

Formulations of lot sizing problems



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Learning objectives

At the end of this chapter you will be able to:

- ▶ write the big-M formulation of the capacitated lot sizing problem and compute a weak and a strong big-M value;
- ▶ write the shortest-path formulation and interpret its variables as the arcs of a network;
- ▶ represent a production plan as a combination of paths of the network;
- ▶ compare the LP bounds of the formulations on the reference instance;
- ▶ write the ILP model of the uncapacitated lot sizing problem;
- ▶ extend the big-M formulation to several products with setup times, to the initial stock, to overtime and to unsatisfied demand, and compute the strong big-M values of each model.

Prerequisites

- ▶ From the module *Integer linear programming*, the chapters «Integer linear programming» and «Primal and dual bounds»: ILP problems, LP relaxations and lower bounds.
- ▶ From this module, the chapter «Lot sizing problems»: the problem, its assumptions, the plans and the reference instance.



Overview of the family

Introduction: overview of the family

- ▶ This chapter writes the models of the lot sizing problems: the capacitated problem and its variants
- ▶ All the models share the same variables and the same kinds of constraints
- ▶ We first look at the family as a whole: its variables, its constraints and the list of its members

Variables, constraints and objective

- ▶ The big- M models of the lot sizing problems use, for every period $t \in T$, an integer **production variable** x_t , an integer **inventory variable** s_t and a binary **setup variable** y_t .
- ▶ The **flow balancing constraints** link the inventories of consecutive periods to the production and to the demand; the setup variables are linked to the production by big- M constraints.
- ▶ The objective function minimizes the total cost of setups, production and holding. The shortest-path formulation of the CLS replaces the production and inventory variables with the arcs of a network.

Flow models with setups

- ▶ The lot sizing models are **flow models over the periods**, with binary setup variables that switch the production on.
- ▶ The members of the family add products, setup times, an initial stock, overtime or unsatisfied demand to the same balance and big- M constraints.

The members of the family

The members of the family, with the names of their models and the sections that write them:

Problem	Models	Formulation
<i>Section «A formulation with a big-M link»</i>		
CLS	ILP_{CLSP}	big-M formulation
<i>Section «A shortest-path formulation»</i>		
CLS	MIP_{CLSP}^{SP}	shortest-path formulation
<i>Section «ILP models for the variants»</i>		
ULS	ILP_{ULS}	big-M, no capacity
MLS	ILP_{CLSP}^P	big-M, setup times
LS-SI	ILP_{CLSP}^{SI}	big-M, initial stock variables
LS-OT	ILP_{CLSP}^{OT}	big-M, overtime variables
LS-UD	ILP_{CLSP}^{UD}	big-M, unsatisfied demand variables

Summary

- ▶ Every big- M model of the family uses production variables x_t , inventory variables s_t and setup variables y_t
- ▶ The flow balancing constraints link inventories, production and demand; big- M constraints link the setups to the production
- ▶ The variants add products, setup times, an initial stock, overtime or unsatisfied demand to the same constraints

ILP models for the Capacitated Lot Sizing Problem



Introduction: ILP models for the CLS

- ▶ We now start from the capacitated lot sizing problem (CLS), the main member of the family
- ▶ It has two different formulations, and we will write both of them
- ▶ Two correct models of the same problem can give very different lower bounds: this is the thread of the next sections

The CLS and its two formulations

- ▶ The Capacitated Lot Sizing Problem and its reference instance are defined in the chapter «[Lot sizing problems](#)».
- ▶ The CLS has two formulations: the big- M formulation ILP_{CLSP} (section «A formulation with a big-M link») and the shortest-path formulation MIP_{CLSP}^{SP} (section «A shortest-path formulation»).
- ▶ The section «Comparing the formulations» compares the bounds of their Linear Programming relaxations on the reference instance.

Summary

- ▶ The CLS has a big- M formulation ILP_{CLSP} and a shortest-path formulation MIP_{CLSP}^{SP}
- ▶ We will compare the bounds of their LP relaxations on the reference instance

A formulation with a big-M link

Introduction: a formulation with a big-M link

- ▶ We write the first ILP model of the CLS, with production, inventory and setup variables
- ▶ The setups are linked to the production by a big-M constraint
- ▶ The value of M matters: we compute a weak and a strong value
- ▶ We close with the model on the reference instance and its optimal solution

Total demands and variables

- ▶ Given a set $T = \{1, 2, \dots, m\}$ of periods with demands d_t , for $j \leq t$ we write $D_{j,t} = \sum_{k=j}^t d_k$ for the total demand of the periods $j, j+1, \dots, t$.
- ▶ We introduce the following **integer variables** and **binary variables**, for every period $t \in T$:

$$x_t \in \mathbb{Z}_{\geq 0} \text{ (production),}$$

$$s_t \in \mathbb{Z}_{\geq 0} \text{ (inventory at the end of the period),}$$

$$y_t = \begin{cases} 1 & \text{if there is a setup in period } t \\ 0 & \text{otherwise} \end{cases}$$

The inventory at the beginning of the horizon is the constant s_0 .

The big-M constraint

- ▶ The setup variables are linked to the quantities by the logical implication

$$x_t > 0 \implies y_t = 1,$$

modeled by the **big-M constraint** $x_t \leq M_t y_t$, where M_t is an upper bound on the production of period t .

- ▶ If the setups are ignored, the best plan produces in every period exactly its demand: it has no inventory and, in the reference instance, cost 0.
- ▶ This is a very weak lower bound: the setup costs must be modeled.

The big-M formulation ILP_{CLSP}

Using these variables, an Integer Linear Programming (ILP) model for the capacitated lot sizing problem (CLS) reads as follows:

$$\min \sum_{t \in T} (sc_t y_t + vc_t x_t + hc_t s_t)$$

$$\text{subject to } s_{t-1} + x_t = d_t + s_t, \quad \forall t \in T,$$

$$x_t \leq M_t y_t, \quad \forall t \in T,$$

$$x_t \leq cap_t, \quad \forall t \in T,$$

$$s_0 = s_m = 0,$$

$$x_t \in \mathbb{Z}_{\geq 0}, \quad \forall t \in T,$$

$$s_t \in \mathbb{Z}_{\geq 0}, \quad \forall t \in T,$$

$$y_t \in \{0, 1\}, \quad \forall t \in T.$$

Description of the model ILP_{CLSP}

- ▶ The linear objective function minimizes the cost of the setups, of the production and of the holding;
- ▶ constraints $s_{t-1} + x_t = d_t + s_t$ are the **flow balancing constraints**: the demand of period t is satisfied by the production of the period and by the inventory of the previous period, and the surplus becomes the inventory at the end of period t (m linear constraints);
- ▶ constraints $x_t \leq M_t y_t$ link the quantities with the setups, and constraints $x_t \leq cap_t$ are the capacity constraints (m linear constraints each);
- ▶ constraint $s_0 = s_m = 0$ imposes an empty inventory at the beginning and at the end of the horizon;
- ▶ finally, the domain constraints define the variables of the ILP model.

This ILP model is referred to as the ILP_{CLSP} model, or **big-M formulation**.

Weak and strong big-M values

- ▶ **A weak value.** Since $s_m = 0$ and the inventories are non-negative, the production of period t cannot exceed the demand of the periods $t, \dots, m$: $M_t^{\text{dem}} = D_{t,m}$ is a valid value.
- ▶ **A second value.** By the capacity constraints $x_t \leq \text{cap}_t$, $M_t^{\text{cap}} = \text{cap}_t$ is also valid.
- ▶ **A strong value.** The best value is the smallest of the two:

$$M_t = \min \{ \text{cap}_t, D_{t,m} \}.$$

Big-M values and LP relaxation

- ▶ **The smaller the big-M value, the stronger the LP relaxation:** the constraint $x_t \leq M_t y_t$ cuts off the fractional values of y_t below x_t/M_t .
- ▶ A value that is too small, however, cuts off feasible plans.

Example: the reference instance (1/2)

We consider $m = 5$ periods, with the following data:

period t	1	2	3	4	5
demand d_t	2	1	3	4	1
capacity cap_t	4	4	4	4	4
setup cost sc_t	50	50	50	50	50
production cost vc_t	0	0	0	0	0
holding cost hc_t	10	10	10	10	10

This is the reference instance of the chapter «[Lot sizing problems](#)».

Example: the reference instance (2/2)

The big-M values are:

period t	1	2	3	4	5
$M_t^{\text{dem}} = D_{t,5}$	11	9	8	5	1
$M_t^{\text{cap}} = \text{cap}_t$	4	4	4	4	4
M_t	4	4	4	4	1

We refer to this CLS instance as the **reference instance**.

Example: the ILP_{CLSP} model (1/3)

The ILP_{CLSP} model with the strong big-M values for the CLS reference instance is ($s_0 = 0$ is a constant, and it does not appear):

$$\begin{aligned} \min \quad & 50 y_1 + 50 y_2 + 50 y_3 + 50 y_4 + 50 y_5 + 10 s_1 + 10 s_2 \\ & + 10 s_3 + 10 s_4 + 10 s_5 \end{aligned}$$

$$\begin{aligned} \text{subject to} \quad & x_1 - s_1 &= 2 \\ & s_1 + x_2 - s_2 &= 1 \\ & s_2 + x_3 - s_3 &= 3 \\ & s_3 + x_4 - s_4 &= 4 \\ & s_4 + x_5 - s_5 &= 1 \end{aligned}$$

Example: the ILP_{CLSP} model (2/3)

$$x_1 - 4y_1 \leq 0$$

$$x_2 - 4y_2 \leq 0$$

$$x_3 - 4y_3 \leq 0$$

$$x_4 - 4y_4 \leq 0$$

$$x_5 - y_5 \leq 0$$

$$x_1 \leq 4$$

$$x_2 \leq 4$$

$$x_3 \leq 4$$

$$x_4 \leq 4$$

$$x_5 \leq 4$$

Example: the ILP_{CLSP} model (3/3)

$$s_5 = 0$$

$$x_1, x_2, x_3, x_4, x_5 \in \mathbb{Z}_{\geq 0}$$

$$s_1, s_2, s_3, s_4, s_5 \in \mathbb{Z}_{\geq 0}$$

$$y_1, y_2, y_3, y_4, y_5 \in \{0, 1\}$$

An optimal solution $(\tilde{\mathbf{y}}, \tilde{\mathbf{x}}, \tilde{\mathbf{s}})$ (ILP_{CLSP}) is:

$$\tilde{\mathbf{y}} = (1, 0, 1, 1, 0), \quad \tilde{\mathbf{x}} = (3, 0, 4, 4, 0), \quad \tilde{\mathbf{s}} = (1, 0, 1, 1, 0)$$

- ▶ The optimal value $z(\text{ILP}_{\text{CLSP}})$ is $3 \cdot 50 + 3 \cdot 10 = 180$, the cost of the optimal plan found by enumeration in the chapter «[Lot sizing problems](#)».
- ▶ The weak big-M values give the same optimal solution.

