

Formulations of coloring problems



Outline

Overview of the family

An ILP model for the Vertex Coloring Problem

A first Linear Programming relaxation

A second Linear Programming relaxation

Typical mistakes

In summary

Learning objectives

At the end of this chapter you will be able to:

- ▶ write the ILP model of the Vertex Coloring Problem and explain the role of the activation variables y_c ;
- ▶ write the two LP relaxations and their duals, and compute their optimal solutions;
- ▶ prove that the optimal value of the LP relaxations is 2 for every graph with an edge;
- ▶ explain why this bound is very weak, and why the model is not enough to color a graph.

Prerequisites

- ▶ From the module *Linear Programming*, the chapter «[Duality theory](#)»: the dual of an LP problem, weak and strong duality
- ▶ From the module *Integer linear programming*, the chapters «[Integer linear programming](#)» and «[Primal and dual bounds](#)»: ILP problems, LP relaxations and bounds
- ▶ From this module, the chapter «[Coloring problems](#)»: the problem, its assumptions and the notation on graphs

Overview of the family

Introduction: Overview of the family

- ▶ In this section we see the common structure of the **ILP models** of the coloring problems
- ▶ All of them use an upper bound k on the number of colors, assignment variables and activation variables
- ▶ Recognizing this structure helps to read every model of the family, and to compare it with the models of the packing problems

The ingredients of the coloring models

- ▶ The ILP models of the coloring problems use an upper bound k on the number of colors and the set $\mathcal{C} = \{1, 2, \dots, k\}$ of the colors that may be used
- ▶ The binary **assignment variables** x_{vc} give a color $c \in \mathcal{C}$ to every vertex $v \in V$, and the binary **activation variables** y_c say whether color c is used
- ▶ Each edge and each color give one constraint, which forbids the same color on the two endpoints of the edge, and the objective function minimizes the number $\sum_{c \in \mathcal{C}} y_c$ of colors used

Coloring models are assignment models

The coloring models are **assignment models**, as the packing models: a vertex is assigned to a color as an item is assigned to a bin, and the edge constraints replace the capacity constraints.

Coloring models	Packing models
a vertex	an item
a color	a bin
the edge constraints	the capacity constraints

The members of the family

- ▶ The table lists the members of the family, with the name of their ILP model and the section that writes it

Problem	Model	Variables	Section
VCP	ILP_{VCP}	binary x_{vc} and y_c	An ILP model for the Vertex Coloring Problem

Summary

The ingredients:

- ▶ An upper bound k on the number of colors and the set $\mathcal{C} = \{1, 2, \dots, k\}$ of the colors
- ▶ Assignment variables x_{vc} and activation variables y_c , all binary
- ▶ One constraint for each edge and each color; the objective function counts the colors used

Assignment models:

- ▶ A vertex is assigned to a color as an item is assigned to a bin
- ▶ The edge constraints replace the capacity constraints of the packing models

An ILP model for the Vertex Coloring Problem

Introduction: An ILP model for the VCP

- ▶ The Vertex Coloring Problem and its assumptions are defined in the chapter «[Coloring problems](#)»
- ▶ In this section we introduce the variables, write the ILP_{VCP} model and describe its objective function and its constraints
- ▶ We explain the link between the assignment variables and the activation variables
- ▶ We write the model for a small graph, a triangle: it is the instance used in the whole chapter

The variables of the model (1/2)

- ▶ Given a graph $G = (V, E)$, for a vertex $v \in V$ we denote by $\delta(v)$ the set of the edges incident to v
- ▶ Given an upper bound $k \in \mathbb{Z}_{\geq 2}$ on the number of colors used in any optimal VCP solution, we introduce the following $k \underbrace{|V|}_{=n}$ binary variables:

$$x_{vc} = \begin{cases} 1 & \text{if vertex } v \text{ takes color } c, \\ 0 & \text{otherwise} \end{cases}$$

$$\forall v \in \underbrace{\{1, 2, \dots, n\}}_{=V}, \quad \forall c \in \underbrace{\{1, 2, \dots, k\}}_{=\mathcal{C}},$$

The variables of the model (2/2)

- ▶ and the following k binary variables:

$$y_c = \begin{cases} 1 & \text{if color } c \text{ is used,} \\ 0 & \text{otherwise} \end{cases} \quad \forall c \in \underbrace{\{1, 2, \dots, k\}}_{=\mathcal{C}}.$$

- ▶ The number n of vertices is a valid upper bound on the minimum number of colors needed to color all the vertices since, in the worst possible case, each vertex takes a different color
- ▶ Also the number of colors used in any feasible VCP solution is a valid upper bound

The ILP_{VCP} model

Using these variables, an Integer Linear Programming (ILP) model for the Vertex Coloring Problem (VCP) reads as follows:

$$\begin{aligned} \min \quad & \sum_{c \in \mathcal{C}} y_c \\ \text{subject to} \quad & \sum_{c \in \mathcal{C}} x_{vc} = 1, & \forall v \in V, \\ & -x_{uc} - x_{vc} + y_c \geq 0, & \forall c \in \mathcal{C}, \forall \{u, v\} \in E, \\ & x_{vc} \in \{0, 1\}, & \forall v \in V, \forall c \in \mathcal{C}, \\ & y_c \in \{0, 1\}, & \forall c \in \mathcal{C}. \end{aligned}$$

This ILP model is referred to as the ILP_{VCP} model.

