

Formulations of knapsack problems



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Learning objectives

At the end of this chapter you will be able to:

- ▶ write the ILP model of the 0/1 Knapsack Problem and explain each of its constraints;
- ▶ write the ILP models of the SSP, the BKP, the UKP, the MDKP, the MKP, the MCKP and the KPS, and write them on a numerical instance;
- ▶ write the two LP relaxations and their duals, and compute their optimal solutions with closed formulas;
- ▶ compute the split item, the residual capacity and the value $z(\text{LP}_{\text{KP}})$;
- ▶ compute the Dantzig and the Martello-Toth upper bounds, rounding them correctly;
- ▶ explain why $z(\text{LP}_{\text{KP}})$ is never larger than twice $z(\text{ILP}_{\text{KP}})$.

Prerequisites

- ▶ From the module *Linear Programming*, the chapter «[Duality theory](#)»: the dual of an LP problem, weak and strong duality.
- ▶ From the module *Integer linear programming*, the chapters «[Integer linear programming](#)» and «[Primal and dual bounds](#)»: ILP problems, LP relaxations and upper bounds.
- ▶ From this module, the chapter «[Knapsack problems](#)»: the 0/1 knapsack problem and its variants, their assumptions and their reference instances.

Overview of the family

Introduction: overview of the family

- ▶ In this chapter we write an ILP model for each knapsack problem: the 0/1 knapsack problem and its variants
- ▶ For the 0/1 knapsack problem we also study two LP relaxations, their duals and two upper bounds
- ▶ First, we see what all the knapsack models have in common: the same variables, the same constraint, the same objective

Variables, constraints and objective

- ▶ The ILP models of the knapsack problems use one variable x_j for each item $j \in \mathcal{I}$: a **binary variable** when the item is selected or not, an **integer variable** when it counts the selected copies of the item.
- ▶ Each capacity gives one **knapsack constraint** on the total weight of the selected items, and the objective function maximizes their total profit.
- ▶ The members of the family add an index to the variables (the knapsack k of item j in the MKP), some constraints on the classes of items (MCKP, KPS) or binary setup variables y_h (KPS).

The common pattern of the knapsack models

The common pattern of the knapsack models is

$$\max \left\{ \sum_{j \in \mathcal{J}} p_j x_j : \sum_{j \in \mathcal{J}} w_j x_j \leq c, \quad x_j \in D_j \quad \forall j \in \mathcal{J} \right\},$$

where the domain D_j is $\{0, 1\}$, $\{0, 1, \dots, b_j\}$ or $\mathbb{Z}_{\geq 0}$.

The members of the family change the domains and add capacities, knapsacks or classes.

The members of the family

The table lists the members of the family, with the name of their ILP model; the following sections write the models in this order.

Problem	Model	Variables
KP	ILP _{KP}	binary x_j
SSP	ILP _{SSP}	binary x_j
BKP	ILP _{BKP}	integer $x_j \in \{0, 1, \dots, b_j\}$
UKP	ILP _{UKP}	integer $x_j \in \mathbb{Z}_{\geq 0}$
MDKP	ILP _{MDKP}	binary x_j , one constraint per capacity
MKP	ILP _{MKP}	binary x_{jk} , item j in knapsack k
MCKP	ILP _{MCKP}	binary x_j , one item per class
KPS	ILP _{KPS}	binary x_j and setup variables y_h

Summary

The family of knapsack models:

- ▶ One variable x_j per item: binary, or integer when it counts the copies of the item
- ▶ One knapsack constraint per capacity; the objective maximizes the total profit

The members of the family:

- ▶ They change the domains D_j and add capacities, knapsacks or classes
- ▶ The KPS also adds binary setup variables y_h

An ILP model for the 0/1 Knapsack Problem

Introduction: an ILP model for the KP

- ▶ The 0/1 Knapsack Problem and its reference instance are defined in the chapter «[Knapsack problems](#)»
- ▶ In this section we write its ILP model: the variables, the objective function and the constraint
- ▶ The whole chapter is built on this model: the relaxations, the duals and the upper bounds start from it

Decision variables

We introduce the following n **binary variables**:

$$x_j = \begin{cases} 1 & \text{if item } j \text{ is selected in the knapsack} \\ 0 & \text{otherwise} \end{cases} \quad \forall j \in \underbrace{\{1, 2, \dots, n\}}_{=\mathcal{I}}.$$

The ILP_{KP} model

Using these variables, an Integer Linear Programming (ILP) model for the 0/1 Knapsack Problem (KP) reads as follows:

$$\begin{aligned} \max \quad & \sum_{j \in \mathcal{J}} p_j x_j \\ \text{subject to} \quad & \sum_{j \in \mathcal{J}} w_j x_j \leq c, \\ & x_j \in \{0, 1\}, \quad \forall j \in \mathcal{J}. \end{aligned}$$

This ILP model is referred to as the ILP_{KP} model.

Description of the ILP_{KP} model

Description of the objective function and constraints of the ILP_{KP} model:

- ▶ the linear objective function maximizes the total profit of the items selected in the knapsack;
- ▶ the capacity constraint ensures that the total weight of the selected items does not exceed the knapsack capacity (one linear constraint);
- ▶ finally, the constraints $x_j \in \{0, 1\}$ define the variables of the ILP model.

Example (1/2)

We consider a knapsack of capacity $c = 9$ and a set $\mathcal{I} = \{1, 2, 3, 4, 5, 6\}$ of 6 items ($n = 6$), with the following profits and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
p_j	6	5	8	9	6	7
w_j	2	3	6	7	5	9

This is the KP instance of the chapter «Knapsack problems», and in this chapter we refer to it as the **reference instance**.

Example (2/2)

The ILP_{KP} model for this KP instance is:

$$\begin{array}{ll}\max & 6x_1 + 5x_2 + 8x_3 + 9x_4 + 6x_5 + 7x_6 \\ \text{subject to} & 2x_1 + 3x_2 + 6x_3 + 7x_4 + 5x_5 + 9x_6 \leq 9 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \in \{0, 1\}\end{array}$$

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{KP}})$ is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 1, \quad \tilde{x}_5 = 0, \quad \tilde{x}_6 = 0$$

The optimal value $z(\text{ILP}_{\text{KP}})$ is 15. The associated optimal solution of the KP is $\tilde{S} = \{1, 4\}$, and it is the unique optimal solution.

Summary

The ILP_{KP} model:

- ▶ n binary variables x_j : item j is selected in the knapsack or not
- ▶ The objective maximizes the total profit; one capacity constraint
- ▶ The constraints $x_j \in \{0, 1\}$ define the variables

In the reference instance:

- ▶ The unique optimal solution is $\tilde{S} = \{1, 4\}$, with $z(\text{ILP}_{\text{KP}}) = 15$

A first Linear Programming relaxation

Introduction: a first LP relaxation

- ▶ Relaxing the constraints $x_j \in \{0, 1\}$ of the ILP_{KP} model gives an LP model: its optimal value is an upper bound on $z(\text{ILP}_{\text{KP}})$
- ▶ In this section we write a first LP relaxation and its dual model
- ▶ We compute an optimal primal solution and an optimal dual solution with closed formulas, and we prove their optimality with LP strong duality

The LP-0_{KP} model

To obtain a Linear Programming (LP) relaxation of the ILP_{KP} model, we relax the constraints $x_j \in \{0, 1\}$, $\forall j \in \mathcal{J}$, with the following ones:

$$x_j \geq 0, \quad \forall j \in \mathcal{J}.$$

This LP model is referred to as the LP-0_{KP} model.

Example (1/2)

The LP-0_{KP} model for the reference KP instance is:

$$\begin{array}{ll}\max & 6x_1 + 5x_2 + 8x_3 + 9x_4 + 6x_5 + 7x_6 \\ \text{subject to} & 2x_1 + 3x_2 + 6x_3 + 7x_4 + 5x_5 + 9x_6 \leq 9 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0\end{array}$$

Example (2/2)

An optimal solution $\tilde{x}(\text{LP-0}_{\text{KP}})$ is:

$$\tilde{x}_1 = \frac{9}{2}, \quad \tilde{x}_2 = \tilde{x}_3 = \tilde{x}_4 = \tilde{x}_5 = \tilde{x}_6 = 0$$

The optimal value $z(\text{LP-0}_{\text{KP}})$ is 27.

Accordingly we have the upper bound:

$$z(\text{ILP}_{\text{KP}}) \leq 27$$

The dual model $D(\text{LP-0}_{\text{KP}})$

Using a non-negative dual variable π for the capacity constraint, the dual model of the LP-0_{KP} model is:

$$\begin{aligned} \min \quad & c \pi \\ \text{subject to} \quad & w_j \pi \geq p_j, \quad \forall j \in \mathcal{J}, \\ & \pi \geq 0. \end{aligned}$$

This LP model is referred to as the $D(\text{LP-0}_{\text{KP}})$ model.

