

Lot sizing problems

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Learning objectives

At the end of this chapter you will be able to:

- ▶ describe a production plan by its quantities, stocks and setups, and compute its cost;
- ▶ state the capacitated lot sizing problem and name its main variants;
- ▶ decide whether a plan is feasible, and count the feasible setup choices of an instance;
- ▶ build a first plan by hand, merging the demands of consecutive periods;
- ▶ compute a lower bound on the cost from the number of setups that the total demand requires.

Prerequisites

- ▶ From this module, the chapter «Combinatorial optimization problems»:
 - ▶ what a combinatorial optimization problem is;
 - ▶ the number of candidate solutions;
 - ▶ primal and dual bounds.

Why capacitated lot sizing?

Introduction: why capacitated lot sizing?

- ▶ We start from a concrete question: a factory must decide in which weeks to produce, and how much
- ▶ The answer is a compromise between the cost of starting the production and the cost of keeping the products in stock
- ▶ This question is the **capacitated lot sizing problem**, a classical problem of production planning and the subject of this chapter

A factory and its weekly demand

- ▶ A factory knows the demand of its product in each of the next weeks
- ▶ Starting the production line costs a fixed amount (the **setup**)
- ▶ The line can make only a limited quantity per week (the **capacity**)
- ▶ What is made in advance is kept in the warehouse, at a cost for every unit and every week
- ▶ In which weeks should the factory produce, and how much?

A compromise

- ▶ The decision is a compromise:
 - ▶ producing every week avoids the warehouse but pays many setups;
 - ▶ producing rarely saves setups but pays the stock;
 - ▶ the capacity limits how much can be anticipated.
- ▶ This is the **capacitated lot sizing problem**

A challenge

- ▶ Look at the data of the reference instance of the chapter, with 5 periods:

period t	1	2	3	4	5
demand d_t	2	1	3	4	1
capacity cap_t	4	4	4	4	4
setup cost sc_t	50	50	50	50	50
production cost vc_t	0	0	0	0	0
holding cost hc_t	10	10	10	10	10

- ▶ Try to find a production plan of cost at most 180
- ▶ Then ask yourself how many setups every plan needs at least

A classical problem

- ▶ Lot sizing has a classical place in the history of production planning
- ▶ In 1958 Wagner and Whitin gave a dynamic programming algorithm for the case without capacity and with demand that varies over time
- ▶ It rests on a property of the optimal plans: **one produces only when the stock is empty**
- ▶ With capacities the property fails, and the problem is NP-hard (see the chapter «Combinatorial optimization problems»)
- ▶ The dynamic programming approach is studied later in the course

Summary

The question:

- ▶ In which periods to produce, and how much, to satisfy a known demand
- ▶ Setups against stock, within the capacity of every period

The history:

- ▶ Without capacity: Wagner and Whitin (1958), one produces only with empty stock
- ▶ With capacity: the property fails, and the problem is NP-hard

Overview of the family



Introduction: overview of the family

- ▶ The lot sizing problems form a **family**: they share the same data and the same kind of solution
- ▶ We first describe what they have in common, then we list the members of the family
- ▶ The capacitated lot sizing problem (CLS) is the reference member: each of the others differs from it in one feature

The common data

- ▶ The **lot sizing problems** share the same data:
 - ▶ a set $T = \{1, 2, \dots, m\}$ of m time periods;
 - ▶ for each period $t \in T$, a *demand* d_t and a production *capacity*;
 - ▶ for each period $t \in T$, three costs: the setup cost sc_t , the variable production cost vc_t and the holding cost hc_t .

Production plan and objective

- ▶ A solution is a **production plan**: the quantity produced in every period
- ▶ The plan determines the stock at the end of each period and the periods with a setup
- ▶ The objective is to satisfy all the demands at the minimum total cost of setups, production and holding

Setups against stock

- ▶ A lot sizing problem balances the setup costs, paid in every period with production, against the holding costs of the stock
- ▶ The members of the family differ:
 - ▶ in the capacities;
 - ▶ in the number of products;
 - ▶ in the way the demand and the capacity can be adjusted.

