

Formulations of clique problems



Outline

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- A second Linear Programming relaxation

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- Stable sets on the complement graph

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 - The vertex-weighted maximum clique problem

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- Typical mistakes

Learning objectives

At the end of this chapter you will be able to:

- ▶ write the edge formulation of the Maximum Clique Problem on the non-edges of G , that is, on the edges of $\overline{G}$;
- ▶ explain why its LP relaxation is weak: the point $x_v = \frac{1}{2}$ is always feasible;
- ▶ write the clique formulation and compare the two LP bounds on a small graph;
- ▶ prove $\omega(G) \leq z(\text{LP}_{\text{MCP}}^{\text{CL}}) \leq \chi(G)$ through LP duality and a coloring formulation;
- ▶ write the edge and the clique formulations of the Maximum Stable Set Problem;
- ▶ write the ILP models of the vertex-weighted and of the edge-weighted maximum clique problems, linearizing the product $x_u x_v$ with the variables y_{uv} .

Prerequisites

- ▶ From this module, the chapter «[Clique problems](#)»: the problem, its assumptions, cliques, stable sets, the complement graph and $\omega(G) = \alpha(\overline{G})$.
- ▶ From this module, the chapters «[Coloring problems](#)» and «[Formulations of coloring problems](#)»: the chromatic number $\chi(G)$ and an ILP model with its LP relaxations.
- ▶ From the module *Integer linear programming*, the chapters «[Primal and dual bounds](#)», «[Ideal formulations](#)» and «[Valid inequalities](#)»: LP relaxations, stronger formulations and valid inequalities.
- ▶ From the module *Linear Programming*, the chapter «[Duality theory](#)»: the dual of an LP problem, weak and strong duality.

Overview of the family

Introduction: Overview of the family

- ▶ In this chapter we write the **ILP models** of the clique problems: the maximum clique problem, the maximum stable set problem and the two weighted variants
- ▶ All these models are built from the same two ingredients: **binary selection variables** on the vertices and **conflict constraints**
- ▶ In this section we see these ingredients, and a table of the members of the family with their models

Variables, constraints and objective

- ▶ The ILP models of the clique problems use one **binary selection variable** x_v for each vertex $v \in V$, equal to 1 if the vertex is in the clique.
- ▶ Two vertices that are not adjacent cannot be both selected: the **conflict constraints** are written on the non-edges of G , that is, on the edges of the complement graph $\overline{G}$, or on its cliques.
- ▶ The objective function maximizes the number of selected vertices or their total weight; the edge-weighted problem adds one variable per edge.

Clique models and stable set models

The clique models select vertices with binary variables and forbid the selection of two non-adjacent vertices.

The same models, written on the edges of G instead of $\overline{G}$, select stable sets.

The members of the family

- ▶ The table lists the members of the family, with the names of their ILP models and the sections that write them.

Problem	Models	Conflict constraints	Section
MCP	ILP_{MCP} (edge formulation)	on the non-edges of G	The edge formulation
MCP	ILP_{MCP}^{CL} (clique formulation)	on the maximal stable sets of G	Clique inequalities
MSSP	$ILP_{MSSP}, ILP_{MSSP}^{CL}$	on the edges and the cliques of G	Stable sets on the complement graph
VWMCP	ILP_{MCP}^{VW}	on the non-edges of G	Weighted variants
EWMC	ILP_{MCP}^{EW}	on the non-edges of G	Weighted variants

Summary: Overview of the family

The ingredients:

- ▶ One binary selection variable x_v per vertex
- ▶ Conflict constraints on the non-edges of G (the edges of $\overline{G}$), or on the cliques of $\overline{G}$

The objective:

- ▶ The number of selected vertices or their total weight; one more variable per edge in the edge-weighted problem

Cliques and stable sets:

- ▶ The same models on the edges of G select stable sets

An ILP model for the Maximum Clique Problem: the edge formulation

Introduction: the edge formulation

- ▶ We write the first ILP model of the Maximum Clique Problem (MCP): the **edge formulation**
- ▶ It has one binary variable per vertex and one constraint per **non-edge** of G
- ▶ It is the starting point of the whole chapter: its LP relaxations, the clique inequalities and the models of the other problems of the family start from it
- ▶ At the end we write it on a small instance, the cycle with a chord, and we solve it

The non-edges of G

- ▶ The problem of this section is the Maximum Clique Problem (MCP), defined in the chapter «Clique problems».
- ▶ Given a simple graph $G = (V, E)$, with $V = \{1, 2, \dots, n\}$ and m edges, we denote by

$$\overline{E} = \{ \{u, v\} : u, v \in V, u \neq v, \{u, v\} \notin E \}$$

the set of the **non-edges** of G , that is, the edges of the complement graph $\overline{G} = (V, \overline{E})$.

The variables

- We introduce the following n **binary variables**:

$$x_v = \begin{cases} 1 & \text{if vertex } v \text{ is selected in the clique} \\ 0 & \text{otherwise} \end{cases} \quad \forall v \in V.$$

The edge formulation

Using these variables, an Integer Linear Programming (ILP) model for the Maximum Clique Problem (MCP) reads as follows:

$$\begin{aligned} \max \quad & \sum_{v \in V} x_v \\ \text{subject to} \quad & x_u + x_v \leq 1, \quad \forall \{u, v\} \in \bar{E}, \\ & x_v \in \{0, 1\}, \quad \forall v \in V. \end{aligned}$$

This ILP model is referred to as the ILP_{MCP} model, or **edge formulation**.

Objective function and constraints of the ILP_{MCP} model

Description of the objective function and constraints of the ILP_{MCP} model:

- ▶ the linear objective function maximizes the cardinality of the selected clique;
- ▶ the constraints $x_u + x_v \leq 1$ ensure that no two non-adjacent vertices are both selected ($|\overline{E}| = \binom{n}{2} - m$ linear constraints, one for each edge of $\overline{G}$);
- ▶ finally, the constraints $x_v \in \{0, 1\}$ define the variables of the ILP model.

Reading the non-edge constraints

- ▶ For every non-edge $\{u, v\}$, the constraint $x_u + x_v \leq 1$ imposes

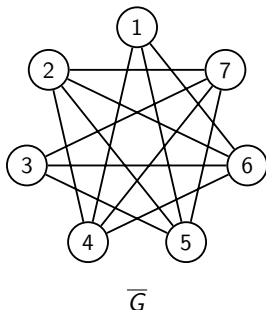
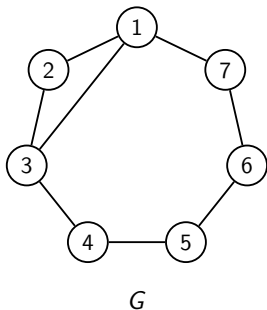
$$x_u = 1 \Rightarrow x_v = 0.$$

The converse is false: $x_u = x_v = 0$ is feasible.

- ▶ Read on $\overline{G}$, the same model is the **edge formulation of the MSSP**: at most one endpoint of every edge of $\overline{G}$ is selected.

Example: an MCP instance (1/2)

- ▶ We consider the graph G with 7 vertices made of the cycle $1, 2, \dots, 7, 1$ and the chord $\{1, 3\}$ (the reference instance of the chapter «Clique problems»), and its complement $\overline{G}$:



Example: an MCP instance (2/2)

- ▶ G has $m = 8$ edges, and $\overline{G}$ has the $\binom{7}{2} - 8 = 13$ non-edges of G :

$$\overline{E} = \{\{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{2, 7\}, \\ \{3, 5\}, \{3, 6\}, \{3, 7\}, \{4, 6\}, \{4, 7\}, \{5, 7\}\}.$$

- ▶ We refer to this MCP instance as **the cycle with a chord**: it accompanies us in the whole chapter.

