $x_1^2 = x_2^2 = 9$. The function has **four** points of minimum, the corners of D , and the minimum is $-\frac{45}{2}$:

$$\arg \min \{ f(\mathbf{x}) \text{ subject to } \mathbf{x} \in D \} = \left\{ \begin{pmatrix} -3 \\ -3 \end{pmatrix}, \begin{pmatrix} -3 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix} \right\}.$$

3. The graph of the function is the surface over the domain D , which is the red square drawn on the bottom plane; the points of the graph corresponding to the four points of minimum are blue and the point corresponding to the point of local minimum that is not global is violet, with a dashed segment that joins them to the bottom plane.



Exercise 5.20 — Optimization problem (E20)

Consider the following function:

$$f : D \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1^2 + x_2^2 - x_1^3, \quad \text{with} \quad D = [-2, 4] \times [-2, 4].$$

Questions

1. Prove, using the definition, that the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a point of local minimum of f in D .

2. Prove that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not a point of (global) minimum of f in D , and find all the points of minimum of f in D , that is, the optimal solutions of the optimization problem $\min\{f(\mathbf{x}) \text{ subject to } \mathbf{x} \in D\}$.
3. Draw the graph of the function in the domain D .

Solution

1. We have $f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = 0$. We use the spherical δ -neighborhood $N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \frac{1}{2}\right)$: for every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in it we have $|x_1| < \frac{1}{2}$, hence $1 - x_1 > \frac{1}{2}$, and then

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) - f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = x_1^2(1 - x_1) + x_2^2 \geq \frac{1}{2}x_1^2 + x_2^2 \geq 0.$$

Therefore,

$$f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \leq f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right), \quad \forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \frac{1}{2}\right) \cap D,$$

and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a point of local minimum of f in D , with local minimum 0.

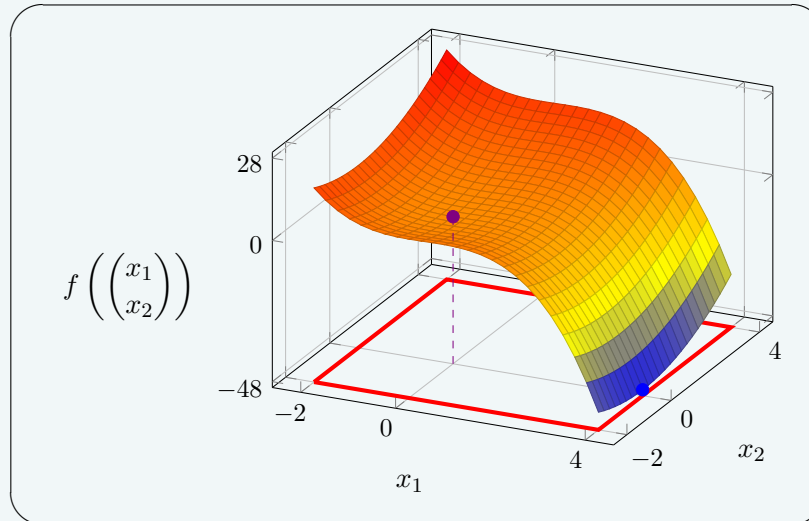
2. We have $f\left(\begin{pmatrix} 4 \\ 0 \end{pmatrix}\right) = 16 - 64 = -48 < 0$, so $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not a point of (global) minimum. Moreover,

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) + 48 = (48 + x_1^2 - x_1^3) + x_2^2 = (4 - x_1)(x_1^2 + 3x_1 + 12) + x_2^2,$$

where $x_1^2 + 3x_1 + 12 = (x_1 + \frac{3}{2})^2 + \frac{39}{4} > 0$. For every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D$ we have $4 - x_1 \geq 0$, so

$f(\mathbf{x}) \geq -48$, with equality only for $x_1 = 4$ and $x_2 = 0$. The point $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ is the unique point of minimum of f in D and the minimum is -48 .

3. The graph of the function is the surface over the domain D , which is the red square drawn on the bottom plane; the point of the graph corresponding to the point of minimum is blue and the point corresponding to the point of local minimum that is not global is violet, with a dashed segment that joins them to the bottom plane.



Exercise 5.21 — Optimization problem (E21)

Consider the following function:

$$f : D \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1^2 + x_2^2 + 1 - x_1x_2^2, \quad \text{with} \quad D = [-1, 5] \times [-3, 3].$$

Questions

1. Prove, using the definition, that the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a point of local minimum of f in D .
2. Prove that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not a point of (global) minimum of f in D , and find all the points of minimum of f in D , that is, the optimal solutions of the optimization problem $\min\{f(\mathbf{x}) \text{ subject to } \mathbf{x} \in D\}$.
3. Draw the graph of the function in the domain D .

Solution

1. We have $f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = 1$. We use the spherical δ -neighborhood $N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \frac{1}{2}\right)$: for every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in it we have $|x_1| < \frac{1}{2}$, hence $1 - x_1 > \frac{1}{2}$, and then

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) - f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = x_1^2 + x_2^2(1 - x_1) \geq x_1^2 + \frac{1}{2}x_2^2 \geq 0.$$

Therefore,

$$f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \leq f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right), \quad \forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \frac{1}{2}\right) \cap D,$$

and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a point of local minimum of f in D , with local minimum 1.

2. We have $f\left(\begin{pmatrix} 5 \\ 3 \end{pmatrix}\right) = 25 + 9 + 1 - 45 = -10 < 1$, so $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not a point of (global) minimum. To find the points of minimum, we check by expanding the products that

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) + \frac{41}{4} = \left(x_1 - \frac{9}{2}\right)^2 + (x_1 - 1)(9 - x_2^2).$$

For every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D$ we have $0 \leq 9 - x_2^2 \leq 9$, and:

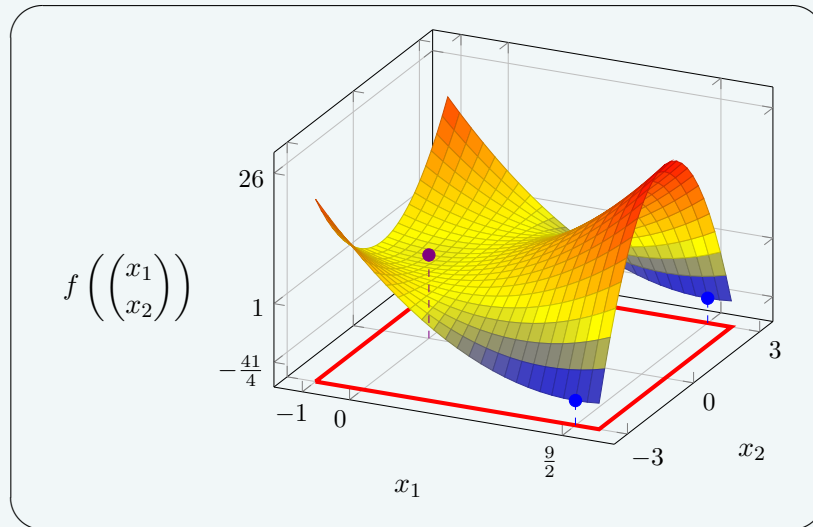
- if $x_1 \geq 1$, both terms are nonnegative, so $f(\mathbf{x}) + \frac{41}{4} \geq 0$, with equality only for $x_1 = \frac{9}{2}$ and $x_2^2 = 9$;
- if $x_1 < 1$, we have $x_1 - 1 < 0$ and then $(x_1 - 1)(9 - x_2^2) \geq 9(x_1 - 1)$, so

$$f(\mathbf{x}) + \frac{41}{4} \geq \left(x_1 - \frac{9}{2}\right)^2 + 9(x_1 - 1) = x_1^2 + \frac{45}{4} > 0.$$

The function has **two** points of minimum and the minimum is $-\frac{41}{4}$:

$$\arg \min \{ f(\mathbf{x}) \text{ subject to } \mathbf{x} \in D \} = \left\{ \begin{pmatrix} \frac{9}{2} \\ -3 \end{pmatrix}, \begin{pmatrix} \frac{9}{2} \\ 3 \end{pmatrix} \right\}.$$

3. The graph of the function is the surface over the domain D , which is the red square drawn on the bottom plane; the points of the graph corresponding to the two points of minimum are blue and the point corresponding to the point of local minimum that is not global is violet, with a dashed segment that joins them to the bottom plane.