The members of the family (1/2)

The table lists the members of the family, with the section that defines each of them.

Problem	Acronym	Difference with respect to the CLS	Section
capacitated lot sizing problem	CLS	one product, capacities cap_t	the CLS
uncapacitated lot sizing problem	ULS	no binding capacity	Variants
several products and setup times	MLS	products sharing the time of each period	Variants

The members of the family (2/2)

Problem	Acronym	Difference with respect to the CLS	Section
initial stock	LS-SI	the initial stock is a decision	Variants
overtime	LS-OT	extra capacity at a higher cost	Variants
unsatisfied demand	LS-UD	part of the demand lost at a penalty	Variants

Summary

The family:

- ▶ Periods, demands, capacities, and setup, production and holding costs
- ▶ A solution is a production plan; the objective is the minimum total cost
- ▶ Setup costs are balanced against holding costs

The members:

- ▶ The CLS and its variants: ULS, MLS, LS-SI, LS-OT and LS-UD

The capacitated lot sizing problem

Introduction: the capacitated lot sizing problem

- ▶ In this section we define the **capacitated lot sizing problem** (CLS)
- ▶ We describe an instance, a feasible production plan and its cost
- ▶ We introduce the **reference instance** of the chapter, used in all the next sections
- ▶ We close with the assumptions that we make on the data

Definition 1: Capacitated lot sizing problem

Given:

1. a set $T = \{1, 2, \dots, m\}$ of m time periods,
2. for each period $t \in T$, a demand $d_t \in \mathbb{Z}_{\geq 1}$ and a capacity $cap_t \in \mathbb{Z}_{\geq 1}$, both counted in units of one product,
3. for each period $t \in T$, three non-negative integer costs: the setup cost sc_t , the variable production cost vc_t per unit produced and the holding cost hc_t per unit kept in stock at the end of the period,

the **capacitated lot sizing problem** (CLS) asks to decide how much to produce in every period, so that every demand is satisfied with what is produced and what is in stock, the capacities are respected, and the total cost of setups, production and holding is minimized.

A CLS instance

► A **CLS** instance is specified by:

1. one integer number:

the number of periods $m \in \mathbb{Z}_{\geq 1}$,

2. five vectors of integer numbers of length m :

$\mathbf{d}$, $\mathbf{cap} \in \mathbb{Z}_{\geq 1}^m$, containing demands and capacities,
 $\mathbf{sc}$, $\mathbf{vc}$, $\mathbf{hc} \in \mathbb{Z}_{\geq 0}^m$, containing the costs.

Production plan and stock

- ▶ A **CLS feasible solution** is a **production plan**, that is a vector of integer quantities

$$\mathbf{x} = (x_1, x_2, \dots, x_m) \in \mathbb{Z}_{\geq 0}^m,$$

where x_t is the quantity produced in period t

- ▶ The stock left at the end of period t is

$$s_t(\mathbf{x}) = \sum_{u=1}^t (x_u - d_u),$$

because the stock of period $t - 1$ plus the production of period t must cover the demand of period t , and what remains is the stock of period t

Feasible plans and their cost

- ▶ The plan is **feasible** if

$$x_t \leq cap_t \quad \text{and} \quad s_t(\mathbf{x}) \geq 0 \quad \forall t \in T, \quad s_m(\mathbf{x}) = 0.$$

- ▶ There is a **setup** in period t when $x_t > 0$: we write $y_t = 1$ in this case and $y_t = 0$ otherwise
- ▶ The **cost** of a plan is

$$c(\mathbf{x}) = \sum_{t \in T} (sc_t y_t + vc_t x_t + hc_t s_t(\mathbf{x})).$$

Formal statement of the CLS

Given a CLS instance, the CLS can be formally stated as:

$$\tilde{\mathbf{x}} \in \arg \min_{\mathbf{x}} \left\{ c(\mathbf{x}) : \begin{array}{l} x_t \leq \text{cap}_t, \ s_t(\mathbf{x}) \geq 0, \ \forall t \in T, \\ s_m(\mathbf{x}) = 0, \ \mathbf{x} \in \mathbb{Z}_{\geq 0}^m \end{array} \right\}.$$

- We denote by $z(I)$ the cost of an optimal plan $\tilde{\mathbf{x}}$ of the instance I

The reference instance

We now fix a small instance with 5 periods.

