

Formulations of routing problems

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Learning objectives

At the end of this chapter you will be able to:

- ▶ write the ILP model of the Travelling Salesman Problem with the position variables of Miller-Tucker-Zemlin (MTZ) and explain how they eliminate subtours;
- ▶ write the ILP model with the subtour elimination constraints of Dantzig-Fulkerson-Johnson (DFJ), and prove that it is correct;
- ▶ count the variables and the constraints of the two models;
- ▶ compare the LP relaxations of the two models on the same instance;
- ▶ write the ILP model of the asymmetric TSP, and explain why the metric TSP has the same models as the TSP.

Prerequisites

- ▶ From the module *Integer linear programming*, the chapters [«Integer linear programming»](#) and [«Primal and dual bounds»](#): ILP problems, LP relaxations and bounds.
- ▶ From this module, the chapter [«Routing problems»](#): the problem, its assumptions, tours, the distance matrix and the reference instance with 5 cities.

Overview of the family

Introduction: Overview of the family

- ▶ The ILP models of the routing problems share the same variables and the same basic constraints
- ▶ We introduce the directed arcs, the arc variables and the degree constraints, common to all the models of the chapter
- ▶ The models differ only in the constraints that eliminate the subtours: this is the key point of the chapter
- ▶ We then list the members of the family, with the names of their ILP models

Cities and arcs

- ▶ Given the set $\mathcal{V} = \{1, 2, \dots, n\}$ of cities, the models of this chapter use the set of directed arcs

$$A = \{(i, j) : i, j \in \mathcal{V}, i \neq j\}.$$

- ▶ Each edge $\{i, j\}$ of the symmetric problem is represented by the two opposite arcs (i, j) and (j, i) , which have the same cost $d_{ij} = d_{ji}$.

Arc variables and degree constraints

- ▶ The ILP models of the routing problems use one binary **arc variable** x_{ij} for each arc $(i, j) \in A$, equal to 1 if the tour goes from city i to city j .
- ▶ The **degree constraints** impose that exactly one arc leaves and exactly one arc enters each city.
- ▶ The models differ in the constraints that eliminate the subtours.

Routing models in one sentence

- ▶ The routing models are assignment models on the arcs, with the degree constraints, plus constraints that forbid the subtours.
- ▶ The objective function minimizes the total distance

$$\sum_{(i,j) \in A} d_{ij} x_{ij}.$$

The members of the family

The table lists the members of the family, with the names of their ILP models and the sections that write them.

Problem	Model	Subtour elimination	Section
TSP	ILP_{TSP}^{MTZ}	position variables u_i	MTZ model
TSP	ILP_{TSP}^{DFJ}	one constraint per subset of cities	DFJ model
metric TSP	$ILP_{TSP}^{MTZ}, ILP_{TSP}^{DFJ}$	the models of the TSP	metric and asymmetric TSP
ATSP	ILP_{ATSP}	the constraints of the TSP models	metric and asymmetric TSP

Summary

What the models share:

- ▶ one binary arc variable x_{ij} for each directed arc $(i, j) \in A$
- ▶ the degree constraints: exactly one arc leaves and exactly one arc enters each city
- ▶ the objective: minimize the total distance $\sum_{(i,j) \in A} d_{ij} x_{ij}$

What makes them different:

- ▶ the constraints that forbid the subtours: position variables (MTZ) or one constraint per subset of cities (DFJ)

ILP models for the Travelling Salesman Problem

Introduction: ILP models for the TSP

- ▶ The Travelling Salesman Problem and its reference instance are defined in the chapter [«Routing problems»](#)
- ▶ Here we write the problem as an ILP model, on the arc variables x_{ij}
- ▶ There are two classical models: they differ in how they forbid the subtours, and their LP relaxations give different lower bounds

Two classical ILP models

- ▶ The TSP has two classical ILP models on the arc variables x_{ij} :
 - ▶ the model of Miller, Tucker and Zemlin (**MTZ**), with position variables;
 - ▶ the model of Dantzig, Fulkerson and Johnson (**DFJ**), with subtour elimination constraints.
- ▶ A later section compares the Linear Programming relaxations of the two models.

Summary

Two ILP models of the TSP:

- ▶ both are written on the arc variables x_{ij}
- ▶ MTZ: position variables
- ▶ DFJ: subtour elimination constraints

Next:

- ▶ the two models one by one, then their LP relaxations

A model with position variables (MTZ)

Introduction: A model with position variables (MTZ)

- ▶ We write the first ILP model of the TSP, due to Miller, Tucker and Zemlin (MTZ)
- ▶ Besides the arc variables, it uses one position variable for each city different from city 1
- ▶ We explain how the position variables eliminate the subtours, we prove that the model is correct, and we write it for the reference instance

Variables of the MTZ model

We introduce the following $n(n - 1)$ **binary variables**:

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i, j) \text{ is traversed by the tour} \\ 0 & \text{otherwise} \end{cases} \quad \forall (i, j) \in A,$$

and the following $n - 1$ **continuous variables**:

$$u_i = \text{position of city } i \text{ after leaving city } 1, \quad \forall i \in \mathcal{V} \setminus \{1\}.$$

The city 1 is used only as a reference city: a tour has no preferred starting city.

The MTZ model

Using these variables, an ILP model for the TSP reads as follows:

$$\begin{aligned} \min \quad & \sum_{i \in \mathcal{V}} \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} d_{ij} x_{ij} \\ \text{subject to} \quad & \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} x_{ij} = 1, & \forall i \in \mathcal{V}, \\ & \sum_{\substack{i \in \mathcal{V} \\ i \neq j}} x_{ij} = 1, & \forall j \in \mathcal{V}, \\ & u_i - u_j + (n-1)x_{ij} \leq n-2, & \forall i, j \in \mathcal{V} \setminus \{1\}, i \neq j, \\ & x_{ij} \in \{0, 1\}, & \forall (i, j) \in A, \\ & 1 \leq u_i \leq n-1, & \forall i \in \mathcal{V} \setminus \{1\}. \end{aligned}$$

This ILP model is referred to as the $\text{ILP}_{\text{TSP}}^{\text{MTZ}}$ model, or **MTZ model**.