Summary

The big-M formulation ILP_{CLSP} :

- ▶ Integer variables x_t , s_t and binary variables y_t ; flow balancing constraints and capacity constraints
- ▶ The link $x_t \leq M_t y_t$ models the implication $x_t > 0 \Rightarrow y_t = 1$

The big-M values:

- ▶ $D_{t,m}$ and cap_t are both valid; the strong value is $M_t = \min\{\text{cap}_t, D_{t,m}\}$
- ▶ The smaller the value, the stronger the LP relaxation

The reference instance: optimal plan $(3, 0, 4, 4, 0)$, of cost 180

The Linear Programming relaxation of the big-M formulation

Introduction: the LP relaxation of the big-M formulation

- ▶ The LP relaxation of an ILP model gives a lower bound on its optimal value
- ▶ In the big-M formulation the bound depends on the values M_t
- ▶ We compute the bound of the three choices M_t^{dem} , M_t^{cap} and M_t on the reference instance

Relaxing the integrality

- ▶ To obtain a Linear Programming (LP) relaxation of the ILP_{CLSP} model, we relax the integrality constraints of the variables into non-negativity constraints and the bounds $y_t \leq 1$.
- ▶ The obtained LP model is written in the next slide.

The LP relaxation of the big-M formulation

$$\min \sum_{t \in T} (sc_t y_t + vc_t x_t + hc_t s_t)$$

$$\text{subject to } s_{t-1} + x_t = d_t + s_t, \quad \forall t \in T,$$

$$x_t \leq M_t y_t, \quad \forall t \in T,$$

$$x_t \leq cap_t, \quad \forall t \in T,$$

$$s_0 = s_m = 0,$$

$$y_t \leq 1, \quad \forall t \in T,$$

$$x_t \geq 0, \quad \forall t \in T,$$

$$s_t \geq 0, \quad \forall t \in T,$$

$$y_t \geq 0, \quad \forall t \in T.$$

Three LP models

- ▶ This LP model is referred to as the LP_{CLSP}^{dem} , LP_{CLSP}^{cap} or LP_{CLSP}^{strong} model, when the big-M values are M_t^{dem} , M_t^{cap} or the strong values M_t .
- ▶ The three models differ only in the coefficients of the constraints $x_t \leq M_t y_t$.
- ▶ We now solve the three of them on the reference instance.

Example: big-M values and LP bounds (1/4)

The LP model for the CLS reference instance is:

$$\begin{aligned} \min \quad & 50 y_1 + 50 y_2 + 50 y_3 + 50 y_4 + 50 y_5 + 10 s_1 + 10 s_2 \\ & + 10 s_3 + 10 s_4 + 10 s_5 \end{aligned}$$

$$\begin{aligned} \text{subject to} \quad & x_1 - s_1 &= 2 \\ & s_1 + x_2 - s_2 &= 1 \\ & s_2 + x_3 - s_3 &= 3 \\ & s_3 + x_4 - s_4 &= 4 \\ & s_4 + x_5 - s_5 &= 1 \end{aligned}$$

Example: big-M values and LP bounds (2/4)

$$x_1 - M_1 y_1 \leq 0$$

$$x_2 - M_2 y_2 \leq 0$$

$$x_3 - M_3 y_3 \leq 0$$

$$x_4 - M_4 y_4 \leq 0$$

$$x_5 - M_5 y_5 \leq 0$$

$$x_1 \leq 4$$

$$x_2 \leq 4$$

$$x_3 \leq 4$$

$$x_4 \leq 4$$

$$x_5 \leq 4$$

Example: big-M values and LP bounds (3/4)

$$s_5 = 0$$

$$y_1 \leq 1$$

$$y_2 \leq 1$$

$$y_3 \leq 1$$

$$y_4 \leq 1$$

$$y_5 \leq 1$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

$$s_1, s_2, s_3, s_4, s_5 \geq 0$$

$$y_1, y_2, y_3, y_4, y_5 \geq 0$$

with the values $M_1, \dots, M_5$ of the table of the reference instance.

Example: big-M values and LP bounds (4/4)

The three choices give the following optimal solutions $(\tilde{\mathbf{y}}, \tilde{\mathbf{x}}, \tilde{\mathbf{s}})$, with $\tilde{s}_5 = 0$:

Model	$\tilde{\mathbf{y}}$	$\tilde{\mathbf{x}}$	$(\tilde{s}_1, \dots, \tilde{s}_4)$	optimal value
$\text{LP}_{\text{CLSP}}^{\text{dem}}$	$(\frac{2}{11}, \frac{1}{9}, \frac{1}{2}, \frac{4}{5}, 0)$	$(2, 1, 4, 4, 0)$	$(0, 0, 1, 1)$	$\frac{9865}{99} \approx 99.65$
$\text{LP}_{\text{CLSP}}^{\text{cap}}$	$(\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{4})$	$(2, 1, 3, 4, 1)$	$(0, 0, 0, 0)$	$\frac{275}{2} = 137.5$
$\text{LP}_{\text{CLSP}}^{\text{strong}}$	$(\frac{1}{2}, \frac{1}{4}, 1, 1, 0)$	$(2, 1, 4, 4, 0)$	$(0, 0, 1, 1)$	$\frac{315}{2} = 157.5$

- ▶ In every solution $\tilde{y}_t = \tilde{x}_t / M_t$: the smaller M_t , the larger the cost of the setups paid by the LP.
- ▶ The integer optimal solution of the ILP_{CLSP} model has value $z(\text{ILP}_{\text{CLSP}}) = 180$.

Summary

- ▶ The LP relaxation of ILP_{CLSP} replaces the integrality of the variables with non-negativity and the bounds $y_t \leq 1$
- ▶ In every LP solution $\tilde{y}_t = \tilde{x}_t/M_t$: the setups are paid only in part
- ▶ On the reference instance: 99.65 with M_t^{dem} , 137.5 with M_t^{cap} , 157.5 with the strong values, against the optimal value 180

A shortest-path formulation

Introduction: a shortest-path formulation

- ▶ We write a second formulation of the CLS, which does not decide the quantities directly
- ▶ Its variables are the arcs of a network: a plan becomes a path, or a combination of paths
- ▶ The setups are linked to the arcs without any big-M value

The variables zv_{tk} and their costs

- ▶ The second formulation does not decide the quantities directly. For every $t, k \in T$, with $k \geq t$, we introduce the following **continuous variable**:

$zv_{tk} \geq 0$, the fraction of the demand $D_{t,k}$ of the periods $t, \dots, k$ that is produced in period t .

- ▶ Producing in period t the demand of the periods $t, \dots, k$ costs the production of $D_{t,k}$ units and the holding of the units that wait for their period:

$$cv_{tk} = vc_t D_{t,k} + \sum_{s=t+1}^k \sum_{u=t}^{s-1} hc_u d_s.$$

The shortest-path formulation MIP_{CLSP}^{SP}

Using these variables and the binary variables y_t , a Mixed Integer Linear Programming (MIP) model for the CLS reads as follows:

$$\min \sum_{t \in T} sc_t y_t + \sum_{t \in T} \sum_{k=t}^m cv_{tk} zv_{tk}$$

$$\text{subject to } \sum_{k=1}^m zv_{1k} = 1,$$

$$\sum_{u=1}^{t-1} zv_{u,t-1} - \sum_{k=t}^m zv_{tk} = 0, \quad \forall t \in T \setminus \{1\},$$

$$\sum_{k=t}^m zv_{tk} \leq y_t, \quad \forall t \in T,$$

$$\sum_{k=t}^m D_{t,k} zv_{tk} \leq cap_t, \quad \forall t \in T,$$

The model $\text{MIP}_{\text{CLSP}}^{\text{SP}}$: domains and description

$$zv_{tk} \geq 0, \quad \forall t, k \in T, \quad k \geq t,$$

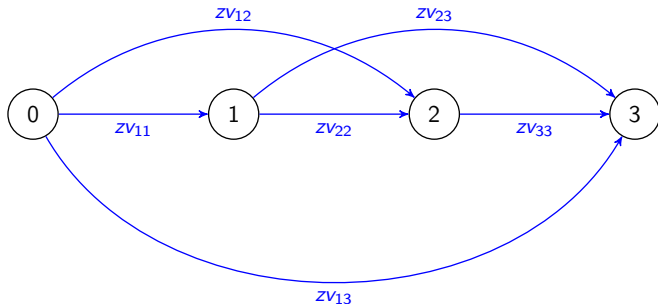
$$y_t \in \{0, 1\}, \quad \forall t \in T.$$

- ▶ The linear objective function minimizes the cost of the setups and of the arcs of the plan;
- ▶ constraints $\sum_k zv_{1k} = 1$ and $\sum_u zv_{u,t-1} - \sum_k zv_{tk} = 0$ are the flow constraints of a network (m linear constraints);
- ▶ constraints $\sum_k zv_{tk} \leq y_t$ force the setup if the product is produced in period t , and constraints $\sum_k D_{t,k} zv_{tk} \leq cap_t$ impose the capacities (m linear constraints each);
- ▶ finally, the domain constraints define the variables of the MIP model.

This MIP model is referred to as the $\text{MIP}_{\text{CLSP}}^{\text{SP}}$ model, or **shortest-path formulation**.

The network of the variables $z_{v_{tk}}$

- ▶ The nodes are $0, 1, \dots, m$, and the variable $z_{v_{tk}}$ is the arc from node $t - 1$ to node k , with cost $c_{v_{tk}}$.
- ▶ A unit of flow leaves node 0 (constraint $\sum_k z_{v_{1k}} = 1$), and the flow is conserved at the other nodes (constraints $\sum_u z_{v_{u,t-1}} - \sum_k z_{v_{tk}} = 0$).



Network of the variables $z_{v_{tk}}$ for $m = 3$: the variable $z_{v_{tk}}$ is the arc from node $t - 1$ to node k .

Paths and plans

- ▶ A path from node 0 to node m is a plan that produces only when the stock is empty: each arc $(t - 1, k)$ is a production in period t for the periods $t, \dots, k$.
- ▶ This is why the formulation is called a **shortest-path formulation**: with the capacities relaxed, the optimal plan is a shortest path.

