

Formulations of scheduling problems



Outline

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A first Linear Programming relaxation

A second Linear Programming relaxation
Some important theoretical properties

Additional considerations

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Learning objectives

At the end of this chapter you will be able to:

- ▶ write the ILP model of $P||C_{\max}$ and explain how the makespan variable is linked to the machine loads;
- ▶ write the LP relaxations and their duals, and prove that their optimal value is $\sum_j t_j/m$;
- ▶ prove that the LP bound is at least $1/m$ of the optimal value, and that the factor is tight;
- ▶ use the valid inequalities and the symmetry-breaking constraints of the model.

Prerequisites

- ▶ From the module *Linear Programming*, the chapter «[Duality theory](#)»: the dual of an LP problem, weak and strong duality.
- ▶ From the module *Integer linear programming*, the chapters «[Integer linear programming](#)» and «[Primal and dual bounds](#)»: ILP problems, LP relaxations and bounds.
- ▶ From this module, the chapter «[Scheduling problems](#)»: the problem, its assumption, the makespan and its reference instance.

Overview of the family

Introduction: Overview of the family

- ▶ In this section we describe what the ILP models of the parallel machine scheduling problems have in common
- ▶ Two kinds of variables: the assignment of the jobs to the machines, and the makespan
- ▶ This structure is the key to read the model of the chapter, its relaxations and its improvements

Assignment variables and makespan variable

- ▶ The ILP models of the parallel machine scheduling problems use the set $\mathcal{H} = \{1, 2, \dots, m\}$ of the machines and one binary **assignment variable** x_{jh} for each job $j \in \mathcal{J}$ and each machine $h \in \mathcal{H}$.
- ▶ One continuous variable $C_{\max}$ represents the makespan: one constraint for each machine imposes that $C_{\max}$ is at least the load of the machine.
- ▶ The objective function minimizes $C_{\max}$, so at the optimum it is equal to the largest load.

Assignment models

The scheduling models are assignment models
of the jobs to the machines,
as the packing models assign items to bins;

the min-max objective is linearized
with the makespan variable $C_{\max}$.

The members of the family

- ▶ The table lists the members of the family, with the name of their ILP model and the section that writes it.

Problem	Model	Variables	Section
$P \parallel C_{\max}$	$ILP_{P \parallel C_{\max}}$	binary x_{jh} and continuous $C_{\max}$	the next one

Summary

The variables:

- ▶ One binary assignment variable x_{jh} for each job j and each machine h
- ▶ One continuous variable $C_{\max}$, the makespan

The constraints and the objective:

- ▶ One constraint for each machine: $C_{\max}$ is at least its load
- ▶ Minimizing $C_{\max}$ makes it equal to the largest load: the min-max objective is linearized

An ILP model for the Scheduling on Identical Parallel Machines Problem



Introduction: An ILP model for $P \parallel C_{\max}$

- ▶ The problem of this section is the scheduling on identical parallel machines ($P \parallel C_{\max}$), defined in the chapter «[Scheduling problems](#)»
- ▶ We write its ILP model, with the assignment variables and the makespan variable
- ▶ We explain how the makespan variable is linked to the loads of the machines
- ▶ Finally, we write and solve the model on the instance of the chapter

The data and the variables

- ▶ Given m identical machines and a set $\mathcal{J} = \{1, 2, \dots, n\}$ of n jobs, with processing times t_j , we denote by $\mathcal{H} = \{1, 2, \dots, m\}$ the set of machines.

We introduce the following nm **binary variables**:

$$x_{jh} = \begin{cases} 1 & \text{if job } j \text{ is assigned to machine } h \\ 0 & \text{otherwise} \end{cases} \quad \forall j \in \mathcal{J}, \forall h \in \mathcal{H},$$

and one **non-negative continuous variable**:

$$C_{\max} = \text{makespan of the schedule.}$$

The ILP $_{P||C_{\max}}$ model

Using these variables, an Integer Linear Programming (ILP) model for the Scheduling on Identical Parallel Machines Problem ($P||C_{\max}$) reads as follows:

$$\begin{aligned} \min \quad & C_{\max} \\ \text{subject to} \quad & \sum_{h \in \mathcal{H}} x_{jh} = 1, & \forall j \in \mathcal{J}, \\ & - \sum_{j \in \mathcal{J}} t_j x_{jh} + C_{\max} \geq 0, & \forall h \in \mathcal{H}, \\ & x_{jh} \in \{0, 1\}, & \forall j \in \mathcal{J}, \forall h \in \mathcal{H}, \\ & C_{\max} \geq 0. \end{aligned}$$

This ILP model is referred to as the ILP $_{P||C_{\max}}$ model.

