

Vertex coloring: limits of approximation

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Learning objectives

At the end of this chapter you will be able to:

- ▶ prove that no polynomial algorithm has a performance ratio below $\frac{4}{3}$ for the VCP, unless $P = NP$;
- ▶ state the result of Zuckerman and explain why the VCP has no constant guarantee;
- ▶ color bipartite graphs and interval graphs optimally, and bound the colors of the greedy coloring by $\Delta(G) + 1$;
- ▶ prove that planar graphs can be colored with 5 colors in polynomial time, and derive an additive guarantee for them.

Prerequisites

- ▶ From this module, the chapter «[Principles of approximation algorithms](#)»: performance ratio, α -approximation algorithm, additive performance guarantee
- ▶ From this module, the chapter «[Coloring problems](#)»: the definition of the VCP, the chromatic number $\chi(G)$, the clique bound $\omega(G) \leq \chi(G)$ and the Petersen graph
- ▶ From the module *Algorithms* ([module home page](#)): breadth-first search, polynomial time and NP-completeness

Why limits of approximation?



Introduction: Why limits of approximation?

- ▶ An approximation algorithm comes with a guarantee: its value is never too far from the optimal one
- ▶ For some problems such a guarantee does not exist, unless $P = NP$
- ▶ The vertex coloring problem is one of them: in this chapter we see why, and what can still be done

The problem of this chapter

- ▶ The problem of this chapter is the [vertex coloring problem \(VCP\)](#): its definition and its assumptions are on its page
- ▶ For the vertex coloring problem we prove the opposite of a guarantee: unless $P = NP$, no polynomial algorithm can have a good guarantee
- ▶ The negative results tell us when we must give up asking

Summary

Guarantees and limits:

- ▶ The positive results bound the gap of an algorithm on every instance
- ▶ The negative results prove that, unless $P = NP$, no polynomial algorithm has a good guarantee

The vertex coloring problem:

- ▶ It is the problem of this chapter, and it is on the negative side

Notation

Introduction: Notation

- ▶ We fix the notation on graphs used in the whole chapter: neighbours, degrees, maximum degree
- ▶ We recall how the quality of a coloring algorithm is measured
- ▶ We note that a trivial ratio exists, and ask how much better a polynomial algorithm can do



Graphs and degrees

- ▶ As on the problem page, $G = (V, E)$ is a simple graph with n vertices and m edges
- ▶ $N(v)$ is the set of the neighbours of v , and $\deg(v) = |N(v)|$ its degree
- ▶ The **maximum degree** is

$$\Delta(G) = \max_{v \in V} \deg(v)$$

- ▶ The optimal value of the VCP is the chromatic number, $z(I) = \chi(G)$

Recall: the performance ratio

- ▶ As in the chapter «Principles of approximation algorithms», $\mathcal{A}(I)$ is the number of colors of the coloring returned by the algorithm $\mathcal{A}$
- ▶ The VCP is a minimization problem, hence

$$\mathcal{A}(I) \geq \chi(G) \quad \text{and} \quad \frac{\mathcal{A}(I)}{\chi(G)} \geq 1$$

for every instance I

The trivial ratio

- ▶ Giving each vertex its own color is always feasible, so

$$\chi(G) \leq n,$$

and the ratio n is trivial

- ▶ This chapter shows that a polynomial algorithm cannot do much better on all graphs
- ▶ And then the classes of graphs on which it can

Summary

Notation:

- ▶ $G = (V, E)$ with n vertices and m edges, degree $\deg(v)$, maximum degree $\Delta(G)$
- ▶ $z(I) = \chi(G)$, and $\mathcal{A}(I)$ is the number of colors used by the algorithm $\mathcal{A}$

The performance ratio:

- ▶ $\frac{\mathcal{A}(I)}{\chi(G)} \geq 1$ for every instance
- ▶ The ratio n is trivial: one color per vertex