Example: the ILP_{MCP} model (1/2)

The ILP_{MCP} model for the cycle with a chord has 7 binary variables and one constraint for each of the 13 edges of $\overline{G}$:

$$\begin{array}{llllllll}
 \max & x_1 & + x_2 & + x_3 & + x_4 & + x_5 & + x_6 & + x_7 \\
 \text{subject to} & x_1 & & & + x_4 & & & \leq 1 \\
 & x_1 & & & & + x_5 & & \leq 1 \\
 & x_1 & & & & & + x_6 & \leq 1 \\
 & & x_2 & & + x_4 & & & \leq 1 \\
 & & x_2 & & & + x_5 & & \leq 1 \\
 & & x_2 & & & & + x_6 & \leq 1 \\
 & & x_2 & & & & & + x_7 \leq 1 \\
 & & & x_3 & & + x_5 & & \leq 1 \\
 & & & x_3 & & & + x_6 & \leq 1 \\
 & & & x_3 & & & & + x_7 \leq 1 \\
 & & & & x_4 & & + x_6 & \leq 1 \\
 & & & & x_4 & & & + x_7 \leq 1 \\
 & & & & & x_5 & & + x_7 \leq 1 \\
 & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 \in \{0, 1\}
 \end{array}$$

Example: the ILP_{MCP} model (2/2)

- ▶ An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{MCP}})$ is:

$$\tilde{x}_1 = \tilde{x}_2 = \tilde{x}_3 = 1, \quad \tilde{x}_4 = \tilde{x}_5 = \tilde{x}_6 = \tilde{x}_7 = 0$$

- ▶ The optimal value $z(\text{ILP}_{\text{MCP}})$ is $3 = \omega(G)$.
- ▶ The associated optimal solution of the MCP is the clique

$$\tilde{K} = \{1, 2, 3\},$$

and it is the unique optimal solution.

Summary: the edge formulation

The ILP_{MCP} model:

- ▶ One binary variable x_v per vertex, the objective $\max \sum_{v \in V} x_v$
- ▶ One constraint $x_u + x_v \leq 1$ per non-edge of G : $\binom{n}{2} - m$ constraints

Reading the model:

- ▶ $x_u = 1 \Rightarrow x_v = 0$ on every non-edge $\{u, v\}$
- ▶ Read on $\overline{G}$, it is the edge formulation of the MSSP

The cycle with a chord:

- ▶ 13 constraints, optimal value $3 = \omega(G)$, unique maximum clique $\{1, 2, 3\}$

A first Linear Programming relaxation



Introduction: A first Linear Programming relaxation

- ▶ We relax the binary constraints of the edge formulation into non-negativity constraints: we obtain a first LP relaxation, the LP-0_{MCP} model
- ▶ Its optimal value is an upper bound on $\omega(G)$, that is, a dual bound for the MCP
- ▶ Without the bounds $x_v \leq 1$ this relaxation can be unbounded: we see when
- ▶ We also write its dual model, which reads as a **fractional edge cover** of the vertices

The $LP-0_{MCP}$ model

To obtain a Linear Programming (LP) relaxation of the ILP_{MCP} model, we relax the constraints $x_v \in \{0, 1\}$ into non-negativity constraints. The obtained LP model reads as follows:

$$\begin{aligned} \max \quad & \sum_{v \in V} x_v \\ \text{subject to} \quad & x_u + x_v \leq 1, \quad \forall \{u, v\} \in \overline{E}, \\ & x_v \geq 0, \quad \forall v \in V. \end{aligned}$$

This LP model is referred to as the $LP-0_{MCP}$ model.

The upper bounds are not implied

- ▶ The upper bounds $x_v \leq 1$ are not implied in LP-0_{MCP} .
- ▶ If a vertex is adjacent to all the others, it belongs to no constraint $x_u + x_v \leq 1$, and LP-0_{MCP} is unbounded.
- ▶ In the cycle with a chord every vertex has a non-edge, and LP-0_{MCP} has an optimal solution.

Example: the LP-0_{MCP} model (1/2)

The LP-0_{MCP} model for the cycle with a chord is:

$$\begin{array}{llllllll}
 \max & x_1 & + x_2 & + x_3 & + x_4 & + x_5 & + x_6 & + x_7 \\
 \text{subject to} & x_1 & & & + x_4 & & & \leq 1 \\
 & x_1 & & & & + x_5 & & \leq 1 \\
 & x_1 & & & & & + x_6 & \leq 1 \\
 & & x_2 & & + x_4 & & & \leq 1 \\
 & & x_2 & & & + x_5 & & \leq 1 \\
 & & x_2 & & & & + x_6 & \leq 1 \\
 & & x_2 & & & & & + x_7 \leq 1 \\
 & & & x_3 & & + x_5 & & \leq 1 \\
 & & & x_3 & & & + x_6 & \leq 1 \\
 & & & x_3 & & & & + x_7 \leq 1 \\
 & & & & x_4 & & + x_6 & \leq 1 \\
 & & & & x_4 & & & + x_7 \leq 1 \\
 & & & & & x_5 & & + x_7 \leq 1 \\
 & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 \geq 0
 \end{array}$$

Example: the LP-0_{MCP} model (2/2)

- ▶ An optimal solution $\tilde{\mathbf{x}}(\text{LP-0}_{\text{MCP}})$ is:

$$\tilde{x}_1 = \tilde{x}_2 = \tilde{x}_3 = \tilde{x}_4 = \tilde{x}_5 = \tilde{x}_6 = \tilde{x}_7 = \frac{1}{2}$$

- ▶ The optimal value $z(\text{LP-0}_{\text{MCP}})$ is $\frac{7}{2}$.
- ▶ Accordingly we have the upper bound:

$$z(\text{ILP}_{\text{MCP}}) \leq \frac{7}{2}$$

From the primal to the dual

Every LP model has a dual model.

We now write the dual of the $LP-0_{MCP}$ model:
it has one variable for each non-edge constraint
and one constraint for each vertex.

The dual model $D(\text{LP-0}_{\text{MCP}})$

Using a non-negative dual variable π_{uv} for each constraint $x_u + x_v \leq 1$, the dual model of the LP-0_{MCP} model is:

$$\begin{aligned} \min \quad & \sum_{\{u,v\} \in \bar{E}} \pi_{uv} \\ \text{subject to} \quad & \sum_{u: \{u,v\} \in \bar{E}} \pi_{uv} \geq 1, \quad \forall v \in V, \\ & \pi_{uv} \geq 0, \quad \forall \{u,v\} \in \bar{E}. \end{aligned}$$

This LP model is referred to as the $D(\text{LP-0}_{\text{MCP}})$ model.

Reading the dual: a fractional edge cover

- ▶ The constraints

$$\sum_{u: \{u,v\} \in \bar{E}} \pi_{uv} \geq 1, \quad \forall v \in V,$$

ask that every vertex is covered by a total weight of at least one on the edges of $\bar{G}$ incident to it.

- ▶ The $D(\text{LP-0}_{\text{MCP}})$ model is a **fractional edge cover** of the vertices with the edges of $\bar{G}$.

