

Clique problems

Outline

Why clique problems?

Overview of the family

The maximum clique problem

Counting the solutions, a first solution and the bounds

Complexity

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Learning objectives

At the end of this chapter you will be able to:

- ▶ state the maximum clique problem and the maximum stable set problem, and pass from one to the other through the complement graph;
- ▶ describe an instance and a feasible solution, and count the candidate solutions of a small graph;
- ▶ find a clique by hand and compute four simple upper bounds on the clique number;
- ▶ state the weighted variants of the problem and its complexity.

Prerequisites

- ▶ From this module, the chapter «[Combinatorial optimization problems](#)»: instances, feasible solutions, primal and dual bounds, and the gap.
- ▶ From this module, the chapter «[Coloring problems](#)»: graphs, neighbourhoods, degrees, the complement graph and the chromatic number $\chi(G)$.

Why clique problems?

Introduction: Why clique problems?

- ▶ A **clique** is a set of vertices of a graph that are **pairwise adjacent**: every two of them are joined by an edge
- ▶ Its opposite is a **stable set**: a set of vertices that are pairwise non-adjacent
- ▶ Many applications look for the **largest** group of objects that are pairwise connected, or pairwise compatible
- ▶ We start from these applications; the rest of the chapter states the problems, bounds their optimal value and discusses how hard they are

Applications of cliques and stable sets

- ▶ **Social networks:** groups of pairwise connected members
- ▶ **Scheduling and planning of timetables:** groups of pairwise compatible tasks or exams
- ▶ **Chemistry and biology:** common substructures of two molecules
- ▶ **Codes that correct errors:** large sets of words pairwise far apart

Summary: Why clique problems?

- ▶ Cliques and stable sets describe groups of objects related **pair by pair**
- ▶ They appear in social networks, in scheduling and timetables, in chemistry and biology, and in the codes that correct errors
- ▶ The common question: how **large** (or how heavy) can such a group be?

Overview of the family

Introduction: Overview of the family

- ▶ In this section we meet the **family** of clique problems: they share the same data and the same kind of solutions
- ▶ We define the two basic objects of the whole chapter: the **clique** and the **clique number** $\omega(G)$
- ▶ A table lists the four members of the family that we study in this chapter

Clique problems: data, solutions, objective

- ▶ The **clique problems** share the same data: a simple undirected graph $G = (V, E)$, with

$$V = \{1, 2, \dots, n\}$$

and m edges, possibly with weights on the vertices or on the edges.

- ▶ A solution is a clique of G , that is, a subset of pairwise adjacent vertices.
- ▶ The objective is to maximize the size of the clique, or its total weight.

Definition 1: clique and clique number

Let $G = (V, E)$ be a simple undirected graph with n vertices and m edges.

1. A **clique** is a subset $K \subseteq V$ of pairwise adjacent vertices: $\{u, v\} \in E$ for all $u, v \in K$ with $u \neq v$. Equivalently, K induces a complete graph.
2. The **clique number** $\omega(G)$ is the largest number of vertices of a clique of G .

- ▶ A clique problem selects a clique of maximum size or of maximum weight.
- ▶ On the complement graph $\overline{G}$ the same problems select stable sets.

The members of the family

- ▶ The table lists the members of the family, with the section that defines each of them.

Problem	Acronym	Objective	Section
maximum clique problem	MCP	maximum number of vertices of a clique	The maximum clique problem
maximum stable set problem	MSSP	maximum number of vertices of a stable set	The maximum clique problem
vertex-weighted maximum clique problem	VWMCP	maximum total vertex weight	Weighted variants
edge-weighted maximum clique problem	EWMCP	maximum total edge weight	Weighted variants

Summary: Overview of the family

Data and solutions:

- ▶ A simple undirected graph $G = (V, E)$, possibly with weights on the vertices or on the edges
- ▶ A solution is a clique: a subset of pairwise adjacent vertices

Clique number:

- ▶ $\omega(G)$ is the largest number of vertices of a clique of G

The family:

- ▶ MCP and MSSP (size), VWMCP and EWMCP (weight)