Example

The dual $D(\text{LP-0}_{\text{KP}})$ model for the reference KP instance is:

$$\begin{array}{llll} \min & 9 & \pi & \\ \text{subject to} & 2 & \pi & \geq 6 \\ & 3 & \pi & \geq 5 \\ & 6 & \pi & \geq 8 \\ & 7 & \pi & \geq 9 \\ & 5 & \pi & \geq 6 \\ & 9 & \pi & \geq 7 \\ & & \pi & \geq 0 \end{array}$$

An optimal solution $\tilde{\pi}(D(\text{LP-0}_{\text{KP}}))$ is:

$$\tilde{\pi} = 3$$

The optimal value $z(D(\text{LP-0}_{\text{KP}}))$ is 27.

Optimal solutions of the primal and the dual

The following proposition provides formulas to compute an optimal solution of the primal $LP-0_{KP}$ model and of the dual $D(LP-0_{KP})$ model.

Proposition 1

If the first item $j = 1$ has the largest efficiency, an optimal solution of the primal $LP-0_{KP}$ model is:

$$\tilde{x}_1 = \frac{c}{w_1} \quad \text{and} \quad \tilde{x}_j = 0, \quad \forall j \in \{2, 3, \dots, n\}, \quad (1)$$

and an optimal solution of the dual $D(LP-0_{KP})$ model is:

$$\tilde{\pi} = \frac{p_1}{w_1}. \quad (2)$$

The optimal value

Corollary 1

The optimal value of the primal LP-0_{KP} and dual $D(\text{LP-0}_{\text{KP}})$ models is:

$$z(\text{LP-0}_{\text{KP}}) = p_1 \frac{c}{w_1} = z(D(\text{LP-0}_{\text{KP}})) \quad (3)$$

- In the reference instance item 1 has the largest efficiency:
 $p_1 \frac{c}{w_1} = 6 \cdot \frac{9}{2} = 27$, the value of the example above

The idea of the proof

The proof proceeds in three steps:

1. We show that (1) is a feasible solution for the primal $LP-0_{KP}$ model;
2. We show that (2) is a feasible solution for the dual $D(LP-0_{KP})$ model;
3. We show that both feasible solutions have the same objective function value.

Therefore, by LP Strong Duality, (1) and (2) are a pair of optimal primal and dual solutions.

Proof (1/3)

1. We have:

$$\sum_{j \in \mathcal{J}} w_j \tilde{x}_j = w_1 \frac{c}{w_1} = c$$

thus the capacity constraint is satisfied.

Since $\tilde{x}_j \geq 0$, for every $j \in \mathcal{J}$, the constraints $x_j \geq 0$ are satisfied.

Therefore, (1) is a feasible solution for the primal LP-0_{KP} model.

Proof (2/3)

2. For every $j \in \mathcal{J}$, we have:

$$w_j \tilde{\pi} = w_j \frac{p_1}{w_1} \geq p_j \quad \text{since} \quad \frac{p_1}{w_1} \geq \frac{p_j}{w_j}$$

thus the constraints $w_j \pi \geq p_j$ are satisfied.

We have $\tilde{\pi} \geq 0$, thus the constraint $\pi \geq 0$ is satisfied.

Therefore, (2) is a feasible solution for the dual $D(\text{LP-0}_{\text{KP}})$ model.

Proof (3/3)

3. For the objective functions, we have:

$$\sum_{j \in \mathcal{J}} p_j \tilde{x}_j = p_1 \frac{c}{w_1} = c \tilde{\pi}$$

thus the primal and dual objective function values are equal.

Therefore, by LP Strong Duality, (1) and (2) are a pair of optimal primal and dual solutions.



Summary

The first LP relaxation:

- ▶ LP-0_{KP} replaces $x_j \in \{0, 1\}$ with $x_j \geq 0$; its dual has one variable π
- ▶ With item 1 of largest efficiency: $\tilde{x}_1 = c/w_1$, $\tilde{\pi} = p_1/w_1$, value $p_1 c/w_1$
- ▶ Optimality follows from primal feasibility, dual feasibility and equal values (strong duality)

In the reference instance:

- ▶ $z(\text{LP-0}_{\text{KP}}) = 27$, hence $z(\text{ILP}_{\text{KP}}) \leq 27$

A second Linear Programming relaxation

Introduction: a second LP relaxation

- ▶ The first relaxation allows many copies of the same item: we now keep the upper bounds $x_j \leq 1$
- ▶ We write the second LP relaxation LP_{KP} and its dual, and we introduce the efficiency of an item, the split item and the residual capacity
- ▶ With them we compute optimal primal and dual solutions with closed formulas
- ▶ Finally we prove that $z(LP_{KP})$ is never larger than twice $z(ILP_{KP})$

The LP_{KP} model

To obtain an LP relaxation of the ILP_{KP} model, we relax the constraints $x_j \in \{0, 1\}$, $\forall j \in \mathcal{J}$, with the following ones:

$$x_j \leq 1, \quad \forall j \in \mathcal{J},$$

$$x_j \geq 0, \quad \forall j \in \mathcal{J}.$$

This LP model is referred to as the LP_{KP} model.

Example (1/2)

The LP_{KP} model for the reference KP instance is:

$$\begin{array}{llllllll} \max & 6 & x_1 + 5 & x_2 + 8 & x_3 + 9 & x_4 + 6 & x_5 + 7 & x_6 \\ \text{subject to} & 2 & x_1 + 3 & x_2 + 6 & x_3 + 7 & x_4 + 5 & x_5 + 9 & x_6 \leq 9 \\ & & x_1 & & & & & \leq 1 \\ & & & x_2 & & & & \leq 1 \\ & & & & x_3 & & & \leq 1 \\ & & & & & x_4 & & \leq 1 \\ & & & & & & x_5 & \leq 1 \\ & & & & & & & x_6 \leq 1 \\ & & x_1, & x_2, & x_3, & x_4, & x_5, & x_6 \geq 0 \end{array}$$

Example (2/2)

An optimal solution $\tilde{\mathbf{x}}(\text{LP}_{\text{KP}})$ is:

$$\tilde{x}_1 = \tilde{x}_2 = 1, \quad \tilde{x}_3 = \frac{2}{3}, \quad \tilde{x}_4 = \tilde{x}_5 = \tilde{x}_6 = 0$$

The optimal value $z(\text{LP}_{\text{KP}})$ is $\frac{49}{3}$.

Accordingly we have the upper bound:

$$z(\text{ILP}_{\text{KP}}) \leq \frac{49}{3}$$

The dual model $D(LP_{KP})$

Using a non-negative dual variable π for the capacity constraint and a non-negative dual variable μ_j for each constraint $x_j \leq 1$, the dual model of the LP_{KP} model is:

$$\min \quad c \pi + \sum_{j \in \mathcal{J}} \mu_j$$

$$\text{subject to} \quad w_j \pi + \mu_j \geq p_j, \quad \forall j \in \mathcal{J},$$

$$\pi \geq 0,$$

$$\mu_j \geq 0, \quad \forall j \in \mathcal{J}.$$

This LP model is referred to as the $D(LP_{KP})$ model.

Example (1/2)

The dual $D(LP_{KP})$ model for the reference KP instance is:

$$\begin{array}{llllllll} \min & 9 & \pi & + \mu_1 & + \mu_2 & + \mu_3 & + \mu_4 & + \mu_5 & + \mu_6 \\ \text{subject to} & 2 & \pi & + \mu_1 & & & & & \geq 6 \\ & 3 & \pi & & + \mu_2 & & & & \geq 5 \\ & 6 & \pi & & & + \mu_3 & & & \geq 8 \\ & 7 & \pi & & & & + \mu_4 & & \geq 9 \\ & 5 & \pi & & & & & + \mu_5 & \geq 6 \\ & 9 & \pi & & & & & & + \mu_6 \geq 7 \\ & & \pi, & \mu_1, & \mu_2, & \mu_3, & \mu_4, & \mu_5, & \mu_6 \geq 0 \end{array}$$

Example (2/2)

An optimal solution $(\tilde{\pi}, \tilde{\mu})(D(\text{LP}_{\text{KP}}))$ is:

$$\tilde{\pi} = \frac{4}{3}, \quad \tilde{\mu}_1 = \frac{10}{3}, \quad \tilde{\mu}_2 = 1, \quad \tilde{\mu}_3 = \tilde{\mu}_4 = \tilde{\mu}_5 = \tilde{\mu}_6 = 0$$

The optimal value $z(D(\text{LP}_{\text{KP}}))$ is $\frac{49}{3}$, the same value as $z(\text{LP}_{\text{KP}})$.