Description of the model

- ▶ The linear objective function $\min C_{\max}$ minimizes the makespan.
- ▶ The constraints $\sum_{h \in \mathcal{H}} x_{jh} = 1$ ensure that each job is assigned to exactly one machine (n linear constraints).
- ▶ The constraints $-\sum_{j \in \mathcal{J}} t_j x_{jh} + C_{\max} \geq 0$ link the assignment variables with the makespan variable. In particular, they ensure that the makespan is greater than or equal to the load of every machine (m linear constraints).
- ▶ Finally, the constraints $x_{jh} \in \{0, 1\}$ and $C_{\max} \geq 0$ define the variables of the ILP model.

How does the makespan variable
become the largest load of a machine?

We look at the link between the variables.

The link between the variables (1/2)

- ▶ For every machine h , the makespan and its load must satisfy the direct relation

$$C_{\max} \geq \sum_{j \in \mathcal{J}} t_j x_{jh},$$

which is imposed by the load constraint of machine h .

- ▶ Since the constraint holds for every machine, it imposes

$$C_{\max} \geq \max_{h \in \mathcal{H}} \left\{ \sum_{j \in \mathcal{J}} t_j x_{jh} \right\}.$$

The link between the variables (2/2)

- ▶ In every optimal solution equality holds.
- ▶ Indeed, if $C_{\max}$ were larger than the maximum load, replacing $C_{\max}$ with the maximum load would keep all the load constraints satisfied and would decrease the objective value by a positive amount, which contradicts optimality.
- ▶ When no job is assigned to a machine, its load is zero.

Example: the $ILP_{P||C_{\max}}$ model (1/3)

- ▶ We consider $m = 3$ machines ($\mathcal{H} = \{1, 2, 3\}$) and a set $\mathcal{J} = \{1, 2, \dots, 7\}$ of 7 jobs ($n = 7$), with the following processing times:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$
t_j	5	5	4	4	3	3	3

- ▶ This is the reference instance of the chapter «Scheduling problems»; we refer to it as the instance of the chapter.
- ▶ The next slide writes the $ILP_{P||C_{\max}}$ model for this instance.

Example: the $ILP_{P||C_{\max}}$ model (2/3)

The $ILP_{P||C_{\max}}$ model for this instance is:

$$\begin{array}{ll} \min & C_{\max} \\ \text{subject to} & x_{11} + x_{12} + x_{13} = 1 \\ & x_{21} + x_{22} + x_{23} = 1 \\ & x_{31} + x_{32} + x_{33} = 1 \\ & x_{41} + x_{42} + x_{43} = 1 \\ & x_{51} + x_{52} + x_{53} = 1 \\ & x_{61} + x_{62} + x_{63} = 1 \\ & x_{71} + x_{72} + x_{73} = 1 \\ & -5x_{11} - 5x_{21} - 4x_{31} - 4x_{41} - 3x_{51} - 3x_{61} - 3x_{71} + C_{\max} \geq 0 \\ & -5x_{12} - 5x_{22} - 4x_{32} - 4x_{42} - 3x_{52} - 3x_{62} - 3x_{72} + C_{\max} \geq 0 \\ & -5x_{13} - 5x_{23} - 4x_{33} - 4x_{43} - 3x_{53} - 3x_{63} - 3x_{73} + C_{\max} \geq 0 \\ & x_{11}, x_{12}, x_{13}, \dots, x_{73} \in \{0, 1\} \\ & C_{\max} \geq 0 \end{array}$$

Example: the $ILP_{P||C_{\max}}$ model (3/3)

An optimal solution $(\tilde{x}, \tilde{C}_{\max})(ILP_{P||C_{\max}})$ is:

$$\tilde{x}_{11} = \tilde{x}_{31} = \tilde{x}_{22} = \tilde{x}_{42} = \tilde{x}_{53} = \tilde{x}_{63} = \tilde{x}_{73} = 1,$$

$$\text{all the other } \tilde{x}_{jh} = 0, \quad \tilde{C}_{\max} = 9$$

- ▶ It assigns the jobs $\{1, 3\}$ to machine 1, $\{2, 4\}$ to machine 2 and $\{5, 6, 7\}$ to machine 3, with loads 9, 9 and 9.
- ▶ The optimal value $z(ILP_{P||C_{\max}})$ is 9, since it attains the lower bound

$$\max \left\{ \max_{j \in \mathcal{J}} t_j, \left\lceil \frac{\sum_{j \in \mathcal{J}} t_j}{m} \right\rceil \right\} = \max\{5, \lceil 27/3 \rceil\} = 9.$$

Summary

The $ILP_{P||C_{\max}}$ model:

- ▶ nm binary assignment variables x_{jh} and one continuous makespan variable $C_{\max}$
- ▶ n assignment constraints: each job on exactly one machine
- ▶ m load constraints: $C_{\max}$ is at least the load of every machine

The link between the variables:

- ▶ At the optimum, $C_{\max}$ is equal to the largest load

The instance of the chapter:

- ▶ Jobs $\{1, 3\}$, $\{2, 4\}$, $\{5, 6, 7\}$, loads 9, 9, 9: $z(ILP_{P||C_{\max}}) = 9$