The objective function and the constraints

Description of the objective function and constraints of the ILP_{VCP} model:

- ▶ the linear objective function $\min \sum_{c \in \mathcal{C}} y_c$ minimizes the total number of used colors;
- ▶ the constraints $\sum_{c \in \mathcal{C}} x_{vc} = 1$ ensure that each vertex takes exactly one color (n linear constraints);
- ▶ the constraints $-x_{uc} - x_{vc} + y_c \geq 0$ ensure that, for each color and each edge, the two adjacent vertices cannot both take that color, and that an unused color is taken by no vertex ($k \underbrace{|E|}_{=m}$ linear constraints);
- ▶ finally, the constraints $x_{vc} \in \{0, 1\}$ and $y_c \in \{0, 1\}$ define the variables of the ILP model.

The edge constraints tie together
the assignment variables x_{vc} and the activation variables y_c .

Let us see this link in detail.

The link between the variables (1/2)

- ▶ If color c is assigned to a vertex, then it is used. The logical implication is

$$(x_{1c} \text{ OR } x_{2c} \text{ OR } \dots \text{ OR } x_{nc}) \implies y_c.$$

Every vertex has an incident edge $\{u, v\}$, and the constraint $x_{uc} + x_{vc} \leq y_c$ forces $y_c = 1$ as soon as $x_{vc} = 1$.

- ▶ If color c is not used, then no vertex takes it. This is the contrapositive of the previous implication. For $y_c = 0$ the edge constraint gives $x_{uc} = x_{vc} = 0$ for every edge $\{u, v\} \in E$.

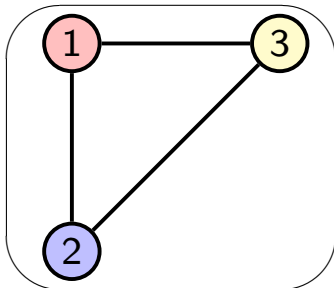
The link between the variables (2/2)

- ▶ If color c is used, then some vertex takes it, but only in an optimal solution.
- ▶ The constraints do not impose it
- ▶ However, if $x_{vc} = 0$ for all v and $y_c = 1$, setting $y_c = 0$ keeps all the constraints satisfied and decreases the objective value by one
- ▶ Hence, in every optimal solution, unused colors are not counted

We now write the ILP_{VCP} model
for a small instance: a triangle.

Example: the ILP_{VCP} model (1/4)

We consider the following graph $G = (V, E)$ with 3 vertices ($n = 3$) and 3 edges ($m = 3$):



The vertex set V and the edge set E are:

$$V = \{ 1, 2, 3 \}, \quad E = \{ \{1,2\}, \{1,3\}, \{2,3\} \}.$$

We refer to this VCP instance as the **triangle** example.

Example: the ILP_{VCP} model (2/4)

The ILP_{VCP} model with $k = 3$ for this instance is:

$$\begin{array}{llllllllllllll}
 \min & & & & & & & & & & & & & & y_1 & + & y_2 & + & y_3 \\
 \text{subject to} & x_{11} & + & x_{12} & + & x_{13} & & & & & & & & & & & & & = & 1 \\
 & & & & & & x_{21} & + & x_{22} & + & x_{23} & & & & & & & & = & 1 \\
 & & & & & & & & & & & x_{31} & + & x_{32} & + & x_{33} & & & = & 1 \\
 & -x_{11} & & & & & -x_{21} & & & & & & & & + & y_1 & & & \geq & 0 \\
 & -x_{11} & & & & & & & & & -x_{31} & & & & + & y_1 & & & \geq & 0 \\
 & & & & & & -x_{21} & & & & -x_{31} & & & + & y_1 & & & \geq & 0 \\
 & & -x_{12} & & & & & -x_{22} & & & & & & & + & y_2 & & & \geq & 0 \\
 & & -x_{12} & & & & & & & & -x_{32} & & & + & y_2 & & & \geq & 0 \\
 & & & & & & & -x_{22} & & & -x_{32} & & & + & y_2 & & & \geq & 0 \\
 & & & & -x_{13} & & & & -x_{23} & & & & & & + & y_3 & & \geq & 0 \\
 & & & & -x_{13} & & & & & & -x_{33} & & & + & y_3 & & \geq & 0 \\
 & & & & & & & -x_{23} & & & -x_{33} & & & + & y_3 & & \geq & 0 \\
 & x_{11}, & x_{12}, & x_{13}, & x_{21}, & x_{22}, & x_{23}, & x_{31}, & x_{32}, & x_{33}, & y_1, & y_2, & y_3 & \in & \{0, 1\}
 \end{array}$$

Example: the ILP_{VCP} model (3/4)

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{VCP}})$ is:

$$\tilde{x}_{11} = 1, \quad \tilde{x}_{12} = \tilde{x}_{13} = \tilde{x}_{21} = 0,$$

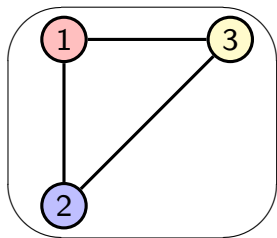
$$\tilde{x}_{22} = 1, \quad \tilde{x}_{23} = \tilde{x}_{31} = \tilde{x}_{32} = 0,$$

$$\tilde{x}_{33} = 1,$$

$$\tilde{y}_1 = \tilde{y}_2 = \tilde{y}_3 = 1$$

The optimal value $z(\text{ILP}_{\text{VCP}})$ is 3.