Exercise 5.22 — Optimization problem (E22)

Consider the following function:

$$f : D \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto -(x_1 - 2)^2 - (x_2 - 2)^2 + 5, \quad \text{with} \quad D = [-4, 4] \times [-4, 4].$$

Questions

1. Prove, using the definition, that the point $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is a point of local maximum of f in D .
2. Prove that $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is also the unique point of (global) maximum of f in D , and find all the points of minimum of f in D .
3. Draw the graph of the function in the domain D .

Solution

1. We have $f\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) = 5$. We use the spherical δ -neighborhood $N\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}, 1\right)$: for every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in it,

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) - f\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) = -((x_1 - 2)^2 + (x_2 - 2)^2) \leq 0.$$

Therefore,

$$f\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) \geq f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right), \quad \forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in N\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}, 1\right) \cap D,$$

and $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is a point of local maximum of f in D , with local maximum 5.

2. We study the minimum and the maximum separately.

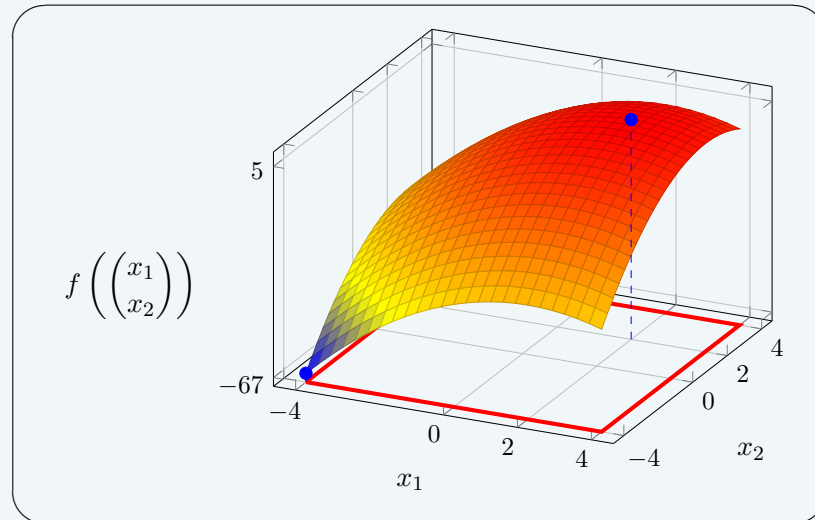
- **Maximum.** The inequality $f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) - f\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) = -((x_1 - 2)^2 + (x_2 - 2)^2) \leq 0$ holds for every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D$, not only in the neighborhood, with equality only for $x_1 = x_2 = 2$. The point $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is the unique point of maximum of f in D and the maximum is 5.

- **Minimum.** For every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D$ we have $-6 \leq x_1 - 2 \leq 2$ and $-6 \leq x_2 - 2 \leq 2$, so $(x_1 - 2)^2 \leq 36$ and $(x_2 - 2)^2 \leq 36$, with equality only for $x_1 = -4$ and $x_2 = -4$, respectively. Therefore,

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \geq -36 - 36 + 5 = -67 = f\left(\begin{pmatrix} -4 \\ -4 \end{pmatrix}\right),$$

and $\begin{pmatrix} -4 \\ -4 \end{pmatrix}$ is the unique point of minimum of f in D , with minimum -67 .

3. The graph of the function is the surface over the domain D , which is the red square drawn on the bottom plane; the points of the graph corresponding to the point of maximum and to the point of minimum are blue, with a dashed segment that joins them to the bottom plane.



Exercise 5.23 — Optimization problem (E23)

Consider the following function:

$$f : D \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto -x_1^2 - x_2^2 + 2 + \frac{1}{10}(x_1^4 + x_2^4), \quad \text{with} \quad D = [-4, 4] \times [-4, 4].$$

Questions

1. Prove, using the definition, that the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a point of local maximum of f in D .

2. Prove that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not a point of (global) maximum of f in D , and find all the points of maximum of f in D , that is, the optimal solutions of the optimization problem $\max\{f(\mathbf{x}) \text{ subject to } \mathbf{x} \in D\}$.
3. Draw the graph of the function in the domain D .

Solution

1. We have $f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = 2$. We use the spherical δ -neighborhood $N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 1\right)$: for every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in it we have $|x_1| < 1$ and $|x_2| < 1$, hence $x_1^4 \leq x_1^2$ and $x_2^4 \leq x_2^2$, and then

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \leq 2 - (x_1^2 + x_2^2) + \frac{1}{10}(x_1^2 + x_2^2) = 2 - \frac{9}{10}(x_1^2 + x_2^2) \leq 2 = f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right).$$

Therefore,

$$f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \geq f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right), \quad \forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 1\right) \cap D,$$

and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a point of local maximum of f in D , with local maximum 2.

2. We write f as a sum of two functions of one variable:

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = 2 - g(x_1) - g(x_2), \quad \text{with} \quad g(t) = t^2 - \frac{t^4}{10}.$$

We have $g(4) = 16 - \frac{256}{10} = -\frac{48}{5}$ and

$$g(t) + \frac{48}{5} = \frac{-t^4 + 10t^2 + 96}{10} = \frac{(16 - t^2)(t^2 + 6)}{10}.$$

For every $t \in [-4, 4]$ we have $t^2 \leq 16$, so $g(t) \geq -\frac{48}{5}$, with equality only for $t = -4$ and $t = 4$. Therefore, for every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D$,

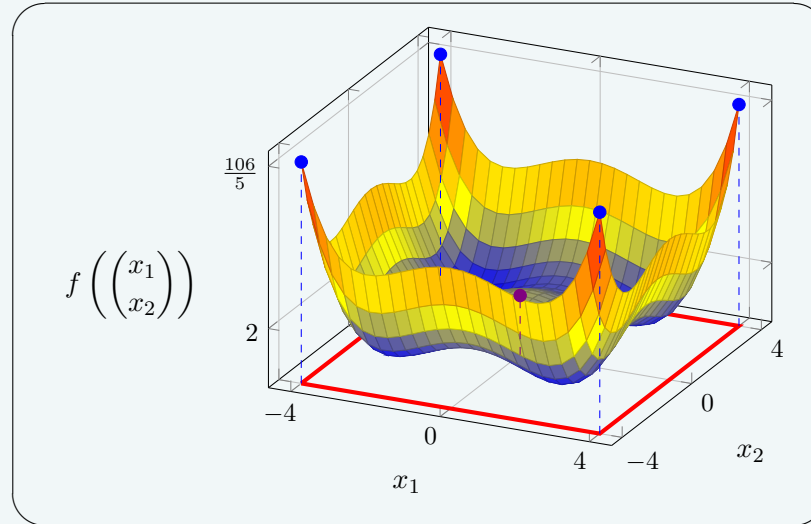
$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \leq 2 + \frac{48}{5} + \frac{48}{5} = \frac{106}{5},$$

with equality only when both x_1 and x_2 are equal to -4 or 4 . The function has **four** points of maximum, the corners of D , and the maximum is $\frac{106}{5}$:

$$\arg \max \{f(\mathbf{x}) \text{ subject to } \mathbf{x} \in D\} = \left\{ \begin{pmatrix} -4 \\ -4 \end{pmatrix}, \begin{pmatrix} -4 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ -4 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix} \right\}.$$

The point of local maximum $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not a point of (global) maximum, since $2 < \frac{106}{5}$.

3. The graph of the function is the surface over the domain D , which is the red square drawn on the bottom plane; the points of the graph corresponding to the four points of maximum are blue and the point corresponding to the point of local maximum that is not global is violet, with a dashed segment that joins them to the bottom plane.



Exercise 5.24 — Optimization problem (E24)

Consider the following function:

$$f : D \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto -x_1^2 - x_2^2 + \frac{1}{2}x_1^2x_2^2, \quad \text{with} \quad D = [-4, 4] \times [-4, 4].$$

Questions

1. Prove, using the definition, that the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a point of local maximum of f in D .
2. Prove that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not a point of (global) maximum of f in D , and find all the points of maximum of f in D , that is, the optimal solutions of the optimization problem $\max\{f(\mathbf{x}) \text{ subject to } \mathbf{x} \in D\}$.
3. Draw the graph of the function in the domain D .