Example: the $D(\text{LP-0}_{\text{MCP}})$ model (1/2)

The dual $D(\text{LP-0}_{\text{MCP}})$ model for the cycle with a chord has one variable for each edge of $\overline{G}$ and one constraint for each vertex:

$$\begin{array}{llll} \min & \pi_{14} + \pi_{15} + \pi_{16} + \pi_{24} + \pi_{25} + \pi_{26} + \pi_{27} \\ & + \pi_{35} + \pi_{36} + \pi_{37} + \pi_{46} + \pi_{47} + \pi_{57} \\ \\ \text{subject to} & \pi_{14} + \pi_{15} + \pi_{16} & \geq & 1 \\ & \pi_{24} + \pi_{25} + \pi_{26} + \pi_{27} & \geq & 1 \\ & \pi_{35} + \pi_{36} + \pi_{37} & \geq & 1 \\ & \pi_{14} + \pi_{24} + \pi_{46} + \pi_{47} & \geq & 1 \\ & \pi_{15} + \pi_{25} + \pi_{35} + \pi_{57} & \geq & 1 \\ & \pi_{16} + \pi_{26} + \pi_{36} + \pi_{46} & \geq & 1 \\ & \pi_{27} + \pi_{37} + \pi_{47} + \pi_{57} & \geq & 1 \\ & \pi_{14}, \pi_{15}, \pi_{16}, \dots, \pi_{57} & \geq & 0 \end{array}$$

Example: the $D(\text{LP-0}_{\text{MCP}})$ model (2/2)

- ▶ An optimal solution $\tilde{\pi}(D(\text{LP-0}_{\text{MCP}}))$ gives value $\frac{1}{2}$ to the 7 edges of the cycle 1, 4, 7, 2, 5, 3, 6, 1 of $\overline{G}$:

$$\tilde{\pi}_{14} = \tilde{\pi}_{47} = \tilde{\pi}_{27} = \tilde{\pi}_{25} = \tilde{\pi}_{35} = \tilde{\pi}_{36} = \tilde{\pi}_{16} = \frac{1}{2},$$

$$\tilde{\pi}_{15} = \tilde{\pi}_{24} = \tilde{\pi}_{26} = \tilde{\pi}_{37} = \tilde{\pi}_{46} = \tilde{\pi}_{57} = 0$$

- ▶ Every vertex lies on two edges of the cycle and receives $\frac{1}{2} + \frac{1}{2} = 1$.
- ▶ The optimal value $z(D(\text{LP-0}_{\text{MCP}}))$ is $\frac{7}{2}$.

Summary: A first Linear Programming relaxation

The $LP-0_{MCP}$ model:

- ▶ The binary constraints become $x_v \geq 0$; the bounds $x_v \leq 1$ are not implied
- ▶ A vertex adjacent to all the others makes it unbounded

Its dual:

- ▶ One variable π_{uv} per edge of $\overline{G}$: a fractional edge cover of the vertices

The cycle with a chord:

- ▶ $x_v = \frac{1}{2}$ is optimal, with value $\frac{7}{2}$, as the dual: $z(ILP_{MCP}) \leq \frac{7}{2}$

A second Linear Programming relaxation

Introduction: A second Linear Programming relaxation

- ▶ We relax the binary constraints into the bounds $0 \leq x_v \leq 1$: we obtain the LP_{MCP} model, which can never be unbounded
- ▶ We write it and its dual on the cycle with a chord
- ▶ Then we prove that its bound can be **arbitrarily weak**: the point $x_v = \frac{1}{2}$ is always feasible
- ▶ This weakness is the reason for the stronger formulation of the next section

The LP_{MCP} model

To obtain an LP relaxation of the ILP_{MCP} model, we relax the constraints $x_v \in \{0, 1\}$ into the bounds $0 \leq x_v \leq 1$. The obtained LP model reads as follows:

$$\begin{aligned} \max \quad & \sum_{v \in V} x_v \\ \text{subject to} \quad & x_u + x_v \leq 1, \quad \forall \{u, v\} \in \overline{E}, \\ & x_v \leq 1, \quad \forall v \in V, \\ & x_v \geq 0, \quad \forall v \in V. \end{aligned}$$

This LP model is referred to as the LP_{MCP} model.

Example: the LP_{MCP} model (1/3)

The LP_{MCP} model for the cycle with a chord is (objective and non-edge constraints):

$$\begin{array}{llllllll} \max & x_1 & + x_2 & + x_3 & + x_4 & + x_5 & + x_6 & + x_7 \\ \text{subject to} & x_1 & & & + x_4 & & & \leq 1 \\ & x_1 & & & & + x_5 & & \leq 1 \\ & x_1 & & & & & + x_6 & \leq 1 \\ & & x_2 & & + x_4 & & & \leq 1 \\ & & x_2 & & & + x_5 & & \leq 1 \\ & & x_2 & & & & + x_6 & \leq 1 \\ & & x_2 & & & & & + x_7 \leq 1 \\ & & & x_3 & & + x_5 & & \leq 1 \\ & & & x_3 & & & + x_6 & \leq 1 \\ & & & x_3 & & & & + x_7 \leq 1 \\ & & & & x_4 & & + x_6 & \leq 1 \\ & & & & x_4 & & & + x_7 \leq 1 \\ & & & & & x_5 & & + x_7 \leq 1 \end{array}$$

Example: the LP_{MCP} model (2/3)

The model continues with the upper bounds and the non-negativity constraints:

$$\begin{array}{rcl} x_1 & & \leq 1 \\ & x_2 & \leq 1 \\ & & x_3 \leq 1 \\ & & & x_4 \leq 1 \\ & & & & x_5 \leq 1 \\ & & & & & x_6 \leq 1 \\ & & & & & & x_7 \leq 1 \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7 & \geq & 0 \end{array}$$

Example: the LP_{MCP} model (3/3)

- ▶ An optimal solution $\tilde{\mathbf{x}}(LP_{MCP})$ is:

$$\tilde{x}_1 = \tilde{x}_2 = \tilde{x}_3 = \tilde{x}_4 = \tilde{x}_5 = \tilde{x}_6 = \tilde{x}_7 = \frac{1}{2}$$

- ▶ The optimal value $z(LP_{MCP})$ is $\frac{7}{2}$.
- ▶ Accordingly we have the upper bound:

$$z(ILP_{MCP}) \leq \frac{7}{2}$$

The dual model $D(\text{LP}_{\text{MCP}})$

Using a non-negative dual variable π_{uv} for each constraint $x_u + x_v \leq 1$ and a non-negative dual variable λ_v for each constraint $x_v \leq 1$, the dual model of the LP_{MCP} model is:

$$\begin{aligned} \min \quad & \sum_{\{u,v\} \in \bar{E}} \pi_{uv} + \sum_{v \in V} \lambda_v \\ \text{subject to} \quad & \sum_{u: \{u,v\} \in \bar{E}} \pi_{uv} + \lambda_v \geq 1, \quad \forall v \in V, \\ & \pi_{uv} \geq 0, \quad \forall \{u,v\} \in \bar{E}, \\ & \lambda_v \geq 0, \quad \forall v \in V. \end{aligned}$$

This LP model is referred to as the $D(\text{LP}_{\text{MCP}})$ model.

Example: the $D(\text{LP}_{\text{MCP}})$ model (1/2)

The dual $D(\text{LP}_{\text{MCP}})$ model for the cycle with a chord is:

$$\begin{array}{ll} \min & \pi_{14} + \pi_{15} + \pi_{16} + \pi_{24} + \pi_{25} + \pi_{26} + \pi_{27} \\ & + \pi_{35} + \pi_{36} + \pi_{37} + \pi_{46} + \pi_{47} + \pi_{57} + \lambda_1 \\ & + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 \\ \\ \text{subject to} & \pi_{14} + \pi_{15} + \pi_{16} + \lambda_1 \geq 1 \\ & \pi_{24} + \pi_{25} + \pi_{26} + \pi_{27} + \lambda_2 \geq 1 \\ & \pi_{35} + \pi_{36} + \pi_{37} + \lambda_3 \geq 1 \\ & \pi_{14} + \pi_{24} + \pi_{46} + \pi_{47} + \lambda_4 \geq 1 \\ & \pi_{15} + \pi_{25} + \pi_{35} + \pi_{57} + \lambda_5 \geq 1 \\ & \pi_{16} + \pi_{26} + \pi_{36} + \pi_{46} + \lambda_6 \geq 1 \\ & \pi_{27} + \pi_{37} + \pi_{47} + \pi_{57} + \lambda_7 \geq 1 \\ \\ & \pi_{14}, \pi_{15}, \pi_{16}, \dots, \pi_{57} \geq 0 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \geq 0 \end{array}$$

Example: the $D(\text{LP}_{\text{MCP}})$ model (2/2)

- ▶ An optimal solution $(\tilde{\pi}, \tilde{\lambda})(D(\text{LP}_{\text{MCP}}))$ is the solution of the $D(\text{LP-0}_{\text{MCP}})$ example with $\tilde{\lambda}_v = 0$ for every $v \in V$:

$$\tilde{\pi}_{14} = \tilde{\pi}_{47} = \tilde{\pi}_{27} = \tilde{\pi}_{25} = \tilde{\pi}_{35} = \tilde{\pi}_{36} = \tilde{\pi}_{16} = \frac{1}{2},$$

all the other $\tilde{\pi}_{uv} = 0$, $\tilde{\lambda}_1 = \tilde{\lambda}_2 = \dots = \tilde{\lambda}_7 = 0$

- ▶ The optimal value $z(D(\text{LP}_{\text{MCP}}))$ is $\frac{7}{2}$.