Example: the ILP_{VCP} model (4/4)



- ▶ The associated optimal solution of the VCP is given by the color classes

$$\tilde{Q}_1 = \{1\}, \quad \tilde{Q}_2 = \{2\}, \quad \tilde{Q}_3 = \{3\}$$

- ▶ The optimal partition is unique, but the 3! permutations of the colors give 6 different optimal solutions

Summary

The ILP_{VCP} model:

- ▶ k n binary variables x_{vc} (vertex v takes color c) and k binary variables y_c (color c is used)
- ▶ n constraints give one color to each vertex; k m edge constraints forbid equal colors on adjacent vertices

The link between the variables:

- ▶ A color taken by a vertex is used; an unused color is taken by no vertex
- ▶ A color that no vertex takes is switched off by the objective function, in every optimal solution

The triangle: $z(\text{ILP}_{\text{VCP}}) = 3$, with 6 optimal solutions that differ by a permutation of the colors

A first Linear Programming relaxation



Introduction: A first Linear Programming relaxation

- ▶ In this section we relax the binary variables of the ILP_{VCP} model into non-negative variables: we obtain the $LP-0_{VCP}$ model
- ▶ Its optimal value is a lower bound on $z(ILP_{VCP})$: we compute it on the triangle
- ▶ We write the dual model $D(LP-0_{VCP})$ and solve it on the triangle
- ▶ Finally we prove, in two ways, that the optimal value is 2 for every graph with an edge

The LP-0_{VCP} model

To obtain a Linear Programming (LP) relaxation of the ILP_{VCP} model, we relax the constraints $x_{vc} \in \{0, 1\}$ and $y_c \in \{0, 1\}$ with the following ones:

$$x_{vc} \geq 0, \quad \forall v \in V, \quad \forall c \in \mathcal{C},$$

$$y_c \geq 0, \quad \forall c \in \mathcal{C}.$$

This LP model is referred to as the LP-0_{VCP} model.

Example: the $LP-0_{VCP}$ model (1/2)

The $LP-0_{VCP}$ model for the triangle example is:

$$\begin{array}{llllllllllll}
 \min & & & & & & & & & & y_1 & + & y_2 & + & y_3 \\
 \text{subject to} & x_{11} & + & x_{12} & + & x_{13} & & & & & & & & & = 1 \\
 & & & & & & x_{21} & + & x_{22} & + & x_{23} & & & & = 1 \\
 & & & & & & & & & & x_{31} & + & x_{32} & + & x_{33} & = 1 \\
 & -x_{11} & & & & & -x_{21} & & & & & + & y_1 & & \geq 0 \\
 & -x_{11} & & & & & & & & & -x_{31} & & + & y_1 & \geq 0 \\
 & & & & & & -x_{21} & & & & -x_{31} & & + & y_1 & \geq 0 \\
 & & -x_{12} & & & & & -x_{22} & & & & & + & y_2 & \geq 0 \\
 & & -x_{12} & & & & & & & & -x_{32} & & + & y_2 & \geq 0 \\
 & & & & & & & -x_{22} & & & -x_{32} & & + & y_2 & \geq 0 \\
 & & & & -x_{13} & & & & -x_{23} & & & & + & y_3 & \geq 0 \\
 & & & & -x_{13} & & & & & & -x_{33} & & + & y_3 & \geq 0 \\
 & & & & & & & -x_{23} & & & -x_{33} & & + & y_3 & \geq 0 \\
 & x_{11}, & x_{12}, & x_{13}, & x_{21}, & x_{22}, & x_{23}, & x_{31}, & x_{32}, & x_{33}, & y_1, & y_2, & y_3 & \geq 0
 \end{array}$$

Example: the LP-0_{VCP} model (2/2)

An optimal solution $\tilde{\mathbf{x}}(\text{LP-0}_{\text{VCP}})$ is:

$$\tilde{x}_{11} = \tilde{x}_{12} = \tilde{x}_{21} = \tilde{x}_{22} = \tilde{x}_{31} = \tilde{x}_{32} = \frac{1}{2},$$

$$\tilde{x}_{13} = \tilde{x}_{23} = \tilde{x}_{33} = 0,$$

$$\tilde{y}_1 = \tilde{y}_2 = 1, \quad \tilde{y}_3 = 0$$

The optimal value $z(\text{LP-0}_{\text{VCP}})$ is 2.

Accordingly we have the lower bound:

$$2 \leq z(\text{ILP}_{\text{VCP}})$$

The dual of the LP-0_{VCP} model: the dual variables

Using

- ▶ a free dual variable μ_v for each constraint $\sum_{c \in \mathcal{C}} x_{vc} = 1$ ($v \in V$)
- ▶ a non-negative dual variable π_{ce} for each constraint $-x_{uc} - x_{vc} + y_c \geq 0$ ($c \in \mathcal{C}$, $e = \{u, v\} \in E$)

the dual model of the LP-0_{VCP} model reads as in the next slide.