Solution

1. We have $f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = 0$. We use the spherical δ -neighborhood $N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 1\right)$: for every

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in it we have $x_2^2 \leq x_1^2 + x_2^2 < 1$, hence $1 - \frac{x_2^2}{2} > \frac{1}{2}$, and then

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) - f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = -x_2^2 - x_1^2 \left(1 - \frac{x_2^2}{2}\right) \leq -x_2^2 - \frac{1}{2}x_1^2 \leq 0.$$

Therefore,

$$f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \geq f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right), \quad \forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 1\right) \cap D,$$

and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a point of local maximum of f in D , with local maximum 0.

2. We have $f\left(\begin{pmatrix} 4 \\ 4 \end{pmatrix}\right) = -16 - 16 + 128 = 96 > 0$, so $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not a point of (global) maximum. To find the points of maximum, we write

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \frac{1}{2}uv - 2, \quad \text{with} \quad u = x_1^2 - 2 \quad \text{and} \quad v = x_2^2 - 2.$$

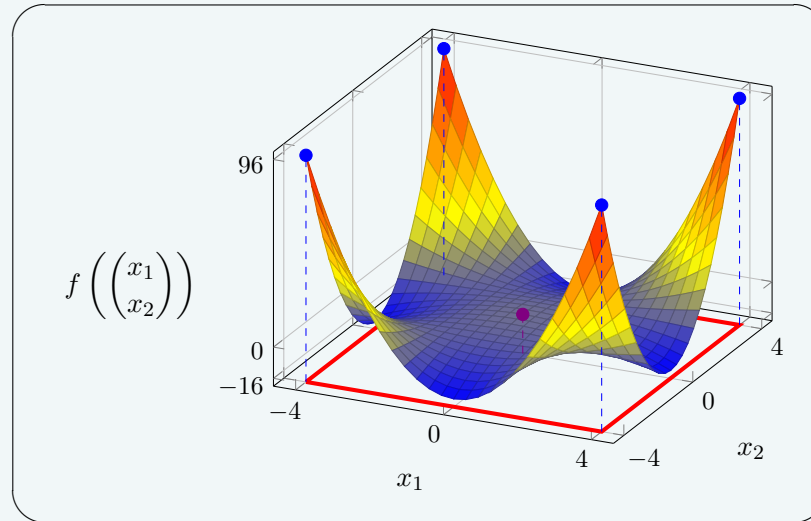
For every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D$ we have $0 \leq x_1^2 \leq 16$ and $0 \leq x_2^2 \leq 16$, so $-2 \leq u \leq 14$ and $-2 \leq v \leq 14$. Then:

- if $u \geq 0$ and $v \geq 0$, we have $uv \leq 14 \cdot 14 = 196$, with equality only for $u = v = 14$;
- if u and v have opposite signs, we have $uv \leq 0$;
- if $u < 0$ and $v < 0$, we have $uv = |u||v| \leq 2 \cdot 2 = 4$.

In all cases $uv \leq 196$, so $f(\mathbf{x}) \leq 98 - 2 = 96$, with equality only for $x_1^2 = x_2^2 = 16$. The function has **four** points of maximum, the corners of D , and the maximum is 96:

$$\arg \max \{ f(\mathbf{x}) \text{ subject to } \mathbf{x} \in D \} = \left\{ \begin{pmatrix} -4 \\ -4 \end{pmatrix}, \begin{pmatrix} -4 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ -4 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix} \right\}.$$

3. The graph of the function is the surface over the domain D , which is the red square drawn on the bottom plane; the points of the graph corresponding to the four points of maximum are blue and the point corresponding to the point of local maximum that is not global is violet, with a dashed segment that joins them to the bottom plane.



Exercise 5.25 — Optimization problem (E25)

Consider the following function:

$$f : D \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto -x_1^2 - x_2^2 + x_1^3, \quad \text{with} \quad D = [-3, 3] \times [-3, 3].$$

Questions

1. Prove, using the definition, that the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a point of local maximum of f in D .
2. Prove that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not a point of (global) maximum of f in D , and find all the points of maximum of f in D , that is, the optimal solutions of the optimization problem $\max\{f(\mathbf{x}) \text{ subject to } \mathbf{x} \in D\}$.
3. Draw the graph of the function in the domain D .

Solution

1. We have $f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = 0$. We use the spherical δ -neighborhood $N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \frac{1}{2}\right)$: for every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in it we have $|x_1| < \frac{1}{2}$, hence

$$x_1^3 = x_1 \cdot x_1^2 \leq |x_1| x_1^2 \leq \frac{1}{2} x_1^2,$$

and then

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) - f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = -x_1^2 - x_2^2 + x_1^3 \leq -x_1^2 - x_2^2 + \frac{1}{2}x_1^2 = -\frac{1}{2}x_1^2 - x_2^2 \leq 0.$$

Therefore,

$$f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \geq f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right), \quad \forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \frac{1}{2}\right) \cap D,$$

and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a point of local maximum of f in D , with local maximum 0.

2. We have $f\left(\begin{pmatrix} 3 \\ 0 \end{pmatrix}\right) = -9 + 27 = 18 > 0$, so $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not a point of (global) maximum.

Moreover, for every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D$ we have $-x_2^2 \leq 0$ and

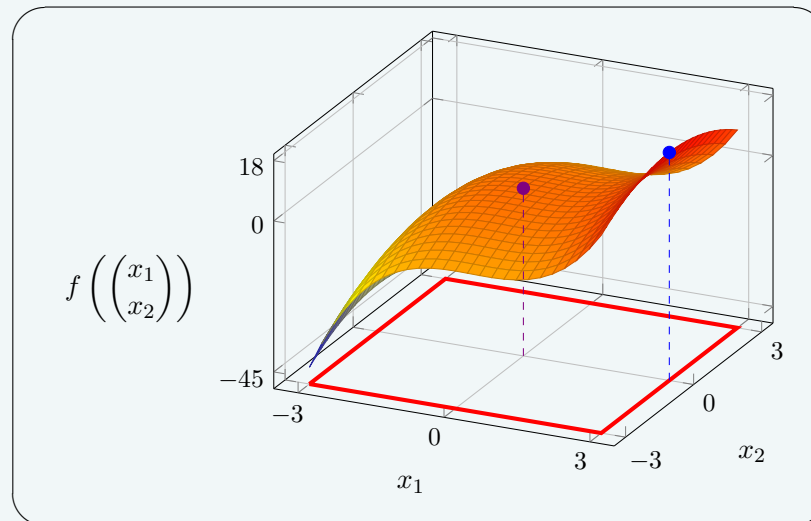
$$18 - (x_1^3 - x_1^2) = (3 - x_1)(x_1^2 + 2x_1 + 6) = (3 - x_1)((x_1 + 1)^2 + 5) \geq 0,$$

since $x_1 \leq 3$. Therefore,

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = (x_1^3 - x_1^2) - x_2^2 \leq 18,$$

with equality only for $x_1 = 3$ and $x_2 = 0$. The point $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ is the unique point of maximum of f in D and the maximum is 18.

3. The graph of the function is the surface over the domain D , which is the red square drawn on the bottom plane; the point of the graph corresponding to the point of maximum is blue and the point corresponding to the point of local maximum that is not global is violet, with a dashed segment that joins them to the bottom plane.



Exercise 5.26 — Optimization problem (E26)

Consider the following function:

$$f : D \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto -x_1^2 - x_2^2 - 1 + x_1 x_2^2, \quad \text{with} \quad D = [-3, 3] \times [-3, 3].$$

Questions

1. Prove, using the definition, that the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a point of local maximum of f in D .
2. Prove that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not a point of (global) maximum of f in D , and find all the points of maximum of f in D , that is, the optimal solutions of the optimization problem $\max\{f(\mathbf{x}) \text{ subject to } \mathbf{x} \in D\}$.
3. Draw the graph of the function in the domain D .