How weak is this bound?

On the cycle with a chord the LP bound is $\frac{7}{2}$, against $\omega(G) = 3$.

The next proposition shows that, on other graphs, the gap can be as large as we want.

Proposition 1: The LP bound can be arbitrarily weak

For every graph G with n vertices, $z(\text{LP}_{\text{MCP}}) \geq n/2$. Consequently, the ratio $z(\text{LP}_{\text{MCP}})/z(\text{ILP}_{\text{MCP}})$ is not bounded over all graphs.

Proof

- ▶ Set $x_v = \frac{1}{2}$ for every vertex. Each constraint $x_u + x_v \leq 1$ becomes

$$\frac{1}{2} + \frac{1}{2} = 1,$$

and the bounds $x_v \leq 1$ and $x_v \geq 0$ hold.

- ▶ So this is a feasible solution of LP_{MCP} of value $n/2$.
- ▶ Graphs without triangles and with an edge have $\omega(G) = 2$ for every n , and for them the ratio is at least

$$\frac{n/2}{2} = \frac{n}{4}.$$



Rounding the bound down

- ▶ The data are integer and the optimal value of the ILP is integer, so $\lfloor z(\text{LP}_{\text{MCP}}) \rfloor$ is also a dual bound.
- ▶ On the cycle with a chord it gives $\lfloor \frac{7}{2} \rfloor = 3 = \omega(G)$ by luck; on a graph without triangles it gives $\lfloor n/2 \rfloor$, against $\omega(G) = 2$.
- ▶ On the cycle with 5 vertices C_5 , where $\omega(C_5) = 2$, the point $x_v = \frac{1}{2}$ gives

$$z(\text{LP}_{\text{MCP}}) = \frac{5}{2}.$$

Summary: A second Linear Programming relaxation

The LP_{MCP} model:

- ▶ The binary constraints become $0 \leq x_v \leq 1$; its dual has the variables π_{uv} and λ_v
- ▶ On the cycle with a chord: $x_v = \frac{1}{2}$, value $\frac{7}{2}$, as the dual

A weak bound:

- ▶ $x_v = \frac{1}{2}$ is always feasible, so $z(LP_{MCP}) \geq n/2$
- ▶ On graphs without triangles $\omega(G) = 2$: the ratio is at least $n/4$
- ▶ On C_5 : $z(LP_{MCP}) = \frac{5}{2}$ against $\omega(C_5) = 2$

Clique inequalities: a stronger formulation

Introduction: Clique inequalities

- ▶ The edge formulation is weak because the point $x_v = \frac{1}{2}$ is always feasible
- ▶ We find **valid inequalities** that cut it off: the **clique inequalities**, one for each clique of $\overline{G}$
- ▶ With them we write a second ILP model, the **clique formulation**, with a stronger LP relaxation
- ▶ On the cycle with a chord the new LP bound closes the gap; on C_5 it does not

Proposition 2: clique inequalities

Let $Q \subseteq V$ be a clique of $\overline{G}$, that is, a stable set of G . Then

$$\sum_{v \in Q} x_v \leq 1$$

*is a **valid inequality** for ILP_{MCP} .*

Proof

- ▶ A clique of G and a stable set Q of G share at most one vertex: two common vertices would be both adjacent and non-adjacent.
- ▶ So every feasible solution of ILP_{MCP} selects at most one vertex of Q :

$$\sum_{v \in Q} x_v \leq 1.$$



Clique inequalities and maximal cliques

- ▶ The inequalities $\sum_{v \in Q} x_v \leq 1$ are called **clique inequalities**, after the clique Q of $\overline{G}$; read on G , they are **stable set inequalities**.
- ▶ The non-edge constraints are the case $|Q| = 2$.
- ▶ If $Q \subseteq Q'$, the inequality of Q' implies the one of Q , since all the variables are non-negative.
- ▶ It is enough to write the inequalities of the **maximal** cliques of $\overline{G}$, whose family we denote by $\mathcal{Q}$.

The clique formulation

Using the variables x_v of the edge formulation, a second ILP model for the Maximum Clique Problem (MCP) reads as follows:

$$\begin{aligned} \max \quad & \sum_{v \in V} x_v \\ \text{subject to} \quad & \sum_{v \in Q} x_v \leq 1, \quad \forall Q \in \mathcal{Q}, \\ & x_v \in \{0, 1\}, \quad \forall v \in V. \end{aligned}$$

This ILP model is referred to as the $\text{ILP}_{\text{MCP}}^{\text{CL}}$ model, or **clique formulation**.

Objective function and constraints of the $\text{ILP}_{\text{MCP}}^{\text{CL}}$ model

Description of the objective function and constraints of the $\text{ILP}_{\text{MCP}}^{\text{CL}}$ model:

- ▶ the linear objective function maximizes the cardinality of the selected clique;
- ▶ the constraints $\sum_{v \in Q} x_v \leq 1$ ensure that at most one vertex of every maximal clique of $\overline{G}$ is selected ($|Q|$ linear constraints);
- ▶ finally, the constraints $x_v \in \{0, 1\}$ define the variables of the ILP model.

Same feasible solutions, many more constraints

- ▶ Every edge of $\overline{G}$ lies in a maximal clique, so each constraint $x_u + x_v \leq 1$ of the edge formulation is implied by a constraint $\sum_{v \in Q} x_v \leq 1$: the two models have the same feasible solutions.
- ▶ The family Q can be very large: Moon and Moser (1965) showed graphs with $3^{n/3}$ maximal cliques.
- ▶ In practice the inequalities are added only when they are violated, as in the chapter «Valid inequalities» of the module *Integer linear programming*.

Example: the ILP_{MCP}^{CL} model (1/2)

The maximal cliques of $\overline{G}$ in the cycle with a chord are the 5 triangles and the 2 edges

$$\mathcal{Q} = \{\{1, 4, 6\}, \{2, 4, 6\}, \{2, 4, 7\}, \{2, 5, 7\}, \{3, 5, 7\}, \{1, 5\}, \{3, 6\}\},$$

and they give 7 constraints instead of 13. The ILP_{MCP}^{CL} model for this MCP instance is:

$$\begin{array}{llllllll} \max & x_1 & + x_2 & + x_3 & + x_4 & + x_5 & + x_6 & + x_7 \\ \text{subject to} & x_1 & & & + x_4 & & + x_6 & \leq 1 \\ & & x_2 & & + x_4 & & + x_6 & \leq 1 \\ & & x_2 & & + x_4 & & & + x_7 \leq 1 \\ & & x_2 & & & + x_5 & & + x_7 \leq 1 \\ & & & x_3 & & + x_5 & & + x_7 \leq 1 \\ & x_1 & & & & + x_5 & & \leq 1 \\ & & & x_3 & & & + x_6 & \leq 1 \\ & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 \in \{0, 1\} \end{array}$$

Example: the $\text{ILP}_{\text{MCP}}^{\text{CL}}$ model (2/2)

- ▶ An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{MCP}}^{\text{CL}})$ is:

$$\tilde{x}_1 = \tilde{x}_2 = \tilde{x}_3 = 1, \quad \tilde{x}_4 = \tilde{x}_5 = \tilde{x}_6 = \tilde{x}_7 = 0$$

- ▶ The optimal value $z(\text{ILP}_{\text{MCP}}^{\text{CL}})$ is $3 = \omega(G)$.
- ▶ The two models have the same optimal solutions, since they have the same feasible solutions: the difference shows up in their LP relaxations.