The $D(\text{LP-0}_{\text{VCP}})$ model

$$\begin{aligned} & \max \quad \sum_{v \in V} \mu_v \\ & \text{subject to} \quad \mu_v - \sum_{e \in \delta(v)} \pi_{ce} \leq 0, \quad \forall v \in V, \forall c \in \mathcal{C}, \\ & \quad \quad \quad \sum_{e \in E} \pi_{ce} \leq 1, \quad \forall c \in \mathcal{C}, \\ & \quad \quad \quad \mu_v \geq 0, \quad \forall v \in V, \\ & \quad \quad \quad \pi_{ce} \geq 0, \quad \forall c \in \mathcal{C}, \forall e \in E. \end{aligned}$$

The obtained LP model is referred to as the $D(\text{LP-0}_{\text{VCP}})$ model.

Example: the $D(\text{LP-0}_{\text{VCP}})$ model (1/3)

The dual $D(\text{LP-0}_{\text{VCP}})$ model for the triangle example is:

$$\begin{array}{ll}
 \max & \mu_1 + \mu_2 + \mu_3 \\
 \text{subject to} & \mu_1 - \pi_{112} - \pi_{113} \\
 & \mu_1 \\
 & \mu_1 \\
 & \mu_2 - \pi_{112} - \pi_{123} \\
 & \mu_2 \\
 & \mu_2 \\
 & \mu_3 - \pi_{113} - \pi_{123} \\
 & \mu_3 \\
 & \mu_3 \\
 & \pi_{112} + \pi_{113} + \pi_{123} \\
 & \mu_1, \mu_2, \mu_3 \\
 & \pi_{112}, \pi_{113}, \pi_{123},
 \end{array}$$

Example: the $D(LP-0_{VCP})$ model (2/3)

Continued: the remaining columns and the right-hand sides.

$$\begin{array}{rll}
 \max & & \\
 \text{subject to} & & \leq 0 \\
 & -\pi_{212} & -\pi_{213} & \leq 0 \\
 & & -\pi_{312} & -\pi_{313} & \leq 0 \\
 & & & & \leq 0 \\
 & -\pi_{212} & & -\pi_{223} & \leq 0 \\
 & & -\pi_{312} & & -\pi_{323} & \leq 0 \\
 & & & & & \leq 0 \\
 & & -\pi_{213} & -\pi_{223} & \leq 0 \\
 & & & & -\pi_{313} & -\pi_{323} & \leq 0 \\
 & & & & & & \leq 1 \\
 & \pi_{212} & +\pi_{213} & +\pi_{223} & \leq 1 \\
 & & & \pi_{312} & +\pi_{313} & +\pi_{323} & \leq 1 \\
 & & & & & & \leq 0 \\
 & \pi_{212}, & \pi_{213}, & \pi_{223}, & \pi_{312}, & \pi_{313}, & \pi_{323} & \geq 0
 \end{array}$$

Example: the $D(\text{LP-0}_{\text{VCP}})$ model (3/3)

The optimal value $z(D(\text{LP-0}_{\text{VCP}}))$ is 2.

An optimal solution $(\tilde{\mu}, \tilde{\pi})$ of $D(\text{LP-0}_{\text{VCP}})$ is:

$$\tilde{\mu}_1 = \tilde{\mu}_2 = 1, \quad \tilde{\mu}_3 = 0,$$

$$\tilde{\pi}_{112} = \tilde{\pi}_{212} = \tilde{\pi}_{312} = 1,$$

$$\text{all the other } \tilde{\pi}_{ce} = 0$$

- The primal and the dual optimal values coincide:
 $z(\text{LP-0}_{\text{VCP}}) = 2 = z(D(\text{LP-0}_{\text{VCP}}))$

On the triangle, both optimal values are 2.

We now show that the same primal and dual solutions are optimal for **every** graph with at least one edge.

Proposition 1: optimal solutions (1/2)

An optimal solution of the primal LP-0_{VCP} model is:

$$\tilde{x}_{v1} = \tilde{x}_{v2} = \frac{1}{2}, \quad \forall v \in V,$$

$$\tilde{x}_{vc} = 0, \quad \forall v \in V, \quad \forall c \in \{3, 4, \dots, k\},$$

$$\tilde{y}_1 = \tilde{y}_2 = 1,$$

$$\tilde{y}_c = 0, \quad \forall c \in \{3, 4, \dots, k\},$$

Proposition: optimal solutions (2/2)

and an optimal solution of the dual $D(\text{LP-0}_{\text{VCP}})$ model is:

$$\tilde{\mu}_u = \tilde{\mu}_v = 1,$$

$$\tilde{\mu}_w = 0, \quad \forall w \in V \setminus \{u, v\},$$

$$\tilde{\pi}_{c\{u,v\}} = 1, \quad \forall c \in \mathcal{C},$$

$$\tilde{\pi}_{ce} = 0, \quad \forall c \in \mathcal{C}, \quad e \in E \setminus \{\{u, v\}\},$$

where $\{u, v\}$ is any edge in E .

Corollary 1

The optimal value of the primal $LP-0_{VCP}$ and dual $D(LP-0_{VCP})$ models is:

$$z(LP-0_{VCP}) = 2 = z(D(LP-0_{VCP}))$$

How the proof works

The proof proceeds in three steps:

1. We show that the primal solution of the proposition is a feasible solution for the primal $LP-0_{VCP}$ model;
2. We show that the dual solution of the proposition is a feasible solution for the dual $D(LP-0_{VCP})$ model;
3. We show that both feasible solutions have the same objective function value.

Therefore, by LP Weak Duality, the two solutions are a pair of optimal primal and dual solutions.