Solution

1. We have $f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = -1$. We use the spherical δ -neighborhood $N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \frac{1}{2}\right)$: for every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in it we have $|x_1| < \frac{1}{2}$, hence

$$x_1 x_2^2 \leq |x_1| x_2^2 \leq \frac{1}{2} x_2^2,$$

and then

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) - f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = -x_1^2 - x_2^2 + x_1 x_2^2 \leq -x_1^2 - x_2^2 + \frac{1}{2} x_2^2 = -x_1^2 - \frac{1}{2} x_2^2 \leq 0.$$

Therefore,

$$f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \geq f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right), \quad \forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \frac{1}{2}\right) \cap D,$$

and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a point of local maximum of f in D , with local maximum -1 .

2. We have $f\left(\begin{pmatrix} 3 \\ 3 \end{pmatrix}\right) = -9 - 9 - 1 + 27 = 8 > -1$, so $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not a point of (global) maximum. To find the points of maximum, we write

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = -x_1^2 - 1 + (x_1 - 1) x_2^2,$$

and we observe that $0 \leq x_2^2 \leq 9$ for every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D$:

- if $x_1 \leq 1$, we have $(x_1 - 1)x_2^2 \leq 0$, so $f(\mathbf{x}) \leq -x_1^2 - 1 \leq -1 < 8$;
- if $x_1 > 1$, we have $(x_1 - 1)x_2^2 \leq 9(x_1 - 1)$, with equality only for $x_2^2 = 9$, so

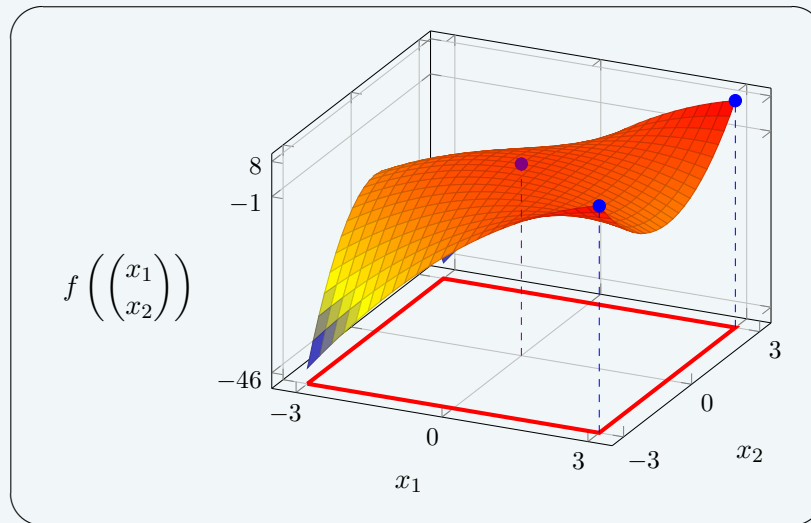
$$f(\mathbf{x}) \leq -x_1^2 + 9x_1 - 10 = 8 - (x_1 - 3)(x_1 - 6) \leq 8,$$

since $x_1 \leq 3$ gives $(x_1 - 3)(x_1 - 6) \geq 0$, with equality only for $x_1 = 3$.

The function has **two** points of maximum and the maximum is 8:

$$\arg \max \{ f(\mathbf{x}) \text{ subject to } \mathbf{x} \in D \} = \left\{ \begin{pmatrix} 3 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix} \right\}.$$

3. The graph of the function is the surface over the domain D , which is the red square drawn on the bottom plane; the points of the graph corresponding to the two points of maximum are blue and the point corresponding to the point of local maximum that is not global is violet, with a dashed segment that joins them to the bottom plane.



Convex and concave optimization problems

6.1 Problems in one variable

In the exercises of this chapter, convexity and concavity of the objective functions are proved only with the definitions of the chapter «Convex and concave functions», using the inequality

$$(\lambda a + (1 - \lambda)b)^2 \leq \lambda a^2 + (1 - \lambda)b^2, \quad \forall a, b \in \mathbb{R}, \forall \lambda \in [0, 1]$$

of the same chapter. The feasible regions are described by constraints $g_i(\mathbf{x}) \leq 0$: when every function g_i is convex, the feasible region is a convex set, by the property of the sets defined by convex functions of the chapter «Convex and concave functions».

Exercise 6.1 — Convex and concave optimization problems (E1)

Consider the following optimization problem:

$$\min \left\{ (x - 3)^2 + 1 \text{ subject to } x^2 - 4 \leq 0 \right\}$$

with objective function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f : x \mapsto (x - 3)^2 + 1$ and feasible region $F = \{x \in \mathbb{R} : x^2 - 4 \leq 0\}$.

Questions

1. Prove that the function $g : \mathbb{R} \rightarrow \mathbb{R}$, $g : x \mapsto x^2 - 4$ is convex in its domain, deduce that the feasible region is a convex set and describe it as an interval.
2. Prove that the objective function is convex in its domain, and deduce that the problem is a convex minimization problem.
3. Draw the graph of the objective function over the feasible region.
4. Prove that the point $x = 2$ is a point of local minimum, and deduce the optimal solution and the optimal value of the problem.

Solution

1. To prove that the function g is convex in its domain, we observe that for every pair of points $p, w \in \mathbb{R}$ and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned}
 g(\lambda p + (1 - \lambda)w) &= (\lambda p + (1 - \lambda)w)^2 - \lambda 4 - (1 - \lambda) 4 \\
 &\quad (\text{since } \lambda + (1 - \lambda) = 1) \\
 &\leq \lambda p^2 + (1 - \lambda)w^2 - \lambda 4 - (1 - \lambda) 4 \\
 &\quad (\text{since } (\lambda a + (1 - \lambda)b)^2 \leq \lambda a^2 + (1 - \lambda)b^2) \\
 &= \lambda(p^2 - 4) + (1 - \lambda)(w^2 - 4) \\
 &= \lambda g(p) + (1 - \lambda) g(w)
 \end{aligned}$$

Therefore,

$$g(\lambda p + (1 - \lambda)w) \leq \lambda g(p) + (1 - \lambda) g(w)$$

By nature, the domain $\mathbb{R}$ is a convex set. Accordingly, the function g is convex in its domain.

The feasible region $F = \{x \in \mathbb{R} : g(x) \leq 0\}$ is defined by a constraint with a convex function, so it is a convex set. Since $x^2 - 4 = (x - 2)(x + 2) \leq 0$ if and only if $-2 \leq x \leq 2$, the feasible region is the interval $F = [-2, 2]$.

2. To prove that the objective function is convex in its domain, we observe that for every pair of points $p, w \in \mathbb{R}$ and every $\lambda \in [0, 1]$, we have:

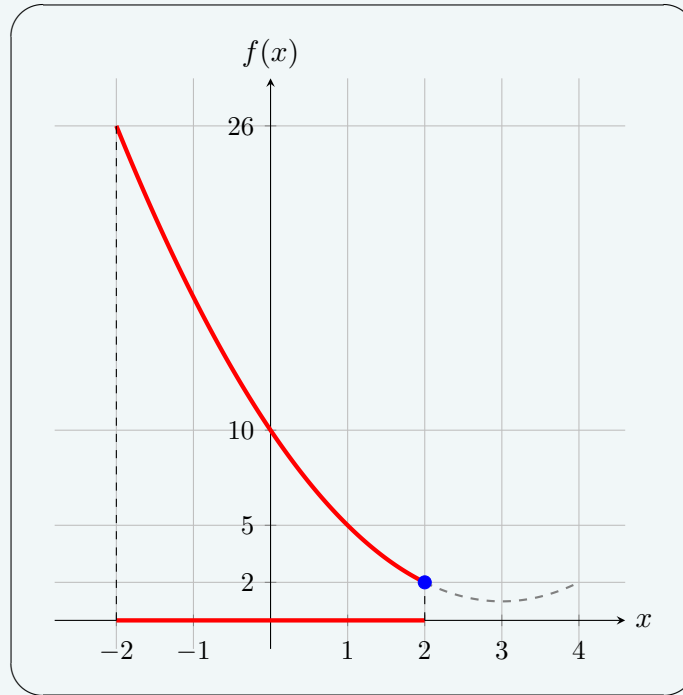
$$\begin{aligned}
 f(\lambda p + (1 - \lambda)w) &= (\lambda(p - 3) + (1 - \lambda)(w - 3))^2 + \lambda + (1 - \lambda) \\
 &\quad (\text{since } \lambda + (1 - \lambda) = 1) \\
 &\leq \lambda(p - 3)^2 + (1 - \lambda)(w - 3)^2 + \lambda + (1 - \lambda) \\
 &\quad (\text{since } (\lambda a + (1 - \lambda)b)^2 \leq \lambda a^2 + (1 - \lambda)b^2) \\
 &= \lambda((p - 3)^2 + 1) + (1 - \lambda)((w - 3)^2 + 1) \\
 &= \lambda f(p) + (1 - \lambda) f(w)
 \end{aligned}$$

Therefore,

$$f(\lambda p + (1 - \lambda)w) \leq \lambda f(p) + (1 - \lambda) f(w)$$

By nature, the domain $\mathbb{R}$ is a convex set. Accordingly, the function is convex in its domain, and in particular over the convex feasible region F : the problem is a convex minimization problem.