The VCP is not approximable

Introduction: The VCP is not approximable

- ▶ We prove that no polynomial algorithm can have a performance ratio below $\frac{4}{3}$, unless $P = NP$
- ▶ We then state much stronger results: not even a ratio of the form $n^{1-\varepsilon}$ is possible
- ▶ The consequence is that the VCP has no constant guarantee: this is why we will look for special classes of graphs

The scheme of the negative results

- ▶ The proof follows the scheme of all the negative results of approximation: an algorithm with a too good guarantee would decide in polynomial time an NP-complete problem
- ▶ The NP-complete problem is **3-colorability**: given a graph G , decide whether

$$\chi(G) \leq 3$$

- ▶ It remains NP-complete on planar graphs of maximum degree 4 (Garey, Johnson and Stockmeyer, 1976)

Theorem 1

Let $\mathcal{A}$ be a polynomial algorithm for the VCP. If $P \neq NP$, there is a graph G such that

$$\frac{\mathcal{A}(I)}{\chi(G)} \geq \frac{4}{3}.$$

Proof (1/2)

Suppose that $\frac{\mathcal{A}(I)}{\chi(G)} < \frac{4}{3}$ for every graph G , and run $\mathcal{A}$ on a graph G .

1. If $\chi(G) \leq 3$, then

$$\mathcal{A}(I) < \frac{4}{3} \chi(G) \leq 4.$$

The number of colors is an integer, hence $\mathcal{A}(I) \leq 3$

2. If $\chi(G) \geq 4$, then

$$\mathcal{A}(I) \geq \chi(G) \geq 4,$$

because $\mathcal{A}$ returns a feasible coloring

Proof (2/2)

- ▶ Hence

$$\chi(G) \leq 3 \quad \text{if and only if} \quad \mathcal{A}(I) \leq 3$$

- ▶ Running $\mathcal{A}$ and counting its colors takes polynomial time
- ▶ So 3-colorability would be in P , and $P = NP$



Remarks on the proof

- ▶ The proof uses only the integrality of $\chi(G)$: a ratio below $\frac{4}{3}$ cannot confuse 3 colors with 4
- ▶ With 2 colors the same argument says nothing, because deciding whether $\chi(G) \leq 2$ is easy (we will see it with the bipartite graphs)
- ▶ The planar graphs of maximum degree 4 are a special case of the VCP: the theorem holds on them too

The theorem excludes only the ratios below $\frac{4}{3}$.

Much more is known,
and we state it without proof.

Theorem 2: Zuckerman, 2007

For every $\varepsilon > 0$, if $P \neq NP$, no polynomial algorithm $\mathcal{A}$ for the VCP satisfies, for every graph G with n vertices,

$$\frac{\mathcal{A}(I)}{\chi(G)} \leq n^{1-\varepsilon}.$$

What the theorem says

- ▶ The ratio n is trivial, so the chromatic number is essentially as hard to approximate as possible
- ▶ The proof uses probabilistically checkable proofs and extractors
- ▶ Earlier, Garey and Johnson (1976) had raised the bound $\frac{4}{3}$ to 2
- ▶ Feige and Kilian (1998) had proved the bound $n^{1-\varepsilon}$ under a stronger assumption than $P \neq NP$

Corollary 1

If $P \neq NP$, the VCP has no α -approximation algorithm, for any constant $\alpha \geq 1$.

Proof

- ▶ Let $\mathcal{A}$ be a polynomial α -approximation algorithm, and take $\varepsilon = \frac{1}{2}$
- ▶ The graphs with $n > \alpha^2$ vertices satisfy

$$\frac{\mathcal{A}(I)}{\chi(G)} \leq \alpha < n^{1/2}$$

- ▶ The graphs with $n \leq \alpha^2$ vertices are solved exactly by enumeration, in a time bounded by a constant that depends only on α , with ratio $1 \leq n^{1/2}$
- ▶ The resulting algorithm is polynomial and has ratio at most $n^{1/2}$ on every graph, against the theorem of Zuckerman



The VCP is not approximable

- ▶ The VCP is not approximable: unless $P = NP$, no polynomial algorithm has a constant performance guarantee
- ▶ Not even a guarantee of the form $n^{1-\varepsilon}$
- ▶ Positive results exist only on special classes of graphs