Setups, quantities and combinations of paths

- ▶ Constraints $\sum_k z v_{tk} \leq y_t$ link the arcs with the setups with the coefficient 1 instead of a big-M value.
- ▶ In constraints $\sum_k D_{t,k} z v_{tk} \leq cap_t$ the quantity produced in period t is $x_t = \sum_k D_{t,k} z v_{tk}$.
- ▶ The paths are the building blocks, and a plan that is not a path is a **combination of paths**: the quantities are integer, but the variables $z v_{tk}$ may be fractional.

Example: MIP_{CLSP}^{SP} model and combinations of paths (1/6)

For the CLS reference instance the demands $D_{t,k}$ and the costs cv_{tk} are (the production cost is zero, so cv_{tk} is only holding cost):

$t \backslash k$	$D_{t,k}$					cv_{tk}				
	1	2	3	4	5	1	2	3	4	5
1	2	3	6	10	11	0	10	70	190	230
2		1	4	8	9		0	30	110	140
3			3	7	8			0	40	60
4				4	5				0	10
5					1					0

Example: $\text{MIP}_{\text{CLSP}}^{\text{SP}}$ model and combinations of paths (2/6)

The $\text{MIP}_{\text{CLSP}}^{\text{SP}}$ model for this instance is:

$$\begin{aligned} \min \quad & 50 y_1 + 50 y_2 + 50 y_3 + 50 y_4 + 50 y_5 \\ & + 10 z_{v_{12}} + 70 z_{v_{13}} + 190 z_{v_{14}} + 230 z_{v_{15}} \\ & + 30 z_{v_{23}} + 110 z_{v_{24}} + 140 z_{v_{25}} + 40 z_{v_{34}} \\ & + 60 z_{v_{35}} + 10 z_{v_{45}} \\ \text{subject to} \quad & z_{v_{11}} + z_{v_{12}} + z_{v_{13}} + z_{v_{14}} + z_{v_{15}} = 1 \\ & z_{v_{11}} - z_{v_{22}} - z_{v_{23}} - z_{v_{24}} - z_{v_{25}} = 0 \\ & z_{v_{12}} + z_{v_{22}} - z_{v_{33}} - z_{v_{34}} - z_{v_{35}} = 0 \\ & z_{v_{13}} + z_{v_{23}} + z_{v_{33}} - z_{v_{44}} - z_{v_{45}} = 0 \\ & z_{v_{14}} + z_{v_{24}} + z_{v_{34}} + z_{v_{44}} - z_{v_{55}} = 0 \end{aligned}$$

Example: $\text{MIP}_{\text{CLSP}}^{\text{SP}}$ model and combinations of paths (3/6)

$$z_{V11} + z_{V12} + z_{V13} + z_{V14} + z_{V15} - y_1 \leq 0$$

$$z_{V22} + z_{V23} + z_{V24} + z_{V25} - y_2 \leq 0$$

$$z_{V33} + z_{V34} + z_{V35} - y_3 \leq 0$$

$$z_{V44} + z_{V45} - y_4 \leq 0$$

$$z_{V55} - y_5 \leq 0$$

$$2 z_{V11} + 3 z_{V12} + 6 z_{V13} + 10 z_{V14} + 11 z_{V15} \leq 4$$

$$z_{V22} + 4 z_{V23} + 8 z_{V24} + 9 z_{V25} \leq 4$$

$$3 z_{V33} + 7 z_{V34} + 8 z_{V35} \leq 4$$

$$4 z_{V44} + 5 z_{V45} \leq 4$$

$$z_{V55} \leq 4$$

Example: $\text{MIP}_{\text{CLSP}}^{\text{SP}}$ model and combinations of paths (4/6)

$$\begin{aligned} z_{v_{11}}, z_{v_{12}}, z_{v_{13}}, \dots, z_{v_{55}} &\geq 0 \\ y_1, y_2, y_3, y_4, y_5 &\in \{0, 1\} \end{aligned}$$

An optimal solution $(\tilde{\mathbf{z}}, \tilde{\mathbf{y}})(\text{MIP}_{\text{CLSP}}^{\text{SP}})$ is:

$$\begin{aligned} \tilde{z}_{v_{12}} = 1, \quad \tilde{z}_{v_{33}} = \frac{4}{5}, \quad \tilde{z}_{v_{35}} = \frac{1}{5}, \quad \tilde{z}_{v_{45}} = \frac{4}{5}, \\ \tilde{\mathbf{y}} = (1, 0, 1, 1, 0), \end{aligned}$$

and all the other variables $z_{v_{tk}}$ are zero.

Example: $\text{MIP}_{\text{CLSP}}^{\text{SP}}$ model and combinations of paths (5/6)

- ▶ It is the optimal plan $\tilde{x} = (3, 0, 4, 4, 0)$, which is not a path: in period 3 it produces 4 units, more than the demand 3 of period 3, and less than the demand 7 of the periods 3 and 4.
- ▶ It is the combination, with weights $\frac{4}{5}$ and $\frac{1}{5}$, of the paths

$$0 \rightarrow 2 \rightarrow 3 \rightarrow 5 \quad (\text{plan } (3, 0, 3, 5, 0)) \quad \text{and}$$

$$0 \rightarrow 2 \rightarrow 5 \quad (\text{plan } (3, 0, 8, 0, 0)).$$

- ▶ The quantities are $x_1 = D_{1,2} z_{v_{12}} = 3$, $x_3 = 3 \cdot \frac{4}{5} + 8 \cdot \frac{1}{5} = 4$ and $x_4 = 5 \cdot \frac{4}{5} = 4$.

Example: $\text{MIP}_{\text{CLSP}}^{\text{SP}}$ model and combinations of paths (6/6)

The optimal value $z(\text{MIP}_{\text{CLSP}}^{\text{SP}})$ is

$$\underbrace{3 \cdot 50}_{\text{setups}} + \underbrace{10 \cdot 1 + 0 \cdot \frac{4}{5} + 60 \cdot \frac{1}{5} + 10 \cdot \frac{4}{5}}_{= 30} = 180.$$

Relaxing the setup variables

- ▶ To obtain an LP relaxation of the $\text{MIP}_{\text{CLSP}}^{\text{SP}}$ model, we relax constraints $y_t \in \{0, 1\}$ into the bounds $0 \leq y_t \leq 1$.
- ▶ The obtained LP model is written in the next slides.

The LP relaxation of the shortest-path formulation

$$\min \sum_{t \in T} sc_t y_t + \sum_{t \in T} \sum_{k=t}^m cv_{tk} zv_{tk}$$

$$\text{subject to } \sum_{k=1}^m zv_{1k} = 1,$$

$$\sum_{u=1}^{t-1} zv_{u,t-1} - \sum_{k=t}^m zv_{tk} = 0, \quad \forall t \in T \setminus \{1\},$$

$$\sum_{k=t}^m zv_{tk} \leq y_t, \quad \forall t \in T,$$

$$\sum_{k=t}^m D_{t,k} zv_{tk} \leq cap_t, \quad \forall t \in T,$$

The model $\text{LP}_{\text{CLSP}}^{\text{SP}}$: bounds and domains

$$\begin{aligned}y_t &\leq 1, & \forall t \in T, \\zv_{tk} &\geq 0, & \forall t, k \in T, k \geq t, \\y_t &\geq 0, & \forall t \in T.\end{aligned}$$

- ▶ This LP model is referred to as the $\text{LP}_{\text{CLSP}}^{\text{SP}}$ model.
- ▶ It differs from the $\text{MIP}_{\text{CLSP}}^{\text{SP}}$ model only in the bounds $0 \leq y_t \leq 1$ in place of the binary constraints $y_t \in \{0, 1\}$.

Example: the LP_{CLSP}^{SP} model (1/4)

The LP_{CLSP}^{SP} model for the CLS reference instance is:

$$\begin{aligned} \min \quad & 50 y_1 + 50 y_2 + 50 y_3 + 50 y_4 + 50 y_5 \\ & + 10 z_{v_{12}} + 70 z_{v_{13}} + 190 z_{v_{14}} + 230 z_{v_{15}} \\ & + 30 z_{v_{23}} + 110 z_{v_{24}} + 140 z_{v_{25}} + 40 z_{v_{34}} \\ & + 60 z_{v_{35}} + 10 z_{v_{45}} \\ \text{subject to} \quad & z_{v_{11}} + z_{v_{12}} + z_{v_{13}} + z_{v_{14}} + z_{v_{15}} = 1 \\ & z_{v_{11}} - z_{v_{22}} - z_{v_{23}} - z_{v_{24}} - z_{v_{25}} = 0 \\ & z_{v_{12}} + z_{v_{22}} - z_{v_{33}} - z_{v_{34}} - z_{v_{35}} = 0 \\ & z_{v_{13}} + z_{v_{23}} + z_{v_{33}} - z_{v_{44}} - z_{v_{45}} = 0 \\ & z_{v_{14}} + z_{v_{24}} + z_{v_{34}} + z_{v_{44}} - z_{v_{55}} = 0 \end{aligned}$$

Example: the LP_{CLSP}^{SP} model (2/4)

$$z_{V11} + z_{V12} + z_{V13} + z_{V14} + z_{V15} - y_1 \leq 0$$

$$z_{V22} + z_{V23} + z_{V24} + z_{V25} - y_2 \leq 0$$

$$z_{V33} + z_{V34} + z_{V35} - y_3 \leq 0$$

$$z_{V44} + z_{V45} - y_4 \leq 0$$

$$z_{V55} - y_5 \leq 0$$

$$2 z_{V11} + 3 z_{V12} + 6 z_{V13} + 10 z_{V14} + 11 z_{V15} \leq 4$$

$$z_{V22} + 4 z_{V23} + 8 z_{V24} + 9 z_{V25} \leq 4$$

$$3 z_{V33} + 7 z_{V34} + 8 z_{V35} \leq 4$$

$$4 z_{V44} + 5 z_{V45} \leq 4$$

$$z_{V55} \leq 4$$

Example: the LP_{CLSP}^{SP} model (3/4)

$$y_1 \leq 1$$

$$y_2 \leq 1$$

$$y_3 \leq 1$$

$$y_4 \leq 1$$

$$y_5 \leq 1$$

$$z_{V_{11}}, z_{V_{12}}, z_{V_{13}}, \dots, z_{V_{55}} \geq 0$$

$$y_1, y_2, y_3, y_4, y_5 \geq 0$$

Example: the $\text{LP}_{\text{CLSP}}^{\text{SP}}$ model (4/4)

An optimal solution $(\tilde{\mathbf{z}}, \tilde{\mathbf{y}})(\text{LP}_{\text{CLSP}}^{\text{SP}})$ is:

$$\tilde{z}_{12} = 1, \quad \tilde{z}_{33} = \frac{4}{5}, \quad \tilde{z}_{35} = \frac{1}{5}, \quad \tilde{z}_{45} = \frac{4}{5},$$
$$\tilde{\mathbf{y}} = (1, 0, 1, \frac{4}{5}, 0),$$

and all the other variables $z_{v_{tk}}$ are zero.