We will use it in all the next sections: to count the solutions, to build a plan by hand and to compute a lower bound.

Example (1/3)

We consider $m = 5$ periods, with the following data, all integer:

period t	1	2	3	4	5
demand d_t	2	1	3	4	1
capacity cap_t	4	4	4	4	4
setup cost sc_t	50	50	50	50	50
production cost vc_t	0	0	0	0	0
holding cost hc_t	10	10	10	10	10

- ▶ The total demand is $2 + 1 + 3 + 4 + 1 = 11$
- ▶ We call this CLS instance the **reference instance** of the chapter

Example (2/3)

The plan $\mathbf{x} = (3, 0, 4, 4, 0)$ is feasible:

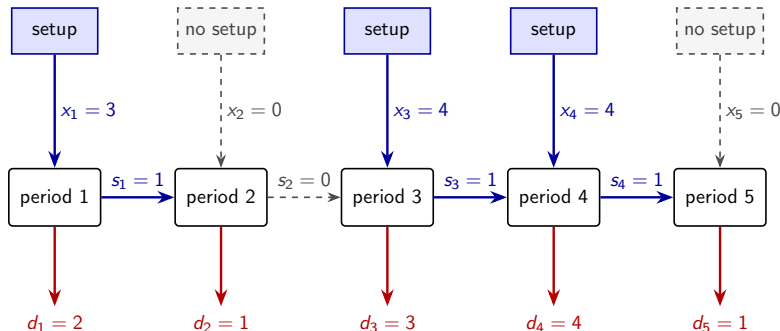
- ▶ the stocks are $s = (1, 0, 1, 1, 0)$, all non-negative, with $s_5 = 0$;
- ▶ every quantity is at most 4;
- ▶ there are 3 setups, in the periods 1, 3 and 4.

Its cost is

$$c(\mathbf{x}) = 3 \cdot 50 + 10 \cdot (1 + 0 + 1 + 1 + 0) = 150 + 30 = 180.$$

Example (3/3)

The plan, period by period from left to right: the production x_t enters from above, through a setup (solid) or not at all (dashed); the stock s_t passes to the next period; the demand d_t leaves from below. Every period has capacity $cap_t = 4$, and the three setups cost 50 each.



Assumptions (1/2)

We make the following assumptions:

1. The stock is empty at the beginning and at the end of the horizon.
 - ▶ Leaving stock at the end only adds cost, so no optimal plan needs it
 - ▶ An initial stock is a variant of the problem (see the section «Variants»)
2. The capacities allow to satisfy the demands, period after period (otherwise there is no feasible plan). Formally:

$$\sum_{u=1}^t cap_u \geq \sum_{u=1}^t d_u, \quad \forall t \in T$$

Assumptions (2/2)

3. The quantities are integer.

- ▶ With integer data, once the periods with a setup are fixed, what remains is a minimum cost flow problem
- ▶ This problem has an integer optimal solution
- ▶ Restricting the quantities to integers does not change the optimal cost

Summary

The CLS:

- ▶ A plan $\mathbf{x}$ gives the stocks $s_t(\mathbf{x})$ and the setups y_t
- ▶ Feasible: $x_t \leq cap_t$ and $s_t(\mathbf{x}) \geq 0$ for every $t \in T$, and $s_m(\mathbf{x}) = 0$
- ▶ Cost: setups, production and holding; $z(I)$ is the optimal cost

The reference instance:

- ▶ 5 periods, total demand 11, capacity 4 in every period
- ▶ The plan $(3, 0, 4, 4, 0)$ is feasible and costs 180

Counting the solutions



Introduction: counting the solutions

- ▶ How many candidate solutions does a CLS instance have?
- ▶ We separate the combinatorial choice, the **periods with a setup**, from the quantities produced
- ▶ On the reference instance we count both, and we find the optimal plan by enumeration
- ▶ The count also shows why enumeration is hopeless on realistic horizons