A first Linear Programming relaxation

Introduction: A first Linear Programming relaxation

- ▶ In this section we relax the binary constraints of the ILP $P_{\|C_{\max}}$ model into non-negativity constraints
- ▶ We are minimizing: the optimal value of the relaxation is a lower bound on the optimal makespan
- ▶ We write the relaxation and its dual, and solve both on the instance of the chapter
- ▶ Finally, we prove that the optimal value of the relaxation is $\sum_j t_j / m$ for every instance

The $\text{LP-0}_{P\|C_{\max}}$ model

To obtain a Linear Programming (LP) relaxation of the $\text{ILP}_{P\|C_{\max}}$ model, we relax the constraints $x_{jh} \in \{0, 1\}$ into non-negativity constraints. The obtained LP model reads as follows:

$$\begin{aligned} \min \quad & C_{\max} \\ \text{subject to} \quad & \sum_{h \in \mathcal{H}} x_{jh} = 1, & \forall j \in \mathcal{J}, \\ & - \sum_{j \in \mathcal{J}} t_j x_{jh} + C_{\max} \geq 0, & \forall h \in \mathcal{H}, \\ & x_{jh} \geq 0, & \forall j \in \mathcal{J}, \forall h \in \mathcal{H}, \\ & C_{\max} \geq 0. \end{aligned}$$

This LP model is referred to as the $\text{LP-0}_{P\|C_{\max}}$ model.

Example: the $LP-0_{P\|C_{\max}}$ model (1/2)

The $LP-0_{P\|C_{\max}}$ model for the instance of the chapter is:

$$\begin{array}{ll} \min & C_{\max} \\ \text{subject to} & x_{11} + x_{12} + x_{13} = 1 \\ & x_{21} + x_{22} + x_{23} = 1 \\ & x_{31} + x_{32} + x_{33} = 1 \\ & x_{41} + x_{42} + x_{43} = 1 \\ & x_{51} + x_{52} + x_{53} = 1 \\ & x_{61} + x_{62} + x_{63} = 1 \\ & x_{71} + x_{72} + x_{73} = 1 \\ & -5x_{11} - 5x_{21} - 4x_{31} - 4x_{41} - 3x_{51} - 3x_{61} - 3x_{71} + C_{\max} \geq 0 \\ & -5x_{12} - 5x_{22} - 4x_{32} - 4x_{42} - 3x_{52} - 3x_{62} - 3x_{72} + C_{\max} \geq 0 \\ & -5x_{13} - 5x_{23} - 4x_{33} - 4x_{43} - 3x_{53} - 3x_{63} - 3x_{73} + C_{\max} \geq 0 \\ & x_{11}, x_{12}, x_{13}, \dots, x_{73} \geq 0 \\ & C_{\max} \geq 0 \end{array}$$

Example: the $\text{LP-}0_{P\|C_{\max}}$ model (2/2)

An optimal solution $(\tilde{\mathbf{x}}, \tilde{C}_{\max})(\text{LP-}0_{P\|C_{\max}})$ is:

$$\tilde{x}_{jh} = \frac{1}{3}, \quad \forall j \in \mathcal{J}, \quad \forall h \in \mathcal{H}, \quad \tilde{C}_{\max} = 9$$

The optimal value $z(\text{LP-}0_{P\|C_{\max}})$ is 9. Accordingly we have the lower bound:

$$9 \leq z(\text{ILP}_{P\|C_{\max}})$$

The dual variables

- ▶ We now write the dual of the $LP-0_{P\|C_{\max}}$ model.
- ▶ We use a **free** dual variable μ_j for each constraint $\sum_{h \in \mathcal{H}} x_{jh} = 1$, that is, for each job $j \in \mathcal{J}$.
- ▶ We use a **non-negative** dual variable π_h for each constraint $-\sum_{j \in \mathcal{J}} t_j x_{jh} + C_{\max} \geq 0$, that is, for each machine $h \in \mathcal{H}$.

The $D(\text{LP-}0_{P\parallel C_{\max}})$ model

Using a free dual variable μ_j for each assignment constraint and a non-negative dual variable π_h for each load constraint, the dual model of the $\text{LP-}0_{P\parallel C_{\max}}$ model is:

$$\begin{aligned} \max \quad & \sum_{j \in \mathcal{J}} \mu_j \\ \text{subject to} \quad & \mu_j - t_j \pi_h \leq 0, \quad \forall j \in \mathcal{J}, \forall h \in \mathcal{H}, \\ & \sum_{h \in \mathcal{H}} \pi_h \leq 1, \\ & \mu_j \begin{matrix} \geq \\ \leq \end{matrix} 0, & \forall j \in \mathcal{J}, \\ & \pi_h \geq 0, & \forall h \in \mathcal{H}. \end{aligned}$$

This LP model is referred to as the $D(\text{LP-}0_{P\parallel C_{\max}})$ model.