The maximum clique problem

Introduction: The maximum clique problem

- ▶ In this section we state the **maximum clique problem** (MCP) and its twin, the **maximum stable set problem** (MSSP)
- ▶ The **complement graph** turns one problem into the other: whatever we learn on one of them holds for the other one
- ▶ We describe an MCP instance and its feasible solutions, and we meet the two **reference instances** of the chapter
- ▶ Finally, we fix the assumptions on the instances

Definition 2: Maximum Clique Problem

Given:

1. a simple graph $G = (V, E)$, where

$$V = \{1, 2, \dots, n\}$$

is the set of n vertices, and

$$E \subseteq \{ \{u, v\} : u, v \in V, u \neq v \}$$

is the set of m edges,

the **Maximum Clique Problem (MCP)** asks to find a clique $K \subseteq V$ of G whose number of vertices is maximum. Its optimal value is the clique number $\omega(G)$.

- ▶ Next to the cliques, the opposite notion: sets of vertices with **no** edge between them.

Definition 3: stable set and Maximum Stable Set Problem

Let $G = (V, E)$ be a graph.

1. A **stable set** (also called **independent set**) is a subset $Q \subseteq V$ of pairwise non-adjacent vertices: $\{u, v\} \notin E$ for all $u, v \in Q$ with $u \neq v$.
2. The **Maximum Stable Set Problem** (MSSP) asks to find a stable set of G whose number of vertices is maximum. Its optimal value is the **stability number** $\alpha(G)$.

- The bridge between cliques and stable sets is the complement graph: it swaps edges and non-edges.

Definition 4: complement graph

The **complement graph** of $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$ with the same vertices and the edge set

$$\overline{E} = \{ \{u, v\} : u, v \in V, u \neq v, \{u, v\} \notin E \}.$$

The pairs in $\overline{E}$ are the **non-edges** of G , and $|\overline{E}| = \binom{n}{2} - m$.

Proposition 1: clique number and stability number

A subset $K \subseteq V$ is a clique of G if and only if it is a stable set of $\overline{G}$. Hence

$$\omega(G) = \alpha(\overline{G}) \quad \text{and} \quad \alpha(G) = \omega(\overline{G}).$$

Proof

- ▶ Two distinct vertices u, v are adjacent in G if and only if they are not adjacent in $\overline{G}$:

$$\{u, v\} \in E \iff \{u, v\} \notin \overline{E}.$$

- ▶ So all the pairs of K are edges of G exactly when no pair of K is an edge of $\overline{G}$.
- ▶ The two problems have the same feasible solutions, hence the same optimal value:

$$\omega(G) = \alpha(\overline{G}).$$

- ▶ The second equality follows since $\overline{\overline{G}} = G$.



MCP and MSSP: the same problem

The MCP on G and the MSSP on $\overline{G}$ are the same problem:
a maximum clique of G is a maximum stable set of $\overline{G}$.

Whatever we learn on one of the two problems
holds for the other one.

MCP instances and feasible solutions

- ▶ An **MCP instance** is specified by:
 1. two integer numbers: the number of vertices $n \in \mathbb{Z}_{\geq 1}$ and the number of edges $m \in \mathbb{Z}_{\geq 1}$;
 2. two vectors of integer numbers of length m : $\mathbf{u} \in \mathbb{Z}_{\geq 1}^m$ and $\mathbf{v} \in \mathbb{Z}_{\geq 1}^m$, with the two endpoints of the edges.
- ▶ An **MCP feasible solution** is a clique $K \subseteq V$. The empty set and a single vertex are cliques, so every instance has feasible solutions.

The MCP in formulas

Given an MCP instance, the MCP can be formally stated as:

$$\tilde{K} \in \arg \max_{K \subseteq V} \left\{ |K| : \{u, v\} \in E, \quad \forall u, v \in K \text{ with } u \neq v \right\},$$

$$|\tilde{K}| = \omega(G).$$

- ▶ The objective counts the vertices of K .
- ▶ The condition asks that every pair of distinct vertices of K is an edge of G .

Two reference instances

Two instances will accompany us in the whole chapter:

a sparse one, **the cycle with a chord**,
and a **dense graph** with 8 vertices.