Summary

No ratio below $\frac{4}{3}$:

- ▶ Otherwise 3-colorability would be decided in polynomial time
- ▶ The proof only uses that the number of colors is an integer

Much stronger results:

- ▶ Zuckerman (2007): not even the ratio $n^{1-\varepsilon}$, unless $P = NP$
- ▶ Hence no α -approximation algorithm for any constant $\alpha \geq 1$

Special cases

Introduction: Special cases

- ▶ On the four classes of this section the VCP is solved exactly, or with a guarantee, in polynomial time
- ▶ The classes are recognized from the graph, or given with it (the intervals of an interval graph)
- ▶ We see, in order: bipartite graphs, graphs of bounded maximum degree, planar graphs and interval graphs
- ▶ Where the general problem gives up, the structure of the graph lets us color well

Definition 1: bipartite graph

*A graph $G = (V, E)$ is **bipartite** if $\chi(G) \leq 2$, i.e., if V can be split into two stable sets.*

Proposition 1

Deciding whether $\chi(G) \leq 2$, and finding a coloring with 2 colors when it exists, takes $O(n + m)$ time.

- ▶ The idea of the proof: a breadth-first search, and the parity of the levels

Proof (1/2)

- ▶ Run a breadth-first search from a vertex of each connected component
 - ▶ Let $\ell(v)$ be the level of v , its distance from the root of its component
 - ▶ Color v with color 1 if $\ell(v)$ is even and with color 2 otherwise
1. If no edge joins two vertices of levels with the same parity, the coloring is feasible

Proof (2/2)

2. Otherwise, let $\{u, v\}$ be such an edge

- ▶ The paths of the search tree from the root to u and to v , with the edge $\{u, v\}$, form a closed walk of odd length

$$\ell(u) + \ell(v) + 1$$

- ▶ A closed walk of odd length contains a cycle of odd length, which needs 3 colors: $\chi(G) \geq 3$

The search and the check of the edges take $O(n + m)$ time. □

Odd cycles as certificates

- ▶ The proof shows the classical characterization: a graph is bipartite if and only if it has no cycle of odd length (Kőnig, 1936)
- ▶ On a graph with at least one edge, the search gives $\chi(G) = 2$, or the certificate $\chi(G) \geq 3$ given by an odd cycle

The greedy coloring

- ▶ The **greedy coloring** *GC* examines the vertices in a given order
- ▶ It gives each vertex the smallest color, among $1, 2, 3, \dots$, not used by its neighbours already colored
- ▶ It runs in $O(n + m)$ time

Proposition 2

For every graph G and every order of the vertices,

$$GC(I) \leq \Delta(G) + 1.$$

Proof

- ▶ When the vertex v is colored, its neighbours already colored use at most

$$\deg(v) \leq \Delta(G)$$

colors

- ▶ One of the colors $1, 2, \dots, \Delta(G) + 1$ is therefore free
- ▶ The smallest free color is at most $\Delta(G) + 1$



Is $\Delta(G) + 1$ a performance guarantee?

- ▶ The bound compares $GC(I)$ with $\Delta(G) + 1$, not with $\chi(G)$
- ▶ On all graphs it is not a performance guarantee, because $\Delta(G)$ can be $n - 1$ while $\chi(G) = 2$ (a star)
- ▶ On the graphs of maximum degree at most a *fixed* constant Δ , it is one
- ▶ Every graph with an edge has $\chi(G) \geq 2$, hence

$$\frac{GC(I)}{\chi(G)} \leq \frac{\Delta + 1}{2}.$$

When the bound is attained

- ▶ The bound $\Delta(G) + 1$ is attained by complete graphs and by cycles of odd length, whatever the order
- ▶ **Remark (Brooks, 1941).** A connected graph that is neither complete nor a cycle of odd length satisfies

$$\chi(G) \leq \Delta(G),$$

and such a coloring can be found in linear time

Planar graphs

- ▶ A graph is **planar** if it can be drawn in the plane without crossing edges
- ▶ Whether a graph is planar, and a drawing when it is, can be found in linear time (Hopcroft and Tarjan, 1974)
- ▶ We first show that a planar graph always has a vertex of small degree, then we use it to color with 5 colors

Lemma 1

Every planar graph with $n \geq 3$ vertices has $m \leq 3n - 6$ edges, and a vertex of degree at most 5.