Definition 1: efficiency

*The **efficiency** of an item $j \in \mathcal{I}$ is defined as the ratio between its profit and its weight:*

$$\frac{p_j}{w_j}$$

- ▶ The efficiency of an item is a measure of its profitability per unit of weight: it indicates how much profit is gained for each unit of weight.

Example

Let us consider the reference KP instance. We have:

Items		$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
Profits	p_j	6	5	8	9	6	7
Weights	w_j	2	3	6	7	5	9
Efficiencies	$\frac{p_j}{w_j}$	3	$\frac{5}{3}$	$\frac{4}{3}$	$\frac{9}{7}$	$\frac{6}{5}$	$\frac{7}{9}$

The items are sorted by non-increasing efficiency, since:

$$3 \geq \frac{5}{3} \geq \frac{4}{3} \geq \frac{9}{7} \geq \frac{6}{5} \geq \frac{7}{9}$$

Sorting the items

Sorting the items by non-increasing efficiency requires a sorting algorithm with running time $O(n \log n)$, which is a polynomial function of the number n of items.

- ▶ From now on, the items are sorted by non-increasing efficiency: the next definitions and formulas need this order

Split item and residual capacity

Definition 2: split item and residual capacity

Given a set $\mathcal{I} = \{1, 2, \dots, n\}$ of n items ordered by non-increasing efficiency, the **split item** is the item

$$s \in \mathcal{I} \quad \text{such that} \quad \sum_{j=1}^{s-1} w_j \leq c \quad \text{and} \quad \sum_{j=1}^s w_j > c$$

and the **residual capacity** is

$$rc = c - \sum_{j=1}^{s-1} w_j$$

Split item and residual capacity: remarks

- ▶ The split item is the first one that does not fit entirely in the knapsack if all the previous ones are selected. It is also called **break item** or **critical item**, and it can be written as

$$s = \min \left\{ j \in \mathcal{J} : \sum_{k=1}^j w_k > c \right\}$$

- ▶ The residual capacity is the knapsack capacity remaining when all the items that precede the split item are selected. It satisfies $0 \leq rc < w_s$.

Example

For the reference KP instance, the split item is $s = 3$ since:

$$\sum_{j=1}^2 w_j = 2 + 3 = 5 \leq 9 \quad \text{and} \quad \sum_{j=1}^3 w_j = 2 + 3 + 6 = 11 > 9$$

and the residual capacity is:

$$rc = c - \sum_{j=1}^2 w_j = 9 - 2 - 3 = 4$$

Closed formulas for LP_{KP} and its dual

- ▶ The following proposition provides formulas to compute an optimal solution of the primal LP_{KP} model and of the dual $D(LP_{KP})$ model.
- ▶ The formulas use the order of the items by non-increasing efficiency, the split item s and the residual capacity rc defined above.

Proposition 2

If the items are sorted by non-increasing efficiency, an optimal solution of the primal LP_{KP} model is:

$$\begin{aligned}\tilde{x}_j &= 1, & \forall j \in \{1, 2, \dots, s-1\}, \\ \tilde{x}_s &= \frac{rc}{w_s}, \\ \tilde{x}_j &= 0, & \forall j \in \{s+1, s+2, \dots, n\},\end{aligned}\tag{4}$$

and an optimal solution of the dual $D(LP_{KP})$ model is:

$$\begin{aligned}\tilde{\pi} &= \frac{p_s}{w_s}, \\ \tilde{\mu}_j &= p_j - w_j \frac{p_s}{w_s}, & \forall j \in \{1, 2, \dots, s-1\}, \\ \tilde{\mu}_j &= 0, & \forall j \in \{s, s+1, \dots, n\}.\end{aligned}\tag{5}$$

The optimal value

Corollary 2

The optimal value of the primal LP_{KP} and dual $D(LP_{KP})$ models is:

$$z(LP_{KP}) = \sum_{j=1}^{s-1} p_j + p_s \frac{rc}{w_s} = z(D(LP_{KP})) \quad (6)$$

- The items before the split item are selected, the split item is taken fractionally, the others are discarded

The idea of the proof

The proof proceeds in three steps:

1. We show that (4) is a feasible solution for the primal LP_{KP} model;
2. We show that (5) is a feasible solution for the dual $D(LP_{KP})$ model;
3. We show that both feasible solutions have the same objective function value.

Therefore, by LP Strong Duality, (4) and (5) are a pair of optimal primal and dual solutions.

Proof (1/4)

1. We have:

$$\sum_{j \in \mathcal{J}} w_j \tilde{x}_j = \sum_{j=1}^{s-1} w_j + w_s \frac{rc}{w_s} = \sum_{j=1}^{s-1} w_j + c - \sum_{j=1}^{s-1} w_j = c$$

thus the capacity constraint is satisfied.

Since $rc < w_s$, we have $0 \leq \tilde{x}_j \leq 1$, for every $j \in \mathcal{J}$.

Accordingly, the constraints $x_j \leq 1$ and $x_j \geq 0$ are satisfied.

Therefore, (4) is a feasible solution for the primal LP_{KP} model.

Proof (2/4)

2. For every $j \in \{1, 2, \dots, s-1\}$, we have:

$$w_j \tilde{\pi} + \tilde{\mu}_j = w_j \frac{p_s}{w_s} + \left(p_j - w_j \frac{p_s}{w_s} \right) = p_j$$

and, for every $j \in \{s, s+1, \dots, n\}$, we have:

$$w_j \tilde{\pi} + \tilde{\mu}_j = w_j \frac{p_s}{w_s} \geq p_j \quad \text{since} \quad \frac{p_s}{w_s} \geq \frac{p_j}{w_j}$$

thus the constraints $w_j \pi + \mu_j \geq p_j$ are satisfied.

Proof (3/4)

We have $\tilde{\pi} \geq 0$. For every $j \in \{1, 2, \dots, s-1\}$, we have:

$$p_j - w_j \frac{p_s}{w_s} \geq 0 \quad \text{since} \quad \frac{p_j}{w_j} \geq \frac{p_s}{w_s}$$

accordingly, $\tilde{\mu}_j \geq 0$ for every $j \in \mathcal{J}$. Thus the constraint $\pi \geq 0$ and the constraints $\mu_j \geq 0$ are satisfied.

Therefore, (5) is a feasible solution for the dual $D(\text{LP}_{\text{KP}})$ model.

Proof (4/4)

3. For the objective functions we have:

$$\begin{aligned}\sum_{j \in \mathcal{J}} p_j \tilde{x}_j &= \sum_{j=1}^{s-1} p_j + p_s \frac{rc}{w_s} \\ &= \sum_{j=1}^{s-1} p_j + \frac{p_s}{w_s} \left(c - \sum_{j=1}^{s-1} w_j \right) \\ &= c \frac{p_s}{w_s} + \sum_{j=1}^{s-1} \left(p_j - w_j \frac{p_s}{w_s} \right) \\ &= c \tilde{\pi} + \sum_{j \in \mathcal{J}} \tilde{\mu}_j\end{aligned}$$

thus the primal and dual objective function values are equal.

Therefore, by LP Strong Duality, (4) and (5) are a pair of optimal primal and dual solutions. □

How tight is $z(\text{LP}_{\text{KP}})$?

In the majority of KP instances the ratio between $z(\text{LP}_{\text{KP}})$ and $z(\text{ILP}_{\text{KP}})$ slightly exceeds one:

$z(\text{LP}_{\text{KP}})$ typically is a tight upper bound for $z(\text{ILP}_{\text{KP}})$.

An alternative proof

- ▶ We report an alternative proof for the formulas (4). We proceed by contradiction, proving that a different solution cannot be optimal.
- ▶ In this proof we assume that items with the same efficiency are merged into a single item. As we are allowed to choose any fraction of an item, this does not change the problem, and we avoid discussing symmetric solutions.

Proof (1/3)

Assume $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ is an optimal solution of LP_{KP} different from $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ determined with formulas (4).

Every optimal solution of LP_{KP} has a total weight equal to c , i.e., the capacity constraint is satisfied at equality (otherwise it can be improved).

Assume that no pair of items $h < k$ exists with $\bar{x}_h < 1$ and $\bar{x}_k > 0$. Then $\bar{\mathbf{x}}$ selects the items in order of efficiency, filling the knapsack item by item up to the weight c : this is exactly $\tilde{\mathbf{x}}$, a contradiction.

Hence such a pair of items $h < k$ exists (the case $k = s$ is included).