The MTZ model: objective function and constraints

Description of the objective function and constraints of the ILP_{TSP}^{MTZ} model:

- ▶ the linear objective function minimizes the total distance travelled by the tour;
- ▶ the first constraints ensure that exactly one arc leaves each city (n linear constraints);
- ▶ the second constraints ensure that exactly one arc enters each city (n linear constraints);
- ▶ the constraints $u_i - u_j + (n - 1)x_{ij} \leq n - 2$ link the arc variables with the position variables and eliminate the subtours that do not contain city 1 ($((n - 1)(n - 2))$ linear constraints);
- ▶ finally, the last two families of constraints define the variables of the ILP model.

How do the position variables eliminate the subtours?

We look at the link between the arc variables
and the position variables.

Explanation of the link between the variables (1/2)

- ▶ For each pair of cities i, j different from city 1, if arc (i, j) is traversed, then the position of city j is at least one unit greater than the position of city i . This logical relation is the implication

$$x_{ij} = 1 \implies u_j \geq u_i + 1.$$

- ▶ Its contrapositive is

$$u_j < u_i + 1 \implies x_{ij} = 0.$$

In particular, an arc from a city to a city in the same or in an earlier position cannot be traversed.

Explanation of the link between the variables (2/2)

- ▶ Both implications are imposed by the linear constraints $u_i - u_j + (n - 1)x_{ij} \leq n - 2$. If $x_{ij} = 1$, then the constraint gives $u_i - u_j + (n - 1) \leq n - 2$, that is $u_j \geq u_i + 1$.
- ▶ Conversely, if $u_j < u_i + 1$, then the constraint gives $(n - 1)x_{ij} \leq n - 2 + u_j - u_i < n - 1$. Since x_{ij} is binary, this forces $x_{ij} = 0$.
- ▶ When $x_{ij} = 0$, the constraint becomes $u_i - u_j \leq n - 2$, which is always satisfied because both positions belong to $[1, n - 1]$. Accordingly, $n - 1$ is the smallest possible big-M coefficient for these bounds.

The link between the variables forbids every subtour
that does not contain city 1.

We now prove that the MTZ model is correct.

Proposition 1: MTZ constraints eliminate subtours

The constraints of the MTZ model are satisfied by an integer solution if and only if its selected arcs form one tour through all the cities.

Proof (1/2)

- ▶ The degree constraints make the selected arcs a collection of vertex-disjoint directed cycles covering all cities.
- ▶ Suppose that one such cycle does not contain city 1 and visits, in order, $i_1, i_2, \dots, i_k$. Since every arc (i_r, i_{r+1}) is selected, the implication above gives

$$u_{i_{r+1}} \geq u_{i_r} + 1 \quad r = 1, \dots, k,$$

where $i_{k+1} = i_1$.

- ▶ Summing the inequalities gives $u_{i_1} \geq u_{i_1} + k$, a contradiction.
- ▶ Thus no cycle can avoid city 1, and the collection is one tour.

Proof (2/2)

- ▶ Conversely, consider any tour and start it from city 1. For every city $i \neq 1$, set u_i equal to the position at which city i is visited after leaving city 1.
- ▶ The bounds $1 \leq u_i \leq n - 1$ hold.
- ▶ If a selected arc (i, j) has $i, j \neq 1$, then city j immediately follows city i , so that $u_j = u_i + 1$ and the constraint $u_i - u_j + (n - 1)x_{ij} \leq n - 2$ holds.
- ▶ If the arc is not selected, the constraint $u_i - u_j + (n - 1)x_{ij} \leq n - 2$ holds by the bounds on the positions.
- ▶ Hence the tour defines a feasible integer solution.



We now write the MTZ model
for the reference TSP instance.

Example: TSP instance

We consider the set $\mathcal{V} = \{1, 2, 3, 4, 5\}$ of 5 cities ($n = 5$), with the distance matrix

d_{ij}	1	2	3	4	5
1	—	5	2	2	9
2	5	—	4	3	4
3	2	4	—	4	7
4	2	3	4	—	7
5	9	4	7	7	—

- ▶ This is the reference instance of the chapter [«Routing problems»](#).
- ▶ We use this TSP instance in all the examples of the chapter.

Example: the $\text{ILP}_{\text{TSP}}^{\text{MTZ}}$ model (1/4)

The $\text{ILP}_{\text{TSP}}^{\text{MTZ}}$ model for the TSP instance has 20 arc variables, 4 position variables, 10 degree constraints and 12 MTZ constraints:

$$\begin{aligned} \min \quad & 5x_{12} + 2x_{13} + 2x_{14} + 9x_{15} + 5x_{21} + 4x_{23} + 3x_{24} \\ & + 4x_{25} + 2x_{31} + 4x_{32} + 4x_{34} + 7x_{35} + 2x_{41} + 3x_{42} \\ & + 4x_{43} + 7x_{45} + 9x_{51} + 4x_{52} + 7x_{53} + 7x_{54} \end{aligned}$$

$$\begin{aligned} \text{subject to} \quad & x_{12} + x_{13} + x_{14} + x_{15} = 1 \\ & x_{21} + x_{23} + x_{24} + x_{25} = 1 \\ & x_{31} + x_{32} + x_{34} + x_{35} = 1 \\ & x_{41} + x_{42} + x_{43} + x_{45} = 1 \\ & x_{51} + x_{52} + x_{53} + x_{54} = 1 \end{aligned}$$