- ▶ These are the arcs of the optimal plan, with the setup of period 4 paid only for the fraction $\frac{4}{5}$.
- ▶ The optimal value $z(\text{LP}_{\text{CLSP}}^{\text{SP}})$ is $180 - \frac{1}{5} \cdot 50 = 170$.

Summary

The shortest-path formulation $\text{MIP}_{\text{CLSP}}^{\text{SP}}$:

- ▶ Continuous variables $z_{v_{tk}}$, the arcs of a network with nodes $0, 1, \dots, m$, with costs $c_{v_{tk}}$
- ▶ A path from 0 to m is a plan; a plan that is not a path is a combination of paths
- ▶ The link $\sum_k z_{v_{tk}} \leq y_t$ has coefficient 1, with no big-M value

The reference instance:

- ▶ The optimal plan $(3, 0, 4, 4, 0)$, of cost 180, combines two paths
- ▶ The LP relaxation $\text{LP}_{\text{CLSP}}^{\text{SP}}$ gives the bound 170

Comparing the formulations



Introduction: comparing the formulations

- ▶ We put side by side the lower bounds of all the relaxations computed so far
- ▶ A stronger bound is closer to the optimal value, but it may come from a larger model
- ▶ The comparison shows what we gain and what we pay with each formulation

Same optimal value, different bounds

- ▶ The formulations have the same optimal value $z(\text{ILP}_{\text{CLSP}}) = z(\text{MIP}_{\text{CLSP}}^{\text{SP}}) = 180$, but their LP relaxations have different optimal values.
- ▶ On the reference instance the lower bounds, computed with the exact simplex algorithm and checked with Gurobi, are given in the next slide.
- ▶ The last column is the gap of the chapter «Combinatorial optimization problems» between the bound and the optimal value; it is undefined for the bound 0.

Lower bounds on the reference instance

Model	Lower bound	Gap $\frac{180-LB}{LB}$
Without setups	0	undefined
$LP_{CLSP}^{dem} (M_t = D_{t,m})$	99.65	80.6%
$LP_{CLSP}^{cap} (M_t = cap_t)$	137.50	30.9%
$LP_{CLSP}^{strong} (M_t = \min\{cap_t, D_{t,m}\})$	157.50	14.3%
LP_{CLSP}^{SP} (shortest path)	170	5.9%

Strength and size

- ▶ The bound 170 of the shortest-path formulation is the optimal value of the problem without capacity (plan $(3, 0, 8, 0, 0)$, of cost 170): the capacity constraints $\sum_k D_{t,k} z_{tk} \leq cap_t$ act only on the quantities, and a combination of paths can satisfy them.
- ▶ The price of the shortest-path formulation is its size: there are $O(m^2)$ variables z_{tk} , against the $O(m)$ variables of the big-M formulation.

Summary

- ▶ The two formulations have the same optimal value 180, but different LP bounds
- ▶ On the reference instance the bound grows from 99.65 (big-M with $D_{t,m}$) to 157.5 (strong big-M) and to 170 (shortest path)
- ▶ The shortest-path formulation is stronger but larger: $O(m^2)$ variables against $O(m)$

ILP models for the variants

Introduction: ILP models for the variants

- ▶ The problems of this section are the variants of the capacitated lot sizing problem defined in the chapter «[Lot sizing problems](#)»
- ▶ We write the ILP model of each variant, in the order of that chapter:
 - ▶ the uncapacitated lot sizing problem;
 - ▶ several products and setup times;
 - ▶ initial stock;
 - ▶ overtime;
 - ▶ unsatisfied demand.
- ▶ Each model keeps the structure of the big-M model of the CLS

The uncapacitated lot sizing problem: variables

- ▶ Given a set $T = \{1, 2, \dots, m\}$ of periods with demands d_t and no binding capacity, we introduce, for every period $t \in T$, the following **integer variables** and **binary variables**:

$$x_t \in \mathbb{Z}_{\geq 0} \text{ (production),}$$

$$s_t \in \mathbb{Z}_{\geq 0} \text{ (inventory at the end of the period),}$$

$$y_t = \begin{cases} 1 & \text{if there is a setup in period } t \\ 0 & \text{otherwise} \end{cases}$$

- ▶ Since $s_m = 0$, the production of period t is at most the demand $D_{t,m}$ of the periods $t, \dots, m$: this is the big-M value of the link between x_t and y_t .

The model ILP_{ULS}

Using these variables, an Integer Linear Programming (ILP) model for the uncapacitated lot sizing problem (ULS) reads as follows:

$$\min \sum_{t \in T} (sc_t y_t + vc_t x_t + hc_t s_t)$$

$$\text{subject to } s_{t-1} + x_t = d_t + s_t, \quad \forall t \in T,$$

$$x_t \leq D_{t,m} y_t, \quad \forall t \in T,$$

$$s_0 = s_m = 0,$$

$$x_t \in \mathbb{Z}_{\geq 0}, \quad \forall t \in T,$$

$$s_t \in \mathbb{Z}_{\geq 0}, \quad \forall t \in T,$$

$$y_t \in \{0, 1\}, \quad \forall t \in T.$$

This ILP model is referred to as the ILP_{ULS} model.

Description of the model ILP_{ULS}

- ▶ The linear objective function minimizes the cost of the setups, of the production and of the holding;
- ▶ constraints $s_{t-1} + x_t = d_t + s_t$ are the flow balancing constraints (m linear constraints);
- ▶ constraints $x_t \leq D_{t,m} y_t$ force a setup in every period with positive production (m linear constraints);
- ▶ constraint $s_0 = s_m = 0$ imposes an empty inventory at the beginning and at the end of the horizon;
- ▶ finally, the domain constraints define the variables of the ILP model.

The ULS and the CLS

- ▶ The ILP_{ULS} model is the ILP_{CLSP} model without the capacity constraints $x_t \leq \text{cap}_t$ and with the big-M values $M_t = D_{t,m}$.
- ▶ The DP algorithm of Wagner and Whitin, in the chapter «DP for the uncapacitated lot sizing problem» later in the course, solves the ULS in polynomial time.

Example: the ILP_{ULS} model (1/3)

We consider the CLS reference instance without the capacities: the demands are $\mathbf{d} = (2, 1, 3, 4, 1)$, the setup costs $sc_t = 50$, the holding costs $hc_t = 10$, and $vc_t = 0$. The big-M values are $D_{t,5} = (11, 9, 8, 5, 1)$, and the ILP_{ULS} model for this ULS instance is:

$$\begin{aligned} \min \quad & 50 y_1 + 50 y_2 + 50 y_3 + 50 y_4 + 50 y_5 + 10 s_1 \\ & + 10 s_2 + 10 s_3 + 10 s_4 + 10 s_5 \\ \text{subject to} \quad & x_1 - s_1 = 2 \\ & s_1 + x_2 - s_2 = 1 \\ & s_2 + x_3 - s_3 = 3 \\ & s_3 + x_4 - s_4 = 4 \\ & s_4 + x_5 - s_5 = 1 \end{aligned}$$

Example: the ILP_{ULS} model (2/3)

$$x_1 - 11 y_1 \leq 0$$

$$x_2 - 9 y_2 \leq 0$$

$$x_3 - 8 y_3 \leq 0$$

$$x_4 - 5 y_4 \leq 0$$

$$x_5 - y_5 \leq 0$$

$$s_5 = 0$$

$$x_1, x_2, x_3, x_4, x_5 \in \mathbb{Z}_{\geq 0}$$

$$s_1, s_2, s_3, s_4, s_5 \in \mathbb{Z}_{\geq 0}$$

$$y_1, y_2, y_3, y_4, y_5 \in \{0, 1\}$$

Example: the ILP_{ULS} model (3/3)

An optimal solution $(\tilde{\mathbf{y}}, \tilde{\mathbf{x}}, \tilde{\mathbf{s}})(\text{ILP}_{\text{ULS}})$ is:

$$\tilde{\mathbf{y}} = (1, 0, 1, 0, 0), \quad \tilde{\mathbf{x}} = (3, 0, 8, 0, 0), \quad \tilde{\mathbf{s}} = (1, 0, 5, 1, 0)$$

- ▶ The optimal value $z(\text{ILP}_{\text{ULS}})$ is $2 \cdot 50 + (1 + 5 + 1) \cdot 10 = 170$, the bound of the shortest-path formulation of the CLS.
- ▶ The optimal solution is not unique: the plan $(3, 0, 3, 5, 0)$, with three setups and the stocks $(1, 0, 0, 1, 0)$, also costs $3 \cdot 50 + 2 \cdot 10 = 170$.

Several products and setup times: data

- ▶ A set P of products shares the resource of each period.
- ▶ We write d_{pt} , sc_{pt} , vc_{pt} and hc_{pt} for the data of product $p \in P$ in period $t \in T$.
- ▶ Producing a unit of p takes the time $vt_{pt} \in \mathbb{Z}_{\geq 1}$, a setup takes the time $st_{pt} \in \mathbb{Z}_{\geq 0}$, and the capacity cap_t is a time.
- ▶ For $j \leq k$ we write $sd_{pjk} = \sum_{u=j}^k d_{pu}$ for the demand of product p in the periods $j, \dots, k$.

Several products and setup times: variables

- ▶ We introduce, for every product $p \in P$ and every period $t \in T$, the following **integer variables** and **binary variables**:

$$x_{pt} \in \mathbb{Z}_{\geq 0} \text{ (production),}$$

$$s_{pt} \in \mathbb{Z}_{\geq 0} \text{ (inventory at the end of the period),}$$

$$y_{pt} = \begin{cases} 1 & \text{if there is a setup of product } p \text{ in period } t \\ 0 & \text{otherwise} \end{cases}$$

- ▶ The inventories at the beginning of the horizon are the constants $s_{p0} = 0$.