Setup choices and production plans

- ▶ The combinatorial part of the decision is the choice of the periods with a setup: there are 2^m subsets of periods
- ▶ Not all of them are feasible, because the periods with a setup must have enough capacity to produce everything
- ▶ For a given choice of the setup periods there may be many production plans, since the quantities can be distributed in many ways
- ▶ The number of feasible plans is larger than the number of feasible setup choices

Counting on the reference instance

Let us count the setup choices and the plans of the reference instance:

5 periods, demands 2, 1, 3, 4, 1, capacity 4 in every period.

Example (1/3)

- ▶ For the reference instance there are $2^5 = 32$ subsets of periods
- ▶ Let k_t be the number of setups in the periods $1, \dots, t$
- ▶ The cumulative demands 2, 3, 6, 10, 11 require, with capacity 4:
 - ▶ at least one setup in the periods 1–2;
 - ▶ two in the periods 1–3;
 - ▶ three in the periods 1–4.

Example (2/3)

- ▶ Since $d_1 > 0$ there is a setup in period 1
- ▶ In the periods 2, 3, 4 we need at least two setups, one of which in the periods 2 or 3:

$$(y_2, y_3, y_4) \in \{(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)\}.$$

- ▶ The period 5 is free, so the feasible setup choices are $4 \cdot 2 = 8$ out of 32

Example (3/3)

By enumeration, these setup choices carry 94 feasible plans. The costs of six of them are:

plan	cost	plan	cost	plan	cost
(3, 0, 4, 4, 0)	180	(4, 0, 3, 4, 0)	200	(2, 1, 4, 4, 0)	220
(3, 0, 3, 4, 1)	210	(4, 0, 4, 3, 0)	210	(2, 1, 3, 4, 1)	250

The plan (3, 0, 4, 4, 0) is the unique optimal plan, and $z(I) = 180$.

What the example shows

- ▶ Without the capacities the same instance has $2^{m-1} = 16$ plans that produce only when the stock is empty, and the optimal cost is 170: here the capacity costs 10
- ▶ The optimal plan $(3, 0, 4, 4, 0)$ produces in period 4 with a stock of 1 unit, so the property of Wagner and Whitin does not hold when the capacities are binding
- ▶ The number of setup choices grows as 2^m : with $m = 52$ weeks it is more than $4 \cdot 10^{15}$

Summary

Setup choices:

- ▶ 2^m subsets of periods, not all feasible
- ▶ On the reference instance: 8 feasible setup choices out of 32

Production plans:

- ▶ Each feasible setup choice carries one or more plans: 94 plans in all
- ▶ The unique optimal plan is $(3, 0, 4, 4, 0)$, with $z(I) = 180$
- ▶ With capacities, an optimal plan may produce while the stock is positive

A first plan by hand

Introduction: a first plan by hand

- ▶ Enumerating all the plans works only on tiny instances
- ▶ Here we build a good plan by hand, with two simple ideas:
lot-for-lot and **merging**
- ▶ The cost of any feasible plan is an upper bound on the optimal cost $z(I)$

Lot-for-lot and merging

- ▶ The simplest feasible plan is **lot-for-lot**: in every period we produce exactly the demand of that period
 - ▶ there is no stock, and there are m setups;
 - ▶ it is feasible when every demand fits in the capacity of its period.
- ▶ A plan is improved by **merging**: the demand of a period is produced in an earlier period
 - ▶ this saves a setup;
 - ▶ and costs the stock of the units anticipated.

Example (1/2)

On the reference instance:

1. the lot-for-lot plan $(2, 1, 3, 4, 1)$ has 5 setups and no stock:
cost $5 \cdot 50 = 250$;
2. producing the demand of period 2 in period 1 saves one setup (-50) and keeps 1 unit for one period $(+10)$: the plan $(3, 0, 3, 4, 1)$ costs 210;

Example (2/2)

3. period 4 cannot produce the demand of period 5 as well, because $4 + 1 > cap_4 = 4$; but period 3 can anticipate one unit of period 4, so that period 4 has room for the demand of period 5: the plan $(3, 0, 4, 4, 0)$ saves a setup (-50) and keeps two more units $(+20)$.