3. We have $f(-2) = 26$, $f(0) = 10$, $f(1) = 5$ and $f(2) = 2$. The graph of the objective function over the feasible region is red, the feasible region is the red part of the x -axis and the optimal solution of the last question is blue; the dashed part of the graph lies outside the feasible region, and its lowest point $x = 3$ is not feasible.



4. In the neighborhood $N((2), 1) = (1, 3)$, the feasible points are those of $N((2), 1) \cap F = (1, 2]$. For every such point we have $-2 < x - 3 \leq -1$, then $(x - 3)^2 \geq 1$ and

$$f(x) = (x - 3)^2 + 1 \geq 2 = f(2), \quad \forall x \in N((2), 1) \cap F = (1, 2].$$

The point $x = 2$ is a point of local minimum. Since the problem is a convex minimization problem, by the proposition «local minima of convex minimization problems» of the chapter «Convex and concave optimization problems», the point $x = 2$ is also a point of minimum. The optimal solution and the optimal value are:

$$\tilde{x}(P) = (2) \quad \text{and} \quad z(P) = f(2) = 2.$$

The optimal solution lies on the boundary of the feasible region: the point $x = 3$, where the objective function attains its smallest value 1 over $\mathbb{R}$, is not feasible.

Exercise 6.2 — Convex and concave optimization problems (E3)

Consider the following objective function and feasible region:

$$f: \mathbb{R} \rightarrow \mathbb{R}, \quad f: x \mapsto 3 - (x - 1)^2 \quad \text{and} \quad F = \{x \in \mathbb{R} : -1 \leq x \leq 4\} = [-1, 4],$$

and the associated optimization problems in minimization and maximization forms:

$$\min \left\{ 3 - (x - 1)^2 \text{ subject to } -1 \leq x \leq 4 \right\}$$

and

$$\max \left\{ 3 - (x - 1)^2 \text{ subject to } -1 \leq x \leq 4 \right\}$$

Questions

1. Prove that the objective function is concave in its domain and that it is not convex over F , using the points $p = -1$, $w = 3$ and the scalar $\lambda = \frac{1}{2}$. Is the minimization problem a convex minimization problem?
2. Draw the graph of the objective function over the feasible region.
3. Prove that $x = -1$ is a point of local minimum, but not a point of minimum. Find the optimal solution and the optimal value of the minimization problem.
4. Explain why the previous question does not contradict the proposition «local minima of convex minimization problems».
5. Prove that $x = 1$ is a point of local maximum, and deduce the optimal solution and the optimal value of the maximization problem.

Solution

1. To prove that the function is concave in its domain, we observe that for every pair of points $p, w \in \mathbb{R}$ and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned} f(\lambda p + (1 - \lambda)w) &= \lambda 3 + (1 - \lambda) 3 - (\lambda(p - 1) + (1 - \lambda)(w - 1))^2 \\ &\quad (\text{since } \lambda + (1 - \lambda) = 1) \\ &\geq \lambda 3 + (1 - \lambda) 3 - \lambda(p - 1)^2 - (1 - \lambda)(w - 1)^2 \\ &\quad (\text{since } (\lambda a + (1 - \lambda)b)^2 \leq \lambda a^2 + (1 - \lambda)b^2, \\ &\quad \text{and } -1 \text{ reverses the inequality}) \\ &= \lambda(3 - (p - 1)^2) + (1 - \lambda)(3 - (w - 1)^2) \\ &= \lambda f(p) + (1 - \lambda)f(w) \end{aligned}$$

Therefore,

$$f(\lambda p + (1 - \lambda)w) \geq \lambda f(p) + (1 - \lambda)f(w)$$

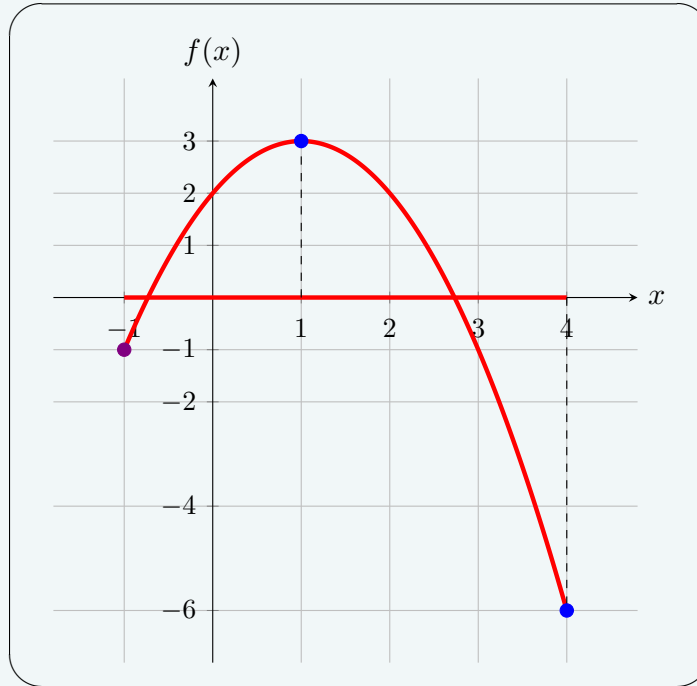
By nature, the domain $\mathbb{R}$ is a convex set. Accordingly, the function is concave in its domain.

With $p = -1$, $w = 3$ and $\lambda = \frac{1}{2}$, both points belong to F , $\lambda p + (1 - \lambda)w = 1$ and

$$f(1) = 3 > -1 = \frac{1}{2} f(-1) + \frac{1}{2} f(3) = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot (-1) :$$

the inequality of the definition of convex function does not hold, and the objective function is not convex over F . The minimization problem is **not** a convex minimization problem, although its feasible region is convex.

2. We have $f(-1) = -1$, $f(1) = 3$, $f(3) = -1$ and $f(4) = -6$. The graph of the objective function over the feasible region is red and the feasible region is the red part of the x -axis; the optimal solutions of the two problems are blue and the point of local minimum that is not a point of minimum is violet.



3. In the neighborhood $N((-1), 1) = (-2, 0)$, the feasible points are those of $N((-1), 1) \cap F = [-1, 0)$. For every such point we have $-2 \leq x - 1 < -1$, then $(x - 1)^2 \leq 4$ and

$$f(x) = 3 - (x - 1)^2 \geq -1 = f(-1), \quad \forall x \in N((-1), 1) \cap F = [-1, 0).$$

The point $x = -1$ is a point of local minimum. It is not a point of minimum, since $4 \in F$ and $f(4) = -6 < -1 = f(-1)$.

For every $x \in F$ we have $-2 \leq x - 1 \leq 3$, then $(x - 1)^2 \leq 9$ and $f(x) = 3 - (x - 1)^2 \geq -6 = f(4)$, with equality only if $x - 1 = 3$, that is, $x = 4$ (the value $x - 1 = -3$ is not feasible). The unique optimal solution and the optimal value of the minimization problem are:

$$\tilde{x}(P) = (4) \quad \text{and} \quad z(P) = -6.$$

4. The proposition «local minima of convex minimization problems» of the chapter «Convex and concave optimization problems» requires an objective function convex over the feasible region. By the first question, the objective function is not convex over F : the minimization problem is not a convex minimization problem, and one of its points of local minimum is not a point of minimum. A method that only explores the points close to $x = -1$ stops at the point of local minimum $x = -1$, with local minimum $f(-1) = -1$, while the optimal value is -6 .