Example: the $D(\text{LP-}0_{P\|C_{\max}})$ model (1/3)

The dual $D(\text{LP-}0_{P\|C_{\max}})$ model for the instance of the chapter is:

$$\begin{array}{llll} \max & \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 & & \\ \text{subject to} & \mu_1 - 5\pi_1 & \leq & 0 \\ & \mu_1 - 5\pi_2 & \leq & 0 \\ & \mu_1 - 5\pi_3 & \leq & 0 \\ & \mu_2 - 5\pi_1 & \leq & 0 \\ & \mu_2 - 5\pi_2 & \leq & 0 \\ & \mu_2 - 5\pi_3 & \leq & 0 \\ & \mu_3 - 4\pi_1 & \leq & 0 \\ & \mu_3 - 4\pi_2 & \leq & 0 \\ & \mu_3 - 4\pi_3 & \leq & 0 \end{array}$$

Example: the $D(\text{LP-}0_{P\|C_{\max}})$ model (2/3)

$$\mu_4 - 4\pi_1 \leq 0$$

$$\mu_4 - 4\pi_2 \leq 0$$

$$\mu_4 - 4\pi_3 \leq 0$$

$$\mu_5 - 3\pi_1 \leq 0$$

$$\mu_5 - 3\pi_2 \leq 0$$

$$\mu_5 - 3\pi_3 \leq 0$$

$$\mu_6 - 3\pi_1 \leq 0$$

$$\mu_6 - 3\pi_2 \leq 0$$

$$\mu_6 - 3\pi_3 \leq 0$$

$$\mu_7 - 3\pi_1 \leq 0$$

$$\mu_7 - 3\pi_2 \leq 0$$

$$\mu_7 - 3\pi_3 \leq 0$$

$$\pi_1 + \pi_2 + \pi_3 \leq 1$$

Example: the $D(\text{LP-}0_{P\|C_{\max}})$ model (3/3)

The domains of the variables are:

$$\begin{array}{rcl} \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7 & \begin{array}{c} \geq \\ \leq \end{array} & 0 \\ \pi_1, \pi_2, \pi_3 & \geq & 0 \end{array}$$

An optimal solution $(\tilde{\mu}, \tilde{\pi})(D(\text{LP-}0_{P\|C_{\max}}))$ is:

$$\begin{aligned} \tilde{\mu}_1 = \tilde{\mu}_2 = \frac{5}{3}, \quad \tilde{\mu}_3 = \tilde{\mu}_4 = \frac{4}{3}, \quad \tilde{\mu}_5 = \tilde{\mu}_6 = \tilde{\mu}_7 = 1, \\ \tilde{\pi}_1 = \tilde{\pi}_2 = \tilde{\pi}_3 = \frac{1}{3} \end{aligned}$$

The optimal value $z(D(\text{LP-}0_{P\|C_{\max}}))$ is 9.

On the instance of the chapter,
the $\text{LP-}0_{P\|C_{\max}}$ model and its dual
have the same optimal value $9 = 27/3$.

The next proposition shows that the optimal value
of the relaxation is $\sum_j t_j/m$ for every instance.

Proposition 1: Optimal value of $\text{LP-0}_{P\parallel C_{\max}}$

For every $P\parallel C_{\max}$ instance,

$$z(\text{LP-0}_{P\parallel C_{\max}}) = \frac{\sum_{j \in \mathcal{J}} t_j}{m}.$$

- The idea of the proof: we build a feasible primal solution and a feasible dual solution with the same objective value.

Proof (1/2)

- ▶ Set $x_{jh} = 1/m$ for every job and every machine, and set

$$C_{\max} = \frac{\sum_{j \in \mathcal{J}} t_j}{m}.$$

- ▶ Each constraint $\sum_{h \in \mathcal{H}} x_{jh} = 1$ is satisfied, and every machine has the displayed load:

$$\sum_{j \in \mathcal{J}} t_j \frac{1}{m} = \frac{\sum_{j \in \mathcal{J}} t_j}{m} = C_{\max}.$$

- ▶ Hence this is a feasible primal solution.

Proof (2/2)

- In the dual, set

$$\pi_h = \frac{1}{m} \quad \forall h \in \mathcal{H}, \quad \mu_j = \frac{t_j}{m} \quad \forall j \in \mathcal{J}.$$

- The constraints $\mu_j - t_j \pi_h \leq 0$ hold at equality, $\sum_h \pi_h = 1$, and the dual objective value is

$$\sum_{j \in \mathcal{J}} \mu_j = \frac{\sum_{j \in \mathcal{J}} t_j}{m}.$$

- By LP Weak Duality, the two solutions are optimal, and the optimal value is the common value of the objective functions.