Proof (1/2)

- ▶ Adding edges to a drawing without creating crossings does not decrease m
- ▶ So we may assume that no such edge can be added: the drawing is connected and every face is a triangle
- ▶ Euler's formula for a connected planar drawing with f faces gives

$$n - m + f = 2$$

- ▶ Every face has 3 edges and every edge borders 2 faces, so

$$3f = 2m$$

Proof (2/2)

- ▶ Substituting $f = \frac{2m}{3}$ in Euler's formula:

$$2 = n - m + f = n - m + \frac{2m}{3} \implies m \leq 3n - 6$$

- ▶ The degrees sum to

$$2m \leq 6n - 12 < 6n,$$

so some vertex has degree at most 5



Theorem 3: Heawood, 1890

Every planar graph can be colored with 5 colors, and such a coloring can be found in polynomial time.

- ▶ The idea of the proof: remove a vertex of degree at most 5, color the rest, and free a color for it by exchanging two colors

Proof (1/3)

- ▶ By induction on n ; graphs with $n \leq 5$ are trivial
 - ▶ Let v be a vertex of degree at most 5 (by the lemma above)
 - ▶ Color $G - v$, which is planar, with 5 colors
1. If the neighbours of v use at most 4 colors, v takes the fifth one
 2. Otherwise v has 5 neighbours $v_1, \dots, v_5$, in this order around v in the drawing, with colors $1, \dots, 5$. Let H_{13} be the graph induced by the vertices of colors 1 and 3

Proof (2/3)

3. If v_1 and v_3 are in different components of H_{13} , exchange the colors 1 and 3 in the component of v_1 : the coloring stays feasible, and color 1 becomes free for v
4. Otherwise a path of colors 1 and 3 joins v_1 to v_3
 - ▶ With v , it closes a cycle that separates v_2 from v_4 in the drawing
 - ▶ A path from v_2 to v_4 must cross that cycle, so v_2 and v_4 are in different components of H_{24}
 - ▶ Exchanging 2 and 4 in the component of v_2 frees color 2 for v

Proof (3/3)

- ▶ Each step uses a search in time

$$O(n + m) = O(n)$$

- ▶ There are n steps: $O(n^2)$ time in total



The four color theorem

- ▶ **Remark (the four color theorem).** Every planar graph can be colored with 4 colors (Appel and Haken, 1977)
- ▶ The proof uses a computer
- ▶ Robertson, Sanders, Seymour and Thomas (1997) simplified it and gave an algorithm that finds such a coloring in $O(n^2)$ time

The algorithm *PL* for planar graphs

The algorithm *PL* for planar graphs combines the previous results:

- ▶ if G has no edge, it uses 1 color;
- ▶ if G is bipartite, it uses 2 colors (by the breadth-first search of the bipartite graphs);
- ▶ otherwise it uses the 5 colors of the theorem of Heawood.

Corollary 2

PL is a polynomial 2-additive approximation algorithm for the VCP on planar graphs:

$$PL(I) - \chi(G) \leq 2.$$

With the four color algorithm in place of the five color one, the additive error is at most 1.

Proof

- ▶ In the first two cases PL is exact
- ▶ In the third case G has a cycle of odd length, so $\chi(G) \geq 3$, while $PL(I) \leq 5$: the additive error is at most

$$5 - 3 = 2$$

- ▶ With 4 colors it is at most

$$4 - 3 = 1$$



The best possible on planar graphs

- ▶ The additive error 1 cannot be improved to 0: deciding whether a planar graph has $\chi(G) \leq 3$ is NP-complete
- ▶ For the same reason, the ratio $\frac{4}{3}$ of the four color version is the best possible on planar graphs (by the theorem of $\frac{4}{3}$)

Definition 2: interval graph

A graph $G = (V, E)$ is an **interval graph** if every vertex v is associated with a closed interval $[a_v, b_v]$ of the real line, and $\{u, v\} \in E$ if and only if the two intervals intersect.