Proof (2/3)

Let:

$$d = \min \{ w_k \bar{x}_k, w_h (1 - \bar{x}_h) \} > 0$$

Construct a new feasible solution $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)$ which is identical to $\bar{\mathbf{x}}$ except that:

$$\hat{x}_k = \bar{x}_k - \frac{d}{w_k} \geq 0 \quad \text{and} \quad \hat{x}_h = \bar{x}_h + \frac{d}{w_h} \leq 1$$

By the definition of d , we have $\hat{x}_k \geq 0$ and $\hat{x}_h \leq 1$, and $\hat{\mathbf{x}}$ is a feasible solution since:

$$\sum_{j \in \mathcal{J}} w_j \hat{x}_j = \sum_{j \in \mathcal{J}} w_j \bar{x}_j + w_h \frac{d}{w_h} - w_k \frac{d}{w_k} = \sum_{j \in \mathcal{J}} w_j \bar{x}_j = c$$

Proof (3/3)

The objective function value of $\hat{\mathbf{x}}$ is:

$$\sum_{j \in \mathcal{J}} p_j \hat{x}_j = \sum_{j \in \mathcal{J}} p_j \bar{x}_j + d \underbrace{\left(\frac{p_h}{w_h} - \frac{p_k}{w_k} \right)}_{>0 \text{ since } \frac{p_h}{w_h} > \frac{p_k}{w_k}} > \sum_{j \in \mathcal{J}} p_j \bar{x}_j$$

Then the objective function value of $\hat{\mathbf{x}}$ is larger than the objective function value of $\bar{\mathbf{x}}$, contradicting the fact that $\bar{\mathbf{x}}$ is an optimal solution.

The strict inequality $p_h/w_h > p_k/w_k$ holds for every $h < k$, since items with the same efficiency have been merged. □

Example (1/2)

For the reference KP instance, we have $s = 3$ and $rc = 4$.

Using formulas (4), we can compute the following optimal solution of the LP_{KP} model:

$$\tilde{x}_1 = \tilde{x}_2 = 1, \quad \tilde{x}_3 = \frac{2}{3}, \quad \tilde{x}_4 = \tilde{x}_5 = \tilde{x}_6 = 0$$

Using formula (6), the optimal value of both LP models, primal and dual, is:

$$\sum_{j=1}^2 p_j + p_s \frac{rc}{w_s} = 6 + 5 + 8 \frac{2}{3} = \frac{49}{3}$$

Example (2/2)

Using formulas (5), we can compute the following optimal solution of the $D(LP_{KP})$ model:

$$\tilde{\pi} = \frac{4}{3},$$

$$\tilde{\mu}_1 = 6 - 2 \frac{4}{3} = \frac{10}{3},$$

$$\tilde{\mu}_2 = 5 - 3 \frac{4}{3} = 1,$$

$$\tilde{\mu}_3 = \tilde{\mu}_4 = \tilde{\mu}_5 = \tilde{\mu}_6 = 0$$

These are the optimal solutions found in the examples of the LP_{KP} and $D(LP_{KP})$ models.

Some important theoretical properties

- ▶ The following two propositions show that the ratio between $z(\text{LP}_{\text{KP}})$ and $z(\text{ILP}_{\text{KP}})$ is at most two for all KP instances. Furthermore, there exist instances for which the value two can be asymptotically achieved as the capacity $c \rightarrow +\infty$.

Proposition 3

For every KP instance, we have:

$$\frac{z(\text{LP}_{\text{KP}})}{z(\text{ILP}_{\text{KP}})} \leq 2$$

Proof (1/3)

We consider two feasible solutions of the ILP_{KP} model:

1. The solution selecting all items preceding the split item:

$$\begin{aligned}\bar{x}_j &= 1, \quad \forall j \in \{1, 2, \dots, s-1\} \quad \text{and} \\ \bar{x}_j &= 0, \quad \forall j \in \{s, s+1, \dots, n\}\end{aligned}$$

which is feasible by definition of the split item, and provides the valid lower bound:

$$\sum_{j=1}^{s-1} p_j \leq z(\text{ILP}_{\text{KP}})$$

Proof (2/3)

2. The solution selecting only the split item:

$$\bar{x}_j = 0, \quad \forall j \in \mathcal{J} \setminus \{s\} \quad \text{and} \quad \bar{x}_s = 1$$

which is feasible by assumption that $w_s \leq c$, and provides the valid lower bound:

$$p_s \leq z(\text{ILP}_{\text{KP}})$$

Proof (3/3)

Adding the two inequalities yields:

$$2 z(\text{ILP}_{\text{KP}}) \geq \sum_{j=1}^{s-1} p_j + p_s \geq z(\text{LP}_{\text{KP}})$$

and thus:

$$\frac{z(\text{LP}_{\text{KP}})}{z(\text{ILP}_{\text{KP}})} \leq 2$$



The ratio two can be approached

Proposition 4

There are KP instances such that:

$$\frac{z(\text{LP}_{\text{KP}})}{z(\text{ILP}_{\text{KP}})} \rightarrow 2 \quad \text{as} \quad c \rightarrow +\infty$$

- ▶ The proof builds a family of instances with two items, each of which fits alone in the knapsack, while the two together do not

Proof (1/3)

Given $\vartheta \in \mathbb{Z}_{\geq 1}$, consider the KP instance with knapsack capacity $c = 2\vartheta$ and $n = 2$ items having the following profits and weights:

$$p_1 = 1, \quad w_1 = \vartheta + 1 \quad \text{and} \quad p_2 = 1, \quad w_2 = \vartheta + 1$$

Since only one item can be selected, an optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{KP}})$ of the ILP_{KP} model is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 0 \quad \text{and} \quad z(\text{ILP}_{\text{KP}}) = 1$$

Proof (2/3)

The split item is $s = 2$, and the residual capacity is:

$$rc = 2\vartheta - (\vartheta + 1) = \vartheta - 1$$

Accordingly, an optimal solution $\tilde{\mathbf{x}}(\text{LP}_{\text{KP}})$ of the LP_{KP} model is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = \frac{\vartheta - 1}{\vartheta + 1}$$

and

$$z(\text{LP}_{\text{KP}}) = 1 + \frac{\vartheta - 1}{\vartheta + 1} = \frac{2\vartheta}{\vartheta + 1}$$

Proof (3/3)

It follows that:

$$\begin{aligned}\lim_{\vartheta \rightarrow +\infty} \frac{z(\text{LP}_{\text{KP}})}{z(\text{ILP}_{\text{KP}})} &= \lim_{\vartheta \rightarrow +\infty} \frac{\frac{2\vartheta}{\vartheta+1}}{1} = \lim_{\vartheta \rightarrow +\infty} \frac{2(\vartheta+1) - 2}{\vartheta+1} \\ &= \lim_{\vartheta \rightarrow +\infty} 2 - \underbrace{\frac{2}{\vartheta+1}}_{\rightarrow 0} = 2\end{aligned}$$

Accordingly, we conclude that:

$$\frac{z(\text{LP}_{\text{KP}})}{z(\text{ILP}_{\text{KP}})} \rightarrow 2 \quad \text{as} \quad c \rightarrow +\infty$$



Summary (1/2)

The second LP relaxation:

- ▶ LP_{KP} replaces $x_j \in \{0, 1\}$ with $0 \leq x_j \leq 1$; its dual has the variables π and μ_j
- ▶ The efficiency of item j is p_j/w_j ; sorting the items takes $O(n \log n)$ time
- ▶ The split item s is the first item that does not fit entirely; the residual capacity is $rc = c - \sum_{j=1}^{s-1} w_j$

Closed formulas:

- ▶ $\tilde{x}_j = 1$ before s , $\tilde{x}_s = rc/w_s$, $\tilde{x}_j = 0$ after s ; $\tilde{\pi} = p_s/w_s$
- ▶ $z(\text{LP}_{\text{KP}}) = \sum_{j=1}^{s-1} p_j + p_s rc/w_s$

Summary (2/2)

Properties:

- ▶ $z(\text{LP}_{\text{KP}})$ is typically a tight upper bound for $z(\text{ILP}_{\text{KP}})$
- ▶ For every instance $z(\text{LP}_{\text{KP}}) \leq 2 z(\text{ILP}_{\text{KP}})$, and the ratio two is approached as $c \rightarrow +\infty$

In the reference instance:

- ▶ $s = 3$, $rc = 4$ and $z(\text{LP}_{\text{KP}}) = \frac{49}{3}$, much better than $z(\text{LP-0}_{\text{KP}}) = 27$

0/1 Knapsack Problem upper bounds



Introduction: upper bounds for the KP

- ▶ The profits are integer, so $z(\text{ILP}_{\text{KP}})$ is integer: an upper bound can be rounded down
- ▶ In this section we see two upper bounds computed with the split item: the Dantzig bound and the Martello-Toth bound
- ▶ Both are computed with closed formulas, from the split item and the residual capacity

The Dantzig upper bound

- The following two propositions require that the items are sorted by non-increasing efficiency.