5. In the neighborhood $N((1), 1) = (0, 2)$, all points are feasible and

$$f(x) = 3 - (x - 1)^2 \leq 3 = f(1), \quad \forall x \in N((1), 1) \cap F = (0, 2).$$

The point $x = 1$ is a point of local maximum. By the first question, the objective function is concave over the convex feasible region F , and the maximization problem is a concave maximization problem. By the corollary «local maxima of concave maximization problems» of the chapter «Convex and concave optimization problems», the point $x = 1$ is also a point of maximum, and the optimal solution and the optimal value of the maximization problem are:

$$\tilde{x}(P) = (1) \quad \text{and} \quad z(P) = 3.$$

6.2 Problems in two variables

Exercise 6.3 — Convex and concave optimization problems (E2)

Consider the following optimization problem:

$$\min \left\{ (x_1 - 2)^2 + (x_2 - 2)^2 \text{ subject to } x_1 + x_2 \leq 2, x_1 \geq 0, x_2 \geq 0 \right\}$$

with objective function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $f : \mathbf{x} \mapsto (x_1 - 2)^2 + (x_2 - 2)^2$ and feasible region F .

Questions

1. Prove that the feasible region is a convex set.
2. Prove that the objective function is convex in its domain, and deduce that the problem is a convex minimization problem.
3. Describe the level sets of the objective function for $k = 2$ and $k = 8$, and draw them together with the feasible region.
4. Prove that $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the unique optimal solution and compute the optimal value.
5. Does the problem have a point of local minimum different from $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$?

Solution

1. The feasible region is the set of the points that satisfy the three constraints

$$g_1(\mathbf{x}) = x_1 + x_2 - 2 \leq 0, \quad g_2(\mathbf{x}) = -x_1 \leq 0, \quad g_3(\mathbf{x}) = -x_2 \leq 0.$$

Each function g_i is linear, and it is convex, since the definition holds with equality: for every pair of points $\mathbf{p}, \mathbf{w} \in \mathbb{R}^2$ and every $\lambda \in [0, 1]$, we have

$$g_1(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) = \lambda (p_1 + p_2 - 2) + (1 - \lambda) (w_1 + w_2 - 2) = \lambda g_1(\mathbf{p}) + (1 - \lambda) g_1(\mathbf{w}),$$

and in the same way for g_2 and g_3 . The feasible region is defined by constraints with convex functions, so it is a convex set: it is the triangle with vertices $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, intersection of three half-planes.

2. To prove that the objective function is convex in its domain, we observe that for every pair of points $\mathbf{p}, \mathbf{w} \in \mathbb{R}^2$ and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned} f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) &= (\lambda(p_1 - 2) + (1 - \lambda)(w_1 - 2))^2 \\ &\quad + (\lambda(p_2 - 2) + (1 - \lambda)(w_2 - 2))^2 \\ &\quad \text{(since } \lambda + (1 - \lambda) = 1 \text{)} \end{aligned}$$

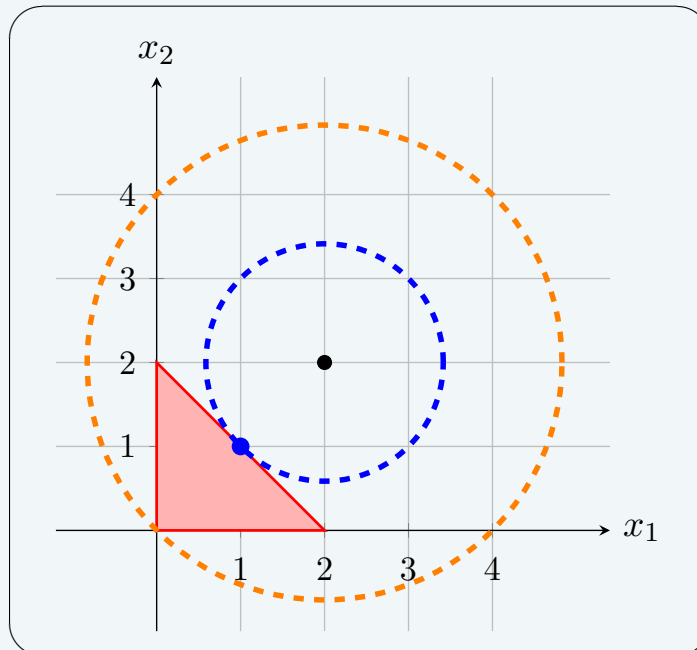
$$\begin{aligned}
 &\leq \lambda(p_1 - 2)^2 + (1 - \lambda)(w_1 - 2)^2 + \lambda(p_2 - 2)^2 + (1 - \lambda)(w_2 - 2)^2 \\
 &\quad (\text{since } (\lambda a + (1 - \lambda)b)^2 \leq \lambda a^2 + (1 - \lambda)b^2, \\
 &\quad \text{twice}) \\
 &= \lambda((p_1 - 2)^2 + (p_2 - 2)^2) + (1 - \lambda)((w_1 - 2)^2 + (w_2 - 2)^2) \\
 &= \lambda f(\mathbf{p}) + (1 - \lambda)f(\mathbf{w})
 \end{aligned}$$

Therefore,

$$f(\lambda \mathbf{p} + (1 - \lambda)\mathbf{w}) \leq \lambda f(\mathbf{p}) + (1 - \lambda)f(\mathbf{w})$$

By nature, the domain $\mathbb{R}^2$ is a convex set. Accordingly, the function is convex in its domain, and in particular over the convex feasible region F : the problem is a convex minimization problem.

3. The level set for k is the set of the points with $(x_1 - 2)^2 + (x_2 - 2)^2 = k$, that is, the circle centered at $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ with radius $\sqrt{k}$. For $k = 2$, the radius is $\sqrt{2}$ and the circle passes through $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, since $1 + 1 = 2$; for $k = 8$, the radius is $2\sqrt{2}$ and the circle passes through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, since $4 + 4 = 8$. In the figure, the feasible region is red, the level set $k = 2$ is blue, the level set $k = 8$ is orange and the optimal solution of the next question is blue: the circle $k = 2$ touches the feasible region only at this point.



4. For every feasible point $\mathbf{x}$, let $a = x_1 - 2$ and $b = x_2 - 2$. Since $x_1 + x_2 \leq 2$, we have $a + b = x_1 + x_2 - 4 \leq -2$, and then $(a + b)^2 \geq 4$. Moreover, $2(a^2 + b^2) - (a + b)^2 = (a - b)^2 \geq 0$. Therefore:

$$f(\mathbf{x}) = a^2 + b^2 \geq \frac{(a + b)^2}{2} \geq \frac{4}{2} = 2 = f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \quad \forall \mathbf{x} \in F,$$

and the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is feasible, since $1 + 1 = 2$. Equality holds only if $a = b$ and $a + b = -2$, that is, $a = b = -1$ and $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The unique optimal solution and the optimal value are:

$$\tilde{\mathbf{x}}(P) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad z(P) = 2.$$

5. No. The problem is a convex minimization problem: by the proposition «local minima of convex minimization problems» of the chapter «Convex and concave optimization problems», every point of local minimum is a point of minimum, that is, an optimal solution, and the optimal solution is unique.

Exercise 6.4 — Convex and concave optimization problems (E4)

Consider the following optimization problem:

$$\max \left\{ 4x_1 + 2x_2 - x_1^2 - x_2^2 \text{ subject to } x_1 + 2x_2 \leq 2, x_1 \geq 0, x_2 \geq 0 \right\}$$

with objective function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $f : \mathbf{x} \mapsto 4x_1 + 2x_2 - x_1^2 - x_2^2$ and feasible region F .

Questions

1. Complete the squares to write the objective function as $f(\mathbf{x}) = c - (x_1 - \alpha_1)^2 - (x_2 - \alpha_2)^2$, with $c, \alpha_1, \alpha_2 \in \mathbb{R}$.
2. Prove that the feasible region is a convex set and that the objective function is concave in its domain. What kind of problem is it?
3. Describe the level sets of the objective function for $k = 2$ and $k = \frac{21}{5}$, and draw them together with the feasible region.
4. Prove that $\mathbf{x} = \begin{pmatrix} 8/5 \\ 1/5 \end{pmatrix}$ is the unique optimal solution and compute the optimal value.
5. Does the problem have a point of local maximum different from $\begin{pmatrix} 8/5 \\ 1/5 \end{pmatrix}$?