On them we will count the solutions, find a clique by hand
and compute the bounds on $\omega(G)$.

Example: a cycle with a chord (1/2)

- ▶ We consider the graph $G = (V, E)$ with 7 vertices ($n = 7$) and 8 edges ($m = 8$): the cycle $1, 2, \dots, 7, 1$ and the chord $\{1, 3\}$.

- ▶ The vertex set and the edge set are:

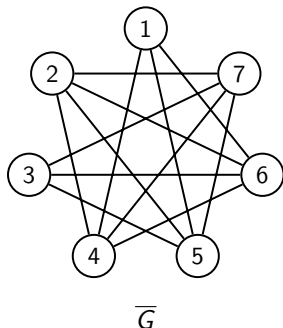
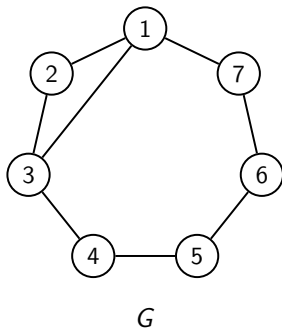
$$V = \{1, \dots, 7\},$$

$$E = \{\{1, 2\}, \{1, 3\}, \{1, 7\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 7\}\}.$$

- ▶ We call this MCP instance **the cycle with a chord**: it is the reference instance of the chapter.

Example: a cycle with a chord (2/2)

- On the right, the complement graph $\overline{G}$, with the $\binom{7}{2} - 8 = 13$ non-edges of G .



Example: a dense graph with 8 vertices (1/2)

- ▶ We consider the graph $G = (V, E)$ with 8 vertices placed on a circle, where every vertex is adjacent to all the others except the opposite one.

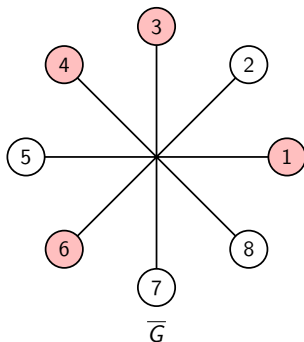
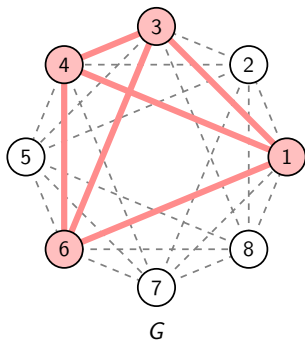
- ▶ It has

$$m = \binom{8}{2} - 4 = 24$$

edges, and its complement has only the 4 edges $\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}$.

- ▶ A maximum clique $\tilde{K} = \{1, 3, 4, 6\}$ is in red in the next figure.
- ▶ We call this MCP instance **the dense graph**.

Example: a dense graph with 8 vertices (2/2)



- In $\overline{G}$ the red vertices form a stable set: they take one vertex from each of the 4 edges.

A third graph: the Petersen graph

The Petersen graph of the chapter «Coloring problems»
has no triangle.

So its largest cliques are its edges, and

$$\omega(G) = 2.$$

Assumptions

- ▶ We consider nontrivial instances: the graph has at least one edge and at least one non-edge. Formally:

$$1 \leq m < \binom{n}{2}.$$

- ▶ The assumption can be made without loss of generality:
 - ▶ if the graph has no edge, every single vertex is a maximum clique and $\omega(G) = 1$;
 - ▶ if the graph is complete, V itself is a clique and $\omega(G) = n$.