Example: the ILP^{MTZ}_{TSP} model (2/4)

$$x_{21} + x_{31} + x_{41} + x_{51} = 1$$

$$x_{12} + x_{32} + x_{42} + x_{52} = 1$$

$$x_{13} + x_{23} + x_{43} + x_{53} = 1$$

$$x_{14} + x_{24} + x_{34} + x_{54} = 1$$

$$x_{15} + x_{25} + x_{35} + x_{45} = 1$$

$$u_2 - u_3 + 4 x_{23} \leq 3$$

$$u_2 - u_4 + 4 x_{24} \leq 3$$

$$u_2 - u_5 + 4 x_{25} \leq 3$$

$$u_3 - u_2 + 4 x_{32} \leq 3$$

$$u_3 - u_4 + 4 x_{34} \leq 3$$

$$u_3 - u_5 + 4 x_{35} \leq 3$$

Example: the ILP^{MTZ}_{TSP} model (3/4)

$$u_4 - u_2 + 4 x_{42} \leq 3$$

$$u_4 - u_3 + 4 x_{43} \leq 3$$

$$u_4 - u_5 + 4 x_{45} \leq 3$$

$$u_5 - u_2 + 4 x_{52} \leq 3$$

$$u_5 - u_3 + 4 x_{53} \leq 3$$

$$u_5 - u_4 + 4 x_{54} \leq 3$$

$$x_{12}, x_{13}, x_{14}, \dots, x_{54} \in \{0, 1\}$$

$$u_2, u_3, u_4, u_5 \geq 1$$

$$u_2, u_3, u_4, u_5 \leq 4$$

Example: the $\text{ILP}_{\text{TSP}}^{\text{MTZ}}$ model (4/4)

An optimal solution $(\tilde{\mathbf{x}}, \tilde{\mathbf{u}})(\text{ILP}_{\text{TSP}}^{\text{MTZ}})$ is:

$$\tilde{x}_{13} = \tilde{x}_{35} = \tilde{x}_{52} = \tilde{x}_{24} = \tilde{x}_{41} = 1, \text{ all the other } \tilde{x}_{ij} = 0,$$

$$\tilde{u}_3 = 1, \quad \tilde{u}_5 = 2, \quad \tilde{u}_2 = 3, \quad \tilde{u}_4 = 4$$

- ▶ The optimal value $z(\text{ILP}_{\text{TSP}}^{\text{MTZ}})$ is 18.
- ▶ The associated optimal solution of the TSP is the tour $\tilde{\sigma} = (1, 3, 5, 2, 4)$, of length

$$L(\tilde{\sigma}) = 2 + 7 + 4 + 3 + 2 = 18;$$

it is the unique optimal tour, up to the direction.

Summary

The MTZ model:

- ▶ $n(n-1)$ binary arc variables and $n-1$ continuous position variables
- ▶ $2n$ degree constraints and $(n-1)(n-2)$ constraints
 $u_i - u_j + (n-1)x_{ij} \leq n-2$

Why it is correct:

- ▶ $x_{ij} = 1$ implies $u_j \geq u_i + 1$, with $n-1$ the smallest big-M coefficient
- ▶ a cycle that avoids city 1 would give $u_{i_1} \geq u_{i_1} + k$

On the reference instance: the optimal tour $(1, 3, 5, 2, 4)$, of length 18.

A model with subtour elimination constraints (DFJ)

Introduction: A model with subtour elimination constraints (DFJ)

- ▶ We write the second ILP model of the TSP, due to Dantzig, Fulkerson and Johnson (DFJ)
- ▶ It uses only the arc variables: the subtours are forbidden directly, with one constraint for every subset of cities
- ▶ We prove that the model is correct, we count its constraints, and we write it for the reference instance

Variables of the DFJ model

- ▶ The second model uses only the $n(n - 1)$ arc variables x_{ij} of the MTZ model:

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i, j) \text{ is traversed by the tour} \\ 0 & \text{otherwise} \end{cases} \quad \forall (i, j) \in A.$$

- ▶ There are no position variables.
- ▶ The subtours are forbidden directly, with one constraint for every subset of cities.

The DFJ model

Using these variables, a second ILP model for the TSP reads as follows:

$$\begin{aligned} \min \quad & \sum_{i \in \mathcal{V}} \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} d_{ij} x_{ij} \\ \text{subject to} \quad & \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} x_{ij} = 1, & \forall i \in \mathcal{V}, \\ & \sum_{\substack{i \in \mathcal{V} \\ i \neq j}} x_{ij} = 1, & \forall j \in \mathcal{V}, \\ & \sum_{i \in S} \sum_{\substack{j \in S \\ j \neq i}} x_{ij} \leq |S| - 1, \quad \forall S \subseteq \mathcal{V} \setminus \{1\}, |S| \geq 2, \\ & x_{ij} \in \{0, 1\}, & \forall (i, j) \in A. \end{aligned}$$

This ILP model is referred to as the $\text{ILP}_{\text{TSP}}^{\text{DFJ}}$ model, or **DFJ model**.

The DFJ model: objective function and constraints

Description of the objective function and constraints of the $\text{ILP}_{\text{TSP}}^{\text{DFJ}}$ model:

- ▶ the objective function and the degree constraints (the first two families) are those of the MTZ model;
- ▶ the third constraints are called **subtour elimination constraints**: a tour cannot contain all the $|S|$ arcs of a cycle on the cities of S , so it uses at most $|S| - 1$ arcs inside S ($2^{n-1} - n$ linear constraints);
- ▶ finally, the last constraints define the variables of the ILP model.

How many subtour elimination constraints?

- ▶ There is one subtour elimination constraint for every subset $S \subseteq \mathcal{V} \setminus \{1\}$ with $|S| \geq 2$.
- ▶ Their number $2^{n-1} - n$ is exponential in n :
 - ▶ for $n = 5$ they are 11;
 - ▶ for $n = 20$ they are 524 268.

Proposition 2: Subtour elimination constraints

The constraints of the DFJ model are satisfied by an integer solution if and only if its selected arcs form one tour through all the cities.

Proof (1/2)

- ▶ Let a tour be given and let $S \subseteq \mathcal{V} \setminus \{1\}$ with $|S| \geq 2$.
- ▶ If the tour used $|S|$ arcs inside S , these arcs would form a cycle on the cities of S , which is not the whole tour since $1 \notin S$: a contradiction.
- ▶ Hence the subtour elimination constraints hold, and the degree constraints hold by definition of a tour.

Proof (2/2)

- ▶ Conversely, the degree constraints make the selected arcs a collection of vertex-disjoint directed cycles.
- ▶ If there were more than one cycle, one of them would not contain city 1. Let S be its set of cities: then $|S| \geq 2$ and the cycle uses $|S|$ arcs inside S , which violates the subtour elimination constraint of S .
- ▶ Hence there is one cycle, a tour.