The LP_{MCP}^{CL} model

To obtain an LP relaxation of the ILP_{MCP}^{CL} model, we relax the constraints $x_v \in \{0, 1\}$ into non-negativity constraints. The obtained LP model reads as follows:

$$\begin{aligned} \max \quad & \sum_{v \in V} x_v \\ \text{subject to} \quad & \sum_{v \in Q} x_v \leq 1, \quad \forall Q \in \mathcal{Q}, \\ & x_v \geq 0, \quad \forall v \in V. \end{aligned}$$

This LP model is referred to as the LP_{MCP}^{CL} model.

A stronger LP relaxation

- ▶ Every vertex lies in some maximal clique of $\overline{G}$, so the constraints $\sum_{v \in Q} x_v \leq 1$ imply $x_v \leq 1$: the bounds $x_v \leq 1$ are not needed.
- ▶ Each constraint $x_u + x_v \leq 1$ of LP_{MCP} is implied by a constraint $\sum_{v \in Q} x_v \leq 1$. Hence the feasible region of $\text{LP}_{\text{MCP}}^{\text{CL}}$ is contained in that of LP_{MCP} , and

$$\omega(G) \leq z(\text{LP}_{\text{MCP}}^{\text{CL}}) \leq z(\text{LP}_{\text{MCP}}).$$

Example: the LP_{MCP}^{CL} model (1/3)

The LP_{MCP}^{CL} model for the cycle with a chord is:

$$\begin{array}{llllllll} \max & x_1 & + x_2 & + x_3 & + x_4 & + x_5 & + x_6 & + x_7 \\ \text{subject to} & x_1 & & & + x_4 & & + x_6 & \leq 1 \\ & & x_2 & & + x_4 & & + x_6 & \leq 1 \\ & & x_2 & & + x_4 & & & + x_7 \leq 1 \\ & & x_2 & & & + x_5 & & + x_7 \leq 1 \\ & & & x_3 & & + x_5 & & + x_7 \leq 1 \\ & x_1 & & & & + x_5 & & \leq 1 \\ & & & x_3 & & & + x_6 & \leq 1 \\ & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 \geq 0 \end{array}$$

Example: the LP_{MCP}^{CL} model (2/3)

- ▶ The point $\bar{x}_v = \frac{1}{2}$ is no longer feasible:

$$\bar{x}_1 + \bar{x}_4 + \bar{x}_6 = \frac{3}{2} > 1.$$

- ▶ Summing the three constraints of $\{1, 4, 6\}$, $\{2, 5, 7\}$ and $\{3, 5, 7\}$ gives

$$\sum_{v=1}^7 x_v + x_5 + x_7 \leq 3 \quad \Longrightarrow \quad \sum_{v=1}^7 x_v \leq 3.$$

Example: the $\text{LP}_{\text{MCP}}^{\text{CL}}$ model (3/3)

- ▶ An optimal solution $\tilde{\mathbf{x}}(\text{LP}_{\text{MCP}}^{\text{CL}})$ is:

$$\tilde{x}_1 = \tilde{x}_2 = \tilde{x}_3 = 1, \quad \tilde{x}_4 = \tilde{x}_5 = \tilde{x}_6 = \tilde{x}_7 = 0$$

- ▶ The optimal value $z(\text{LP}_{\text{MCP}}^{\text{CL}})$ is $3 = \omega(G)$.
- ▶ The LP bound drops from $\frac{7}{2}$ to 3 and closes the gap.

Clique inequalities do not close every gap

- ▶ Clique inequalities do not always close the gap. On C_5 the complement is again a cycle with 5 vertices, whose maximal cliques are its edges: the two formulations coincide and

$$z(\text{LP}_{\text{MCP}}^{\text{CL}}) = \frac{5}{2} > 2 = \omega(C_5).$$

- ▶ If C is a cycle of $\overline{G}$ with an odd number of vertices and no chord, then

$$\sum_{v \in C} x_v \leq \frac{|C| - 1}{2}$$

is a valid inequality (an **odd cycle inequality**): a stable set of $\overline{G}$ takes at most $(|C| - 1)/2$ vertices of C .

- ▶ For C_5 it reads $x_1 + \dots + x_5 \leq 2$ and closes the gap.

Summary: Clique inequalities

Clique inequalities:

- ▶ $\sum_{v \in Q} x_v \leq 1$ for every clique Q of $\overline{G}$; the maximal cliques $\mathcal{Q}$ are enough

The clique formulation:

- ▶ Same feasible solutions as the edge formulation, but $\omega(G) \leq z(\text{LP}_{\text{MCP}}^{\text{CL}}) \leq z(\text{LP}_{\text{MCP}})$
- ▶ $\mathcal{Q}$ can be huge: the inequalities are added only when violated

On the instances:

- ▶ Cycle with a chord: the bound drops from $\frac{7}{2}$ to $3 = \omega(G)$
- ▶ C_5 : the bound stays $\frac{5}{2}$; the odd cycle inequality closes the gap

The relation with vertex coloring

Introduction: The relation with vertex coloring

- ▶ The clique number and the chromatic number are related:
 $\omega(G) \leq \chi(G)$
- ▶ In this section we place the LP bound of the clique formulation between the two numbers
- ▶ The tool is **LP duality**: the dual of $\text{LP}_{\text{MCP}}^{\text{CL}}$ is the LP relaxation of a coloring formulation, written on the maximal stable sets of G
- ▶ We check the chain of bounds on the cycle with a chord and on C_5

Cliques and colorings

- ▶ In a coloring of G the vertices of a clique take pairwise different colors, hence (chapter «Coloring problems»)

$$\omega(G) \leq \chi(G).$$

- ▶ Here we place the LP bound of the clique formulation between the two numbers.

The variables of a coloring formulation

- ▶ The maximal cliques of $\overline{G}$ are the **maximal stable sets** of G , and a color class is a stable set.
- ▶ We introduce the following $|Q|$ **binary variables**:

$$y_Q = \begin{cases} 1 & \text{if the stable set } Q \text{ is a color class} \\ 0 & \text{otherwise} \end{cases} \quad \forall Q \in \mathcal{Q}.$$

The stable set formulation of the VCP

Using these variables, an ILP model for the Vertex Coloring Problem (VCP) reads as follows:

$$\begin{aligned} & \min \sum_{Q \in \mathcal{Q}} y_Q \\ & \text{subject to} \quad \sum_{Q \in \mathcal{Q}: v \in Q} y_Q \geq 1, \quad \forall v \in V, \\ & \quad \quad \quad y_Q \in \{0, 1\}, \quad \forall Q \in \mathcal{Q}. \end{aligned}$$

This ILP model is referred to as the $\text{ILP}_{\text{VCP}}^{\text{ST}}$ model, or **stable set formulation** of the VCP.

Objective function and constraints of the $\text{ILP}_{\text{VCP}}^{\text{ST}}$ model

Description of the objective function and constraints of the $\text{ILP}_{\text{VCP}}^{\text{ST}}$ model:

- ▶ the linear objective function minimizes the number of color classes;
- ▶ the constraints $\sum_{Q \in \mathcal{Q}: v \in Q} y_Q \geq 1$ ensure that every vertex belongs to at least one chosen stable set (n linear constraints);
- ▶ finally, the constraints $y_Q \in \{0, 1\}$ define the variables of the ILP model.

Its optimal value is $\chi(G)$: a vertex covered by two chosen sets is kept in one of them, and every subset of a stable set is stable, so the chosen sets give a coloring with the same number of colors.