Summary: The maximum clique problem (1/2)

Two twin problems:

- ▶ MCP: a clique with the largest number of vertices, optimal value $\omega(G)$
- ▶ MSSP: a stable set with the largest number of vertices, optimal value $\alpha(G)$

Complement graph:

- ▶ $\overline{G}$ has the non-edges of G as edges: $|\overline{E}| = \binom{n}{2} - m$
- ▶ A clique of G is a stable set of $\overline{G}$: $\omega(G) = \alpha(\overline{G})$ and $\alpha(G) = \omega(\overline{G})$

Summary: The maximum clique problem (2/2)

Instances and solutions:

- ▶ An instance: n , m and the endpoints $\mathbf{u}$, $\mathbf{v}$ of the edges
- ▶ A feasible solution: a clique (the empty set and a single vertex are cliques)

Reference instances:

- ▶ The cycle with a chord ($n = 7$, $m = 8$) and the dense graph ($n = 8$, $m = 24$)

Assumptions:

- ▶ Nontrivial instances: $1 \leq m < \binom{n}{2}$

Counting the solutions, a first solution and the bounds

Introduction: Counting the solutions, a first solution and the bounds

- ▶ In this section we work by hand on the two reference instances
- ▶ We count the candidate solutions and the cliques, and we see how few subsets are cliques
- ▶ We build a first clique by hand: a **primal bound** on $\omega(G)$
- ▶ We prove four simple **upper bounds** on $\omega(G)$: together with the primal bound, they certify how far we are from the optimum

Candidate solutions: the cycle with a chord

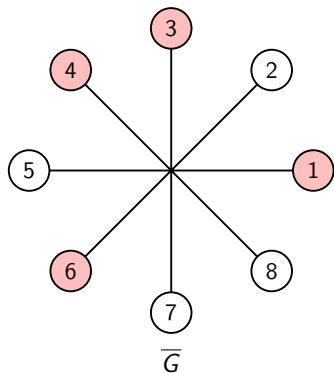
- ▶ The candidate solutions are the subsets of V : there are 2^n of them.
- ▶ For the cycle with a chord we have $2^7 = 128$ subsets, and only 17 of them are cliques:

number of vertices	0	1	2	3	total
cliques	1	7	8	1	17

- ▶ The only clique with 3 vertices is $\{1, 2, 3\}$, so the optimal solution is unique and

$$\omega(G) = 3.$$

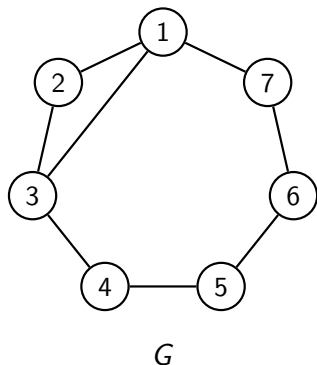
Candidate solutions: the dense graph



- ▶ For the dense graph we have $2^8 = 256$ subsets and 81 cliques.
- ▶ A clique takes at most one vertex from each pair $\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}$.
- ▶ The maximum cliques take exactly one, so there are $2^4 = 16$ of them and

$$\omega(G) = 4.$$

A first solution, by hand



- ▶ We start from a vertex of maximum degree and add, one at a time, a vertex adjacent to all the vertices already chosen.
- ▶ In the cycle with a chord, vertex 1 has degree 3; we add its neighbour 2, and then 3, which is adjacent to both.
- ▶ No other vertex is adjacent to 1, 2 and 3, so we stop with $K = \{1, 2, 3\}$.

- ▶ K is a **primal bound**: $LB = 3 \leq \omega(G)$.

Simple upper bounds

- ▶ The MCP is a maximization problem: a dual bound is an **upper bound** on $\omega(G)$.
- ▶ We denote by $\deg(v)$ the degree of a vertex and by Δ the maximum degree of the graph.
- ▶ We now see four simple upper bounds: the **degree bound**, the **edge bound**, the **degree count bound** and the **coloring bound**.

- ▶ The first bound looks at the largest degree of the graph.

Observation 1: degree bound

For every graph $G = (V, E)$ we have $\omega(G) \leq \Delta + 1$.

Proof

- ▶ In a clique K , every vertex is adjacent to the other $|K| - 1$ vertices of K , so its degree is at least $|K| - 1$:

$$|K| - 1 \leq \deg(v) \leq \Delta \quad \forall v \in K.$$

- ▶ Hence $|K| - 1 \leq \Delta$ for every clique, and in particular for a maximum one:

$$\omega(G) \leq \Delta + 1.$$



- The second bound looks at the number of edges of the graph.

Observation 2: edge bound

For every graph $G = (V, E)$ we have $\omega(G)(\omega(G)-1)/2 \leq m$.