Example: the ILP^{DFJ}_{TSP} model (1/4)

The ILP^{DFJ}_{TSP} model for the TSP instance has 20 arc variables, 10 degree constraints and 11 subtour elimination constraints, one for every subset $S \subseteq \{2, 3, 4, 5\}$ with at least two cities:

$$\begin{aligned} \min \quad & 5x_{12} + 2x_{13} + 2x_{14} + 9x_{15} + 5x_{21} + 4x_{23} + 3x_{24} \\ & + 4x_{25} + 2x_{31} + 4x_{32} + 4x_{34} + 7x_{35} + 2x_{41} + 3x_{42} \\ & + 4x_{43} + 7x_{45} + 9x_{51} + 4x_{52} + 7x_{53} + 7x_{54} \end{aligned}$$

$$\begin{aligned} \text{subject to} \quad & x_{12} + x_{13} + x_{14} + x_{15} = 1 \\ & x_{21} + x_{23} + x_{24} + x_{25} = 1 \\ & x_{31} + x_{32} + x_{34} + x_{35} = 1 \\ & x_{41} + x_{42} + x_{43} + x_{45} = 1 \\ & x_{51} + x_{52} + x_{53} + x_{54} = 1 \end{aligned}$$

Example: the ILP^{DFJ}_{TSP} model (2/4)

$$x_{21} + x_{31} + x_{41} + x_{51} = 1$$

$$x_{12} + x_{32} + x_{42} + x_{52} = 1$$

$$x_{13} + x_{23} + x_{43} + x_{53} = 1$$

$$x_{14} + x_{24} + x_{34} + x_{54} = 1$$

$$x_{15} + x_{25} + x_{35} + x_{45} = 1$$

$$x_{23} + x_{32} \leq 1$$

$$x_{24} + x_{42} \leq 1$$

$$x_{25} + x_{52} \leq 1$$

$$x_{34} + x_{43} \leq 1$$

$$x_{35} + x_{53} \leq 1$$

$$x_{45} + x_{54} \leq 1$$

Example: the ILP^{DFJ}_{TSP} model (3/4)

$$x_{23} + x_{24} + x_{32} + x_{34} + x_{42} + x_{43} \leq 2$$

$$x_{23} + x_{25} + x_{32} + x_{35} + x_{52} + x_{53} \leq 2$$

$$x_{24} + x_{25} + x_{42} + x_{45} + x_{52} + x_{54} \leq 2$$

$$x_{34} + x_{35} + x_{43} + x_{45} + x_{53} + x_{54} \leq 2$$

$$\begin{aligned} x_{23} + x_{24} + x_{25} + x_{32} + x_{34} + x_{35} \\ + x_{42} + x_{43} + x_{45} + x_{52} + x_{53} + x_{54} \leq 3 \end{aligned}$$

$$x_{12}, x_{13}, x_{14}, \dots, x_{54} \in \{0, 1\}$$

Example: the $\text{ILP}_{\text{TSP}}^{\text{DFJ}}$ model (4/4)

- ▶ The subtour $2 \rightarrow 5 \rightarrow 2$ together with the subtour $1 \rightarrow 3 \rightarrow 4 \rightarrow 1$ satisfies the degree constraints but violates the constraint $x_{25} + x_{52} \leq 1$ of $S = \{2, 5\}$.
- ▶ An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{TSP}}^{\text{DFJ}})$ is the tour $\tilde{\sigma} = (1, 3, 5, 2, 4)$:

$$\tilde{x}_{13} = \tilde{x}_{35} = \tilde{x}_{52} = \tilde{x}_{24} = \tilde{x}_{41} = 1, \quad \text{all the other } \tilde{x}_{ij} = 0$$

- ▶ The optimal value $z(\text{ILP}_{\text{TSP}}^{\text{DFJ}})$ is 18.

Summary

The DFJ model:

- ▶ only the $n(n - 1)$ arc variables, no position variables
- ▶ $2n$ degree constraints and $2^{n-1} - n$ subtour elimination constraints, one for every $S \subseteq \mathcal{V} \setminus \{1\}$ with $|S| \geq 2$

Why it is correct:

- ▶ a tour uses at most $|S| - 1$ arcs inside S
- ▶ a collection of two or more cycles has a cycle without city 1, which violates the constraint of its set of cities

On the reference instance: 11 subtour elimination constraints, optimal value 18.

Linear Programming relaxations

Introduction: Linear Programming relaxations

- ▶ We relax the binary constraints of the models into non-negativity constraints, and obtain LP models
- ▶ The optimal value of each relaxation is a lower bound on the optimal value of the TSP
- ▶ We compare three relaxations on the reference instance: the assignment relaxation, the relaxation of the MTZ model and the relaxation of the DFJ model

The assignment relaxation

Keeping only the degree constraints of the two models, and relaxing the binary constraints into non-negativity constraints, we obtain the following LP model:

$$\begin{aligned} \min \quad & \sum_{i \in \mathcal{V}} \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} d_{ij} x_{ij} \\ \text{subject to} \quad & \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} x_{ij} = 1, \quad \forall i \in \mathcal{V}, \\ & \sum_{\substack{i \in \mathcal{V} \\ i \neq j}} x_{ij} = 1, \quad \forall j \in \mathcal{V}, \\ & x_{ij} \geq 0, \quad \forall (i, j) \in A. \end{aligned}$$

This LP model is referred to as the LP-0_{TSP} model, or **assignment relaxation**.