- Interval graphs describe requests over time that conflict when they overlap: lectures that need a room, reservations that need a resource. A color is a room.

Proposition 3

On an interval graph, the greedy coloring with the vertices in order of non-decreasing left endpoint a_v uses $\omega(G)$ colors. It is therefore optimal: $GC(I) = \omega(G) = \chi(G)$.

Proof (1/2)

- ▶ Let v be a vertex that receives color k
- ▶ The colors $1, \dots, k - 1$ are used by neighbours u of v colored before v , so with

$$a_u \leq a_v$$

1. Since $[a_u, b_u]$ intersects $[a_v, b_v]$ and $a_u \leq a_v$, we have

$$b_u \geq a_v :$$

the point a_v lies in the interval of each of these neighbours, and in the interval of v

Proof (2/2)

2. The k intervals that contain a_v pairwise intersect, so their vertices form a clique and

$$k \leq \omega(G)$$

Hence, by the clique bound,

$$GC(I) \leq \omega(G) \leq \chi(G) \leq GC(I).$$



The order matters

- ▶ Sorting the intervals takes $O(n \log n)$ time, and the coloring $O(n + m)$
- ▶ The order matters: with another order the greedy coloring may use more than $\omega(G)$ colors on an interval graph (see the example on six intervals in the next section)

Summary (1/2)

Bipartite graphs:

- ▶ A breadth-first search decides $\chi(G) \leq 2$ in $O(n + m)$ time
- ▶ A conflict gives an odd cycle, the certificate of $\chi(G) \geq 3$

Graphs of bounded maximum degree:

- ▶ The greedy coloring uses at most $\Delta(G) + 1$ colors, in any order
- ▶ With a fixed Δ , the ratio is at most $\frac{\Delta+1}{2}$

Summary (2/2)

Planar graphs:

- ▶ A vertex of degree at most 5 always exists, and 5 colors suffice (Heawood)
- ▶ PL has additive error at most 2, or 1 with the four color theorem

Interval graphs:

- ▶ The greedy coloring by left endpoint uses $\omega(G)$ colors, hence it is optimal
- ▶ The order matters

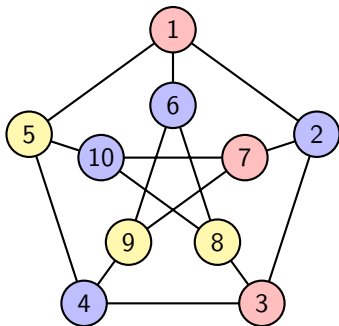
Examples

Introduction: Examples

- ▶ We apply the special cases to two small graphs
- ▶ First the Petersen graph, the reference instance of the VCP: is it bipartite, and how good is the greedy coloring?
- ▶ Then an interval graph with six intervals, to see that the order of the greedy coloring matters

Example: the special cases on the Petersen graph (1/3)

The reference instance of the VCP is the Petersen graph, with $n = 10$, $m = 15$ and $\Delta(G) = 3$.



Example: the special cases on the Petersen graph (2/3)

Is it bipartite?

- ▶ The search from vertex 1 gives the levels 0 to vertex 1, 1 to the vertices 2, 5, 6, and 2 to the other six
- ▶ The edge $\{3, 4\}$ joins two vertices of level 2
- ▶ With the paths 1, 2, 3 and 1, 5, 4 it closes the odd cycle 1, 2, 3, 4, 5, so

$$\chi(G) \geq 3$$

Example: the special cases on the Petersen graph (3/3)

Greedy coloring.