Proof

- ▶ A clique with k vertices contains all the pairs of its vertices as edges:

$$\binom{k}{2} = \frac{k(k-1)}{2}.$$

- ▶ These are edges of G , so $k(k-1)/2 \leq m$; with $k = \omega(G)$ we get

$$\omega(G)(\omega(G)-1)/2 \leq m.$$



- ▶ The third bound counts the vertices of large degree.

Observation 3: degree count bound

Let $k^{\max}$ be the largest integer k such that at least k vertices of G have degree at least $k - 1$. Then $\omega(G) \leq k^{\max}$.

Proof

- ▶ A clique with k vertices has k vertices, each of degree at least $k - 1$ (proof of the degree bound).
- ▶ So $k = \omega(G)$ satisfies the condition, and

$$\omega(G) \leq k^{\max}.$$



- ▶ The fourth bound comes from the colorings of the graph.

Observation 4: coloring bound

For every graph $G = (V, E)$ we have $\omega(G) \leq \chi(G)$. In particular, the number of colors of any feasible coloring of G is an upper bound on $\omega(G)$.

Proof

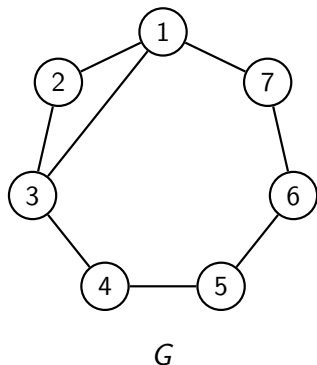
- ▶ The vertices of a clique are pairwise adjacent, so they take pairwise different colors.
- ▶ A coloring with k colors has

$$k \geq |K| \quad \text{for every clique } K.$$

- ▶ This is the clique bound of the chapter «Coloring problems», read from the side of the clique.

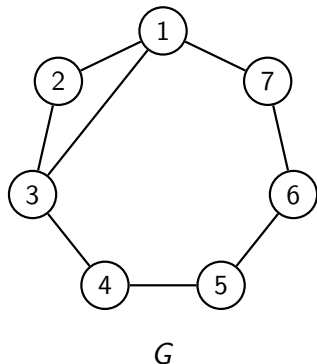


Bounds on the cycle with a chord (1/2)



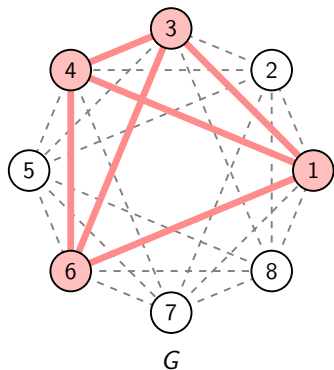
- ▶ Vertices 1 and 3 have degree 3, the other five have degree 2.
- ▶ The degree bound gives $\Delta + 1 = 4$.
- ▶ The edge bound gives 4 too, since
$$4 \cdot 3/2 = 6 \leq 8 < 10 = 5 \cdot 4/2.$$

Bounds on the cycle with a chord (2/2)



- ▶ Only 2 vertices have degree at least 3, so $k = 4$ fails the condition of the degree count bound, while $k = 3$ satisfies it: $k^{\max} = 3$.
- ▶ The coloring $\{1, 4, 6\}, \{2, 5, 7\}, \{3\}$ gives 3 as well.
- ▶ The gap with $LB = 3$ is zero.

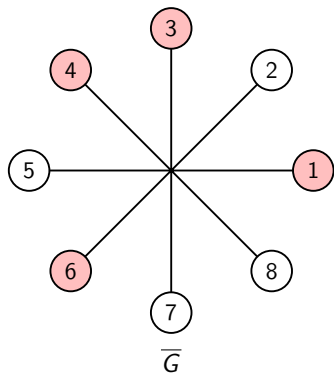
Bounds on the dense graph (1/2)



- ▶ Every vertex has degree 6: the degree bound and the degree count bound give 7.
- ▶ The edge bound gives 7, since

$$7 \cdot 6/2 = 21 \leq 24 < 28.$$

Bounds on the dense graph (2/2)



- ▶ The four pairs $\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}$ are stable sets, so they color the graph with 4 colors: the coloring bound gives $4 = \omega(G)$.
- ▶ On dense graphs the bounds based on degrees are weak, and a good coloring is much better.