Proposition 5: Dantzig upper bound

For every KP instance, we have:

$$z(\text{ILP}_{\text{KP}}) \leq \underbrace{\sum_{j=1}^{s-1} p_j}_{\text{UB}_D} + \left\lfloor p_s \frac{rc}{w_s} \right\rfloor \quad (7)$$

*and UB_D is called the **Dantzig upper bound** for the KP.*

Proof

We have:

$$\lfloor z(\text{LP}_{\text{KP}}) \rfloor = \left\lfloor \sum_{j=1}^{s-1} p_j + p_s \frac{rc}{w_s} \right\rfloor = \sum_{j=1}^{s-1} p_j + \left\lfloor p_s \frac{rc}{w_s} \right\rfloor$$

which is a valid upper bound for $z(\text{ILP}_{\text{KP}})$ since $p_j \in \mathbb{Z}_{\geq 1}$, for every $j \in \mathcal{I}$.



The Martello-Toth upper bound: the items $s - 1$ and $s + 1$

- ▶ The Martello-Toth bound uses the items $s - 1$ and $s + 1$.
- ▶ Since $w_1 \leq c$ and $\sum_j w_j > c$ we have $s \geq 2$, so the item $s - 1$ always exists.
- ▶ If $s = n$ the item $s + 1$ does not exist, and we set $p_{n+1}/w_{n+1} := 0$: in this case $UB_0 = \sum_{j=1}^{s-1} p_j$.

Proposition 6: Martello-Toth upper bound

For every KP instance, we have:

$$z(\text{ILP}_{\text{KP}}) \leq \text{UB}_{\text{MT}} = \max \left\{ \text{UB}_0, \text{UB}_1 \right\} \quad (8)$$

where

$$\text{UB}_0 = \sum_{j=1}^{s-1} p_j + \left\lfloor rc \frac{p_{s+1}}{w_{s+1}} \right\rfloor$$

$$\text{UB}_1 = \sum_{j=1}^{s-1} p_j + \left\lfloor p_s - (w_s - rc) \frac{p_{s-1}}{w_{s-1}} \right\rfloor$$

and UB_{MT} is called the **Martello-Toth upper bound** for the KP.

Proof

Since x_s cannot take a fractional value, an optimal solution $\tilde{x}(\text{ILP}_{\text{KP}})$ can be obtained from an optimal solution $\tilde{x}(\text{LP}_{\text{KP}})$ either without the split item s or with the split item s and hence removing at least one of the first $s - 1$ items.

1. If the split item is out, we have $z(\text{ILP}_{\text{KP}}) \leq \text{UB}_0$ since the residual capacity rc is filled with items having the largest possible efficiency $\frac{p_{s+1}}{w_{s+1}}$
2. If the split item is in, we have $z(\text{ILP}_{\text{KP}}) \leq \text{UB}_1$ since it is supposed that the item to be removed has exactly the minimum necessary weight $w_s - rc$ and the smallest possible efficiency $\frac{p_{s-1}}{w_{s-1}}$



Example (1/3)

For the reference KP instance, using formula (7), we can compute the Dantzig upper bound:

$$\text{UB}_D = \sum_{j=1}^2 p_j + \left\lfloor p_3 \frac{rc}{w_3} \right\rfloor = 6 + 5 + \left\lfloor 8 \frac{2}{3} \right\rfloor = 11 + \left\lfloor \frac{16}{3} \right\rfloor = 16$$

Example (2/3)

Using formula (8), we can compute the Martello-Toth upper bound:

$$UB_0 = \sum_{j=1}^2 p_j + \left\lfloor rc \frac{p_4}{w_4} \right\rfloor = 6 + 5 + \left\lfloor 4 \frac{9}{7} \right\rfloor = 11 + \left\lfloor \frac{36}{7} \right\rfloor = 16$$

$$\begin{aligned} UB_1 &= \sum_{j=1}^2 p_j + \left\lfloor p_3 - (w_3 - rc) \frac{p_2}{w_2} \right\rfloor \\ &= 6 + 5 + \left\lfloor 8 - (6 - 4) \frac{5}{3} \right\rfloor = 11 + \left\lfloor \frac{14}{3} \right\rfloor = 15 \end{aligned}$$

Example (3/3)

The Martello-Toth upper bound is:

$$UB_{MT} = \max \left\{ UB_0, UB_1 \right\} = \max \left\{ 16, 15 \right\} = 16$$

For this KP instance, the Martello-Toth upper bound is as strong as the Dantzig upper bound.

Summary

Upper bounds for the KP (items sorted by non-increasing efficiency):

- ▶ Dantzig: $UB_D = \lfloor z(LP_{KP}) \rfloor$, valid because the profits are integer
- ▶ Martello-Toth: $UB_{MT} = \max\{UB_0, UB_1\}$, with the split item out (UB_0) or in (UB_1)

In the reference instance:

- ▶ $UB_D = 16$, $UB_0 = 16$, $UB_1 = 15$, $UB_{MT} = 16$, while $z(ILP_{KP}) = 15$

An ILP model for the Subset Sum Problem

Introduction: an ILP model for the SSP

- ▶ The Subset Sum Problem and its reference instance are defined in the chapter «[Knapsack problems](#)»
- ▶ In this section we write its ILP model, with the same variables as the ILP_{KP} model
- ▶ We will see that it is the ILP_{KP} model with the profits equal to the weights

Decision variables

We introduce the following n **binary variables**:

$$x_j = \begin{cases} 1 & \text{if item } j \text{ is selected in the subset} \\ 0 & \text{otherwise} \end{cases} \quad \forall j \in \mathcal{I}.$$

The ILP_{SSP} model

Using these variables, an Integer Linear Programming (ILP) model for the Subset Sum Problem (SSP) reads as follows:

$$\begin{aligned} \max \quad & \sum_{j \in \mathcal{J}} w_j x_j \\ \text{subject to} \quad & \sum_{j \in \mathcal{J}} w_j x_j \leq c, \\ & x_j \in \{0, 1\}, \quad \forall j \in \mathcal{J}. \end{aligned}$$

This ILP model is referred to as the ILP_{SSP} model.

Description of the ILP_{SSP} model

Description of the objective function and constraints of the ILP_{SSP} model:

- ▶ the linear objective function maximizes the total weight of the items selected in the subset;
- ▶ the capacity constraint ensures that the total weight of the selected items does not exceed the target value (one linear constraint);
- ▶ finally, the constraints $x_j \in \{0, 1\}$ define the variables of the ILP model.

The ILP_{SSP} model is the ILP_{KP} model with $p_j = w_j$, for every $j \in \mathcal{J}$: the answer to the SSP is **yes** if and only if $z(\text{ILP}_{\text{SSP}}) = c$.

Example (1/2)

We consider a target value $c = 9$ and a set $\mathcal{I} = \{1, 2, 3, 4, 5, 6\}$ of 6 items ($n = 6$), with the following weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
w_j	2	3	6	7	5	9

This is the SSP instance of the chapter «Knapsack problems».

Example (2/2)

The ILP_{SSP} model for this SSP instance is:

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 + 6x_3 + 7x_4 + 5x_5 + 9x_6 \\ \text{subject to} \quad & 2x_1 + 3x_2 + 6x_3 + 7x_4 + 5x_5 + 9x_6 \leq 9 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \in \{0, 1\} \end{aligned}$$

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{SSP}})$ is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 1, \quad \tilde{x}_5 = 0, \quad \tilde{x}_6 = 0$$

The optimal value $z(\text{ILP}_{\text{SSP}})$ is $9 = c$, so the answer to the SSP is **yes**. The associated subset is $\tilde{S} = \{1, 4\}$. The optimal solution is not unique: the subsets $\{2, 3\}$ and $\{6\}$ also reach the target value.

Summary

The ILP_{SSP} model:

- ▶ The ILP_{KP} model with $p_j = w_j$: maximize the total weight of the subset, without exceeding c
- ▶ The answer is **yes** if and only if $z(\text{ILP}_{\text{SSP}}) = c$

In the example:

- ▶ $z(\text{ILP}_{\text{SSP}}) = 9 = c$: the answer is yes, with the subsets $\{1, 4\}$, $\{2, 3\}$ and $\{6\}$

An ILP model for the Bounded Knapsack Problem



Introduction: an ILP model for the BKP

- ▶ The Bounded Knapsack Problem and its reference instance are defined in the chapter «[Knapsack problems](#)»
- ▶ Item j is available in b_j copies: the variables are no longer binary, they count the copies
- ▶ In this section we write its ILP model with integer variables

Decision variables

We introduce the following n **integer variables**:

x_j = number of copies of item j inserted in the knapsack $\forall j \in \mathcal{I}$.

The ILP_{BKP} model

Using these variables, an Integer Linear Programming (ILP) model for the Bounded Knapsack Problem (BKP) reads as follows:

$$\begin{aligned} \max \quad & \sum_{j \in \mathcal{J}} p_j x_j \\ \text{subject to} \quad & \sum_{j \in \mathcal{J}} w_j x_j \leq c, \\ & x_j \in \{0, 1, \dots, b_j\}, \quad \forall j \in \mathcal{J}. \end{aligned}$$

This ILP model is referred to as the ILP_{BKP} model.