Proof (1/4)

1. For every $v \in V$, we have:

$$\sum_{c \in \mathcal{C}} \tilde{x}_{vc} = \frac{1}{2} + \frac{1}{2} = 1$$

Thus the constraints $\sum_{c \in \mathcal{C}} x_{vc} = 1$ are satisfied.

For $c \in \{1, 2\}$ and every $\{u, v\} \in E$, we have:

$$\tilde{y}_c - \tilde{x}_{uc} - \tilde{x}_{vc} = 1 - \frac{1}{2} - \frac{1}{2} = 0$$

and for $c \in \{3, 4, \dots, k\}$ we have $\tilde{y}_c - \tilde{x}_{uc} - \tilde{x}_{vc} = 0 - 0 - 0 = 0$.

Thus the constraints $-x_{uc} - x_{vc} + y_c \geq 0$ are satisfied.

Proof (2/4)

We have $\tilde{x}_{vc} \geq 0$ and $\tilde{y}_c \geq 0$, for every $v \in V$ and every $c \in \mathcal{C}$, thus the constraints $x_{vc} \geq 0$ and $y_c \geq 0$ are satisfied.

Therefore, the primal solution provides a feasible solution for the primal LP-0_{VCP} model.

2. For every $c \in \mathcal{C}$, we have:

$$\sum_{e \in E} \tilde{\pi}_{ce} = 1$$

Thus the constraints $\sum_{e \in E} \pi_{ce} \leq 1$ are satisfied.

Proof (3/4)

For every $c \in \mathcal{C}$ and $w \in \{u, v\}$, we have:

$$\tilde{\mu}_w - \sum_{e \in \delta(w)} \tilde{\pi}_{ce} = 1 - 1 = 0$$

For every $c \in \mathcal{C}$ and $w \in V \setminus \{u, v\}$, we have $\tilde{\mu}_w = 0$ and $\tilde{\pi}_{ce} = 0$ for every $e \in \delta(w)$ (since $\tilde{\pi}_{ce}$ is nonzero only for $e = \{u, v\}$, which is not incident to w), so

$$\tilde{\mu}_w - \sum_{e \in \delta(w)} \tilde{\pi}_{ce} = 0 - 0 = 0.$$

Thus the constraints $\mu_v - \sum_{e \in \delta(v)} \pi_{ce} \leq 0$ are satisfied.

Proof (4/4)

We have $\tilde{\mu}_v \geq 0$, $\forall v \in V$ and $\tilde{\pi}_{ce} \geq 0$ for every $c \in \mathcal{C}$ and every $e \in E$. Thus the constraints $\mu_v \geq 0$ and $\pi_{ce} \geq 0$ are satisfied.

Therefore, the dual solution provides a feasible solution for the dual $D(\text{LP-0}_{\text{VCP}})$ model.

3. For the objective functions we have:

$$\sum_{c \in \mathcal{C}} \tilde{y}_c = 1 + 1 = \sum_{v \in V} \tilde{\mu}_v$$

thus the primal and dual objective function values are equal.

Therefore, by LP Weak Duality, the two solutions are a pair of optimal primal and dual solutions. □

An alternative proof

- ▶ This is an alternative proof that does not rely on the dual formulation but instead reasons directly on the minimum objective function value imposed by the constraints

Proposition 2

An optimal solution of the primal LP-0_{VCP} model is:

$$\begin{aligned}\tilde{x}_{v1} &= \tilde{x}_{v2} = \frac{1}{2}, & \forall v \in V, \\ \tilde{x}_{vc} &= 0, & \forall v \in V, \forall c \in \{3, 4, \dots, k\}, \\ \tilde{y}_1 &= \tilde{y}_2 = 1, \\ \tilde{y}_c &= 0, & \forall c \in \{3, 4, \dots, k\},\end{aligned}$$

and the optimal value is $z(\text{LP-0}_{\text{VCP}}) = 2$.

Proof (1/4)

Summing the edge constraints $-x_{uc} - x_{vc} + y_c \geq 0$ over all $c \in \mathcal{C}$, for any edge $\{u, v\} \in E$, gives

$$\begin{aligned} \sum_{c \in \mathcal{C}} y_c - \sum_{c \in \mathcal{C}} x_{uc} - \sum_{c \in \mathcal{C}} x_{vc} &\geq 0 \\ \Rightarrow \sum_{c \in \mathcal{C}} y_c &\geq \underbrace{\sum_{c \in \mathcal{C}} x_{uc}}_{=1} + \underbrace{\sum_{c \in \mathcal{C}} x_{vc}}_{=1} = 2, \end{aligned}$$

where both sums are equal to 1 by the constraints $\sum_{c \in \mathcal{C}} x_{uc} = 1$.

Proof (2/4)

Hence

$$\sum_{c \in \mathcal{C}} y_c \geq 2.$$

Therefore every feasible solution of LP-0_{VCP} satisfies $\sum_{c \in \mathcal{C}} y_c \geq 2$.

Now consider the point defined in the proposition. For every $v \in V$, we have

$$\sum_{c \in \mathcal{C}} \tilde{x}_{vc} = \frac{1}{2} + \frac{1}{2} = 1,$$

so the constraints $\sum_{c \in \mathcal{C}} x_{vc} = 1$ hold.

Proof (3/4)

Moreover, for every $\{u, v\} \in E$ and $c \in \{1, 2\}$,

$$\tilde{y}_c - \tilde{x}_{uc} - \tilde{x}_{vc} = 1 - \frac{1}{2} - \frac{1}{2} = 0,$$

so the constraints $-x_{uc} - x_{vc} + y_c \geq 0$ hold with equality, while for all $c \geq 3$ both sides are zero.