The model ILP_{CLSP}^P : objective and constraints

Using these variables, an Integer Linear Programming (ILP) model for the lot sizing problem with several products and setup times (MLS) reads as follows:

$$\min \sum_{p \in P} \sum_{t \in T} (sc_{pt} y_{pt} + vc_{pt} x_{pt} + hc_{pt} s_{pt})$$

$$\text{subject to } s_{p,t-1} + x_{pt} = d_{pt} + s_{pt}, \quad \forall p \in P, \forall t \in T,$$

$$\sum_{p \in P} (st_{pt} y_{pt} + vt_{pt} x_{pt}) \leq cap_t, \quad \forall t \in T,$$

$$x_{pt} \leq M_{pt} y_{pt}, \quad \forall p \in P, \forall t \in T,$$

$$s_{p0} = s_{pm} = 0, \quad \forall p \in P,$$

The model $\text{ILP}_{\text{CLSP}}^P$: domains of the variables

$$x_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T,$$

$$s_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T,$$

$$y_{pt} \in \{0, 1\}, \quad \forall p \in P, \forall t \in T.$$

This ILP model is referred to as the $\text{ILP}_{\text{CLSP}}^P$ model.

Description of the model ILP_{CLSP}^P

- ▶ The linear objective function minimizes the cost of the setups, of the production and of the holding of all the products;
- ▶ constraints $s_{p,t-1} + x_{pt} = d_{pt} + s_{pt}$ are the flow balancing constraints of every product ($|P| m$ linear constraints);
- ▶ constraints $\sum_p (st_{pt} y_{pt} + vt_{pt} x_{pt}) \leq cap_t$ impose that the setup times and the production times of each period do not exceed its capacity (m linear constraints);
- ▶ constraints $x_{pt} \leq M_{pt} y_{pt}$ link the quantities with the setups ($|P| m$ linear constraints);
- ▶ constraints $s_{p0} = s_{pm} = 0$ impose an empty inventory at the beginning and at the end of the horizon;
- ▶ finally, the domain constraints define the variables of the ILP model.

Strong big-M values and products

- ▶ A setup of p in period t leaves the time $cap_t - st_{pt}$ for the production, and $s_{pm} = 0$ bounds the production by the remaining demand. The strong big-M value is

$$M_{pt} = \min \left\{ \left\lfloor \frac{cap_t - st_{pt}}{vt_{pt}} \right\rfloor, sd_{ptm} \right\}.$$

- ▶ The products interact only through the capacity constraints $\sum_p (st_{pt} y_{pt} + vt_{pt} x_{pt}) \leq cap_t$: without them, the model splits into $|P|$ independent single-product models.

Example: the ILP_{CLSP}^P model (1/5)

We consider $|P| = 2$ products and $m = 3$ periods, with capacities $cap_t = 9$, production times $vt_{pt} = 1$, setup times $st_{pt} = 1$, holding costs $hc_{pt} = 5$ and production costs $vc_{pt} = 0$. The setup costs are $sc_{1t} = 30$ and $sc_{2t} = 40$, and the demands are:

period t	product 1			product 2		
	1	2	3	1	2	3
d_{pt}	3	2	4	2	4	1
sd_{ptm}	9	6	4	7	5	1
M_{pt}	8	6	4	7	5	1

Example: the ILP_{CLSP}^P model (2/5)

The ILP_{CLSP}^P model with the strong big-M values for this instance is:

$$\begin{aligned} \min \quad & 30 y_{11} + 30 y_{12} + 30 y_{13} + 40 y_{21} + 40 y_{22} + 40 y_{23} \\ & + 5 s_{11} + 5 s_{12} + 5 s_{13} + 5 s_{21} + 5 s_{22} + 5 s_{23} \end{aligned}$$

$$\begin{aligned} \text{subject to} \quad & x_{11} - s_{11} &= 3 \\ & s_{11} + x_{12} - s_{12} &= 2 \\ & s_{12} + x_{13} - s_{13} &= 4 \\ & x_{21} - s_{21} &= 2 \\ & s_{21} + x_{22} - s_{22} &= 4 \\ & s_{22} + x_{23} - s_{23} &= 1 \end{aligned}$$

Example: the ILP_{CLSP}^P model (3/5)

$$x_{11} + x_{21} + y_{11} + y_{21} \leq 9$$

$$x_{12} + x_{22} + y_{12} + y_{22} \leq 9$$

$$x_{13} + x_{23} + y_{13} + y_{23} \leq 9$$

$$x_{11} - 8 y_{11} \leq 0$$

$$x_{12} - 6 y_{12} \leq 0$$

$$x_{13} - 4 y_{13} \leq 0$$

$$x_{21} - 7 y_{21} \leq 0$$

$$x_{22} - 5 y_{22} \leq 0$$

$$x_{23} - y_{23} \leq 0$$

Example: the $\text{ILP}_{\text{CLSP}}^P$ model (4/5)

$$s_{13} = 0$$

$$s_{23} = 0$$

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \in \mathbb{Z}_{\geq 0}$$

$$s_{11}, s_{12}, s_{13}, s_{21}, s_{22}, s_{23} \in \mathbb{Z}_{\geq 0}$$

$$y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23} \in \{0, 1\}$$

An optimal solution $(\tilde{y}, \tilde{x}, \tilde{s})(\text{ILP}_{\text{CLSP}}^P)$ is:

$$\tilde{y}_{11} = \tilde{y}_{13} = \tilde{y}_{21} = \tilde{y}_{22} = 1, \quad \tilde{y}_{12} = \tilde{y}_{23} = 0,$$

$$\tilde{x}_{11} = 5, \quad \tilde{x}_{13} = 4, \quad \tilde{x}_{21} = 2, \quad \tilde{x}_{22} = 5, \quad \tilde{x}_{12} = \tilde{x}_{23} = 0,$$

$$\tilde{s}_{11} = 2, \quad \tilde{s}_{22} = 1, \quad \text{all the other } \tilde{s}_{pt} = 0$$

Example: the $\text{ILP}_{\text{CLSP}}^P$ model (5/5)

- ▶ The optimal value $z(\text{ILP}_{\text{CLSP}}^P)$ is $2 \cdot 30 + 2 \cdot 40 + (2 + 1) \cdot 5 = 155$, and the optimal solution is unique (Gurobi).
- ▶ Period 1 uses its whole capacity, $5 + 2 + 1 + 1 = 9$.
- ▶ Without the capacity constraints product 2 would be produced only in period 1, with value 140: this plan needs the time $5 + 7 + 1 + 1 = 14 > 9$.

Initial stock: data and variables

- ▶ The data are those of the lot sizing problem with several products and setup times. In addition, the initial inventory of product p is a decision, bought at the cost fc_p per unit.
- ▶ We introduce the integer variables x_{pt} , s_{pt} and the binary variables y_{pt} of the $\text{ILP}_{\text{CLSP}}^P$ model, and, for every product $p \in P$, the following **integer variable**:

$si_p \in \mathbb{Z}_{\geq 0}$ (inventory of product p
at the beginning of the horizon).

The model ILP_{CLSP}^{SI}: objective and constraints

Using these variables, an Integer Linear Programming (ILP) model for the lot sizing problem with initial stock (LS-SI) reads as follows:

$$\begin{aligned} \min \quad & \sum_{p \in P} f_{Cp} si_p + \\ & \sum_{p \in P} \sum_{t \in T} (sc_{pt} y_{pt} + vc_{pt} x_{pt} + hc_{pt} s_{pt}) \end{aligned}$$

$$\text{subject to} \quad si_p + x_{p1} = d_{p1} + s_{p1}, \quad \forall p \in P,$$

$$s_{p,t-1} + x_{pt} = d_{pt} + s_{pt}, \quad \forall p \in P, \forall t \in T \setminus \{1\},$$

$$\sum_{p \in P} (st_{pt} y_{pt} + vt_{pt} x_{pt}) \leq cap_t, \quad \forall t \in T,$$

$$x_{pt} \leq M_{pt} y_{pt}, \quad \forall p \in P, \forall t \in T,$$

$$s_{pm} = 0, \quad \forall p \in P,$$

The model $\text{ILP}_{\text{CLSP}}^{\text{SI}}$: domains of the variables

$$x_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T,$$

$$s_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T,$$

$$si_p \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P,$$

$$y_{pt} \in \{0, 1\}, \quad \forall p \in P, \forall t \in T.$$

- ▶ This ILP model is referred to as the $\text{ILP}_{\text{CLSP}}^{\text{SI}}$ model.
- ▶ It is the $\text{ILP}_{\text{CLSP}}^P$ model with the variables si_p in place of the constants $s_{p0} = 0$, and with their cost in the objective function; the strong big-M values M_{pt} are the same.

Description of the model ILP_{CLSP}^{SI}

- ▶ The linear objective function minimizes the cost of the initial inventory, of the setups, of the production and of the holding;
- ▶ constraints $si_p + x_{p1} = d_{p1} + s_{p1}$ and $s_{p,t-1} + x_{pt} = d_{pt} + s_{pt}$ are the flow balancing constraints of every product, with the initial inventory si_p in the first period ($|P| m$ linear constraints);
- ▶ constraints $\sum_p (st_{pt} y_{pt} + vt_{pt} x_{pt}) \leq cap_t$ impose that the setup times and the production times of each period do not exceed its capacity (m linear constraints); constraints $x_{pt} \leq M_{pt} y_{pt}$ link the quantities with the setups ($|P| m$ linear constraints);
- ▶ constraints $s_{pm} = 0$ impose an empty inventory at the end of the horizon ($|P|$ linear constraints);
- ▶ finally, the domain constraints define the variables of the ILP model.