Summary: Counting the solutions, a first solution and the bounds (1/2)

Candidate solutions:

- ▶ The 2^n subsets of V ; few of them are cliques
- ▶ Cycle with a chord: 17 cliques out of 128 subsets, unique optimum $\{1, 2, 3\}$
- ▶ Dense graph: 81 cliques out of 256 subsets, 16 maximum cliques

A first solution, by hand:

- ▶ Start from a vertex of maximum degree, add vertices adjacent to all the chosen ones
- ▶ The clique found is a primal bound: $LB \leq \omega(G)$

Summary: Counting the solutions, a first solution and the bounds (2/2)

Four upper bounds:

- ▶ Degree bound $\Delta + 1$ and edge bound $\omega(G)(\omega(G) - 1)/2 \leq m$
- ▶ Degree count bound $k^{\max}$ and coloring bound: any coloring

On the reference instances:

- ▶ Cycle with a chord: $LB = 3$ and $k^{\max} = 3$, a coloring with 3 colors: gap zero
- ▶ Dense graph: the degree bounds give 7, a coloring gives $4 = \omega(G)$

Complexity

Introduction: Complexity

- ▶ How hard is the MCP? We look at its **decision version**, at the **enumeration** of the subsets and at **approximation**
- ▶ The answer is negative on all three fronts: the problem is NP-hard, enumeration is exponential, and even approximating it is hard
- ▶ This motivates the bounds of this chapter, and the formulations and algorithms that come later in the course

Decision version and NP-hardness

- ▶ The **decision version** of the MCP asks, for a graph G and an integer k , whether G has a clique with k vertices.
- ▶ Karp (1972) included it among his 21 NP-complete problems, so the MCP is NP-hard.
- ▶ By the proposition on the clique number and the stability number, the same holds for the MSSP.

Enumeration

- ▶ Listing all the subsets of V and checking the pairs of each one takes

$$O(2^n n^2) \text{ steps.}$$

- ▶ For a *fixed* k , the $\binom{n}{k}$ subsets with k vertices are checked in

$$O(n^k k^2) \text{ steps.}$$

- ▶ This is a polynomial for every fixed k , but the exponent grows with the size of the clique.

Hard to approximate

- ▶ The MCP is also hard to approximate.
- ▶ Håstad (1999) proved that, for every $\varepsilon > 0$, no polynomial algorithm guarantees a clique with at least

$$\frac{\omega(G)}{n^{1-\varepsilon}}$$

vertices, unless every problem in NP can be solved in randomized polynomial time.

Summary: Complexity

NP-hardness:

- ▶ The decision version of the MCP is one of Karp's 21 NP-complete problems (1972)
- ▶ The MCP and the MSSP are NP-hard

Enumeration:

- ▶ $O(2^n n^2)$ steps in general, $O(n^k k^2)$ for a fixed clique size k

Approximation:

- ▶ No polynomial algorithm guarantees $\omega(G)/n^{1-\varepsilon}$ vertices (Håstad, 1999), unless NP problems are solvable in randomized polynomial time

Weighted variants

Introduction: Weighted variants

- ▶ Often the vertices or the edges of the graph have a **weight**: we look for the heaviest clique, not for the largest one
- ▶ Two variants: weights on the vertices (VWMCP) and weights on the edges (EWMCP)
- ▶ With unit weights both variants give back the MCP; with general weights the optimal clique can change, as the dense graph shows

Vertex weights and edge weights

- ▶ Let $\mathbf{w} \in \mathbb{Z}_{\geq 1}^n$ be a **vertex-weight vector**, with a weight w_v for each vertex $v \in V$.
- ▶ Let $\mathbf{r} \in \mathbb{Z}_{\geq 1}^m$ be an **edge-weight vector**, with a weight r_{uv} for each edge $\{u, v\} \in E$.
- ▶ For a clique K we define

$$w(K) = \sum_{v \in K} w_v \quad \text{and} \quad r(K) = \sum_{\{u,v\} \in E[K]} r_{uv},$$

where $E[K]$ is the set of the edges with both endpoints in K .