Summary

The relaxation $LP-0_{P\|C_{\max}}$:

- ▶ The binary constraints become $x_{jh} \geq 0$
- ▶ On the instance of the chapter: $\tilde{x}_{jh} = 1/3$, $\tilde{C}_{\max} = 9$, so $9 \leq z(ILP_{P\|C_{\max}})$

The dual $D(LP-0_{P\|C_{\max}})$:

- ▶ A free variable μ_j for each job, a non-negative variable π_h for each machine
- ▶ On the instance of the chapter: optimal value 9

The optimal value:

- ▶ $z(LP-0_{P\|C_{\max}}) = \sum_j t_j/m$, with primal $x_{jh} = 1/m$ and dual $\pi_h = 1/m$, $\mu_j = t_j/m$

A second Linear Programming relaxation



Introduction: A second Linear Programming relaxation

- ▶ In this section we relax the binary constraints into the bounds $0 \leq x_{jh} \leq 1$
- ▶ We show that the upper bounds do not change the relaxation, and we write its dual
- ▶ Then we study the quality of the LP bound: how far it can be from the optimal value, and an example of integrality gap

The $LP_{P||C_{\max}}$ model

To obtain an LP relaxation of the $ILP_{P||C_{\max}}$ model, we relax the constraints $x_{jh} \in \{0, 1\}$ into the bounds $0 \leq x_{jh} \leq 1$. The obtained LP model reads as follows:

$$\begin{aligned} \min \quad & C_{\max} \\ \text{subject to} \quad & \sum_{h \in \mathcal{H}} x_{jh} = 1, & \forall j \in \mathcal{J}, \\ & - \sum_{j \in \mathcal{J}} t_j x_{jh} + C_{\max} \geq 0, & \forall h \in \mathcal{H}, \\ & x_{jh} \leq 1, & \forall j \in \mathcal{J}, \forall h \in \mathcal{H}, \\ & x_{jh} \geq 0, & \forall j \in \mathcal{J}, \forall h \in \mathcal{H}, \\ & C_{\max} \geq 0. \end{aligned}$$

This LP model is referred to as the $LP_{P||C_{\max}}$ model.

The upper bounds do not change the relaxation

- ▶ The constraints $x_{jh} \leq 1$ do not change the relaxation: by the constraints $\sum_{h \in \mathcal{H}} x_{jh} = 1$ and $x_{jh} \geq 0$, each x_{jh} is at most one.
- ▶ Thus

$$z(\text{LP}_{P \parallel C_{\max}}) = z(\text{LP-0}_{P \parallel C_{\max}}) = \frac{\sum_{j \in \mathcal{J}} t_j}{m}.$$

Example: the $LP_{P||C_{\max}}$ model (1/2)

The $LP_{P||C_{\max}}$ model for the instance of the chapter is:

$$\begin{array}{ll} \min & C_{\max} \\ \text{subject to} & x_{11} + x_{12} + x_{13} = 1 \\ & x_{21} + x_{22} + x_{23} = 1 \\ & x_{31} + x_{32} + x_{33} = 1 \\ & x_{41} + x_{42} + x_{43} = 1 \\ & x_{51} + x_{52} + x_{53} = 1 \\ & x_{61} + x_{62} + x_{63} = 1 \\ & x_{71} + x_{72} + x_{73} = 1 \\ & -5x_{11} - 5x_{21} - 4x_{31} - 4x_{41} - 3x_{51} - 3x_{61} - 3x_{71} + C_{\max} \geq 0 \\ & -5x_{12} - 5x_{22} - 4x_{32} - 4x_{42} - 3x_{52} - 3x_{62} - 3x_{72} + C_{\max} \geq 0 \\ & -5x_{13} - 5x_{23} - 4x_{33} - 4x_{43} - 3x_{53} - 3x_{63} - 3x_{73} + C_{\max} \geq 0 \\ & x_{11}, x_{12}, x_{13}, \dots, x_{73} \leq 1 \\ & x_{11}, x_{12}, x_{13}, \dots, x_{73} \geq 0 \\ & C_{\max} \geq 0 \end{array}$$

Example: the $LP_{P\|C_{\max}}$ model (2/2)

An optimal solution $(\tilde{\mathbf{x}}, \tilde{C}_{\max})(LP_{P\|C_{\max}})$ is the solution of the $LP-0_{P\|C_{\max}}$ example:

$$\tilde{x}_{jh} = \frac{1}{3}, \quad \forall j \in \mathcal{J}, \quad \forall h \in \mathcal{H}, \quad \tilde{C}_{\max} = 9$$

The optimal value $z(LP_{P\|C_{\max}})$ is 9. Accordingly we have the same lower bound.

The dual variables

- ▶ We now write the dual of the $LP_{P||C_{\max}}$ model.
- ▶ We use a **free** dual variable μ_j for each constraint $\sum_{h \in \mathcal{H}} x_{jh} = 1$.
- ▶ We use a **non-negative** dual variable π_h for each constraint $-\sum_{j \in \mathcal{J}} t_j x_{jh} + C_{\max} \geq 0$.
- ▶ We use a **non-positive** dual variable λ_{jh} for each constraint $x_{jh} \leq 1$.