Finally, the constraints $x_{vc} \geq 0$ and $y_c \geq 0$ are satisfied by construction.

Hence the point defined in the proposition is a feasible solution for LP-0_{VCP} .

Proof (4/4)

The objective value of this point is

$$\sum_{c \in \mathcal{C}} \tilde{y}_c = 1 + 1 = 2,$$

which matches the lower bound $\sum_{c \in \mathcal{C}} y_c \geq 2$.

Therefore the point is an optimal solution with objective value $z(\text{LP-0}_{\text{VCP}}) = 2$. □

Summary

The LP-0_{VCP} model:

- ▶ The binary domains are replaced by $x_{vc} \geq 0$ and $y_c \geq 0$
- ▶ Its dual $D(\text{LP-0}_{\text{VCP}})$ has a free variable μ_v for each vertex and a non-negative variable π_{ce} for each color and each edge

The optimal value is 2 for every graph with an edge:

- ▶ Primal: $\frac{1}{2}$ on colors 1 and 2 for all the vertices, $y_1 = y_2 = 1$
- ▶ Dual: one edge $\{u, v\}$, with $\mu_u = \mu_v = 1$ and $\pi_{c\{u,v\}} = 1$ for every color
- ▶ Alternative proof: summing the edge constraints of one edge gives $\sum_{c \in \mathcal{C}} y_c \geq 2$

A second Linear Programming relaxation

Introduction: A second Linear Programming relaxation

- ▶ In this section we write a second LP relaxation, the LP_{VCP} model, which keeps the upper bounds $y_c \leq 1$
- ▶ We solve it and its dual $D(LP_{VCP})$ on the triangle
- ▶ We prove that, also for this relaxation, the optimal value is 2 for every graph with an edge: adding constraints does not improve the bound

The LP_{VCP} model

To obtain a Linear Programming (LP) relaxation of the ILP_{VCP} model, we relax the constraints $x_{vc} \in \{0, 1\}$ and $y_c \in \{0, 1\}$ with the following ones:

$$x_{vc} \geq 0, \quad \forall v \in V, \quad \forall c \in \mathcal{C},$$

$$y_c \leq 1, \quad \forall c \in \mathcal{C},$$

$$y_c \geq 0, \quad \forall c \in \mathcal{C}.$$

This LP model is referred to as the LP_{VCP} model.

Example: the LP_{VCP} model (1/2)

The LP_{VCP} model for the triangle example is:

$$\begin{array}{llllllllllll}
 \min & & & & & & & & & & y_1 & + & y_2 & + & y_3 \\
 \text{subject to} & x_{11} & + & x_{12} & + & x_{13} & & & & & & & & & = & 1 \\
 & & & & & & x_{21} & + & x_{22} & + & x_{23} & & & & & = & 1 \\
 & & & & & & & & & & x_{31} & + & x_{32} & + & x_{33} & & = & 1 \\
 & -x_{11} & & & & & -x_{21} & & & & & & + & y_1 & & & \geq & 0 \\
 & -x_{11} & & & & & & & & & -x_{31} & & + & y_1 & & & \geq & 0 \\
 & & & & & & -x_{21} & & & & -x_{31} & & + & y_1 & & & \geq & 0 \\
 & & & -x_{12} & & & -x_{22} & & & & & & + & y_2 & & & \geq & 0 \\
 & & & -x_{12} & & & & & & & -x_{32} & & + & y_2 & & & \geq & 0 \\
 & & & & & & -x_{22} & & & & -x_{32} & & + & y_2 & & & \geq & 0 \\
 & & & & & -x_{13} & & & -x_{23} & & & & + & y_3 & & \geq & 0 \\
 & & & & & -x_{13} & & & & & -x_{33} & & + & y_3 & & \geq & 0 \\
 & & & & & & & & -x_{23} & & -x_{33} & & + & y_3 & & \geq & 0 \\
 & & & & & & & & & & & y_1 & & & & \leq & 1 \\
 & & & & & & & & & & & & y_2 & & & \leq & 1 \\
 & & & & & & & & & & & & & y_3 & & \leq & 1 \\
 & x_{11}, & x_{12}, & x_{13}, & x_{21}, & x_{22}, & x_{23}, & x_{31}, & x_{32}, & x_{33}, & y_1, & y_2, & y_3 & \geq & 0
 \end{array}$$

Example: the LP_{VCP} model (2/2)

An optimal solution $\tilde{\mathbf{x}}(LP_{VCP})$ is:

$$\tilde{x}_{11} = \tilde{x}_{12} = \tilde{x}_{21} = \tilde{x}_{22} = \tilde{x}_{31} = \tilde{x}_{32} = \frac{1}{2},$$

$$\tilde{x}_{13} = \tilde{x}_{23} = \tilde{x}_{33} = 0,$$

$$\tilde{y}_1 = \tilde{y}_2 = 1, \quad \tilde{y}_3 = 0$$

The optimal value $z(LP_{VCP})$ is 2. Accordingly we have the same lower bound:

$$2 \leq z(ILP_{VCP})$$

The dual of the LP_{VCP} model: the dual variables

Using

- ▶ a free dual variable μ_v for each constraint $\sum_{c \in \mathcal{C}} x_{vc} = 1$ ($v \in V$)
- ▶ a non-negative dual variable π_{ce} for each constraint $-x_{uc} - x_{vc} + y_c \geq 0$ ($c \in \mathcal{C}$, $e = \{u, v\} \in E$)
- ▶ a non-positive dual variable λ_c for each constraint $y_c \leq 1$ ($\forall c \in \mathcal{C}$)

the dual model of the LP_{VCP} model reads as in the next slide.