Definition 5: weighted maximum clique problems

1. Given a graph G and a vertex-weight vector $\mathbf{w}$, the **vertex-weighted maximum clique problem** (VWMCP) asks to find a clique K of G with maximum total vertex weight $w(K)$.
2. Given a graph G and an edge-weight vector $\mathbf{r}$, the **edge-weighted maximum clique problem** (EWMCP) asks to find a clique K of G with maximum total edge weight $r(K)$.

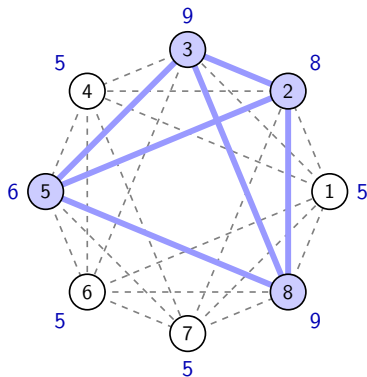
Unit weights give back the MCP

- ▶ With all the weights $w_v = 1$ the first problem is the MCP.
- ▶ The edge-weighted problem rewards the cliques with many heavy edges.
- ▶ With $r_{uv} = 1$ it is again the MCP, since

$$r(K) = \binom{|K|}{2}$$

grows with $|K|$.

Example: weights on the vertices (1/2)



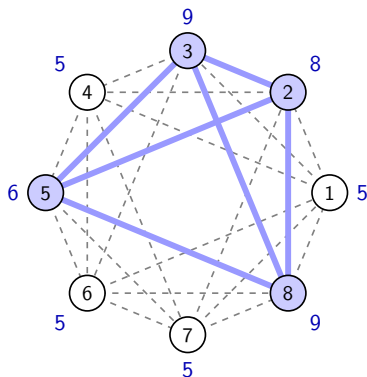
- ▶ We add to the dense graph the vertex weights in blue.
- ▶ The vertex-weighted maximum clique is $\{2, 3, 5, 8\}$, with weight

$$8 + 9 + 6 + 9 = 32 :$$

it is unique.

- ▶ It is not the maximum clique in red of the dense graph, whose weight is $5 + 9 + 5 + 5 = 24$.

Example: weights on the vertices (2/2)



- ▶ A clique takes at most one vertex from each pair $\{1, 5\}$, $\{2, 6\}$, $\{3, 7\}$, $\{4, 8\}$.
- ▶ Taking the heavier one of each pair gives

$$6 + 8 + 9 + 9 = 32,$$

and these four vertices form a clique.

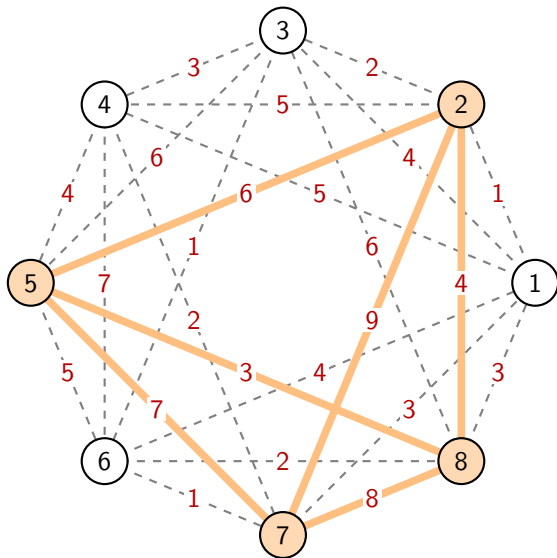
Example: weights on the edges (1/2)

- ▶ We add to the dense graph the edge weights in red.
- ▶ The edge-weighted maximum clique is $\{2, 5, 7, 8\}$, in orange in the next figure: its six edges weigh

$$6 + 4 + 9 + 7 + 3 + 8 = 37.$$

- ▶ The enumeration of the 16 maximum cliques shows that it is the unique optimal solution.