Example: the ILP^{SI}_{CLSP} model (1/6)

We consider $|P| = 2$ products and $m = 3$ periods, with capacities $cap_1 = 2500$, $cap_2 = 2100$, $cap_3 = 2500$, the initial inventory costs $fc_1 = fc_2 = 3000$ and the setup costs $sc_{pt} = 10000$. The other data are:

period t	product 1			product 2		
	1	2	3	1	2	3
d_{pt}	100	200	10	10	100	150
vt_{pt}	20	10	10	10	5	5
st_{pt}	100	100	100	110	110	110
hc_{pt}	110	100	100	105	105	105
vc_{pt}	1100	1000	1000	900	800	800

Example: the ILP^{SI}_{CLSP} model (2/6)

The big-M values are:

period t	1	2	3	1	2	3
sd_{ptm}	310	210	10	260	250	150
$\lfloor (cap_t - st_{pt})/vt_{pt} \rfloor$	120	200	240	239	398	478
M_{pt}	120	200	10	239	250	150

Example: the ILP_{CLSP}^{SI} model (3/6)

The ILP_{CLSP}^{SI} model with the strong big-M values for this instance is:

$$\begin{aligned} \min \quad & 3000 s_{i_1} + 3000 s_{i_2} + 10000 y_{11} + 10000 y_{12} \\ & + 10000 y_{13} + 10000 y_{21} + 10000 y_{22} + 10000 y_{23} \\ & + 1100 x_{11} + 1000 x_{12} + 1000 x_{13} + 900 x_{21} \\ & + 800 x_{22} + 800 x_{23} + 110 s_{11} + 100 s_{12} \\ & + 100 s_{13} + 105 s_{21} + 105 s_{22} + 105 s_{23} \end{aligned}$$

$$\begin{aligned} \text{subject to} \quad & s_{i_1} + x_{11} - s_{11} &= 100 \\ & s_{11} + x_{12} - s_{12} &= 200 \\ & s_{12} + x_{13} - s_{13} &= 10 \\ & s_{i_2} + x_{21} - s_{21} &= 10 \\ & s_{21} + x_{22} - s_{22} &= 100 \\ & s_{22} + x_{23} - s_{23} &= 150 \end{aligned}$$

Example: the ILP^{SI}_{CLSP} model (4/6)

$$20 x_{11} + 10 x_{21} + 100 y_{11} + 110 y_{21} \leq 2500$$

$$10 x_{12} + 5 x_{22} + 100 y_{12} + 110 y_{22} \leq 2100$$

$$10 x_{13} + 5 x_{23} + 100 y_{13} + 110 y_{23} \leq 2500$$

$$x_{11} - 120 y_{11} \leq 0$$

$$x_{12} - 200 y_{12} \leq 0$$

$$x_{13} - 10 y_{13} \leq 0$$

$$x_{21} - 239 y_{21} \leq 0$$

$$x_{22} - 250 y_{22} \leq 0$$

$$x_{23} - 150 y_{23} \leq 0$$

Example: the ILP^{SI}_{CLSP} model (5/6)

$$s_{13} = 0$$

$$s_{23} = 0$$

$$s_{i_1}, s_{i_2} \in \mathbb{Z}_{\geq 0}$$

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \in \mathbb{Z}_{\geq 0}$$

$$s_{11}, s_{12}, s_{13}, s_{21}, s_{22}, s_{23} \in \mathbb{Z}_{\geq 0}$$

$$y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23} \in \{0, 1\}$$

Example: the ILP_{CLSP}^{SI} model (6/6)

An optimal solution of the ILP_{CLSP}^{SI} model is:

$$\tilde{x}_{11} = 59, \quad \tilde{x}_{12} = 200, \quad \tilde{x}_{13} = 10,$$

$$\tilde{x}_{21} = 110, \quad \tilde{x}_{22} = 0, \quad \tilde{x}_{23} = 150,$$

$$\tilde{s}i_1 = 41, \quad \tilde{s}i_2 = 0, \quad \tilde{s}_{21} = 100, \quad \text{all the other } \tilde{s}_{pt} = 0,$$

with a setup in every period except for product 2 in period 2.

- ▶ The optimal value $z(\text{ILP}_{\text{CLSP}}^{\text{SI}})$ is 677 400, and the optimal solution is unique (Gurobi).
- ▶ Product 1 starts with 41 units bought at the beginning of the horizon, because period 1 has no time to produce all its demand.

The LP relaxation LP_{CLSP}^{SI} : objective and constraints

To obtain an LP relaxation of the ILP_{CLSP}^{SI} model, we relax the integrality constraints of the variables into non-negativity constraints and the bounds $y_{pt} \leq 1$. The obtained LP model reads as follows:

$$\begin{aligned} \min \quad & \sum_{p \in P} f_{cp} s_{ip} + \\ & \sum_{p \in P} \sum_{t \in T} (sc_{pt} y_{pt} + vc_{pt} x_{pt} + hc_{pt} s_{pt}) \end{aligned}$$

$$\text{subject to} \quad s_{ip} + x_{p1} = d_{p1} + s_{p1}, \quad \forall p \in P,$$

$$s_{p,t-1} + x_{pt} = d_{pt} + s_{pt}, \quad \forall p \in P, \forall t \in T \setminus \{1\},$$

$$\sum_{p \in P} (st_{pt} y_{pt} + vt_{pt} x_{pt}) \leq cap_t, \quad \forall t \in T,$$

$$x_{pt} \leq M_{pt} y_{pt}, \quad \forall p \in P, \forall t \in T,$$

$$s_{pm} = 0, \quad \forall p \in P,$$

The LP relaxation LP_{CLSP}^{SI} : bounds and domains

$$y_{pt} \leq 1, \quad \forall p \in P, \forall t \in T,$$

$$x_{pt} \geq 0, \quad \forall p \in P, \forall t \in T,$$

$$s_{pt} \geq 0, \quad \forall p \in P, \forall t \in T,$$

$$y_{pt} \geq 0, \quad \forall p \in P, \forall t \in T,$$

$$s_{ip} \geq 0, \quad \forall p \in P.$$

This LP model is referred to as the LP_{CLSP}^{SI} model.

A shortest-path formulation with initial stock

- ▶ In the shortest-path formulation each product has its own network, with an additional continuous variable $w_{pt} \geq 0$, the arc from node 0 to node t , for the fraction of the initial inventory that covers the demands of the first t periods.
- ▶ The costs of the arcs are

$$cv_{ptk} = vc_{pt} sd_{ptk} + \sum_{s=t+1}^k \sum_{u=t}^{s-1} hc_{pu} d_{ps},$$

$$ci_{pt} = fc_p sd_{p1t} + \sum_{s=2}^t \sum_{u=1}^{s-1} hc_{pu} d_{ps}.$$

The model $MIP_{CLSP}^{SP,SI}$: objective and constraints

Using these variables, a MIP model for the lot sizing problem with initial stock (LS-SI) reads as follows:

$$\min \sum_{p \in P} \sum_{t \in T} (s_{cpt} y_{pt} + c_{ipt} w_{pt}) +$$

$$\sum_{p \in P} \sum_{t \in T} \sum_{k=t}^m c_{vptk} z_{vptk}$$

$$\text{subject to } \sum_{k=1}^m (w_{pk} + z_{vp1k}) = 1, \quad \forall p \in P,$$

$$w_{p,t-1} + \sum_{u=1}^{t-1} z_{vp,u,t-1} - \sum_{k=t}^m z_{vptk} = 0, \quad \forall p \in P, \forall t \in T \setminus \{1\},$$

$$\sum_{k=t}^m z_{vptk} \leq y_{pt}, \quad \forall p \in P, \forall t \in T,$$

$$\sum_{p \in P} s_{tpt} y_{pt} + \sum_{p \in P} \sum_{k=t}^m v_{tpt} s_{dptk} z_{vptk} \leq cap_t, \quad \forall t \in T,$$

The model $MIP_{CLSP}^{SP,SI}$: integrality and domains

$$\sum_{k=t}^m sd_{ptk} zv_{ptk} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T,$$

$$\sum_{t \in T} sd_{p1t} w_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P,$$

$$zv_{ptk} \geq 0, \quad \forall p \in P, \forall t, k \in T, k \geq t,$$

$$w_{pt} \geq 0, \quad \forall p \in P, \forall t \in T,$$

$$y_{pt} \in \{0, 1\}, \quad \forall p \in P, \forall t \in T.$$

- ▶ This MIP model is referred to as the $MIP_{CLSP}^{SP,SI}$ model.
- ▶ Relaxing constraints $y_{pt} \in \{0, 1\}$ into $0 \leq y_{pt} \leq 1$ and removing the integrality constraints of the quantities gives the LP model referred to as the $LP_{CLSP}^{SP,SI}$ model.

Description of the model $MIP^{SP,SI}_{CLSP}$

- ▶ The linear objective function minimizes the cost of the setups, of the initial inventory and of the arcs of the plans of all the products;
- ▶ constraints $\sum_k (w_{pk} + z v_{p1k}) = 1$ and $w_{p,t-1} + \sum_u z v_{p,u,t-1} - \sum_k z v_{ptk} = 0$ are the flow constraints of the network of every product, with the arcs w_{pt} of the initial inventory leaving node 0 ($|P| m$ linear constraints);
- ▶ constraints $\sum_k z v_{ptk} \leq y_{pt}$ force the setup of product p in period t if it is produced ($|P| m$ linear constraints), and constraints $\sum_p s t_{pt} y_{pt} + \sum_p \sum_k v t_{pt} s d_{ptk} z v_{ptk} \leq cap_t$ impose the capacities, setup times included (m linear constraints);
- ▶ the integrality constraints keep integer the quantities and the initial inventories;
- ▶ finally, the domain constraints define the variables of the MIP model.

Integer quantities

- ▶ The quantity produced is $x_{pt} = \sum_{k=t}^m sd_{ptk} zv_{ptk}$, and the initial inventory is $si_p = \sum_{t \in T} sd_{p1t} w_{pt}$: the integrality constraints of the quantities keep them integer, as in the ILP_{CLSP}^{SI} model.
- ▶ With one product and $vt_t = 1$ these constraints are not needed: once the setups are fixed, what remains is a minimum cost flow problem, which has an integer optimal solution.
- ▶ With several products and production times $vt_{pt} > 1$, a fractional quantity can use the capacity better. Without the integrality constraints of the quantities the model is only a relaxation of the problem, and its optimal value can be smaller than $z(ILP_{CLSP}^{SI})$ (next example).

Example: lower bounds with two products (1/2)

- ▶ We consider the instance of the $\text{ILP}_{\text{CLSP}}^{\text{SI}}$ example.
- ▶ Ignoring the setups (in the costs and in the capacities), the optimal value is 575400, attained by the integer solution

$$\begin{aligned}x_{11} &= 120, & x_{12} &= 160, & x_{13} &= 10, \\x_{21} &= 10, & x_{22} &= 100, & x_{23} &= 150, & s_{11} &= 40, & s_{i1} &= 20.\end{aligned}$$

Example: lower bounds with two products (2/2)

The lower bounds and the optimal value of the formulations are:

Model	Value
Without setups	575 400.00
LP_{CLSP}^{SI} with $M_{pt} = sd_{ptm}$	638 882.20
LP_{CLSP}^{SI} with the strong M_{pt}	651 267.37
$LP_{CLSP}^{SP,SI}$	668 928.57
$MIP_{CLSP}^{SP,SI}$ without integrality constraints	676 450.00
ILP_{CLSP}^{SI} and $MIP_{CLSP}^{SP,SI}$ (integer optimum)	677 400

Without the integrality constraints of the quantities the shortest-path model reaches 676 450 with the same setups and the fractional quantities $x_{11} = 59.5$ and $s_{i1} = 40.5$: period 1 then uses its whole capacity, $100 + 20 \cdot 59.5 + 110 + 10 \cdot 110 = 2500$.