The $D(\text{LP}_{\text{VCP}})$ model

$$\begin{aligned} \max \quad & \sum_{v \in V} \mu_v + \sum_{c \in \mathcal{C}} \lambda_c \\ \text{subject to} \quad & \mu_v - \sum_{e \in \delta(v)} \pi_{ce} \leq 0, \quad \forall v \in V, \forall c \in \mathcal{C}, \\ & \sum_{e \in E} \pi_{ce} + \lambda_c \leq 1, \quad \forall c \in \mathcal{C}, \\ & \mu_v \geq 0, \quad \forall v \in V, \\ & \pi_{ce} \geq 0, \quad \forall c \in \mathcal{C}, \forall e \in E, \\ & \lambda_c \leq 0, \quad \forall c \in \mathcal{C}. \end{aligned}$$

The obtained LP model is referred to as the $D(\text{LP}_{\text{VCP}})$ model.

Example: the $D(\text{LP}_{\text{VCP}})$ model (1/3)

The dual $D(\text{LP}_{\text{VCP}})$ model for the triangle example is:

$$\begin{array}{ll}
 \max & \mu_1 + \mu_2 + \mu_3 \\
 \text{subject to} & \mu_1 \quad \quad \quad -\pi_{112} \quad -\pi_{113} \\
 & \mu_1 \quad \quad \quad \quad \quad \quad -\pi_{212} \quad -\pi_{213} \\
 & \mu_1 \\
 & \mu_2 \quad \quad -\pi_{112} \quad \quad \quad -\pi_{123} \\
 & \mu_2 \quad \quad \quad \quad \quad \quad -\pi_{212} \quad \quad \quad -\pi_{223} \\
 & \mu_2 \\
 & \mu_3 \quad \quad \quad -\pi_{113} \quad -\pi_{123} \\
 & \mu_3 \quad \quad \quad \quad \quad \quad -\pi_{213} \quad -\pi_{223} \\
 & \mu_3 \\
 & \quad \quad \pi_{112} \quad +\pi_{113} \quad +\pi_{123} \\
 & \quad \quad \quad \quad \quad \quad \pi_{212} \quad +\pi_{213} \quad +\pi_{223} \\
 & \mu_1, \mu_2, \mu_3 \\
 & \pi_{112}, \pi_{113}, \pi_{123}, \pi_{212}, \pi_{213}, \pi_{223},
 \end{array}$$

Example: the $D(\text{LP}_{\text{VCP}})$ model (2/3)

Continued: the remaining columns and the right-hand sides.

$$\begin{array}{rcll}
 \max & +\lambda_1 & +\lambda_2 & +\lambda_3 \\
 \text{subject to} & & & \leq 0 \\
 & & & \leq 0 \\
 & -\pi_{312} & -\pi_{313} & \leq 0 \\
 & & & \leq 0 \\
 & & & \leq 0 \\
 & -\pi_{312} & & -\pi_{323} & \leq 0 \\
 & & & \leq 0 \\
 & & & \leq 0 \\
 & & -\pi_{313} & -\pi_{323} & \leq 0 \\
 & & & +\lambda_1 & \leq 1 \\
 & & & +\lambda_2 & \leq 1 \\
 & \pi_{312} & +\pi_{313} & +\pi_{323} & +\lambda_3 & \leq 1 \\
 & & & & & \geq 0 \\
 & & & & & \leq 0 \\
 & \pi_{312}, & \pi_{313}, & \pi_{323} & & \geq 0 \\
 & & & \lambda_1, & \lambda_2, & \lambda_3 & \leq 0
 \end{array}$$

Example: the $D(\text{LP}_{\text{VCP}})$ model (3/3)

The optimal value $z(D(\text{LP}_{\text{VCP}}))$ is 2.

An optimal solution $(\tilde{\mu}, \tilde{\pi}, \tilde{\lambda})$ of $D(\text{LP}_{\text{VCP}})$ is:

$$\tilde{\mu}_1 = \tilde{\mu}_2 = 1, \quad \tilde{\mu}_3 = 0,$$

$$\tilde{\pi}_{112} = \tilde{\pi}_{212} = \tilde{\pi}_{312} = 1,$$

$$\text{all the other } \tilde{\pi}_{ce} = 0,$$

$$\tilde{\lambda}_1 = 0, \quad \tilde{\lambda}_2 = 0, \quad \tilde{\lambda}_3 = 0$$

Valid constraints that are not needed

- ▶ The constraints:

$$x_{vc} \leq 1, \quad \forall v \in V, \forall c \in \mathcal{C},$$

are also valid since, for every $v \in V$ and every $c \in \mathcal{C}$, we have:

$$x_{vc} \in \{0, 1\} \quad \Rightarrow \quad x_{vc} \leq 1$$

- ▶ But they are implied by the constraints $\sum_{c \in \mathcal{C}} x_{vc} = 1$

Proposition 3: optimal solutions (1/2)