Solution

1. Since $x_1^2 - 4x_1 = (x_1 - 2)^2 - 4$ and $x_2^2 - 2x_2 = (x_2 - 1)^2 - 1$, we have:

$$f(\mathbf{x}) = -(x_1^2 - 4x_1) - (x_2^2 - 2x_2) = 5 - (x_1 - 2)^2 - (x_2 - 1)^2,$$

that is, $c = 5$, $\alpha_1 = 2$ and $\alpha_2 = 1$. The objective function is at most 5, and equals 5 only at $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, which is not feasible, since $2 + 2 \cdot 1 = 4 > 2$.

2. The feasible region is the set of the points that satisfy the three constraints

$$g_1(\mathbf{x}) = x_1 + 2x_2 - 2 \leq 0, \quad g_2(\mathbf{x}) = -x_1 \leq 0, \quad g_3(\mathbf{x}) = -x_2 \leq 0.$$

Each function g_i is linear, and it is convex, since the definition holds with equality: for every pair of points $\mathbf{p}, \mathbf{w} \in \mathbb{R}^2$ and every $\lambda \in [0, 1]$, we have

$$g_1(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) = \lambda(p_1 + 2p_2 - 2) + (1 - \lambda)(w_1 + 2w_2 - 2) = \lambda g_1(\mathbf{p}) + (1 - \lambda) g_1(\mathbf{w}),$$

and in the same way for g_2 and g_3 . The feasible region is defined by constraints with convex functions, so it is a convex set: it is the triangle with vertices $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

To prove that the objective function is concave in its domain, we observe that for every pair of points $\mathbf{p}, \mathbf{w} \in \mathbb{R}^2$ and every $\lambda \in [0, 1]$, we have:

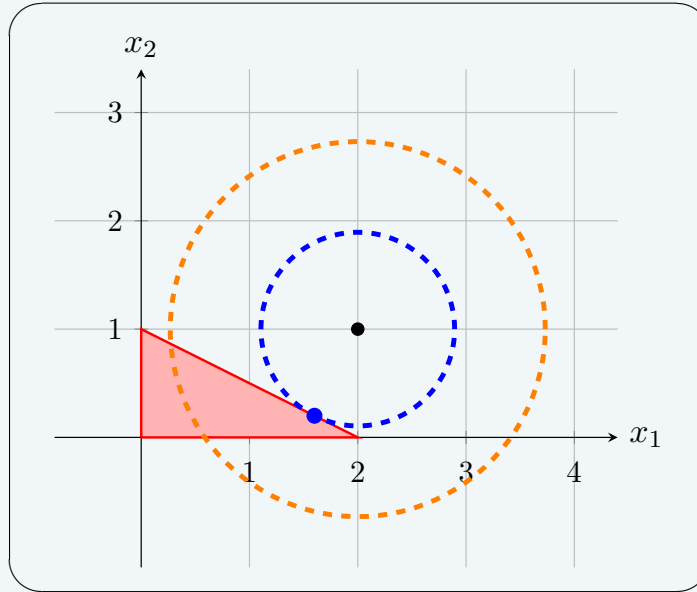
$$\begin{aligned} f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) &= \lambda 5 + (1 - \lambda) 5 - (\lambda(p_1 - 2) + (1 - \lambda)(w_1 - 2))^2 \\ &\quad - (\lambda(p_2 - 1) + (1 - \lambda)(w_2 - 1))^2 \\ &\quad (\text{since } \lambda + (1 - \lambda) = 1) \\ &\geq \lambda 5 + (1 - \lambda) 5 - \lambda(p_1 - 2)^2 - (1 - \lambda)(w_1 - 2)^2 \\ &\quad - \lambda(p_2 - 1)^2 - (1 - \lambda)(w_2 - 1)^2 \\ &\quad (\text{since } (\lambda a + (1 - \lambda)b)^2 \leq \lambda a^2 + (1 - \lambda)b^2, \text{ twice,} \\ &\quad \text{and } -1 \text{ reverses the inequality}) \\ &= \lambda(5 - (p_1 - 2)^2 - (p_2 - 1)^2) \\ &\quad + (1 - \lambda)(5 - (w_1 - 2)^2 - (w_2 - 1)^2) \\ &= \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w}) \end{aligned}$$

Therefore,

$$f(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) \geq \lambda f(\mathbf{p}) + (1 - \lambda) f(\mathbf{w})$$

By nature, the domain $\mathbb{R}^2$ is a convex set. Accordingly, the function is concave in its domain, and in particular over the convex feasible region F : the problem is a **concave maximization problem**.

3. The level set for $k < 5$ is the set of the points with $(x_1 - 2)^2 + (x_2 - 1)^2 = 5 - k$, that is, the circle centered at $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ with radius $\sqrt{5 - k}$. For $k = 2$, the radius is $\sqrt{3}$; for $k = \frac{21}{5}$, the radius is $\sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$ and the circle passes through $\begin{pmatrix} 8/5 \\ 1/5 \end{pmatrix}$, since $\left(\frac{2}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = \frac{4}{5}$. In the figure, the feasible region is red, the level set $k = 2$ is orange, the level set $k = \frac{21}{5}$ is blue and the optimal solution of the next question is blue: the circle $k = \frac{21}{5}$ touches the feasible region only at this point, on the side $x_1 + 2x_2 = 2$.



4. For every feasible point $\mathbf{x}$, let $a = x_1 - 2$ and $b = x_2 - 1$, so that $f(\mathbf{x}) = 5 - (a^2 + b^2)$. Since $x_1 + 2x_2 \leq 2$, we have $a + 2b = x_1 + 2x_2 - 4 \leq -2$, and then $(a + 2b)^2 \geq 4$. Moreover, $5(a^2 + b^2) - (a + 2b)^2 = 4a^2 - 4ab + b^2 = (2a - b)^2 \geq 0$. Therefore:

$$a^2 + b^2 \geq \frac{(a + 2b)^2}{5} \geq \frac{4}{5} \quad \text{and} \quad f(\mathbf{x}) = 5 - (a^2 + b^2) \leq 5 - \frac{4}{5} = \frac{21}{5}, \quad \forall \mathbf{x} \in F.$$

Equality holds only if $b = 2a$ and $a + 2b = -2$, that is, $5a = -2$, $a = -\frac{2}{5}$, $b = -\frac{4}{5}$ and $\mathbf{x} = \begin{pmatrix} 8/5 \\ 1/5 \end{pmatrix}$. This point is feasible, since $\frac{8}{5} + \frac{2}{5} = 2$, $\frac{8}{5} \geq 0$ and $\frac{1}{5} \geq 0$, and

$$f\left(\begin{pmatrix} 8/5 \\ 1/5 \end{pmatrix}\right) = \frac{32}{5} + \frac{2}{5} - \frac{64}{25} - \frac{1}{25} = \frac{34}{5} - \frac{13}{5} = \frac{21}{5}.$$

The unique optimal solution and the optimal value are:

$$\tilde{\mathbf{x}}(P) = \begin{pmatrix} 8/5 \\ 1/5 \end{pmatrix} \quad \text{and} \quad z(P) = \frac{21}{5}.$$

5. No. The problem is a concave maximization problem: by the corollary «local maxima of concave maximization problems» of the chapter «Convex and concave optimization problems», every point of local maximum is a point of maximum, that is, an optimal solution, and the optimal solution is unique.

Exercise 6.5 — Convex and concave optimization problems (E5)

Consider the objective function

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, f : \mathbf{x} \mapsto (x_1 - 3)^2 + (x_2 - 3)^2$$

and the two optimization problems:

$$(P_1) \quad \min \left\{ f(\mathbf{x}) \text{ subject to } x_1^2 + x_2^2 - 2 \leq 0 \right\}$$

and

$$(P_2) \quad \min \left\{ f(\mathbf{x}) \text{ subject to } x_1^2 + x_2^2 - 2 \geq 0 \right\}$$

with feasible regions F_1 and F_2 , respectively.