Description of the ILP_{BKP} model

Description of the objective function and constraints of the ILP_{BKP} model:

- ▶ the linear objective function maximizes the total profit of the item copies selected in the knapsack;
- ▶ the capacity constraint ensures that the total weight of the selected item copies does not exceed the knapsack capacity (one linear constraint);
- ▶ finally, the constraints $x_j \in \{0, 1, \dots, b_j\}$ define the variables of the ILP model: each item j is taken at most b_j times.

Example (1/2)

We consider a knapsack of capacity $c = 9$ and a set $\mathcal{I} = \{1, 2, 3, 4, 5, 6\}$ of 6 items ($n = 6$), with the following profits, weights and available copies:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
p_j	6	5	8	9	6	7
w_j	2	3	6	7	5	9
b_j	2	2	1	1	1	1

This is the BKP instance of the chapter «Knapsack problems».

Example (2/2)

The ILP_{BKP} model for this BKP instance is:

$$\begin{array}{llllllll} \max & 6 & x_1 + 5 & x_2 + 8 & x_3 + 9 & x_4 + 6 & x_5 + 7 & x_6 \\ \text{subject to} & 2 & x_1 + 3 & x_2 + 6 & x_3 + 7 & x_4 + 5 & x_5 + 9 & x_6 \leq 9 \\ & x_1, & x_2 & & & & & \in \{0, 1, 2\} \\ & & & x_3, & x_4, & x_5, & x_6 & \in \{0, 1\} \end{array}$$

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{BKP}})$ is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 1, \quad \tilde{x}_6 = 0$$

The optimal value $z(\text{ILP}_{\text{BKP}})$ is 18. The associated optimal solution of the BKP takes two copies of item 1 and one copy of item 5, and it is the unique optimal solution.

Summary

The ILP_{BKP} model:

- ▶ n integer variables x_j : the number of copies of item j in the knapsack
- ▶ The same objective and capacity constraint as the ILP_{KP} model, with $x_j \in \{0, 1, \dots, b_j\}$

In the example:

- ▶ Two copies of item 1 and one of item 5, with $z(\text{ILP}_{\text{BKP}}) = 18$

An ILP model for the Unbounded Knapsack Problem

Introduction: an ILP model for the UKP

- ▶ The Unbounded Knapsack Problem and its reference instance are defined in the chapter «[Knapsack problems](#)»
- ▶ Each item is available in an unlimited number of copies: the variables count the copies, with no upper bound
- ▶ In this section we write its ILP model with non-negative integer variables

Decision variables

We introduce the following n **integer variables**:

x_j = number of copies of item j inserted in the knapsack $\forall j \in \mathcal{I}$.

The ILP_{UKP} model

Using these variables, an Integer Linear Programming (ILP) model for the Unbounded Knapsack Problem (UKP) reads as follows:

$$\begin{aligned} & \max \sum_{j \in \mathcal{J}} p_j x_j \\ & \text{subject to} \quad \sum_{j \in \mathcal{J}} w_j x_j \leq c, \\ & \quad x_j \in \mathbb{Z}_{\geq 0}, \quad \forall j \in \mathcal{J}. \end{aligned}$$

This ILP model is referred to as the ILP_{UKP} model.

Description of the ILP_{UKP} model

Description of the objective function and constraints of the ILP_{UKP} model:

- ▶ the linear objective function maximizes the total profit of the item copies selected in the knapsack;
- ▶ the capacity constraint ensures that the total weight of the selected item copies does not exceed the knapsack capacity (one linear constraint);
- ▶ finally, the constraints $x_j \in \mathbb{Z}_{\geq 0}$ define the variables of the ILP model; the capacity constraint already gives $x_j \leq \hat{b}_j = \lfloor c/w_j \rfloor$, for every $j \in \mathcal{I}$.

Example (1/2)

We consider a knapsack of capacity $c = 9$ and a set $\mathcal{J} = \{1, 2, 3, 4, 5, 6\}$ of 6 items ($n = 6$), each available in an unlimited number of copies, with the following profits and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
p_j	6	5	8	9	6	7
w_j	2	3	6	7	5	9

This is the UKP instance of the chapter «Knapsack problems».

Example (2/2)

The ILP_{UKP} model for this UKP instance is:

$$\begin{array}{ll}\max & 6x_1 + 5x_2 + 8x_3 + 9x_4 + 6x_5 + 7x_6 \\ \text{subject to} & 2x_1 + 3x_2 + 6x_3 + 7x_4 + 5x_5 + 9x_6 \leq 9 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \in \mathbb{Z}_{\geq 0}\end{array}$$

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{UKP}})$ is:

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0, \quad \tilde{x}_6 = 0$$

The optimal value $z(\text{ILP}_{\text{UKP}})$ is 24. The associated optimal solution of the UKP takes four copies of item 1, and it is the unique optimal solution.

Summary

The ILP_{UKP} model:

- ▶ n integer variables $x_j \in \mathbb{Z}_{\geq 0}$: the number of copies of item j in the knapsack
- ▶ The capacity constraint already gives $x_j \leq \hat{b}_j = \lfloor c/w_j \rfloor$

In the example:

- ▶ Four copies of item 1, with $z(\text{ILP}_{\text{UKP}}) = 24$

An ILP model for the Multidimensional Knapsack Problem

Introduction: an ILP model for the MDKP

- ▶ The Multidimensional Knapsack Problem and its reference instance are defined in the chapter «[Knapsack problems](#)»
- ▶ The knapsack has k dimensions, each with its capacity c_d , $d \in \mathcal{D}$
- ▶ In this section we write its ILP model: binary variables and one capacity constraint per dimension

Decision variables

We introduce the following n **binary variables**:

$$x_j = \begin{cases} 1 & \text{if item } j \text{ is selected in the knapsack} \\ 0 & \text{otherwise} \end{cases} \quad \forall j \in \mathcal{I}.$$

The ILP_{MDKP} model

Using these variables, an Integer Linear Programming (ILP) model for the Multidimensional Knapsack Problem (MDKP) reads as follows:

$$\begin{aligned} & \max \sum_{j \in \mathcal{J}} p_j x_j \\ & \text{subject to } \sum_{j \in \mathcal{J}} w_{dj} x_j \leq c_d, \quad \forall d \in \mathcal{D}, \\ & \quad x_j \in \{0, 1\}, \quad \forall j \in \mathcal{J}. \end{aligned}$$

This ILP model is referred to as the ILP_{MDKP} model.

Description of the ILP_{MDKP} model

Description of the objective function and constraints of the ILP_{MDKP} model:

- ▶ the linear objective function maximizes the total profit of the items selected in the knapsack;
- ▶ the capacity constraints ensure that, in every dimension d , the total weight of the selected items does not exceed the capacity c_d (k linear constraints, one for each dimension);
- ▶ finally, the constraints $x_j \in \{0, 1\}$ define the variables of the ILP model.

Example (1/2)

We consider a 2-dimensional knapsack ($k = 2$) with capacities $c_1 = 10$ and $c_2 = 12$, and a set $\mathcal{J} = \{1, 2, 3, 4\}$ of 4 items ($n = 4$), with the following profits and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
p_j	6	7	5	9
w_{1j}	4	5	3	7
w_{2j}	6	4	5	3

This is the MDKP instance of the chapter «Knapsack problems».

Example (2/2)

The ILP_{MDKP} model for this MDKP instance is:

$$\begin{aligned} \max \quad & 6x_1 + 7x_2 + 5x_3 + 9x_4 \\ \text{subject to} \quad & 4x_1 + 5x_2 + 3x_3 + 7x_4 \leq 10 \\ & 6x_1 + 4x_2 + 5x_3 + 3x_4 \leq 12 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\} \end{aligned}$$

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{MDKP}})$ is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = 1, \quad \tilde{x}_4 = 1$$

The optimal value $z(\text{ILP}_{\text{MDKP}})$ is 14. The associated optimal solution of the MDKP is $\tilde{S} = \{3, 4\}$, and it is the unique optimal solution.