The upper bounds $x_{ij} \leq 1$ are not needed

- ▶ Adding the upper bounds $x_{ij} \leq 1$ does not change this relaxation, nor the two relaxations below.
- ▶ For every arc (i, j) , non-negativity and the degree equation of the arcs leaving city i give

$$x_{ij} \leq \sum_{\substack{k \in \mathcal{V} \\ k \neq i}} x_{ik} = 1.$$

Example: the LP-0_{TSP} model (1/3)

The LP-0_{TSP} model for the TSP instance is:

$$\begin{aligned} \min \quad & 5x_{12} + 2x_{13} + 2x_{14} + 9x_{15} + 5x_{21} + 4x_{23} + 3x_{24} \\ & + 4x_{25} + 2x_{31} + 4x_{32} + 4x_{34} + 7x_{35} + 2x_{41} + 3x_{42} \\ & + 4x_{43} + 7x_{45} + 9x_{51} + 4x_{52} + 7x_{53} + 7x_{54} \end{aligned}$$

$$\begin{aligned} \text{subject to} \quad & x_{12} + x_{13} + x_{14} + x_{15} = 1 \\ & x_{21} + x_{23} + x_{24} + x_{25} = 1 \\ & x_{31} + x_{32} + x_{34} + x_{35} = 1 \\ & x_{41} + x_{42} + x_{43} + x_{45} = 1 \\ & x_{51} + x_{52} + x_{53} + x_{54} = 1 \end{aligned}$$

Example: the LP-0_{TSP} model (2/3)

$$x_{21} + x_{31} + x_{41} + x_{51} = 1$$

$$x_{12} + x_{32} + x_{42} + x_{52} = 1$$

$$x_{13} + x_{23} + x_{43} + x_{53} = 1$$

$$x_{14} + x_{24} + x_{34} + x_{54} = 1$$

$$x_{15} + x_{25} + x_{35} + x_{45} = 1$$

$$x_{12}, x_{13}, x_{14}, \dots, x_{54} \geq 0$$

Example: the LP-0_{TSP} model (3/3)

- ▶ An optimal solution $\tilde{\mathbf{x}}(\text{LP-0}_{\text{TSP}})$ is:

$$\tilde{x}_{13} = \tilde{x}_{34} = \tilde{x}_{41} = \tilde{x}_{25} = \tilde{x}_{52} = 1, \text{ all the other } \tilde{x}_{ij} = 0$$

- ▶ It is integer, but it is made of the two subtours
 $1 \rightarrow 3 \rightarrow 4 \rightarrow 1$ and $2 \rightarrow 5 \rightarrow 2$.
- ▶ The optimal value $z(\text{LP-0}_{\text{TSP}})$ is 16. Accordingly we have the lower bound:

$$16 \leq z(\text{ILP}_{\text{TSP}}^{\text{MTZ}}) = z(\text{ILP}_{\text{TSP}}^{\text{DFJ}})$$

To obtain an LP relaxation of the $\text{ILP}_{\text{TSP}}^{\text{MTZ}}$ model,
we relax the binary constraints
into non-negativity constraints;
the position variables keep their bounds.

The relaxation of the MTZ model

The obtained LP model reads as follows:

$$\begin{aligned} \min \quad & \sum_{i \in \mathcal{V}} \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} d_{ij} x_{ij} \\ \text{subject to} \quad & \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} x_{ij} = 1, & \forall i \in \mathcal{V}, \\ & \sum_{\substack{i \in \mathcal{V} \\ i \neq j}} x_{ij} = 1, & \forall j \in \mathcal{V}, \\ & u_i - u_j + (n-1)x_{ij} \leq n-2, \quad \forall i, j \in \mathcal{V} \setminus \{1\}, \quad i \neq j, \\ & x_{ij} \geq 0, & \forall (i, j) \in A, \\ & 1 \leq u_i \leq n-1, & \forall i \in \mathcal{V} \setminus \{1\}. \end{aligned}$$

This LP model is referred to as the $\text{LP}_{\text{TSP}}^{\text{MTZ}}$ model.

Proposition 3: A weak region of the MTZ relaxation

Let (x, u) satisfy the degree equations, with $x_{ij} \geq 0$. If

$$x_{ij} \leq \frac{n-2}{n-1} \quad \forall i, j \in \mathcal{V} \setminus \{1\}, i \neq j,$$

then (x, u) is feasible for $\text{LP}_{\text{TSP}}^{\text{MTZ}}$ after setting $u_i = 1$ for every $i \neq 1$.

Proof

- ▶ With $u_i = u_j = 1$, each constraint $u_i - u_j + (n - 1)x_{ij} \leq n - 2$ becomes

$$(n - 1)x_{ij} \leq n - 2,$$

which is exactly the assumed upper bound.

- ▶ The bounds $1 \leq u_i \leq n - 1$ on the position variables also hold.



What the proposition tells us

- ▶ The proposition shows why the compact MTZ formulation can have a weak lower bound.
- ▶ Many fractional solutions of the assignment relaxation already satisfy the ordering constraints.

Example: the LP_{TSP}^{MTZ} model (1/4)

The LP_{TSP}^{MTZ} model for the TSP instance is:

$$\begin{aligned} \min \quad & 5x_{12} + 2x_{13} + 2x_{14} + 9x_{15} + 5x_{21} + 4x_{23} + 3x_{24} \\ & + 4x_{25} + 2x_{31} + 4x_{32} + 4x_{34} + 7x_{35} + 2x_{41} + 3x_{42} \\ & + 4x_{43} + 7x_{45} + 9x_{51} + 4x_{52} + 7x_{53} + 7x_{54} \end{aligned}$$

$$\begin{aligned} \text{subject to} \quad & x_{12} + x_{13} + x_{14} + x_{15} = 1 \\ & x_{21} + x_{23} + x_{24} + x_{25} = 1 \\ & x_{31} + x_{32} + x_{34} + x_{35} = 1 \\ & x_{41} + x_{42} + x_{43} + x_{45} = 1 \\ & x_{51} + x_{52} + x_{53} + x_{54} = 1 \end{aligned}$$