Example: weights on the edges (2/2)



Summary: Weighted variants

Weights:

- ▶ Vertex weights $\mathbf{w}$, with $w(K) = \sum_{v \in K} w_v$
- ▶ Edge weights $\mathbf{r}$, with $r(K)$ the total weight of the edges of $E[K]$

Two problems:

- ▶ VWMCP: maximum $w(K)$; EWMCP: maximum $r(K)$
- ▶ With unit weights both are the MCP

On the dense graph:

- ▶ VWMCP: $\{2, 3, 5, 8\}$ with 32; EWMCP: $\{2, 5, 7, 8\}$ with 37; both unique

Remarks

Introduction: Remarks

- ▶ Two remarks close the chapter
- ▶ The first one links cliques and stable sets to the **colorings** of the previous chapter
- ▶ The second one comes back to the **applications** that opened the chapter

Cliques, stable sets and colorings

- ▶ In a coloring, every color class is a stable set.
- ▶ The MSSP is the problem of the largest color class.
- ▶ The clique number is the dual bound of the chapter «Coloring problems»:

$$\omega(G) \leq \chi(G).$$

Applications of cliques and stable sets, seen again

- ▶ Groups of pairwise connected members in **social networks**.
- ▶ Groups of pairwise compatible tasks or exams, in **scheduling** and in the **planning of timetables**.
- ▶ Common substructures of two molecules, in **chemistry** and in **biology**.
- ▶ **Codes that correct errors**, as large sets of words pairwise far apart.

Summary: Remarks

Colorings:

- ▶ Every color class is a stable set; the MSSP looks for the largest one
- ▶ The clique number is a dual bound for the coloring problem

Applications:

- ▶ Social networks, scheduling and timetables, chemistry and biology, codes that correct errors

Typical mistakes

Introduction: Typical mistakes

- ▶ Four mistakes that come up often in the exercises on cliques
- ▶ They concern the words (maximal or maximum), the complement graph, and the direction of the bounds
- ▶ Checking them is a quick way to avoid losing points

Observation 5: typical mistakes (1/2)

- ▶ *Confusing a maximal clique with a maximum clique.* A clique is maximal when no vertex can be added to it; it is maximum when no clique is larger. In the cycle with a chord, $\{4, 5\}$ is maximal but not maximum.
- ▶ *Building the complement graph with the edges of G .* The edges of $\overline{G}$ are exactly the pairs that are not edges of G : there are $\binom{n}{2} - m$ of them.

Observation: typical mistakes (2/2)

- ▶ *Using a lower bound as an upper bound. A clique found by hand gives $LB \leq \omega(G)$; the degree bounds and a coloring give $UB \geq \omega(G)$. In this chapter the problem is a maximization.*
- ▶ *Believing that the coloring bound is tight. On the cycle with 5 vertices $\omega(G) = 2$ and $\chi(G) = 3$.*

Summary: Typical mistakes

- ▶ Maximal (no vertex can be added) is not maximum (no larger clique)
- ▶ The edges of $\overline{G}$ are the non-edges of G
- ▶ A clique gives a lower bound, degrees and colorings give upper bounds
- ▶ The coloring bound is not always tight: $\omega(G) < \chi(G)$ on the cycle with 5 vertices

Summary (1/2)

- ▶ The MCP looks for the largest set of pairwise adjacent vertices; its optimal value is the clique number $\omega(G)$.
- ▶ A clique of G is a stable set of $\overline{G}$: the MCP on G is the MSSP on $\overline{G}$, and $\omega(G) = \alpha(\overline{G})$.
- ▶ A clique found by hand is a primal bound; the degree bound, the edge bound, the degree count bound and the number of colors of a coloring are dual bounds.
- ▶ The MCP is NP-hard and hard to approximate; its weighted variants weigh the vertices or the edges.

Summary (2/2)

Dashboard of the reference instances.

instance (max)	lower bound	upper bound	from
the cycle with a chord	3 (by hand)	3 (degree count, coloring)	this chapter
the dense graph	4 (by hand)	4 (coloring)	this chapter