The $D(\text{LP}_{P\|C_{\max}})$ model

Using these dual variables, the dual model of the $\text{LP}_{P\|C_{\max}}$ model is:

$$\begin{aligned} \max \quad & \sum_{j \in \mathcal{J}} \mu_j + \sum_{j \in \mathcal{J}} \sum_{h \in \mathcal{H}} \lambda_{jh} \\ \text{subject to} \quad & \mu_j - t_j \pi_h + \lambda_{jh} \leq 0, \quad \forall j \in \mathcal{J}, \forall h \in \mathcal{H}, \\ & \sum_{h \in \mathcal{H}} \pi_h \leq 1, \\ & \mu_j \geq 0, \quad \forall j \in \mathcal{J}, \\ & \pi_h \geq 0, \quad \forall h \in \mathcal{H}, \\ & \lambda_{jh} \leq 0, \quad \forall j \in \mathcal{J}, \forall h \in \mathcal{H}. \end{aligned}$$

This LP model is referred to as the $D(\text{LP}_{P\|C_{\max}})$ model.

Example: the $D(\text{LP}_{P\|C_{\max}})$ model (1/3)

The dual $D(\text{LP}_{P\|C_{\max}})$ model for the instance of the chapter is:

$$\begin{aligned} \max \quad & \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 + \lambda_{11} \\ & + \lambda_{12} + \lambda_{13} + \lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{31} + \lambda_{32} + \lambda_{33} \\ & + \lambda_{41} + \lambda_{42} + \lambda_{43} + \lambda_{51} + \lambda_{52} + \lambda_{53} + \lambda_{61} + \lambda_{62} \\ & + \lambda_{63} + \lambda_{71} + \lambda_{72} + \lambda_{73} \\ \text{subject to} \quad & \mu_1 - 5\pi_1 + \lambda_{11} \leq 0 \\ & \mu_1 - 5\pi_2 + \lambda_{12} \leq 0 \\ & \mu_1 - 5\pi_3 + \lambda_{13} \leq 0 \\ & \mu_2 - 5\pi_1 + \lambda_{21} \leq 0 \\ & \mu_2 - 5\pi_2 + \lambda_{22} \leq 0 \\ & \mu_2 - 5\pi_3 + \lambda_{23} \leq 0 \\ & \mu_3 - 4\pi_1 + \lambda_{31} \leq 0 \\ & \mu_3 - 4\pi_2 + \lambda_{32} \leq 0 \\ & \mu_3 - 4\pi_3 + \lambda_{33} \leq 0 \end{aligned}$$

Example: the $D(\text{LP}_{P\|C_{\max}})$ model (2/3)

$$\mu_4 - 4\pi_1 + \lambda_{41} \leq 0$$

$$\mu_4 - 4\pi_2 + \lambda_{42} \leq 0$$

$$\mu_4 - 4\pi_3 + \lambda_{43} \leq 0$$

$$\mu_5 - 3\pi_1 + \lambda_{51} \leq 0$$

$$\mu_5 - 3\pi_2 + \lambda_{52} \leq 0$$

$$\mu_5 - 3\pi_3 + \lambda_{53} \leq 0$$

$$\mu_6 - 3\pi_1 + \lambda_{61} \leq 0$$

$$\mu_6 - 3\pi_2 + \lambda_{62} \leq 0$$

$$\mu_6 - 3\pi_3 + \lambda_{63} \leq 0$$

$$\mu_7 - 3\pi_1 + \lambda_{71} \leq 0$$

$$\mu_7 - 3\pi_2 + \lambda_{72} \leq 0$$

$$\mu_7 - 3\pi_3 + \lambda_{73} \leq 0$$

$$\pi_1 + \pi_2 + \pi_3 \leq 1$$

Example: the $D(\text{LP}_{P\|C_{\max}})$ model (3/3)

The domains of the variables are:

$$\begin{array}{ll} \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7 & \begin{array}{l} \geq \\ \leq \end{array} 0 \\ \pi_1, \pi_2, \pi_3 & \geq 0 \\ \lambda_{11}, \lambda_{12}, \lambda_{13}, \dots, \lambda_{73} & \leq 0 \end{array}$$

An optimal solution $(\tilde{\mu}, \tilde{\pi}, \tilde{\lambda})(D(\text{LP}_{P\|C_{\max}}))$ is the solution of the $D(\text{LP-0}_{P\|C_{\max}})$ example with $\tilde{\lambda}_{jh} = 0$ for every job and every machine:

$$\begin{aligned} \tilde{\mu}_1 = \tilde{\mu}_2 = \frac{5}{3}, \quad \tilde{\mu}_3 = \tilde{\mu}_4 = \frac{4}{3}, \quad \tilde{\mu}_5 = \tilde{\mu}_6 = \tilde{\mu}_7 = 1, \\ \tilde{\pi}_1 = \tilde{\pi}_2 = \tilde{\pi}_3 = \frac{1}{3}, \quad \tilde{\lambda}_{jh} = 0 \end{aligned}$$

The optimal value $z(D(\text{LP}_{P\|C_{\max}}))$ is 9.