Example: the LP_{TSP}^{MTZ} model (2/4)

$$x_{21} + x_{31} + x_{41} + x_{51} = 1$$

$$x_{12} + x_{32} + x_{42} + x_{52} = 1$$

$$x_{13} + x_{23} + x_{43} + x_{53} = 1$$

$$x_{14} + x_{24} + x_{34} + x_{54} = 1$$

$$x_{15} + x_{25} + x_{35} + x_{45} = 1$$

$$u_2 - u_3 + 4x_{23} \leq 3$$

$$u_2 - u_4 + 4x_{24} \leq 3$$

$$u_2 - u_5 + 4x_{25} \leq 3$$

$$u_3 - u_2 + 4x_{32} \leq 3$$

$$u_3 - u_4 + 4x_{34} \leq 3$$

$$u_3 - u_5 + 4x_{35} \leq 3$$

Example: the $\text{LP}_{\text{TSP}}^{\text{MTZ}}$ model (3/4)

$$u_4 - u_2 + 4x_{42} \leq 3$$

$$u_4 - u_3 + 4x_{43} \leq 3$$

$$u_4 - u_5 + 4x_{45} \leq 3$$

$$u_5 - u_2 + 4x_{52} \leq 3$$

$$u_5 - u_3 + 4x_{53} \leq 3$$

$$u_5 - u_4 + 4x_{54} \leq 3$$

$$x_{12}, x_{13}, x_{14}, \dots, x_{54} \geq 0$$

$$u_2, u_3, u_4, u_5 \geq 1$$

$$u_2, u_3, u_4, u_5 \leq 4$$

Example: the LP_{TSP}^{MTZ} model (4/4)

An optimal solution $(\tilde{\mathbf{x}}, \tilde{\mathbf{u}})(LP_{TSP}^{MTZ})$ is:

$$\tilde{x}_{13} = 1, \quad \tilde{x}_{31} = \tilde{x}_{34} = \tilde{x}_{41} = \frac{1}{2}, \quad \tilde{x}_{25} = \tilde{x}_{52} = \frac{3}{4},$$

$$\tilde{x}_{24} = \tilde{x}_{42} = \tilde{x}_{45} = \tilde{x}_{54} = \frac{1}{4}, \quad \text{all the other } \tilde{x}_{ij} = 0,$$

$$\tilde{u}_2 = \tilde{u}_3 = \tilde{u}_4 = \tilde{u}_5 = 1$$

- ▶ All the arcs between the cities 2, 3, 4, 5 have value at most $\frac{3}{4} = \frac{n-2}{n-1}$, as in the proposition on the weak region of the MTZ relaxation.
- ▶ The optimal value $z(LP_{TSP}^{MTZ})$ is 17. Accordingly we have the lower bound:

$$17 \leq z(ILP_{TSP}^{MTZ})$$

To obtain an LP relaxation of the $\text{ILP}_{\text{TSP}}^{\text{DFJ}}$ model,
we relax the binary constraints
into non-negativity constraints.

The relaxation of the DFJ model

The obtained LP model reads as follows:

$$\begin{aligned} \min \quad & \sum_{i \in \mathcal{V}} \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} d_{ij} x_{ij} \\ \text{subject to} \quad & \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} x_{ij} = 1, & \forall i \in \mathcal{V}, \\ & \sum_{\substack{i \in \mathcal{V} \\ i \neq j}} x_{ij} = 1, & \forall j \in \mathcal{V}, \\ & \sum_{i \in S} \sum_{\substack{j \in S \\ j \neq i}} x_{ij} \leq |S| - 1, \quad \forall S \subseteq \mathcal{V} \setminus \{1\}, |S| \geq 2, \\ & x_{ij} \geq 0, & \forall (i, j) \in A. \end{aligned}$$

This LP model is referred to as the $\text{LP}_{\text{TSP}}^{\text{DFJ}}$ model.

Compact or strong?

- ▶ The MTZ model is valuable because it has only $O(n^2)$ variables and constraints.
- ▶ The DFJ model has a stronger relaxation, but exponentially many constraints.
- ▶ In practice, the subtour elimination constraints are added only when they are violated by the current solution of the relaxation.

Example: the LP_{TSP}^{DFJ} model (1/4)

The LP_{TSP}^{DFJ} model for the TSP instance is:

$$\begin{aligned} \min \quad & 5x_{12} + 2x_{13} + 2x_{14} + 9x_{15} + 5x_{21} + 4x_{23} + 3x_{24} \\ & + 4x_{25} + 2x_{31} + 4x_{32} + 4x_{34} + 7x_{35} + 2x_{41} + 3x_{42} \\ & + 4x_{43} + 7x_{45} + 9x_{51} + 4x_{52} + 7x_{53} + 7x_{54} \end{aligned}$$

$$\begin{aligned} \text{subject to} \quad & x_{12} + x_{13} + x_{14} + x_{15} = 1 \\ & x_{21} + x_{23} + x_{24} + x_{25} = 1 \\ & x_{31} + x_{32} + x_{34} + x_{35} = 1 \\ & x_{41} + x_{42} + x_{43} + x_{45} = 1 \\ & x_{51} + x_{52} + x_{53} + x_{54} = 1 \end{aligned}$$

Example: the LP_{TSP}^{DFJ} model (2/4)

$$x_{21} + x_{31} + x_{41} + x_{51} = 1$$

$$x_{12} + x_{32} + x_{42} + x_{52} = 1$$

$$x_{13} + x_{23} + x_{43} + x_{53} = 1$$

$$x_{14} + x_{24} + x_{34} + x_{54} = 1$$

$$x_{15} + x_{25} + x_{35} + x_{45} = 1$$

$$x_{23} + x_{32} \leq 1$$

$$x_{24} + x_{42} \leq 1$$

$$x_{25} + x_{52} \leq 1$$

$$x_{34} + x_{43} \leq 1$$

$$x_{35} + x_{53} \leq 1$$

$$x_{45} + x_{54} \leq 1$$

Example: the LP_{TSP}^{DFJ} model (3/4)

$$x_{23} + x_{24} + x_{32} + x_{34} + x_{42} + x_{43} \leq 2$$

$$x_{23} + x_{25} + x_{32} + x_{35} + x_{52} + x_{53} \leq 2$$

$$x_{24} + x_{25} + x_{42} + x_{45} + x_{52} + x_{54} \leq 2$$

$$x_{34} + x_{35} + x_{43} + x_{45} + x_{53} + x_{54} \leq 2$$

$$\begin{aligned} x_{23} + x_{24} + x_{25} + x_{32} + x_{34} + x_{35} \\ + x_{42} + x_{43} + x_{45} + x_{52} + x_{53} + x_{54} \leq 3 \end{aligned}$$

$$x_{12}, x_{13}, x_{14}, \dots, x_{54} \geq 0$$

Example: the $\text{LP}_{\text{TSP}}^{\text{DFJ}}$ model (4/4)