The cost is

$$210 - 50 + 20 = 180.$$

Is 180 the optimal cost?

- ▶ We have found the plan of cost 180 by hand
- ▶ But we cannot yet tell whether a cheaper plan exists
- ▶ We need a **lower bound**

Summary

Lot-for-lot:

- ▶ Produce in every period its own demand: no stock, m setups
- ▶ On the reference instance: cost 250

Merging:

- ▶ Produce a demand earlier: one setup less, some stock more
- ▶ On the reference instance: $250 \rightarrow 210 \rightarrow 180$

A combinatorial lower bound

Introduction: a combinatorial lower bound

- ▶ A feasible plan gives an upper bound on $z(I)$; to judge its quality we need a lower bound
- ▶ We derive a simple combinatorial bound: we count the setups that the total demand requires
- ▶ On the reference instance, the bound and the plan found by hand give a gap of 20%

Total demand and number of setups

- ▶ Let $D = \sum_{t \in T} d_t$ be the total demand
- ▶ Let k be the smallest number of periods whose capacities sum up to at least D
- ▶ The number k is found by taking the periods with the largest capacities

A lower bound on the cost

Observation 1

For every CLS instance, every feasible plan has at least k setups, and

$z(I) \geq$ the sum of the k smallest setup costs.

Why the bound holds

- ▶ The plan must produce D units in the periods with a setup, and a period without setup produces nothing
- ▶ The capacities of the periods with a setup sum up to at least D , hence they are at least k
- ▶ The costs of production and holding are non-negative
- ▶ Dropping them, and paying only k setups, the cheapest ones, gives a bound on the cost of every plan

Example (1/2)

- ▶ For the reference instance we have $D = 11$ and all capacities equal to 4, so

$$k = \lceil 11/4 \rceil = 3.$$

- ▶ All the setup costs are 50, hence

$$z(I) \geq 3 \cdot 50 = 150.$$

Example (2/2)

- ▶ Together with the plans of the previous section we obtain

$$150 \leq z(I) \leq 180,$$

an absolute gap of 30, that is a relative gap of $\frac{30}{150} = 20\%$

- ▶ The enumeration of the 94 plans proves that 180 is optimal

Summary

The bound:

- ▶ k is the smallest number of periods whose capacities cover the total demand D
- ▶ Every feasible plan has at least k setups
- ▶ $z(I) \geq$ the sum of the k smallest setup costs

On the reference instance:

- ▶ $k = 3$ and $150 \leq z(I) \leq 180$: a relative gap of 20%

Variants

Introduction: variants

- ▶ Each variant keeps the data of the CLS (periods, demands, costs) and changes one feature of the problem
- ▶ The definitions below give only what changes
- ▶ We see five variants: without capacity, with several products and setup times, with initial stock, with overtime and with unsatisfied demand

The uncapacitated lot sizing problem

Definition 2: Uncapacitated lot sizing problem

*The **uncapacitated lot sizing problem (ULS)** is the CLS in which the capacities are not binding, for example $cap_t \geq \sum_{u \in T} d_u$ for every $t \in T$.*

- The ULS is solved in polynomial time by dynamic programming

Several products and setup times

Definition 3: Lot sizing with several products and setup times

*In the **lot sizing problem with several products and setup times** (MLS) a set P of products shares the resource of each period: d_{pt} is the demand of product $p \in P$ in period t , and cap_t is the time available in period t . Producing a unit of p takes vt_{pt} units of time, and a setup of p costs sc_{pt} and takes st_{pt} units of time.*

Definition 4: Lot sizing with initial stock

In the lot sizing problem with initial stock (LS-SI) the quantity si_p of product p available at the beginning of the horizon is a decision, bought at the cost fc_p per unit.

Definition 5: Lot sizing with overtime

In the lot sizing problem with overtime (LS-OT) the capacity is not fixed: working time in normal hours costs cw_t per unit up to cap_t^N , and overtime costs co_t per unit up to cap_t^O .