Overtime: data and variables

- ▶ The data are those of the lot sizing problem with several products and setup times, but the time of each period is not a fixed capacity: in period t the working time in normal hours costs cw_t per unit, up to cap_t^N , and the overtime costs co_t per unit, up to cap_t^O .
- ▶ We introduce the integer variables x_{pt} , s_{pt} and the binary variables y_{pt} of the ILP_{CLSP}^P model, and, for every period $t \in T$, the following **integer variables**:

$$\begin{aligned}\mu_t &\in \mathbb{Z}_{\geq 0} \text{ (working time in normal hours),} \\ \nu_t &\in \mathbb{Z}_{\geq 0} \text{ (overtime).}\end{aligned}$$

Since the times vt_{pt} and st_{pt} are integer, the working times can be taken integer.

The model ILP_{CLSP}^{OT}: objective and constraints

Using these variables, an Integer Linear Programming (ILP) model for the lot sizing problem with overtime (LS-OT) reads as follows:

$$\begin{aligned} \min \quad & \sum_{p \in P} \sum_{t \in T} (s_{cpt} y_{pt} + v_{cpt} x_{pt} + h_{cpt} s_{pt}) + \\ & \sum_{t \in T} (cw_t \mu_t + co_t \nu_t) \end{aligned}$$

$$\text{subject to} \quad s_{p,t-1} + x_{pt} = d_{pt} + s_{pt}, \quad \forall p \in P, \forall t \in T,$$

$$\sum_{p \in P} (st_{pt} y_{pt} + vt_{pt} x_{pt}) \leq \mu_t + \nu_t, \quad \forall t \in T,$$

$$\mu_t \leq cap_t^N, \quad \forall t \in T,$$

$$\nu_t \leq cap_t^O, \quad \forall t \in T,$$

$$x_{pt} \leq M_{pt} y_{pt}, \quad \forall p \in P, \forall t \in T,$$

$$s_{p0} = s_{pm} = 0, \quad \forall p \in P,$$

The model $\text{ILP}_{\text{CLSP}}^{\text{OT}}$: domains of the variables

$$x_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T,$$

$$s_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T,$$

$$\mu_t \in \mathbb{Z}_{\geq 0}, \quad \forall t \in T,$$

$$\nu_t \in \mathbb{Z}_{\geq 0}, \quad \forall t \in T,$$

$$y_{pt} \in \{0, 1\}, \quad \forall p \in P, \forall t \in T.$$

This ILP model is referred to as the $\text{ILP}_{\text{CLSP}}^{\text{OT}}$ model.

Description of the model ILP_{CLSP}^{OT}

- ▶ The linear objective function adds to the costs of the ILP_{CLSP}^P model the cost of the working time in normal hours and in overtime;
- ▶ constraints $s_{p,t-1} + x_{pt} = d_{pt} + s_{pt}$ are the flow balancing constraints of every product ($|P| m$ linear constraints);
- ▶ constraints $\sum_p (st_{pt} y_{pt} + vt_{pt} x_{pt}) \leq \mu_t + \nu_t$ impose that the setup times and the production times of each period are covered by the working time, in normal hours and in overtime (m linear constraints);
- ▶ constraints $\mu_t \leq cap_t^N$ and $\nu_t \leq cap_t^O$ bound the two working times (m linear constraints each);
- ▶ constraints $x_{pt} \leq M_{pt} y_{pt}$ link the quantities with the setups ($|P| m$ linear constraints);
- ▶ constraints $s_{p0} = s_{pm} = 0$ impose an empty inventory at the beginning and at the end of the horizon;
- ▶ finally, the domain constraints define the variables of the ILP model.

Strong big-M values with overtime

- ▶ The time available in period t is at most $cap_t^N + cap_t^O$, so the strong big-M value is

$$M_{pt} = \min \left\{ \left\lfloor \frac{cap_t^N + cap_t^O - st_{pt}}{vt_{pt}} \right\rfloor, sd_{ptm} \right\}.$$

- ▶ With $co_t > cw_t$ an optimal solution uses overtime only when the normal hours of the period are exhausted.

Example: the ILP^{OT}_{CLSP} model (1/5)

- ▶ We consider the CLS reference instance, with one product ($|P| = 1$, and we drop the index p), production times $vt_t = 1$ and setup times $st_t = 0$.
- ▶ The capacity $cap_t = 4$ becomes the normal hours $cap_t^N = 4$, at the cost $cw_t = 1$, and one unit of overtime is available in every period: $cap_t^O = 1$, at the cost $co_t = 6$.
- ▶ The big-M values are $M_t = \min\{5, D_{t,5}\}$, that is $(5, 5, 5, 5, 1)$.

The ILP^{OT}_{CLSP} model for this instance is:

$$\begin{aligned} \min \quad & 50 y_1 + 50 y_2 + 50 y_3 + 50 y_4 + 50 y_5 + 10 s_1 \\ & + 10 s_2 + 10 s_3 + 10 s_4 + 10 s_5 + \mu_1 + \mu_2 \\ & + \mu_3 + \mu_4 + \mu_5 + 6 \nu_1 + 6 \nu_2 + 6 \nu_3 \\ & + 6 \nu_4 + 6 \nu_5 \end{aligned}$$

Example: the ILP^{OT}_{CLSP} model (2/5)

$$\text{subject to } x_1 - s_1 = 2$$

$$s_1 + x_2 - s_2 = 1$$

$$s_2 + x_3 - s_3 = 3$$

$$s_3 + x_4 - s_4 = 4$$

$$s_4 + x_5 - s_5 = 1$$

$$x_1 - \mu_1 - \nu_1 \leq 0$$

$$x_2 - \mu_2 - \nu_2 \leq 0$$

$$x_3 - \mu_3 - \nu_3 \leq 0$$

$$x_4 - \mu_4 - \nu_4 \leq 0$$

$$x_5 - \mu_5 - \nu_5 \leq 0$$

Example: the ILP^{OT}_{CLSP} model (3/5)

$$x_1 - 5y_1 \leq 0$$

$$x_2 - 5y_2 \leq 0$$

$$x_3 - 5y_3 \leq 0$$

$$x_4 - 5y_4 \leq 0$$

$$x_5 - y_5 \leq 0$$

$$\mu_1 \leq 4$$

$$\mu_2 \leq 4$$

$$\mu_3 \leq 4$$

$$\mu_4 \leq 4$$

$$\mu_5 \leq 4$$

Example: the ILP^{OT}_{CLSP} model (4/5)

$$\begin{array}{ll}\nu_1 & \leq 1 \\ \nu_2 & \leq 1 \\ \nu_3 & \leq 1 \\ \nu_4 & \leq 1 \\ \nu_5 & \leq 1 \\ s_5 & = 0 \\ x_1, x_2, x_3, x_4, x_5 & \in \mathbb{Z}_{\geq 0} \\ s_1, s_2, s_3, s_4, s_5 & \in \mathbb{Z}_{\geq 0} \\ \mu_1, \mu_2, \mu_3, \dots, \nu_5 & \in \mathbb{Z}_{\geq 0} \\ y_1, y_2, y_3, y_4, y_5 & \in \{0, 1\}\end{array}$$

Example: the $\text{ILP}_{\text{CLSP}}^{\text{OT}}$ model (5/5)

An optimal solution of the $\text{ILP}_{\text{CLSP}}^{\text{OT}}$ model is:

$$\begin{aligned}\tilde{\mathbf{y}} &= (1, 0, 1, 1, 0), & \tilde{\mathbf{x}} &= (3, 0, 3, 5, 0), & \tilde{\mathbf{s}} &= (1, 0, 0, 1, 0), \\ \tilde{\boldsymbol{\mu}} &= (3, 0, 3, 4, 0), & \tilde{\boldsymbol{\nu}} &= (0, 0, 0, 1, 0)\end{aligned}$$

- ▶ The optimal value $z(\text{ILP}_{\text{CLSP}}^{\text{OT}})$ is $3 \cdot 50 + 2 \cdot 10 + 10 \cdot 1 + 1 \cdot 6 = 186$, and the optimal solution is unique (Gurobi).
- ▶ One unit of overtime in period 4, at the cost 6, replaces one unit stored from period 3 to period 4, at the cost 10: the optimal plan $(3, 0, 4, 4, 0)$ of the CLS costs 191 here.

Unsatisfied demand: data and variables

- ▶ The data are those of the lot sizing problem with several products and setup times. In addition, a part of the demand d_{pt} may be left unsatisfied, at the penalty cu_{pt} per unit.
- ▶ We introduce the integer variables x_{pt} , s_{pt} and the binary variables y_{pt} of the $\text{ILP}_{\text{CLSP}}^P$ model, and, for every product $p \in P$ and every period $t \in T$, the following **integer variable**:
 $\varepsilon_{pt} \in \mathbb{Z}_{\geq 0}$ (demand of product p in period t left unsatisfied).

The model ILP_{CLSP}^{UD} : objective and constraints

Using these variables, an Integer Linear Programming (ILP) model for the lot sizing problem with unsatisfied demand (LS-UD) reads as follows:

$$\begin{aligned} \min \quad & \sum_{p \in P} \sum_{t \in T} (sc_{pt} y_{pt} + vc_{pt} x_{pt} + hc_{pt} s_{pt}) + \\ & \sum_{p \in P} \sum_{t \in T} cu_{pt} \varepsilon_{pt} \end{aligned}$$

$$\text{subject to} \quad s_{p,t-1} + x_{pt} = d_{pt} - \varepsilon_{pt} + s_{pt}, \quad \forall p \in P, \forall t \in T,$$

$$\sum_{p \in P} (st_{pt} y_{pt} + vt_{pt} x_{pt}) \leq cap_t, \quad \forall t \in T,$$

$$x_{pt} \leq M_{pt} y_{pt}, \quad \forall p \in P, \forall t \in T,$$

$$\varepsilon_{pt} \leq d_{pt}, \quad \forall p \in P, \forall t \in T,$$

$$s_{p0} = s_{pm} = 0, \quad \forall p \in P,$$

The model $\text{ILP}_{\text{CLSP}}^{\text{UD}}$: domains of the variables

$$x_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T,$$

$$s_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T,$$

$$\varepsilon_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T,$$

$$y_{pt} \in \{0, 1\}, \quad \forall p \in P, \forall t \in T.$$

This ILP model is referred to as the $\text{ILP}_{\text{CLSP}}^{\text{UD}}$ model.