The dual model $D(LP_{MCP}^{CL})$

Using a non-negative dual variable π_Q for each constraint $\sum_{v \in Q} x_v \leq 1$ of the LP_{MCP}^{CL} model, its dual model is:

$$\begin{aligned} \min \quad & \sum_{Q \in \mathcal{Q}} \pi_Q \\ \text{subject to} \quad & \sum_{Q \in \mathcal{Q}: v \in Q} \pi_Q \geq 1, \quad \forall v \in V, \\ & \pi_Q \geq 0, \quad \forall Q \in \mathcal{Q}. \end{aligned}$$

This LP model is referred to as the $D(LP_{MCP}^{CL})$ model.

Two models that look alike

Compare the dual $D(LP_{MCP}^{CL})$ with the stable set formulation of the VCP:

the same objective, the same constraints,
only the domains of the variables differ.

The next proposition turns this remark into a chain of bounds.

Proposition 3: clique bound from LP duality

The dual $D(\text{LP}_{\text{MCP}}^{\text{CL}})$ model is the LP relaxation of the $\text{ILP}_{\text{VCP}}^{\text{ST}}$ model, with the variables π_Q in place of y_Q . Hence

$$\omega(G) \leq z(\text{LP}_{\text{MCP}}^{\text{CL}}) \leq \chi(G).$$

Proof

- ▶ Relaxing the constraints $y_Q \in \{0, 1\}$ into $y_Q \geq 0$ gives exactly the $D(\text{LP}_{\text{MCP}}^{\text{CL}})$ model, with y_Q in place of π_Q .
- ▶ By strong duality

$$z(\text{LP}_{\text{MCP}}^{\text{CL}}) = z(D(\text{LP}_{\text{MCP}}^{\text{CL}})).$$

- ▶ The first inequality holds because $\text{LP}_{\text{MCP}}^{\text{CL}}$ is a relaxation of the MCP.
- ▶ The second one holds because $D(\text{LP}_{\text{MCP}}^{\text{CL}})$ is a relaxation of the $\text{ILP}_{\text{VCP}}^{\text{ST}}$ model, a minimization problem with optimal value $\chi(G)$.



The fractional chromatic number

- ▶ The common value $z(\text{LP}_{\text{MCP}}^{\text{CL}})$ is called the **fractional chromatic number** of G .
- ▶ The clique formulation of the MCP and the stable set formulation of the VCP are two faces of the same LP.

Example: the $D(LP_{MCP}^{CL})$ model (1/2)

The dual $D(LP_{MCP}^{CL})$ model for the cycle with a chord has one variable for each maximal stable set of G (π_{146} for $Q = \{1, 4, 6\}$, and so on):

$$\begin{array}{llllllll}
 \min & \pi_{146} & + \pi_{246} & + \pi_{247} & + \pi_{257} & + \pi_{357} & + \pi_{15} & + \pi_{36} \\
 \text{subject to} & \pi_{146} & & & & & + \pi_{15} & \geq 1 \\
 & & \pi_{246} & + \pi_{247} & + \pi_{257} & & & \geq 1 \\
 & & & & & \pi_{357} & + \pi_{36} & \geq 1 \\
 & \pi_{146} & + \pi_{246} & + \pi_{247} & & & & \geq 1 \\
 & & & & \pi_{257} & + \pi_{357} & + \pi_{15} & \geq 1 \\
 & \pi_{146} & + \pi_{246} & & & & + \pi_{36} & \geq 1 \\
 & & & \pi_{247} & + \pi_{257} & + \pi_{357} & & \geq 1 \\
 & \pi_{146}, & \pi_{246}, & \pi_{247}, & \pi_{257}, & \pi_{357}, & \pi_{15}, & \pi_{36} \geq 0
 \end{array}$$

Example: the $D(LP_{MCP}^{CL})$ model (2/2)

- ▶ An optimal solution $\tilde{\pi}(D(LP_{MCP}^{CL}))$ gives value 1 to the multipliers used in the LP_{MCP}^{CL} example (the constraints of $\{1, 4, 6\}$, $\{2, 5, 7\}$ and $\{3, 5, 7\}$):

$$\tilde{\pi}_{146} = \tilde{\pi}_{257} = \tilde{\pi}_{357} = 1, \quad \tilde{\pi}_{246} = \tilde{\pi}_{247} = \tilde{\pi}_{15} = \tilde{\pi}_{36} = 0$$

- ▶ The optimal value $z(D(LP_{MCP}^{CL}))$ is 3.

Example: the chain $\omega \leq z(\text{LP}_{\text{MCP}}^{\text{CL}}) \leq \chi(1/3)$

On the cycle with a chord, the $\text{ILP}_{\text{VCP}}^{\text{ST}}$ model is:

$$\begin{array}{llllllll}
 \min & y_{146} & + y_{246} & + y_{247} & + y_{257} & + y_{357} & + y_{15} & + y_{36} \\
 \text{subject to} & y_{146} & & & & & + y_{15} & \geq 1 \\
 & & y_{246} & + y_{247} & + y_{257} & & & \geq 1 \\
 & & & & & y_{357} & + y_{36} & \geq 1 \\
 & y_{146} & + y_{246} & + y_{247} & & & & \geq 1 \\
 & & & & y_{257} & + y_{357} & + y_{15} & \geq 1 \\
 & y_{146} & + y_{246} & & & & + y_{36} & \geq 1 \\
 & & & y_{247} & + y_{257} & + y_{357} & & \geq 1 \\
 & y_{146}, & y_{246}, & y_{247}, & y_{257}, & y_{357}, & y_{15}, & y_{36} \in \{0,1\}
 \end{array}$$

Example: the chain $\omega \leq z(\text{LP}_{\text{MCP}}^{\text{CL}}) \leq \chi$ (2/3)

- The integer point

$$\tilde{y}_{146} = \tilde{y}_{257} = \tilde{y}_{357} = 1,$$

all the other variables equal to 0, is the solution of the $D(\text{LP}_{\text{MCP}}^{\text{CL}})$ example.

- It is a coloring with 3 colors, the classes $\{1, 4, 6\}$, $\{2, 5, 7\}$ and $\{3\}$.
- Hence $\chi(G) = 3$, and the chain reads

$$3 \leq 3 \leq 3.$$

Example: the chain $\omega \leq z(\text{LP}_{\text{MCP}}^{\text{CL}}) \leq \chi$ (3/3)

- ▶ On C_5 , with vertices $1, \dots, 5$, the maximal stable sets are the 5 non-edges.
- ▶ Setting $\pi_Q = \frac{1}{2}$ on each of them gives every vertex a total of 1, so the dual has value $\frac{5}{2}$, as the primal.
- ▶ The chain reads

$$\omega(C_5) = 2 < z(\text{LP}_{\text{MCP}}^{\text{CL}}) = \frac{5}{2} < \chi(C_5) = 3.$$

Summary: The relation with vertex coloring

The stable set formulation of the VCP:

- ▶ One binary variable y_Q per maximal stable set of G , optimal value $\chi(G)$

LP duality:

- ▶ The dual of $\text{LP}_{\text{MCP}}^{\text{CL}}$ is the LP relaxation of $\text{ILP}_{\text{VCP}}^{\text{ST}}$, hence

$$\omega(G) \leq z(\text{LP}_{\text{MCP}}^{\text{CL}}) \leq \chi(G)$$

- ▶ The common value is the fractional chromatic number of G

On the instances:

- ▶ Cycle with a chord: $3 \leq 3 \leq 3$; C_5 : $2 < \frac{5}{2} < 3$

Stable sets on the complement graph

Introduction: Stable sets on the complement graph

- ▶ The Maximum Stable Set Problem (MSSP) looks for a largest set of pairwise non-adjacent vertices
- ▶ Its ILP models are the clique models read on the complement graph: this is the identity $\alpha(G) = \omega(\overline{G})$ written as a model
- ▶ We write the **edge formulation** and the **clique formulation** of the MSSP, and we compare them on the cycle with a chord