- ▶ In the order $1, 2, \dots, 10$, GC gives the colors

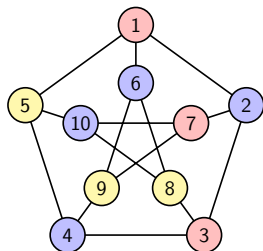
1, 2, 1, 2, 3, 2, 1, 3, 3, 2

(red, blue, yellow in the figure)

- ▶ $GC(I) = 3 \leq \Delta(G) + 1 = 4$
- ▶ Together,

$$3 \leq \chi(G) \leq GC(I) = 3 :$$

the greedy coloring is optimal on this instance

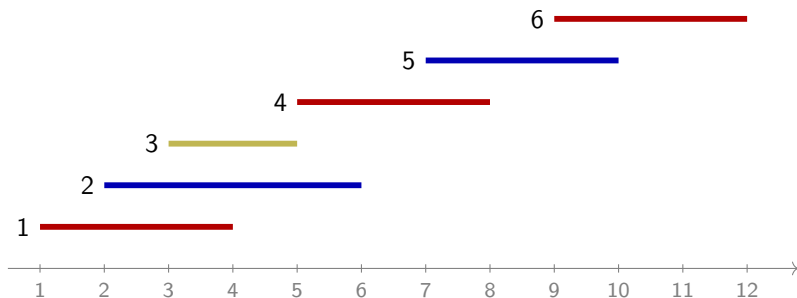


On the Petersen graph every order would be fine.

On an interval graph, instead,
the order of the greedy coloring matters:
let us see it on six intervals.

Example: the greedy coloring on an interval graph (1/3)

Six intervals, with the edges between intersecting intervals:



$$[a_v, b_v] : [1, 4], [2, 6], [3, 5], [5, 8], [7, 10], [9, 12],$$

$$E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}, \{5, 6\}\}.$$

Example: the greedy coloring on an interval graph (2/3)

- ▶ The left endpoints are already in increasing order

- ▶ GC gives the colors

1, 2, 3, 1, 2, 1

(red, blue, yellow in the figure): $GC(I) = 3$

- ▶ The intervals 1, 2, 3 contain the point 3, so $\{1, 2, 3\}$ is a clique and $\omega(G) = 3$
- ▶ The coloring is optimal, as the proposition on interval graphs guarantees

Example: the greedy coloring on an interval graph (3/3)

Another order.

- ▶ With the order 1, 2, 3, 5, 4, 6 the colors are 1, 2, 3, then 1 to vertex 5
- ▶ Vertex 4 gets color 4: its neighbours 2, 3, 5 use the colors 1, 2, 3
- ▶ 4 colors, one more than $\chi(G)$

Summary

The Petersen graph:

- ▶ The breadth-first search finds the odd cycle 1, 2, 3, 4, 5:
 $\chi(G) \geq 3$
- ▶ The greedy coloring in the order 1, 2, ..., 10 uses 3 colors: it is optimal

The interval graph:

- ▶ By left endpoint, the greedy coloring uses $\omega(G) = 3$ colors
- ▶ With the order 1, 2, 3, 5, 4, 6 it uses 4

Sources

Introduction: Sources

- ▶ We list the papers where the results of this chapter were proved
- ▶ First the negative results on the approximation of the chromatic number
- ▶ Then the positive results on the special classes of graphs

Sources (1/2)

- ▶ M.R. Garey, D.S. Johnson, L. Stockmeyer, Some simplified NP-complete graph problems, *Theoretical Computer Science* 1(3):237–267, 1976.
- ▶ M.R. Garey, D.S. Johnson, The complexity of near-optimal graph coloring, *Journal of the ACM* 23(1):43–49, 1976.
- ▶ U. Feige, J. Kilian, Zero knowledge and the chromatic number, *Journal of Computer and System Sciences* 57(2):187–199, 1998.
- ▶ D. Zuckerman, Linear degree extractors and the inapproximability of max clique and chromatic number, *Theory of Computing* 3:103–128, 2007.