Definition 6: Lot sizing with unsatisfied demand

In the lot sizing problem with unsatisfied demand (LS-UD) a part of the demand may be left unsatisfied, at a penalty cu_{pt} per unit.

The variants change one feature of the CLS:

- ▶ ULS: no binding capacity, solved in polynomial time by dynamic programming
- ▶ MLS: several products share the time of each period, with setup times
- ▶ LS-SI: the initial stock is a decision
- ▶ LS-OT: extra capacity in overtime, at a higher cost
- ▶ LS-UD: part of the demand may be lost, at a penalty

Remarks

Introduction: remarks

- ▶ Two remarks place the CLS in context
- ▶ The first one is historical: the algorithm of Wagner and Whitin for the case without capacity
- ▶ The second one is about complexity: what changes when the capacities are binding

Remarks

- ▶ In 1958 Wagner and Whitin gave a dynamic programming algorithm for the case without capacity and with demand that varies over time
- ▶ It rests on a property of the optimal plans: one produces only when the stock is empty
- ▶ With capacities the property fails, and the problem is NP-hard (chapter «[Combinatorial optimization problems](#)»)

Summary

Without capacity:

- ▶ Wagner and Whitin (1958): dynamic programming
- ▶ Optimal plans produce only when the stock is empty

With capacity:

- ▶ The property fails, and the problem is NP-hard

Typical mistakes

Introduction: typical mistakes

- ▶ We collect five mistakes that are easy to make when we check a plan or compute its cost
- ▶ Each one comes with the rule that avoids it
- ▶ They are a quick checklist for the exercises

Observation 2: Typical mistakes (1/2)

- ▶ ***Producing late.*** The production of a period cannot serve the demand of an earlier period: the stock $s_t(\mathbf{x})$ must be non-negative in every period.
- ▶ ***Charging a setup when nothing is produced.*** A setup cost is paid only in the periods with $x_t > 0$.
- ▶ ***Forgetting the stock of the last periods.*** The holding cost is paid on the stock at the end of every period, including the periods in which the plan produces nothing.

Observation 3: Typical mistakes (2/2)

- ▶ ***Believing that one produces only with empty stock.*** The property of Wagner and Whitin holds without capacities; with capacities the optimal plan of the reference instance violates it.
- ▶ ***Counting 2^m feasible plans.*** The 2^m subsets of periods are the setup choices: some are infeasible, and each feasible one carries one or more plans.

Summary

When checking a plan:

- ▶ The stock is non-negative in every period: no production late
- ▶ A setup is paid only where $x_t > 0$
- ▶ The holding cost is paid in every period, also without production

When reasoning on the problem:

- ▶ With capacities, one may produce while the stock is positive
- ▶ 2^m counts the setup choices, not the feasible plans

In summary

Introduction: in summary

- ▶ We collect the key ideas of the chapter
- ▶ We close with the dashboard of the bounds obtained on the reference instance

Summary (1/3)

- ▶ The capacitated lot sizing problem asks for the production quantities of every period, with capacities, setups and stock, at the minimum total cost; a solution is a production plan
- ▶ The combinatorial part is the choice of the periods with a setup: there are 2^m choices, only some of them feasible, and each carries one or more plans
- ▶ Lot-for-lot gives a plan quickly, merging the demands improves it; the number of setups that the total demand requires gives a lower bound

Summary (2/3)

- ▶ With capacities, optimal plans may produce while the stock is positive; without capacities they never do
- ▶ The variants add products, setup times, initial stock, overtime or unsatisfied demand
- ▶ Their models are given later in the course, in the chapter «Formulations of lot sizing problems»

Summary (3/3)

Dashboard of the CLS instance (minimization: the lower bound is a dual bound, the upper bound is the cost of a plan):

Chapter	Lower bound	Upper bound	From
This chapter	150	250	three setups at least; lot-for-lot
This chapter, merging	150	180	the plan (3, 0, 4, 4, 0)
This chapter, enumeration	180	180	the 94 feasible plans