The MSSP and its variables

- ▶ The problem of this section is the Maximum Stable Set Problem (MSSP), defined in the chapter «[Clique problems](#)».
- ▶ Given a simple graph $G = (V, E)$, with $V = \{1, 2, \dots, n\}$ and m edges, we introduce the following n **binary variables**:

$$x_v = \begin{cases} 1 & \text{if vertex } v \text{ is selected in the stable set} \\ 0 & \text{otherwise} \end{cases} \quad \forall v \in V.$$

The edge formulation of the MSSP

Using these variables, an Integer Linear Programming (ILP) model for the Maximum Stable Set Problem (MSSP) reads as follows:

$$\begin{aligned} \max \quad & \sum_{v \in V} x_v \\ \text{subject to} \quad & x_u + x_v \leq 1, \quad \forall \{u, v\} \in E, \\ & x_v \in \{0, 1\}, \quad \forall v \in V. \end{aligned}$$

This ILP model is referred to as the ILP_{MSSP} model, or **edge formulation** of the MSSP.

Objective function and constraints of the ILP_{MSSP} model

Description of the objective function and constraints of the ILP_{MSSP} model:

- ▶ the linear objective function maximizes the cardinality of the selected stable set;
- ▶ the constraints $x_u + x_v \leq 1$ ensure that no two adjacent vertices are both selected (m linear constraints, one for each edge of G);
- ▶ finally, the constraints $x_v \in \{0, 1\}$ define the variables of the ILP model.

The ILP_{MSSP} model on G is the ILP_{MCP} model on $\overline{G}$: the non-edges of $\overline{G}$ are the edges of G . This is the identity $\alpha(G) = \omega(\overline{G})$ written as a model.

Example: the ILP_{MSSP} model (1/2)

We look for a maximum stable set of the cycle with a chord. The ILP_{MSSP} model has 7 binary variables and one constraint for each of the 8 edges of G :

$$\begin{array}{llllllll} \max & x_1 & + x_2 & + x_3 & + x_4 & + x_5 & + x_6 & + x_7 \\ \text{subject to} & x_1 & + x_2 & & & & & \leq 1 \\ & x_1 & & + x_3 & & & & \leq 1 \\ & x_1 & & & & & + x_7 & \leq 1 \\ & & x_2 & + x_3 & & & & \leq 1 \\ & & & x_3 & + x_4 & & & \leq 1 \\ & & & & x_4 & + x_5 & & \leq 1 \\ & & & & & x_5 & + x_6 & \leq 1 \\ & & & & & & x_6 & + x_7 \leq 1 \\ & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 \in \{0, 1\} \end{array}$$

Example: the ILP_{MSSP} model (2/2)

- ▶ An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{MSSP}})$ is:

$$\tilde{x}_1 = \tilde{x}_4 = \tilde{x}_6 = 1, \quad \tilde{x}_2 = \tilde{x}_3 = \tilde{x}_5 = \tilde{x}_7 = 0$$

- ▶ The optimal value $z(\text{ILP}_{\text{MSSP}})$ is $3 = \alpha(G)$.
- ▶ The optimal solution is not unique: the maximum stable sets of G are

$$\{1, 4, 6\}, \{2, 4, 6\}, \{2, 4, 7\}, \{2, 5, 7\}, \{3, 5, 7\},$$

the five triangles of $\overline{G}$.

Clique inequalities for the MSSP

- ▶ A stable set and a clique of G share at most one vertex, so for every clique K of G the **clique inequality**

$$\sum_{v \in K} x_v \leq 1$$

is valid, and it implies the constraints $x_u + x_v \leq 1$ of the edges of K .

- ▶ As for the MCP, it is enough to write the inequalities of the family $\mathcal{K}$ of the **maximal** cliques of G .

The clique formulation of the MSSP

Using the variables x_v of the edge formulation of the MSSP, a second ILP model for the Maximum Stable Set Problem (MSSP) reads as follows:

$$\begin{aligned} & \max \sum_{v \in V} x_v \\ & \text{subject to} \quad \sum_{v \in K} x_v \leq 1, \quad \forall K \in \mathcal{K}, \\ & \quad \quad \quad x_v \in \{0, 1\}, \quad \forall v \in V. \end{aligned}$$

This ILP model is referred to as the $\text{ILP}_{\text{MSSP}}^{\text{CL}}$ model, or **clique formulation** of the MSSP.

Objective function and constraints of the $\text{ILP}_{\text{MSSP}}^{\text{CL}}$ model

Description of the objective function and constraints of the $\text{ILP}_{\text{MSSP}}^{\text{CL}}$ model:

- ▶ the linear objective function maximizes the cardinality of the selected stable set;
- ▶ the constraints $\sum_{v \in K} x_v \leq 1$ ensure that at most one vertex of every maximal clique of G is selected ($|\mathcal{K}|$ linear constraints);
- ▶ finally, the constraints $x_v \in \{0, 1\}$ define the variables of the ILP model.

The $\text{ILP}_{\text{MSSP}}^{\text{CL}}$ model on G is the $\text{ILP}_{\text{MCP}}^{\text{CL}}$ model on $\overline{G}$, and its LP relaxation is at least as strong as that of the ILP_{MSSP} model.

Example: the $\text{ILP}_{\text{MSSP}}^{\text{CL}}$ model (1/2)

The maximal cliques of the cycle with a chord are the triangle $\{1, 2, 3\}$ and the five edges $\{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 7\}, \{1, 7\}$. The $\text{ILP}_{\text{MSSP}}^{\text{CL}}$ model replaces the three constraints of the triangle with one constraint, and has 6 constraints:

$$\begin{array}{llllllll} \max & x_1 & + x_2 & + x_3 & + x_4 & + x_5 & + x_6 & + x_7 \\ \text{subject to} & x_1 & + x_2 & + x_3 & & & & \leq 1 \\ & & & x_3 & + x_4 & & & \leq 1 \\ & & & & x_4 & + x_5 & & \leq 1 \\ & & & & & x_5 & + x_6 & \leq 1 \\ & & & & & & x_6 & + x_7 \leq 1 \\ & & & & & & x_1 & + x_7 \leq 1 \\ & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 \in \{0, 1\} \end{array}$$

Example: the $\text{ILP}_{\text{MSSP}}^{\text{CL}}$ model (2/2)

- ▶ The optimal solution $\tilde{x}_1 = \tilde{x}_4 = \tilde{x}_6 = 1$ of the ILP_{MSSP} example is optimal here too, with value

$$z(\text{ILP}_{\text{MSSP}}^{\text{CL}}) = 3.$$

- ▶ The LP relaxations differ (Gurobi):
 - ▶ the edge formulation, with $0 \leq x_v \leq 1$, has optimal value $\frac{7}{2}$ at $x_v = \frac{1}{2}$;
 - ▶ the clique formulation has optimal value 3, attained by $x_2 = x_5 = x_7 = 1$.