Sources (2/2)

- ▶ R.L. Brooks, On colouring the nodes of a network, *Proceedings of the Cambridge Philosophical Society* 37(2):194–197, 1941.
- ▶ P.J. Heawood, Map-colour theorem, *Quarterly Journal of Pure and Applied Mathematics* 24, 1890.
- ▶ J. Hopcroft, R. Tarjan, Efficient planarity testing, *Journal of the ACM* 21(4):549–568, 1974.
- ▶ K. Appel, W. Haken, Every planar map is four colorable, *Illinois Journal of Mathematics* 21, 1977.
- ▶ N. Robertson, D. Sanders, P. Seymour, R. Thomas, The four-colour theorem, *Journal of Combinatorial Theory, Series B* 70(1):2–44, 1997.

Summary

Negative results:

- ▶ Garey, Johnson and Stockmeyer; Garey and Johnson; Feige and Kilian; Zuckerman

Positive results:

- ▶ Brooks; Heawood; Hopcroft and Tarjan; Appel and Haken; Robertson, Sanders, Seymour and Thomas

Typical mistakes

Introduction: Typical mistakes

- ▶ In this section we list the most common mistakes on the approximation of the VCP
- ▶ They concern the meaning of the bounds, the four color theorem, the order of the intervals and the odd cycle
- ▶ Recognizing them helps to state the results correctly

Observation 1: typical mistakes (1/2)

- ▶ **Calling the bound $\Delta(G) + 1$ a performance guarantee.** It compares the greedy coloring with $\Delta(G) + 1$, not with $\chi(G)$; it becomes a constant ratio only on graphs of bounded maximum degree.
- ▶ **Reading the theorem of $\frac{4}{3}$ as an upper bound.** It says that ratios below $\frac{4}{3}$ are impossible, not that a ratio of $\frac{4}{3}$ is achievable: by the result of Zuckerman, no constant ratio is.
- ▶ **Using the four color theorem as an exact algorithm.** It colors every planar graph with at most 4 colors, but deciding whether 3 colors suffice remains NP-complete on planar graphs.

Observation: typical mistakes (2/2)

- ▶ *Sorting the intervals by right endpoint, or by length.* The proposition on interval graphs uses the order of the left endpoints: the colors used before v all contain the point a_v .
- ▶ *Forgetting the odd cycle.* A breadth-first search that finds a conflict proves $\chi(G) \geq 3$ only through the odd cycle it closes; the cycle is the certificate.

Summary

The meaning of the results:

- ▶ $\Delta(G) + 1$ is not a ratio with $\chi(G)$, except on graphs of bounded degree
- ▶ The theorem of $\frac{4}{3}$ is a lower bound on the ratio, not an achievable ratio
- ▶ Four colors always suffice on planar graphs, but three are hard to decide

The details of the algorithms:

- ▶ Interval graphs: sort by left endpoint
- ▶ Bipartite test: the odd cycle is the certificate

In summary

Introduction: In summary

- ▶ We collect the key ideas of the chapter in a few points
- ▶ We close with a table of the results on the classes of graphs, and with the dashboard of the reference instance

Summary (1/3)

- ▶ Unless $P = NP$, no polynomial algorithm colors every graph with fewer than $\frac{4}{3}\chi(G)$ colors: it would decide 3-colorability
- ▶ By the result of Zuckerman, not even the ratio $n^{1-\epsilon}$ is possible: the VCP is not approximable
- ▶ Bipartite graphs and interval graphs are colored optimally in polynomial time
- ▶ On graphs of bounded degree the greedy coloring has the ratio $\frac{\Delta+1}{2}$
- ▶ On planar graphs the additive error is at most 2, or 1 with the four color theorem

Summary (2/3)

graphs	algorithm	result
all graphs	any polynomial	no constant ratio, unless $P = NP$
bipartite	breadth-first search	exact, $O(n + m)$
maximum degree $\leq \Delta$	greedy coloring	at most $\Delta + 1$ colors
planar	PL	additive error ≤ 2 (or ≤ 1)
interval	greedy by left endpoint	exact, $\omega(G)$ colors

Summary (3/3)

Dashboard of the reference instance.

Instance	Lower bound	Heuristic value	Optimal value
Petersen graph	3, odd cycle	<i>GC</i> 3	3