Some important theoretical properties

How good is the LP bound $\sum_j t_j/m$?

We compare it with the optimal value $z(\text{ILP}_{P\|C_{\max}})$:
first a worst-case ratio, valid for every instance,
then instances that attain it,
and finally an example of integrality gap.

Proposition 2: Worst-case ratio of the LP bound

For every $P \parallel C_{\max}$ instance,

$$\frac{z(\text{LP}_{P \parallel C_{\max}})}{z(\text{ILP}_{P \parallel C_{\max}})} \geq \frac{1}{m}.$$

Proof

- ▶ Assigning every job to one machine gives a feasible schedule of makespan $\sum_j t_j$.
- ▶ Hence $z(\text{ILP}_{P\|C_{\max}}) \leq \sum_j t_j$.
- ▶ By Proposition 1,

$$\frac{z(\text{LP}_{P\|C_{\max}})}{z(\text{ILP}_{P\|C_{\max}})} \geq \frac{\frac{\sum_j t_j}{m}}{\sum_j t_j} = \frac{1}{m}.$$



Proposition 3: The ratio $1/m$ is asymptotically attained

For a fixed number $m \geq 2$ of machines, there are instances for which

$$\frac{z(\text{LP}_{P \parallel C_{\max}})}{z(\text{ILP}_{P \parallel C_{\max}})} \longrightarrow \frac{1}{m}.$$

- The idea of the proof: one very long job forces a large makespan, while the LP bound spreads it over the m machines.

Proof (1/2)

- ▶ For $M \in \mathbb{Z}_{\geq 2}$, consider one job of processing time M and m jobs of processing time one.
- ▶ The long job alone forces $z(\text{ILP}_{P\parallel C_{\max}}) \geq M$.
- ▶ The schedule that assigns the long job alone to machine 1 and the m unit jobs to the other $m - 1$ machines has loads at most $2 \leq M$, hence the makespan M is feasible.
- ▶ Thus

$$z(\text{ILP}_{P\parallel C_{\max}}) = M.$$

Proof (2/2)

- ▶ Proposition 1 gives

$$z(\text{LP}_{P\|C_{\max}}) = \frac{M + m}{m}.$$

- ▶ Therefore,

$$\lim_{M \rightarrow +\infty} \frac{z(\text{LP}_{P\|C_{\max}})}{z(\text{ILP}_{P\|C_{\max}})} = \lim_{M \rightarrow +\infty} \left(\frac{1}{m} + \frac{1}{M} \right) = \frac{1}{m}.$$



Example: an integrality gap

- ▶ Consider two machines and three jobs with processing times $(3, 3, 2)$.
- ▶ Proposition 1 gives $z(\text{LP}_{P\parallel C_{\max}}) = 8/2 = 4$.
- ▶ In every integer schedule one machine processes at least two of the three jobs, so its load is at least $3 + 2 = 5$; the schedule $\{3, 2\}, \{3\}$ reaches it, so $z(\text{ILP}_{P\parallel C_{\max}}) = 5$.
- ▶ The fractional assignment

$$x_{j1} = x_{j2} = \frac{1}{2} \quad \forall j \in \{1, 2, 3\}$$

has load four on both machines, whereas the integer optimum has loads five and three.

Summary (1/2)

The relaxation $LP_{P\|C_{\max}}$:

- ▶ The binary constraints become $0 \leq x_{jh} \leq 1$
- ▶ The upper bounds are implied:
$$z(LP_{P\|C_{\max}}) = z(LP-0_{P\|C_{\max}}) = \sum_j t_j/m$$

The dual $D(LP_{P\|C_{\max}})$:

- ▶ One more non-positive variable λ_{jh} for each upper bound
- ▶ On the instance of the chapter: the dual solution of $D(LP-0_{P\|C_{\max}})$ with $\tilde{\lambda}_{jh} = 0$, optimal value 9

Summary (2/2)

The quality of the LP bound:

- ▶ For every instance, $z(\text{LP}_{P\|C_{\max}})/z(\text{ILP}_{P\|C_{\max}}) \geq 1/m$
- ▶ The ratio $1/m$ is asymptotically attained: one long job of length M and m unit jobs, with $M \rightarrow +\infty$

An integrality gap:

- ▶ Two machines and jobs $(3, 3, 2)$: LP bound 4, optimal value 5

Additional considerations

Introduction: Additional considerations

- ▶ In this section we add constraints to the $ILP_{P||C_{\max}}$ model
- ▶ Two **valid inequalities**: they strengthen the model, since the LP bound alone ignores the longest job and the integrality of the makespan
- ▶ **Optimality-preserving constraints**: the machines are identical, and these constraints reduce the equivalent relabellings of the machines
- ▶ We also see when these constraints remove nothing

A valid inequality for each job

- ▶ For every job j , the inequality

$$C_{\max} \geq t_j$$

is a **valid inequality** for the ILP $_{P||C_{\max}}$ model.