Questions

1. Prove that the function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$, $g : \mathbf{x} \mapsto x_1^2 + x_2^2 - 2$ is convex in its domain, and deduce that F_1 is a convex set.
2. Prove that F_2 is not a convex set, using the points $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ and the scalar $\lambda = \frac{1}{2}$.
3. Prove that the objective function is convex in its domain. Which of the two problems is a convex minimization problem?
4. Describe the level sets of the objective function for $k = 8$ and $k = 18$, and draw them together with the feasible region F_1 .
5. Prove that $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the unique optimal solution of (P_1) and compute the optimal value.

Solution

1. To prove that the function g is convex in its domain, we observe that for every pair of points $\mathbf{p}, \mathbf{w} \in \mathbb{R}^2$ and every $\lambda \in [0, 1]$, we have:

$$\begin{aligned} g(\lambda \mathbf{p} + (1 - \lambda) \mathbf{w}) &= (\lambda p_1 + (1 - \lambda) w_1)^2 + (\lambda p_2 + (1 - \lambda) w_2)^2 - \lambda 2 - (1 - \lambda) 2 \\ &\quad (\text{since } \lambda + (1 - \lambda) = 1) \\ &\leq \lambda p_1^2 + (1 - \lambda) w_1^2 + \lambda p_2^2 + (1 - \lambda) w_2^2 - \lambda 2 - (1 - \lambda) 2 \\ &\quad (\text{since } (\lambda a + (1 - \lambda) b)^2 \leq \lambda a^2 + (1 - \lambda) b^2, \\ &\quad \text{twice}) \end{aligned}$$

$$\begin{aligned}
&= \lambda(p_1^2 + p_2^2 - 2) + (1 - \lambda)(w_1^2 + w_2^2 - 2) \\
&= \lambda g(\mathbf{p}) + (1 - \lambda)g(\mathbf{w})
\end{aligned}$$

Therefore,

$$g(\lambda \mathbf{p} + (1 - \lambda)\mathbf{w}) \leq \lambda g(\mathbf{p}) + (1 - \lambda)g(\mathbf{w})$$

By nature, the domain $\mathbb{R}^2$ is a convex set. Accordingly, the function g is convex in its domain. The feasible region $F_1 = \{\mathbf{x} \in \mathbb{R}^2 : g(\mathbf{x}) \leq 0\}$ is defined by a constraint with a convex function, so it is a convex set: it is the disk centered at the origin with radius $\sqrt{2}$.

2. The two points belong to F_2 , since $2^2 + 0^2 - 2 = 2 \geq 0$ and $(-2)^2 + 0^2 - 2 = 2 \geq 0$. With $\lambda = \frac{1}{2}$, their convex combination is

$$\frac{1}{2} \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{with} \quad 0^2 + 0^2 - 2 = -2 < 0,$$

and it does not belong to F_2 . The set F_2 (the points outside the open disk of radius $\sqrt{2}$) is not convex.

3. To prove that the objective function is convex in its domain, we observe that for every pair of points $\mathbf{p}, \mathbf{w} \in \mathbb{R}^2$ and every $\lambda \in [0, 1]$, we have:

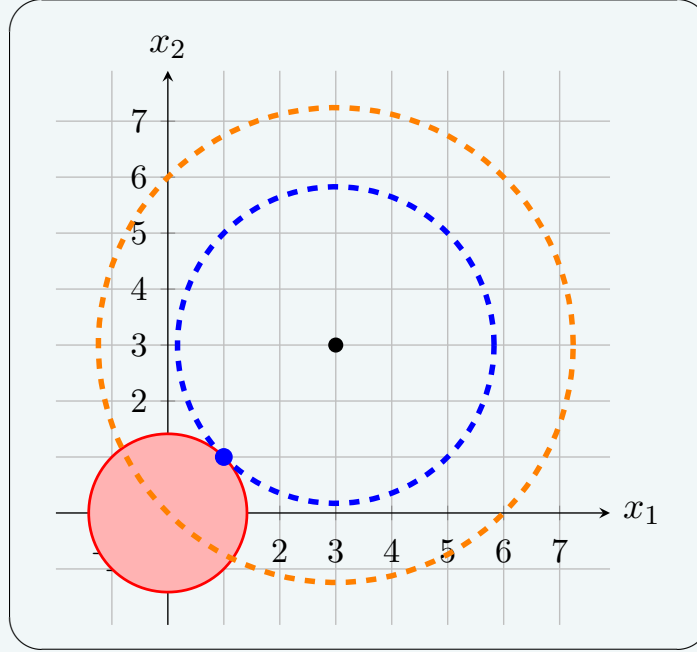
$$\begin{aligned}
f(\lambda \mathbf{p} + (1 - \lambda)\mathbf{w}) &= (\lambda(p_1 - 3) + (1 - \lambda)(w_1 - 3))^2 \\
&\quad + (\lambda(p_2 - 3) + (1 - \lambda)(w_2 - 3))^2 \\
&\quad \text{(since } \lambda + (1 - \lambda) = 1\text{)} \\
&\leq \lambda(p_1 - 3)^2 + (1 - \lambda)(w_1 - 3)^2 + \lambda(p_2 - 3)^2 + (1 - \lambda)(w_2 - 3)^2 \\
&\quad \text{(since } (\lambda a + (1 - \lambda)b)^2 \leq \lambda a^2 + (1 - \lambda)b^2, \\
&\quad \text{twice)} \\
&= \lambda((p_1 - 3)^2 + (p_2 - 3)^2) + (1 - \lambda)((w_1 - 3)^2 + (w_2 - 3)^2) \\
&= \lambda f(\mathbf{p}) + (1 - \lambda)f(\mathbf{w})
\end{aligned}$$

Therefore,

$$f(\lambda \mathbf{p} + (1 - \lambda)\mathbf{w}) \leq \lambda f(\mathbf{p}) + (1 - \lambda)f(\mathbf{w})$$

By nature, the domain $\mathbb{R}^2$ is a convex set. Accordingly, the function is convex in its domain. The problem (P_1) is a convex minimization problem, since its feasible region is convex and the objective function is convex over it. The problem (P_2) is not a convex optimization problem, since its feasible region is not convex: the definition requires a convex feasible region, whatever the objective function.

4. The level set for k is the set of the points with $(x_1 - 3)^2 + (x_2 - 3)^2 = k$, that is, the circle centered at $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ with radius $\sqrt{k}$. For $k = 8$, the radius is $2\sqrt{2}$ and the circle passes through $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, since $4 + 4 = 8$; for $k = 18$, the radius is $3\sqrt{2}$ and the circle passes through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, since $9 + 9 = 18$. In the figure, the feasible region F_1 is red, the level set $k = 8$ is blue, the level set $k = 18$ is orange and the optimal solution of the next question is blue: the circle $k = 8$ touches the disk F_1 only at this point.



5. We write the objective function as

$$f(\mathbf{x}) = x_1^2 - 6x_1 + 9 + x_2^2 - 6x_2 + 9 = q - 6s + 18, \quad \text{with} \quad q = x_1^2 + x_2^2, \quad s = x_1 + x_2.$$

Since $2q - s^2 = (x_1 - x_2)^2 \geq 0$, we have $q \geq \frac{s^2}{2}$. For every $\mathbf{x} \in F_1$ we have $q \leq 2$, then $s^2 \leq 2q \leq 4$ and $s \leq 2$. Therefore:

$$\begin{aligned} f(\mathbf{x}) &= q - 6s + 18 \geq \frac{s^2}{2} - 6s + 18 = \frac{(s-6)^2}{2} \\ &\geq \frac{(2-6)^2}{2} = 8 = f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \quad \forall \mathbf{x} \in F_1, \end{aligned}$$

where the last inequality holds since $6 - s \geq 4 > 0$. The point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is feasible, since $1 + 1 - 2 = 0$. Equality holds only if $x_1 = x_2$ and $s = 2$, that is, $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The unique optimal solution and the optimal value of (P_1) are:

$$\tilde{\mathbf{x}}(P_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad z(P_1) = 8.$$

By the proposition «local minima of convex minimization problems» of the chapter «Convex and concave optimization problems», (P_1) has no other point of local minimum.