Description of the model $\text{ILP}_{\text{CLSP}}^{\text{UD}}$

- ▶ The linear objective function adds to the costs of the $\text{ILP}_{\text{CLSP}}^P$ model the penalty of the unsatisfied demand;
- ▶ constraints $s_{p,t-1} + x_{pt} = d_{pt} - \varepsilon_{pt} + s_{pt}$ are the flow balancing constraints, with the demand reduced by its unsatisfied part ($|P| m$ linear constraints);
- ▶ constraints $\sum_p (st_{pt} y_{pt} + vt_{pt} x_{pt}) \leq \text{cap}_t$ are the capacity constraints and constraints $x_{pt} \leq M_{pt} y_{pt}$ link the quantities with the setups (m and $|P| m$ linear constraints);
- ▶ constraints $\varepsilon_{pt} \leq d_{pt}$ ensure that the unsatisfied demand does not exceed the demand ($|P| m$ linear constraints);
- ▶ constraints $s_{p0} = s_{pm} = 0$ impose an empty inventory at the beginning and at the end of the horizon;
- ▶ finally, the domain constraints define the variables of the ILP model.

Big-M values with unsatisfied demand

- ▶ The big-M values of the $\text{ILP}_{\text{CLSP}}^P$ model remain valid, since the production can only decrease.
- ▶ With $\varepsilon_{pt} = 0$ for every product and period the model is the $\text{ILP}_{\text{CLSP}}^P$ model.

Example: the $\text{ILP}_{\text{CLSP}}^{\text{UD}}$ model (1/4)

We consider the CLS reference instance, with one product ($|P| = 1$, and we drop the index p), production times $vt_t = 1$, setup times $st_t = 0$ and the strong big-M values $M_t = (4, 4, 4, 4, 1)$. Each unit of unsatisfied demand costs $cu_t = 18$. The $\text{ILP}_{\text{CLSP}}^{\text{UD}}$ model for this instance is:

$$\begin{aligned} \min \quad & 50 y_1 + 50 y_2 + 50 y_3 + 50 y_4 + 50 y_5 + 10 s_1 \\ & + 10 s_2 + 10 s_3 + 10 s_4 + 10 s_5 + 18 \varepsilon_1 + 18 \varepsilon_2 \\ & + 18 \varepsilon_3 + 18 \varepsilon_4 + 18 \varepsilon_5 \end{aligned}$$

Example: the ILP^{UD}_{CLSP} model (2/4)

$$\begin{array}{llll} \text{subject to} & x_1 - s_1 + \varepsilon_1 & = & 2 \\ & s_1 + x_2 - s_2 + \varepsilon_2 & = & 1 \\ & s_2 + x_3 - s_3 + \varepsilon_3 & = & 3 \\ & s_3 + x_4 - s_4 + \varepsilon_4 & = & 4 \\ & s_4 + x_5 - s_5 + \varepsilon_5 & = & 1 \\ & x_1 & \leq & 4 \\ & x_2 & \leq & 4 \\ & x_3 & \leq & 4 \\ & x_4 & \leq & 4 \\ & x_5 & \leq & 4 \end{array}$$

Example: the ILP^{UD}_{CLSP} model (3/4)

$$x_1 - 4y_1 \leq 0$$

$$x_2 - 4y_2 \leq 0$$

$$x_3 - 4y_3 \leq 0$$

$$x_4 - 4y_4 \leq 0$$

$$x_5 - y_5 \leq 0$$

$$\varepsilon_1 \leq 2$$

$$\varepsilon_2 \leq 1$$

$$\varepsilon_3 \leq 3$$

$$\varepsilon_4 \leq 4$$

$$\varepsilon_5 \leq 1$$

Example: the $\text{ILP}_{\text{CLSP}}^{\text{UD}}$ model (4/4)

$$s_5 = 0$$

$$x_1, x_2, x_3, x_4, x_5 \in \mathbb{Z}_{\geq 0}$$

$$s_1, s_2, s_3, s_4, s_5 \in \mathbb{Z}_{\geq 0}$$

$$\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5 \in \mathbb{Z}_{\geq 0}$$

$$y_1, y_2, y_3, y_4, y_5 \in \{0, 1\}$$

An optimal solution of the $\text{ILP}_{\text{CLSP}}^{\text{UD}}$ model is:

$$\tilde{\mathbf{y}} = (0, 0, 1, 1, 0), \quad \tilde{\mathbf{x}} = (0, 0, 3, 4, 0),$$

$$\tilde{\mathbf{s}} = (0, 0, 0, 0, 0), \quad \tilde{\boldsymbol{\varepsilon}} = (2, 1, 0, 0, 1)$$

Example: the $\text{ILP}_{\text{CLSP}}^{\text{UD}}$ model, the optimal value

- ▶ The optimal value $z(\text{ILP}_{\text{CLSP}}^{\text{UD}})$ is $2 \cdot 50 + 4 \cdot 18 = 172$, and the optimal solution is unique (Gurobi).
- ▶ The demands of the periods 1, 2 and 5 are not satisfied: their 4 units cost 72, and the plan saves one setup and all the holding costs of the optimal plan $(3, 0, 4, 4, 0)$ of the CLS, of cost 180.

Summary (1/2)

The uncapacitated lot sizing problem:

- ▶ The big-M model without capacities and with $M_t = D_{t,m}$
- ▶ On the reference instance its optimal value is 170, the bound of the shortest-path formulation of the CLS

Several products and setup times:

- ▶ The products share the capacity with their setup times; they interact only through the capacity constraints
- ▶ Strong big-M: $M_{pt} = \min\{\lfloor (cap_t - st_{pt})/vt_{pt} \rfloor, sd_{ptm}\}$

Summary (2/2)

Initial stock: the variables sj_p , at the cost fc_p ; with several products the shortest-path model needs the integrality of the quantities ($676\,450 < 677\,400$)

Overtime: the working times μ_t and ν_t replace the fixed capacity; the big-M values use $cap_t^N + cap_t^O$; on the reference instance the optimal value is 186

Unsatisfied demand: the variables $\varepsilon_{pt} \leq d_{pt}$, with the penalty cu_{pt} ; on the reference instance the optimal value is 172

Typical mistakes

Introduction: typical mistakes

- ▶ We collect the mistakes that are most frequent when writing the lot sizing models
- ▶ They concern the big-M values, the capacity constraints, the costs of the arcs and the variants
- ▶ Each one comes with the reason why it is a mistake

Observation 1: Typical mistakes (1/3)

- ▶ ***Taking a big-M value that is too small.*** A value smaller than the production of a feasible plan cuts off that plan: the values $D_{t,m}$ and cap_t are both valid.
- ▶ ***Taking a big-M value that is too large.*** Every valid value gives a correct model, but a large value gives a weak relaxation: the setup variables become almost zero in the LP solutions, as in the bound 99.65 of the reference instance.
- ▶ ***Forgetting the setup times in the capacity constraints.*** With several products the constraints contain $st_{pt} y_{pt}$ besides $vt_{pt} x_{pt}$.

Observation: Typical mistakes (2/3)

- ▶ **Counting the holding cost of the demand of the production period.** In cv_{tk} the units for period t are not stored: the holding sum starts from the demand of period $t + 1$.
- ▶ **Believing that the shortest-path formulation needs binary paths.** The variables zv_{tk} are continuous: a feasible plan that is not a path is a combination of paths.
- ▶ **Forgetting the integrality of the quantities with several products.** With production times $vt_{pt} > 1$ the continuous variables zv_{ptk} can give fractional quantities: the shortest-path model needs the integrality constraints of the quantities, otherwise it is only a relaxation ($676\,450 < 677\,400$).

Observation: Typical mistakes (3/3)

- ▶ *Keeping the fixed capacity in the model with overtime.* The time of period t is bounded by the working time $\mu_t + \nu_t$, and the big- M values use $cap_t^N + cap_t^O$.
- ▶ *Leaving the unsatisfied demand unbounded.* Without $\varepsilon_{pt} \leq d_{pt}$ the balance constraints would allow a negative net demand, that is, inventory created from nothing at the price of the penalty.

Summary

Big-M values:

- ▶ Too small cuts off feasible plans; too large gives a weak relaxation

Shortest-path formulation:

- ▶ The holding sum of cv_{tk} starts from period $t + 1$; the variables zv_{tk} are continuous
- ▶ With several products the integrality of the quantities is needed

Variants:

- ▶ Setup times in the capacity constraints; working time $\mu_t + \nu_t$ with overtime; $\varepsilon_{pt} \leq d_{pt}$

In summary

Introduction: in summary

- ▶ We collect the key ideas of the chapter
- ▶ We close with the dashboard of the lower bounds computed on the reference instance

Summary (1/3)

- ▶ Without the setups the CLS is a simple flow problem; the setups need the binary variables y_t and a link with the quantities.
- ▶ In the big-M formulation the link $x_t \leq M_t y_t$ is correct for every valid M_t , but the LP bound improves when M_t decreases: the strong value is $\min\{cap_t, D_{t,m}\}$.
- ▶ The shortest-path formulation writes the plans as paths in a network, with the link $\sum_k z v_{tk} \leq y_t$ and no big-M value: its relaxation is the strongest.

Summary (2/3)

- ▶ The two formulations have the same optimal value; they differ in the lower bound and in the size of the model.
- ▶ The ULS model is the big-M model without capacities and with $M_t = D_{t,m}$: on the reference instance its optimal value is 170.
- ▶ The variants keep the structure of the big-M model: several products share the capacity with their setup times, the initial stock adds the variables si_p , overtime adds the working times μ_t, ν_t , and the unsatisfied demand adds the variables ε_{pt} . On the reference instance, overtime gives 186 and unsatisfied demand 172.

Summary (3/3)

Dashboard of the reference instance. Lower bounds on the optimal value computed in this chapter:

Bound	Value	From
Without setups	0	lot-for-lot plan
Big-M, $M_t = D_{t,m}$	99.65	$z(\text{LP}_{\text{CLSP}}^{\text{dem}}) = \frac{9865}{99}$
Big-M, $M_t = \text{cap}_t$	137.5	$z(\text{LP}_{\text{CLSP}}^{\text{cap}})$
Strong big-M	157.5	$z(\text{LP}_{\text{CLSP}}^{\text{strong}})$
Shortest path	170	$z(\text{LP}_{\text{CLSP}}^{\text{SP}})$
Optimal value	180	plan (3, 0, 4, 4, 0)