Summary

The ILP_{MDKP} model:

- ▶ n binary variables x_j , as in the ILP_{KP} model
- ▶ One capacity constraint for each dimension $d \in \mathcal{D}$: k linear constraints

In the example:

- ▶ The unique optimal solution is $\tilde{S} = \{3, 4\}$, with $z(\text{ILP}_{\text{MDKP}}) = 14$

An ILP model for the Multiple Knapsack Problem

Introduction: an ILP model for the MKP

- ▶ The Multiple Knapsack Problem and its reference instance are defined in the chapter «[Knapsack problems](#)»
- ▶ There are m knapsacks $k \in \mathcal{K}$: we must decide not only whether an item is selected, but also in which knapsack
- ▶ In this section we write its ILP model, with one binary variable for each item and each knapsack

Decision variables

We introduce the following $n m$ **binary variables**:

$$x_{jk} = \begin{cases} 1 & \text{if item } j \text{ is inserted in knapsack } k \\ 0 & \text{otherwise} \end{cases} \quad \forall j \in \mathcal{I}, \forall k \in \mathcal{K}.$$

The ILP_{MKP} model

Using these variables, an Integer Linear Programming (ILP) model for the Multiple Knapsack Problem (MKP) reads as follows:

$$\begin{aligned} \max \quad & \sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{J}} p_j x_{jk} \\ \text{subject to} \quad & \sum_{j \in \mathcal{J}} w_j x_{jk} \leq c_k, \quad \forall k \in \mathcal{K}, \\ & \sum_{k \in \mathcal{K}} x_{jk} \leq 1, \quad \forall j \in \mathcal{J}, \\ & x_{jk} \in \{0, 1\}, \quad \forall j \in \mathcal{J}, \forall k \in \mathcal{K}. \end{aligned}$$

This ILP model is referred to as the ILP_{MKP} model.

Description of the ILP_{MKP} model

Description of the objective function and constraints of the ILP_{MKP} model:

- ▶ the linear objective function maximizes the total profit of the items inserted in the knapsacks;
- ▶ the capacity constraints ensure that, in every knapsack k , the total weight of the assigned items does not exceed the capacity c_k (m linear constraints, one for each knapsack);
- ▶ the constraints $\sum_{k \in \mathcal{K}} x_{jk} \leq 1$ ensure that each item is inserted in at most one knapsack (n linear constraints, one for each item);
- ▶ finally, the constraints $x_{jk} \in \{0, 1\}$ define the variables of the ILP model.

Example (1/3)

We consider $m = 2$ knapsacks with capacities $c_1 = 10$ and $c_2 = 12$, and a set $\mathcal{J} = \{1, 2, 3, 4, 5\}$ of 5 items ($n = 5$), with the following profits and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
p_j	6	8	5	7	9
w_j	4	6	5	3	7

This is the MKP instance of the chapter «Knapsack problems». The model has 10 variables, so we write one constraint per line.

Example (2/3)

The ILP_{MKP} model for this MKP instance is:

$$\begin{aligned} \max \quad & 6x_{11} + 8x_{21} + 5x_{31} + 7x_{41} + 9x_{51} \\ & + 6x_{12} + 8x_{22} + 5x_{32} + 7x_{42} + 9x_{52} \\ \text{subject to} \quad & 4x_{11} + 6x_{21} + 5x_{31} + 3x_{41} + 7x_{51} \leq 10 \\ & 4x_{12} + 6x_{22} + 5x_{32} + 3x_{42} + 7x_{52} \leq 12 \\ & x_{11} + x_{12} \leq 1 \\ & x_{21} + x_{22} \leq 1 \\ & x_{31} + x_{32} \leq 1 \\ & x_{41} + x_{42} \leq 1 \\ & x_{51} + x_{52} \leq 1 \\ & x_{11}, x_{21}, x_{31}, x_{41}, x_{51}, x_{12}, x_{22}, x_{32}, x_{42}, x_{52} \in \{0, 1\} \end{aligned}$$

Example (3/3)

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{MKP}})$ is:

$$\tilde{x}_{11} = \tilde{x}_{21} = 1, \quad \tilde{x}_{42} = \tilde{x}_{52} = 1,$$

and all the other variables equal to 0.

The optimal value $z(\text{ILP}_{\text{MKP}})$ is 30.

The associated optimal solution of the MKP is $(\tilde{S}_1, \tilde{S}_2) = (\{1, 2\}, \{4, 5\})$. The optimal solution is not unique: $(\{2, 4\}, \{1, 5\})$ and $(\{4, 5\}, \{1, 2\})$ have the same value.

Summary

The ILP_{MKP} model:

- ▶ $n m$ binary variables x_{jk} : item j in knapsack k
- ▶ One capacity constraint per knapsack (m) and one assignment constraint per item (n): each item in at most one knapsack

In the example:

- ▶ $(\tilde{S}_1, \tilde{S}_2) = (\{1, 2\}, \{4, 5\})$, with $z(\text{ILP}_{\text{MKP}}) = 30$; the optimal solution is not unique

An ILP model for the Multiple-Choice Knapsack Problem

Introduction: an ILP model for the MCKP

- ▶ The Multiple-Choice Knapsack Problem and its reference instance are defined in the chapter «[Knapsack problems](#)»
- ▶ The items are partitioned into q classes $\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_q$, and exactly one item must be selected from each class
- ▶ In this section we write its ILP model: the ILP_{KP} model plus one equality per class

Decision variables

We introduce the following n **binary variables**:

$$x_j = \begin{cases} 1 & \text{if item } j \text{ is selected in the knapsack} \\ 0 & \text{otherwise} \end{cases} \quad \forall j \in \mathcal{I}.$$

The ILP_{MCKP} model

Using these variables, an Integer Linear Programming (ILP) model for the Multiple-Choice Knapsack Problem (MCKP) reads as follows:

$$\begin{aligned} & \max \sum_{j \in \mathcal{J}} p_j x_j \\ & \text{subject to} \quad \sum_{j \in \mathcal{J}} w_j x_j \leq c, \\ & \quad \sum_{j \in \mathcal{J}_h} x_j = 1, \quad \forall h \in \{1, 2, \dots, q\}, \\ & \quad x_j \in \{0, 1\}, \quad \forall j \in \mathcal{J}. \end{aligned}$$

This ILP model is referred to as the ILP_{MCKP} model.

Description of the ILP_{MCKP} model

Description of the objective function and constraints of the ILP_{MCKP} model:

- ▶ the linear objective function maximizes the total profit of the items selected in the knapsack;
- ▶ the capacity constraint ensures that the total weight of the selected items does not exceed the knapsack capacity (one linear constraint);
- ▶ the constraints $\sum_{j \in \mathcal{J}_h} x_j = 1$ ensure that exactly one item is selected from each class (q linear constraints, one for each class);
- ▶ finally, the constraints $x_j \in \{0, 1\}$ define the variables of the ILP model.

Example (1/2)

We consider a knapsack of capacity $c = 15$ and a set $\mathcal{I} = \{1, 2, 3, 4, 5, 6\}$ of 6 items ($n = 6$), partitioned into $q = 3$ classes $\mathcal{I}_1 = \{1, 2\}$, $\mathcal{I}_2 = \{3, 4\}$ and $\mathcal{I}_3 = \{5, 6\}$, with the following profits and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
p_j	6	8	5	7	4	9
w_j	4	6	5	8	3	7

This is the MCKP instance of the chapter «Knapsack problems».

Example (2/2)

The ILP_{MCKP} model for this MCKP instance is:

$$\begin{array}{llllllll} \max & 6 & x_1 + 8 & x_2 + 5 & x_3 + 7 & x_4 + 4 & x_5 + 9 & x_6 \\ \text{subject to} & 4 & x_1 + 6 & x_2 + 5 & x_3 + 8 & x_4 + 3 & x_5 + 7 & x_6 \leq 15 \\ & & x_1 & + x_2 & & & & = 1 \\ & & & & x_3 & + x_4 & & = 1 \\ & & & & & & x_5 & + x_6 = 1 \\ & & x_1, & x_2, & x_3, & x_4, & x_5, & x_6 \in \{0, 1\} \end{array}$$

An optimal solution $\tilde{x}(\text{ILP}_{\text{MCKP}})$ is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 1, \quad \tilde{x}_5 = 1, \quad \tilde{x}_6 = 0$$

The optimal value $z(\text{ILP}_{\text{MCKP}})$ is 17. The associated optimal solution of the MCKP is $\tilde{S} = \{1, 4, 5\}$. The optimal solution is not unique: the subset $\{2, 3, 5\}$ has the same value.