- ▶ An optimal solution $\tilde{\mathbf{x}}(\text{LP}_{\text{TSP}}^{\text{DFJ}})$ is the tour $\tilde{\sigma} = (1, 3, 5, 2, 4)$:

$$\tilde{x}_{13} = \tilde{x}_{35} = \tilde{x}_{52} = \tilde{x}_{24} = \tilde{x}_{41} = 1, \text{ all the other } \tilde{x}_{ij} = 0$$

- ▶ The optimal value $z(\text{LP}_{\text{TSP}}^{\text{DFJ}})$ is 18: on this instance the relaxation closes the gap.

Example: comparing the relaxations

For the TSP instance, the three relaxations give the following lower bounds on the optimal value 18:

Relaxation	Optimal value
$LP-0_{TSP}$ (assignment relaxation)	16
LP_{TSP}^{MTZ}	17
LP_{TSP}^{DFJ}	18

- ▶ The ordering constraints raise the bound from 16 to 17.
- ▶ The subtour elimination constraints close the gap on this instance.

Summary

Three LP relaxations:

- ▶ assignment relaxation: only the degree constraints, $x_{ij} \geq 0$; the bounds $x_{ij} \leq 1$ are implied
- ▶ MTZ relaxation: weak, since every solution with $x_{ij} \leq \frac{n-2}{n-1}$ between the cities different from 1 is feasible with $u_i = 1$
- ▶ DFJ relaxation: stronger, with exponentially many constraints, added when violated

On the reference instance:

- ▶ lower bounds 16, 17 and 18 on the optimal value 18

ILP models for the metric and asymmetric TSP

Introduction: ILP models for the metric and asymmetric TSP

- ▶ The problems of this section are the asymmetric travelling salesman problem (ATSP) and the metric TSP, defined in the chapter [«Routing problems»](#)
- ▶ We write the ILP model of the ATSP, where d_{ij} may differ from d_{ji}
- ▶ We see why the models of the TSP are also models of the ATSP and of the metric TSP

Variables of the ATSP model

- ▶ Given the set $\mathcal{V} = \{1, 2, \dots, n\}$ of cities and the set A of directed arcs, each arc (i, j) has its own distance d_{ij} , which may differ from d_{ji} .
- ▶ We introduce the following $n(n - 1)$ **binary variables**:

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i, j) \text{ is traversed by the tour} \\ 0 & \text{otherwise} \end{cases} \quad \forall (i, j) \in A.$$

The ATSP model

Using these variables, an ILP model for the ATSP reads as follows:

$$\begin{aligned} \min \quad & \sum_{i \in \mathcal{V}} \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} d_{ij} x_{ij} \\ \text{subject to} \quad & \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} x_{ij} = 1, & \forall i \in \mathcal{V}, \\ & \sum_{\substack{i \in \mathcal{V} \\ i \neq j}} x_{ij} = 1, & \forall j \in \mathcal{V}, \\ & \sum_{i \in S} \sum_{\substack{j \in S \\ j \neq i}} x_{ij} \leq |S| - 1, \quad \forall S \subseteq \mathcal{V} \setminus \{1\}, |S| \geq 2, \\ & x_{ij} \in \{0, 1\}, & \forall (i, j) \in A. \end{aligned}$$

This ILP model is referred to as the ILP_{ATSP} model.

The ATSP model: objective function and constraints

Description of the objective function and constraints of the ILP_{ATSP} model:

- ▶ the linear objective function minimizes the total distance travelled along the arcs of the tour, each arc with its own distance;
- ▶ the first two families of constraints ensure that exactly one arc leaves and exactly one arc enters each city ($2n$ linear constraints);
- ▶ the third constraints are the subtour elimination constraints ($2^{n-1} - n$ linear constraints);
- ▶ finally, the last constraints define the variables of the ILP model.

The ATSP model and the TSP models

- ▶ The ILP_{ATSP} model is the $\text{ILP}_{\text{TSP}}^{\text{DFJ}}$ model with the distances of the arcs: its proof of correctness, the proposition on the subtour elimination constraints, never uses $d_{ij} = d_{ji}$.
- ▶ For the same reason the $\text{ILP}_{\text{TSP}}^{\text{MTZ}}$ model is also an ILP model of the ATSP.
- ▶ A tour and the same tour travelled backwards use opposite arcs: in the ATSP they are different solutions, with different lengths in general.

The metric TSP

- ▶ The metric TSP, defined in the chapter [«Routing problems»](#), has the same ILP models as the TSP.
- ▶ The triangle inequality is a property of the distances, not a constraint of the model.
- ▶ The reference TSP instance is metric, so the $\text{ILP}_{\text{TSP}}^{\text{MTZ}}$ and $\text{ILP}_{\text{TSP}}^{\text{DFJ}}$ models written for it are already models of a metric TSP instance.

Example: the ILP_{ATSP} model (1/4)

We consider the set $\mathcal{V} = \{1, 2, 3, 4\}$ of 4 cities ($n = 4$), with the asymmetric distance matrix

d_{ij}	1	2	3	4
1	—	2	7	7
2	2	—	4	2
3	9	7	—	1
4	2	4	1	—

For instance, $d_{13} = 7$ and $d_{31} = 9$.