- ▶ In every integer schedule, job j is assigned to one machine, whose load and therefore makespan are at least t_j (n linear constraints).

A valid inequality from the average load

- ▶ The inequality

$$C_{\max} \geq \left\lceil \frac{\sum_{j \in \mathcal{J}} t_j}{m} \right\rceil$$

is also a **valid inequality** for the $\text{ILP}_{P \parallel C_{\max}}$ model.

- ▶ The m machine loads sum to $\sum_j t_j$, so their maximum is at least their average and is an integer in every integer schedule (one linear constraint).

Optimality-preserving constraints

- ▶ The machines are identical. The constraints

$$\sum_{j \in \mathcal{J}} t_j x_{jh} \geq \sum_{j \in \mathcal{J}} t_j x_{j,h+1} \quad \forall h \in \{1, 2, \dots, m-1\}$$

are **optimality-preserving constraints**.

- ▶ They are not valid for every feasible assignment, but every schedule can be relabelled so that its machine loads are non-increasing, without changing the makespan.

When the symmetry is not broken

- ▶ These constraints remove the equivalent relabellings of the machines only when the loads are distinct.
- ▶ With equal loads they remove none.
- ▶ In the instance of the chapter all the optimal schedules have loads 9, 9, 9, and all the 12 optimal solutions of the model satisfy them.

Summary

Valid inequalities:

- ▶ $C_{\max} \geq t_j$ for every job j (n constraints)
- ▶ $C_{\max} \geq \lceil \sum_j t_j / m \rceil$ (one constraint)

Optimality-preserving constraints:

- ▶ Non-increasing machine loads: not valid for every assignment, but every schedule can be relabelled to satisfy them
- ▶ They remove the relabellings only when the loads are distinct

Typical mistakes

Introduction: Typical mistakes

- ▶ In this section we collect the most common mistakes on the formulations of $P||C_{\max}$
- ▶ They concern the LP bound, the rounding, the symmetry constraints, the direction of the load constraints and the indices of the variables
- ▶ Recognizing them helps to write and use the model correctly

Typical mistakes (1/2)

Observation 1: Warning

- ▶ **Using the LP bound alone.** The value $\sum_j t_j/m$ ignores the longest job: with one very long job it is much smaller than the optimal value. The valid inequality $C_{\max} \geq t_j$ must be added.
- ▶ **Forgetting to round up.** All the processing times are integer, so $C_{\max}$ is an integer in every integer schedule: the lower bound is $\lceil \sum_j t_j/m \rceil$.
- ▶ **Believing that the symmetry constraints remove all the equivalent solutions.** They remove the relabellings of the machines only when the loads are distinct; with equal loads they remove none.

Typical mistakes (2/2)

Observation: Warning

- ▶ **Forgetting the direction of the load constraints.**
The constraints $-\sum_{j \in \mathcal{J}} t_j x_{jh} + C_{\max} \geq 0$ bound $C_{\max}$ from below: it is the minimization that makes $C_{\max}$ equal to the largest load.
- ▶ **Mixing the indices.** *The variable x_{jh} has the job j first and the machine h second.*

Summary

Bounds:

- ▶ The LP bound alone ignores the longest job: add $C_{\max} \geq t_j$
- ▶ Round up: the lower bound is $\lceil \sum_j t_j / m \rceil$

The model:

- ▶ The symmetry constraints do not remove relabellings with equal loads
- ▶ The load constraints bound $C_{\max}$ from below; the minimization does the rest
- ▶ In x_{jh} : job first, machine second

Summary of the chapter (1/2)

- ▶ The ILP $_{P\parallel C_{\max}}$ model has $n \cdot m$ assignment variables x_{jh} and the makespan variable $C_{\max}$, linked by one load constraint for each machine.
- ▶ The LP relaxation has optimal value $z(\text{LP}_{P\parallel C_{\max}}) = \sum_j t_j/m$, attained by assigning every job with value $1/m$ to every machine; the dual solution is $\tilde{\pi}_h = 1/m$ and $\tilde{\mu}_j = t_j/m$.
- ▶ The ratio between $z(\text{LP}_{P\parallel C_{\max}})$ and $z(\text{ILP}_{P\parallel C_{\max}})$ is at least $1/m$, and $1/m$ is asymptotically attained.
- ▶ The valid inequalities $C_{\max} \geq t_j$ and $C_{\max} \geq \lceil \sum_j t_j/m \rceil$ strengthen the model.

Summary of the chapter (2/2)

Dashboard of the instance of the chapter ($m = 3$, $t = (5, 5, 4, 4, 3, 3, 3)$). Lower bounds on the optimal value 9:

Bound	Value	From
LP relaxation	9	$\sum_j t_j / m = 27/3$
Valid inequalities	9	$\max\{5, \lceil 27/3 \rceil\}$
Optimal value	9	loads 9, 9, 9

On the instance of the integrality gap ($m = 2$, $t = (3, 3, 2)$) the LP bound is 4 and the optimal value is 5.