The ILP_{MCKP} model:

- ▶ n binary variables x_j and one capacity constraint, as in the ILP_{KP} model
- ▶ One **equality** per class: exactly one item from each class (q linear constraints)

In the example:

- ▶ $\tilde{S} = \{1, 4, 5\}$, with $z(\text{ILP}_{\text{MCKP}}) = 17$; the subset $\{2, 3, 5\}$ has the same value

An ILP model for the Knapsack Problem with Setups

Introduction: an ILP model for the KPS

- ▶ The Knapsack Problem with Setups and its reference instance are defined in the chapter «[Knapsack problems](#)»
- ▶ The items are partitioned into q classes; selecting an item of class h requires activating the class, which costs f_h and uses the capacity s_h
- ▶ In this section we write its ILP model, with binary variables for the items and binary setup variables for the classes

Decision variables

We introduce the following n **binary variables**:

$$x_j = \begin{cases} 1 & \text{if item } j \text{ is selected in the knapsack} \\ 0 & \text{otherwise} \end{cases} \quad \forall j \in \mathcal{I}.$$

We introduce the following q **binary variables**:

$$y_h = \begin{cases} 1 & \text{if class } h \text{ is activated (its setup is paid)} \\ 0 & \text{otherwise} \end{cases} \quad \forall h \in \{1, 2, \dots, q\}.$$

The ILP_{KPS} model

Using these variables, an Integer Linear Programming (ILP) model for the Knapsack Problem with Setups (KPS) reads as follows:

$$\begin{aligned} \max \quad & \sum_{j \in \mathcal{J}} p_j x_j - \sum_{h=1}^q f_h y_h \\ \text{subject to} \quad & \sum_{j \in \mathcal{J}} w_j x_j + \sum_{h=1}^q s_h y_h \leq c, \\ & x_j \leq y_h, \quad \forall h \in \{1, 2, \dots, q\}, \forall j \in \mathcal{J}_h, \\ & x_j \in \{0, 1\}, \quad \forall j \in \mathcal{J}, \\ & y_h \in \{0, 1\}, \quad \forall h \in \{1, 2, \dots, q\}. \end{aligned}$$

This ILP model is referred to as the ILP_{KPS} model.

Description of the ILP_{KPS} model

Description of the objective function and constraints of the ILP_{KPS} model:

- ▶ the linear objective function maximizes the total revenue of the selected items minus the setup costs of the activated classes;
- ▶ the capacity constraint ensures that the total weight of the selected items plus the setup weights of the activated classes does not exceed the knapsack capacity (one linear constraint);
- ▶ the constraints $x_j \leq y_h$ ensure that an item can be selected only if its class is activated (n linear constraints, one for each item);
- ▶ finally, the constraints $x_j \in \{0, 1\}$ and $y_h \in \{0, 1\}$ define the variables of the ILP model.

Example (1/3)

We consider a knapsack of capacity $c = 20$ and a set $\mathcal{I} = \{1, 2, 3, 4, 5, 6\}$ of 6 items ($n = 6$), partitioned into $q = 2$ classes $\mathcal{I}_1 = \{1, 2, 3\}$ and $\mathcal{I}_2 = \{4, 5, 6\}$, with setup costs $f_1 = 3$, $f_2 = 4$, setup weights $s_1 = 2$, $s_2 = 3$, and the following revenues and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
p_j	6	8	5	7	9	4
w_j	4	5	3	6	7	2

This is the KPS instance of the chapter «Knapsack problems».

Example (2/3)

The ILP_{KPS} model for this KPS instance is:

$$\begin{array}{llllllllll}
 \max & 6 & x_1 + 8 & x_2 + 5 & x_3 + 7 & x_4 + 9 & x_5 + 4 & x_6 - 3 & y_1 - 4 & y_2 \\
 \text{subject to} & 4 & x_1 + 5 & x_2 + 3 & x_3 + 6 & x_4 + 7 & x_5 + 2 & x_6 + 2 & y_1 + 3 & y_2 \leq 20 \\
 & & x_1 & & & & & & - y_1 & \leq 0 \\
 & & & x_2 & & & & & - y_1 & \leq 0 \\
 & & & & x_3 & & & & - y_1 & \leq 0 \\
 & & & & & x_4 & & & - y_2 & \leq 0 \\
 & & & & & & x_5 & & - y_2 & \leq 0 \\
 & & & & & & & x_6 & - y_2 & \leq 0 \\
 & & x_1, & x_2, & x_3, & x_4, & x_5, & x_6 & & \in \{0, 1\} \\
 & & & & & & & & y_1, & y_2 \in \{0, 1\}
 \end{array}$$

Example (3/3)

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{KPS}})$, $\tilde{\mathbf{y}}(\text{ILP}_{\text{KPS}})$ is:

$$\tilde{x}_1 = \tilde{x}_2 = \tilde{x}_3 = 1, \quad \tilde{x}_4 = \tilde{x}_5 = 0, \quad \tilde{x}_6 = 1, \quad \tilde{y}_1 = \tilde{y}_2 = 1$$

The optimal value $z(\text{ILP}_{\text{KPS}})$ is 16.

The associated optimal solution of the KPS is $\tilde{S} = \{1, 2, 3, 6\}$. The optimal solution is not unique: the subsets $\{1, 2, 3\}$ and $\{4, 5, 6\}$ have the same value.

Summary

The ILP_{KPS} model:

- ▶ n binary variables x_j for the items and q binary setup variables y_h for the classes
- ▶ The objective subtracts the setup costs; the capacity constraint adds the setup weights
- ▶ The constraints $x_j \leq y_h$ allow an item only if its class is activated

In the example:

- ▶ $\tilde{S} = \{1, 2, 3, 6\}$, with $z(\text{ILP}_{\text{KPS}}) = 16$; the subsets $\{1, 2, 3\}$ and $\{4, 5, 6\}$ have the same value

Typical mistakes

Introduction: typical mistakes

- ▶ We collect the most frequent mistakes on the formulations of the knapsack problems
- ▶ Four concern the LP relaxations and the upper bounds of the KP, two the models of the variants
- ▶ Recognizing them avoids wrong bounds and wrong models

Typical mistakes (1/3)

Observation 1: the order and the relaxations

- ▶ **Forgetting to sort the items.** The formulas (4), (5) and the bounds hold only if the items are sorted by non-increasing efficiency: with another order the vector (4) is feasible but not optimal.
- ▶ **Using $z(\text{LP-0}_{\text{KP}})$ as the bound.** The first relaxation has no upper bounds on the variables, and it is much weaker: in the reference instance it gives 27, against $\frac{49}{3}$ for the second relaxation.

Typical mistakes (2/3)

Observation: the bounds and the split item

- ▶ ***Rounding when the profits are not integer.** The bounds UB_D and UB_{MT} round down the fractional part: this is valid only because all the profits are integer.*
- ▶ ***Confusing the split item with the last selected item.** The split item s is the first item that does not fit: in the reference instance it is $s = 3$, while the items 1 and 2 are selected entirely.*

Typical mistakes (3/3)

Observation: the optimum and the variants

- ▶ *Assuming that the LP optimum is unique. With items of equal efficiency the LP_{KP} problem has other optimal solutions; the formulas (4) give one of them.*
- ▶ *Copying the ILP_{KP} model to the variants. In the BKP and in the UKP the variables are integer, not binary; in the MCKP the class constraints $\sum_{j \in \mathcal{J}_h} x_j = 1$ are equalities; in the KPS, without the constraints $x_j \leq y_h$ the model selects items with $y_h = 0$ and never pays a setup.*

Typical mistakes:

- ▶ Sort the items by non-increasing efficiency before using the formulas and the bounds
- ▶ $LP-0_{KP}$ is much weaker than LP_{KP} ; round down only with integer profits
- ▶ The split item is the first item that does not fit; the LP optimum may not be unique
- ▶ Each variant changes the ILP_{KP} model: integer variables, equalities, setup constraints

In summary

Introduction: in summary

- ▶ We collect the key ideas of the chapter
- ▶ We close with the dashboard of the reference instance: the upper bounds computed in this chapter

Summary (1/3)

- ▶ The ILP_{KP} model has one binary variable per item, one capacity constraint and the objective of maximizing the total profit.
- ▶ The relaxation LP_{KP} is solved by sorting the items by non-increasing efficiency: the items before the split item are selected, the split item is taken fractionally, the others are discarded.
- ▶ The dual of LP_{KP} has the optimal solution $\tilde{\pi} = p_s/w_s$, and $z(LP_{KP})$ is never larger than twice $z(ILP_{KP})$.

Summary (2/3)

- ▶ Rounding down $z(\text{LP}_{\text{KP}})$ gives the Dantzig bound; the Martello-Toth bound considers separately the split item out and in.
- ▶ Each variant of the chapter «[Knapsack problems](#)» changes one ingredient of the ILP_{KP} model: integer variables (BKP, UKP), one capacity constraint per dimension (MDKP), one variable per item and knapsack (MKP), one equality per class (MCKP), setup variables y_h with $x_j \leq y_h$ (KPS); the SSP is the KP with $p_j = w_j$.

Summary (3/3)

Dashboard of the reference instance. Upper bounds on $z(\text{ILP}_{\text{KP}})$ computed in this chapter:

Bound	Value	From
First relaxation	27	$z(\text{LP-0}_{\text{KP}})$, item 1 with $\frac{9}{2}$ copies
Second relaxation	$\frac{49}{3}$	$z(\text{LP}_{\text{KP}})$, split item $s = 3$
Dantzig bound	16	$\lfloor \frac{49}{3} \rfloor$
Martello-Toth bound	16	$\max\{16, 15\}$
Optimal value	15	solution $\{1, 4\}$ of the ILP_{KP} model

The gap between the best upper bound (16) and the optimal value (15) is closed by the exact algorithms later in the course.