Example: the ILP_{ATSP} model (2/4)

The ILP_{ATSP} model for this ATSP instance has 12 arc variables, 8 degree constraints and 4 subtour elimination constraints, one for every subset $S \subseteq \{2, 3, 4\}$ with at least two cities:

$$\begin{aligned} \min \quad & 2x_{12} + 7x_{13} + 7x_{14} + 2x_{21} + 4x_{23} + 2x_{24} \\ & + 9x_{31} + 7x_{32} + x_{34} + 2x_{41} + 4x_{42} + x_{43} \\ \text{subject to} \quad & x_{12} + x_{13} + x_{14} = 1 \\ & x_{21} + x_{23} + x_{24} = 1 \\ & x_{31} + x_{32} + x_{34} = 1 \\ & x_{41} + x_{42} + x_{43} = 1 \end{aligned}$$

Example: the ILP_{ATSP} model (3/4)

$$x_{21} + x_{31} + x_{41} = 1$$

$$x_{12} + x_{32} + x_{42} = 1$$

$$x_{13} + x_{23} + x_{43} = 1$$

$$x_{14} + x_{24} + x_{34} = 1$$

$$x_{23} + x_{32} \leq 1$$

$$x_{24} + x_{42} \leq 1$$

$$x_{34} + x_{43} \leq 1$$

$$x_{23} + x_{24} + x_{32} + x_{34} + x_{42} + x_{43} \leq 2$$

$$x_{12}, x_{13}, x_{14}, \dots, x_{43} \in \{0, 1\}$$

Example: the ILP_{ATSP} model (4/4)

- ▶ The two subtours $1 \rightarrow 2 \rightarrow 1$ and $3 \rightarrow 4 \rightarrow 3$, of total length $2 + 2 + 1 + 1 = 6$, satisfy the degree constraints but violate the constraint $x_{34} + x_{43} \leq 1$ of $S = \{3, 4\}$.
- ▶ An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{ATSP}})$ is the tour $\tilde{\sigma} = (1, 2, 3, 4)$:

$$\tilde{x}_{12} = \tilde{x}_{23} = \tilde{x}_{34} = \tilde{x}_{41} = 1, \quad \text{all the other } \tilde{x}_{ij} = 0$$

- ▶ The optimal value $z(\text{ILP}_{\text{ATSP}})$ is $2 + 4 + 1 + 2 = 9$, and the optimal tour is unique.
- ▶ The same tour travelled backwards, $(1, 4, 3, 2)$, has length $7 + 1 + 7 + 2 = 17$.

Summary

The ATSP:

- ▶ the ILP_{ATSP} model is the DFJ model with $d_{ij} \neq d_{ji}$ allowed; the MTZ model is also a model of the ATSP
- ▶ a tour and its reverse are different solutions: 9 and 17 in the example

The metric TSP:

- ▶ same models as the TSP: the triangle inequality is a property of the data, not a constraint

Typical mistakes

Introduction: Typical mistakes

- ▶ We collect the mistakes that are made most often when writing and using the routing models
- ▶ Each of them comes from a model or a result of this chapter
- ▶ Check them before solving the exercises

Typical mistakes (1/4)

Observation 1: Writing the MTZ constraints for the city 1

The MTZ constraints are written only for the pairs of cities different from city 1: the city 1 has no position variable.

Observation 2: Using a big-M smaller than $n - 1$

Written as $u_i - u_j + Mx_{ij} \leq M - 1$, the constraint with $M < n - 1$ cuts off feasible tours: when $x_{ij} = 0$ it requires $u_i - u_j \leq M - 1 < n - 2$, while two positions can differ by $n - 2$; $M = n - 1$ is the smallest valid value.

Typical mistakes (2/4)

Observation 3: Taking $S = \mathcal{V}$ in the subtour elimination constraints

A tour uses $|\mathcal{V}|$ arcs inside $\mathcal{V}$: the constraint would cut off every tour. The sets S are proper subsets of the cities, with at least two cities.

Observation 4: Trusting the MTZ bound

By the proposition on the weak region of the MTZ relaxation, many fractional solutions of the assignment relaxation are feasible for the MTZ relaxation: the compact formulation is not a strong one.

Typical mistakes (3/4)

Observation 5: Listing all the subtour elimination constraints

Their number is $2^{n-1} - n$: they are not written all at once, but added when violated.

Observation 6: Identifying a tour with its reverse in the ATSP

With asymmetric distances the two directions of a tour are different solutions: in the ATSP example of this chapter they have lengths 9 and 17.

Typical mistakes (4/4)

Observation 7: Adding the triangle inequality as a constraint

In the metric TSP the triangle inequality holds for the data d_{ij} ; the variables of the model are the same as in the TSP.

Summary

Writing the models:

- ▶ MTZ constraints only for the cities different from 1, with coefficient $n - 1$
- ▶ subtour elimination constraints only for proper subsets S with at least two cities, added when violated

Using the models:

- ▶ the MTZ bound is weak
- ▶ in the ATSP a tour and its reverse are different solutions
- ▶ the triangle inequality is a property of the data, not a constraint

In summary

Introduction: In summary

- ▶ We collect the key ideas of the chapter on the formulations of routing problems
- ▶ The dashboard gathers the lower bounds computed on the reference instance

Summary (1/3)

- ▶ The two models have the same arc variables x_{ij} and the same degree constraints; they differ in how they forbid the subtours.
- ▶ The MTZ model adds $n - 1$ position variables and $(n - 1)(n - 2)$ constraints: it is compact, but its relaxation is weak.
- ▶ The DFJ model adds $2^{n-1} - n$ subtour elimination constraints: the relaxation is stronger, and the constraints are generated when needed.

Summary (2/3)

- ▶ Both models are correct: an integer solution is feasible if and only if its arcs form one tour.
- ▶ Both models are written on arcs, so they are also models of the ATSP: the ILP_{ATSP} model is the DFJ model with $d_{ij} \neq d_{ji}$ allowed. The metric TSP has the same models as the TSP.
- ▶ From the module *Integer linear programming*, the chapter [«Branch-and-cut»](#) shows how the subtour elimination constraints are added only when they are violated.

Summary (3/3)

Dashboard of the instance of the chapter. Lower bounds on the optimal value 18 computed in this chapter:

Bound	Value	From
Assignment relaxation	16	$z(\text{LP-0}_{\text{TSP}})$
MTZ relaxation	17	$z(\text{LP}_{\text{TSP}}^{\text{MTZ}})$
DFJ relaxation	18	$z(\text{LP}_{\text{TSP}}^{\text{DFJ}})$
Optimal value	18	tour $\sigma = (1, 3, 5, 2, 4)$