An optimal solution of the primal LP_{VCP} model is:

$$\tilde{x}_{v1} = \tilde{x}_{v2} = \frac{1}{2}, \quad \forall v \in V,$$

$$\tilde{x}_{vc} = 0, \quad \forall v \in V, \quad \forall c \in \{3, 4, \dots, k\},$$

$$\tilde{y}_1 = \tilde{y}_2 = 1,$$

$$\tilde{y}_c = 0, \quad \forall c \in \{3, 4, \dots, k\},$$

Proposition: optimal solutions (2/2)

and an optimal solution of the dual $D(\text{LP}_{\text{VCP}})$ model is:

$$\tilde{\mu}_u = \tilde{\mu}_v = 1,$$

$$\tilde{\mu}_w = 0, \quad \forall w \in V \setminus \{u, v\},$$

$$\tilde{\pi}_{c\{u,v\}} = 1, \quad \forall c \in \mathcal{C},$$

$$\tilde{\pi}_{ce} = 0, \quad \forall c \in \mathcal{C}, \quad e \in E \setminus \{\{u, v\}\},$$

$$\tilde{\lambda}_c = 0, \quad \forall c \in \mathcal{C},$$

where $\{u, v\}$ is any edge in E .

Proof

The proof is similar to the previous one.

In addition it is necessary to observe that:

$$\tilde{y}_c \leq 1, \quad \forall c \in \mathcal{C}.$$

Accordingly, also the constraints $y_c \leq 1$ are satisfied.



Corollary 2

The optimal value of the primal LP_{VCP} and dual $D(LP_{VCP})$ models is:

$$z(LP_{VCP}) = 2 = z(D(LP_{VCP}))$$

Summary

The LP_{VCP} model:

- ▶ The binary domains are replaced by $x_{vc} \geq 0$, $y_c \leq 1$ and $y_c \geq 0$
- ▶ Its dual $D(LP_{VCP})$ has, in addition, a non-positive variable λ_c for each color
- ▶ The constraints $x_{vc} \leq 1$ are valid, but implied by
$$\sum_{c \in \mathcal{C}} x_{vc} = 1$$

The same bound:

- ▶ The same primal solution, and the same dual solution with $\lambda_c = 0$, are optimal
- ▶ $z(LP_{VCP}) = 2 = z(D(LP_{VCP}))$ for every graph with an edge

Typical mistakes

Introduction: Typical mistakes

- ▶ In this section we list the most common mistakes on the ILP_{VCP} model and its relaxations
- ▶ They concern the choice of k , the LP bound, the activation variables, the symmetry and the dual solution
- ▶ Recognizing them helps to write and to use the model correctly

Observation 1: typical mistakes (1/2)

- ▶ **Choosing k too small.** The number k of colors must be at least the chromatic number of the graph: with fewer colors the model has no feasible solution. The value $k = n$ is always valid.
- ▶ **Trusting the LP bound.** The optimal value of both relaxations is 2 for every graph with at least one edge, even for the complete graph on n vertices, whose optimal value is n .

Observation: typical mistakes (2/2)

- ▶ **Believing that the model counts only the colors that are really used.** The constraints allow $y_c = 1$ for a color that no vertex takes: it is the objective function that switches such colors off.
- ▶ **Ignoring the symmetry.** Permuting the colors of an optimal solution gives another optimal solution: a coloring that uses q of the k colors has $k!/(k - q)!$ equivalent copies ($k!$ when it uses all the colors).
- ▶ **Writing the dual solution with a wrong edge.** The dual values $\tilde{\pi}_{ce}$ are non-zero only for one edge $e = \{u, v\}$, and for every color c : all the other edges have value zero.

Summary

When writing the model:

- ▶ Choose k at least equal to the chromatic number: $k = n$ is always valid
- ▶ The objective function, not the constraints, switches off the unused colors
- ▶ Permuting the colors gives equivalent copies of the same coloring

When using the relaxations:

- ▶ The LP bound is 2 for every graph with an edge, even for the complete graph
- ▶ In the dual solution, only one edge has non-zero values $\tilde{\pi}_{ce}$

In summary

Introduction: In summary

- ▶ We collect the key ideas of the chapter in a few points
- ▶ We close with the dashboard of the triangle example: the bounds on $z(\text{ILP}_{\text{VCP}})$ computed in this chapter

Summary (1/2)

- ▶ The ILP_{VCP} model has $k \cdot n$ variables x_{vc} (vertex v takes color c) and k variables y_c (color c is used); the edge constraints forbid equal colors on adjacent vertices
- ▶ The two LP relaxations have the same optimal value 2 for every graph with an edge: put $\frac{1}{2}$ on two colors for all the vertices
- ▶ The dual solution selects one edge $\{u, v\}$ and gives value 1 to μ_u , μ_v and $\pi_{c\{u,v\}}$ for every color
- ▶ The bound 2 is very weak: a formulation that must color graphs needs additional valid inequalities

Summary (2/2)

Dashboard of the triangle example. Bounds on $z(\text{ILP}_{\text{VCP}})$ computed in this chapter:

Bound	Value	From
First relaxation	2	$z(\text{LP-0}_{\text{VCP}})$
Second relaxation	2	$z(\text{LP}_{\text{VCP}})$
Optimal value	3	color classes $\{1\}$, $\{2\}$, $\{3\}$

The gap between 2 and 3 is small here, but it grows with the size of the complete graph: 2 against n .