Summary: Stable sets on the complement graph

Edge formulation of the MSSP:

- ▶ One constraint $x_u + x_v \leq 1$ per edge of G : it is the ILP_{MCP} model on $\overline{G}$

Clique formulation of the MSSP:

- ▶ One constraint $\sum_{v \in K} x_v \leq 1$ per maximal clique K of G : a stronger LP relaxation

The cycle with a chord:

- ▶ $\alpha(G) = 3$, with five maximum stable sets
- ▶ LP bounds: $\frac{7}{2}$ with the edge formulation, 3 with the clique formulation

ILP models for the weighted variants



Introduction: ILP models for the weighted variants

- ▶ The problems of this section are the vertex-weighted (VWMCP) and the edge-weighted (EWMCP) maximum clique problems, defined in the chapter «[Clique problems](#)», with their reference instances
- ▶ With weights on the vertices only the objective function changes
- ▶ With weights on the edges, the weight of an edge is collected only if both its endpoints are selected: we must **linearize** the product $x_u x_v$ with new variables y_{uv}

The vertex-weighted maximum clique problem

- ▶ Given a simple graph $G = (V, E)$ and a vertex-weight vector $\mathbf{w} \in \mathbb{Z}_{\geq 1}^n$, we use the n **binary variables** of the edge formulation:

$$x_v = \begin{cases} 1 & \text{if vertex } v \text{ is selected in the clique} \\ 0 & \text{otherwise} \end{cases} \quad \forall v \in V.$$

The $\text{ILP}_{\text{MCP}}^{\text{VW}}$ model

Using these variables, an Integer Linear Programming (ILP) model for the vertex-weighted maximum clique problem (VWMCP) reads as follows:

$$\begin{aligned} \max \quad & \sum_{v \in V} w_v x_v \\ \text{subject to} \quad & x_u + x_v \leq 1, \quad \forall \{u, v\} \in \overline{E}, \\ & x_v \in \{0, 1\}, \quad \forall v \in V. \end{aligned}$$

This ILP model is referred to as the $\text{ILP}_{\text{MCP}}^{\text{VW}}$ model.

Objective function and constraints of the $\text{ILP}_{\text{MCP}}^{\text{VW}}$ model

Description of the objective function and constraints of the $\text{ILP}_{\text{MCP}}^{\text{VW}}$ model:

- ▶ the linear objective function maximizes the total weight $w(K)$ of the selected clique;
- ▶ the constraints $x_u + x_v \leq 1$ ensure that no two non-adjacent vertices are both selected ($|\overline{E}| = \binom{n}{2} - m$ linear constraints);
- ▶ finally, the constraints $x_v \in \{0, 1\}$ define the variables of the ILP model.

Only the objective function changes: the feasible solutions are the cliques of G , as in the ILP_{MCP} model, and with $w_v = 1$ for every vertex the two models coincide. The clique inequalities of $\overline{G}$ are valid here too.

Example: the ILP^{VW}_{MCP} model (1/3)

- ▶ We consider the graph G with 8 vertices on a circle, where every vertex is adjacent to all the others except the opposite one.
- ▶ G has 24 edges, and $\overline{G}$ has the 4 edges $\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}$.
- ▶ The vertex weights are:

	$v = 1$	$v = 2$	$v = 3$	$v = 4$	$v = 5$	$v = 6$	$v = 7$	$v = 8$
w_v	5	8	9	5	6	5	5	9

- ▶ This is the instance of the VWMCP of the chapter «Clique problems».

Example: the ILP_{MCP}^{VW} model (2/3)

The ILP_{MCP}^{VW} model has 8 binary variables and one constraint for each of the 4 edges of $\overline{G}$:

$$\begin{array}{llllllll} \max & 5x_1 & +8x_2 & +9x_3 & +5x_4 & +6x_5 & +5x_6 & +5x_7 & +9x_8 \\ \text{subject to} & x_1 & & & & +x_5 & & & \leq 1 \\ & & x_2 & & & & +x_6 & & \leq 1 \\ & & & x_3 & & & & +x_7 & \leq 1 \\ & & & & x_4 & & & +x_8 & \leq 1 \\ & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7, & x_8 \in \{0,1\} \end{array}$$

Example: the $\text{ILP}_{\text{MCP}}^{\text{VW}}$ model (3/3)

- ▶ An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{MCP}}^{\text{VW}})$ is:

$$\tilde{x}_2 = \tilde{x}_3 = \tilde{x}_5 = \tilde{x}_8 = 1, \quad \tilde{x}_1 = \tilde{x}_4 = \tilde{x}_6 = \tilde{x}_7 = 0$$

- ▶ The optimal value $z(\text{ILP}_{\text{MCP}}^{\text{VW}})$ is

$$8 + 9 + 6 + 9 = 32,$$

as stated in the chapter «Clique problems».

- ▶ The associated optimal solution is the clique $\tilde{K} = \{2, 3, 5, 8\}$, and it is the unique optimal solution: each constraint keeps the heavier vertex of its pair.

The edge-weighted maximum clique problem

- ▶ Given a simple graph $G = (V, E)$ and an edge-weight vector $\mathbf{r} \in \mathbb{Z}_{\geq 1}^m$, the weight of an edge is collected only if both its endpoints are selected: the product $x_u x_v$ is not linear.
- ▶ We introduce the n **binary variables** x_v of the edge formulation and the following m **binary variables**:

$$y_{uv} = \begin{cases} 1 & \text{if both endpoints of edge } \{u, v\} \\ & \text{are selected in the clique} \\ 0 & \text{otherwise} \end{cases} \quad \forall \{u, v\} \in E.$$

The ILP_{MCP}^{EW} model

Using these variables, an Integer Linear Programming (ILP) model for the edge-weighted maximum clique problem (EWMCP) reads as follows:

$$\begin{aligned} \max \quad & \sum_{\{u,v\} \in E} r_{uv} y_{uv} \\ \text{subject to} \quad & y_{uv} \leq x_u, \quad \forall \{u, v\} \in E, \\ & y_{uv} \leq x_v, \quad \forall \{u, v\} \in E, \\ & x_u + x_v \leq 1, \quad \forall \{u, v\} \in \bar{E}, \\ & x_v \in \{0, 1\}, \quad \forall v \in V, \\ & y_{uv} \in \{0, 1\}, \quad \forall \{u, v\} \in E. \end{aligned}$$

This ILP model is referred to as the ILP_{MCP}^{EW} model.

Objective function and constraints of the $\text{ILP}_{\text{MCP}}^{\text{EW}}$ model

Description of the objective function and constraints of the $\text{ILP}_{\text{MCP}}^{\text{EW}}$ model:

- ▶ the linear objective function maximizes the total weight $r(K)$ of the edges with both endpoints in the selected clique;
- ▶ the constraints $y_{uv} \leq x_u$ and $y_{uv} \leq x_v$ allow $y_{uv} = 1$ only if both endpoints are selected ($2m$ linear constraints);
- ▶ the constraints $x_u + x_v \leq 1$ ensure that the selected vertices form a clique ($\binom{n}{2} - m$ linear constraints);
- ▶ finally, the constraints $x_v \in \{0, 1\}$ and $y_{uv} \in \{0, 1\}$ define the variables of the ILP model.

Linearizing the product $x_u x_v$

- ▶ The constraints $y_{uv} \leq x_u$ and $y_{uv} \leq x_v$ model the implication

$$y_{uv} = 1 \Rightarrow x_u = x_v = 1.$$

- ▶ The converse implication $x_u = x_v = 1 \Rightarrow y_{uv} = 1$, that is

$$y_{uv} \geq x_u + x_v - 1,$$

is not needed: since $r_{uv} > 0$, in every optimal solution y_{uv} takes the largest value allowed, $\min\{x_u, x_v\}$.

Example: the ILP_{MCP}^{EW} model (1/4)

- ▶ We consider the graph G with 8 vertices of the ILP_{MCP}^{VW} example, with the edge weights of the instance of the EWMCP of the chapter «Clique problems»; they are the coefficients of the objective function below.
- ▶ The ILP_{MCP}^{EW} model has $8 + 24 = 32$ binary variables, 48 constraints $y_{uv} \leq x_u$, $y_{uv} \leq x_v$ and 4 non-edge constraints.

The objective function is:

$$\begin{aligned} \max \quad & y_{12} + 4y_{13} + 5y_{14} + 4y_{16} + 3y_{17} + 3y_{18} \\ & + 2y_{23} + 5y_{24} + 6y_{25} + 9y_{27} + 4y_{28} + 3y_{34} \\ & + 6y_{35} + y_{36} + 6y_{38} + 4y_{45} + 7y_{46} + 2y_{47} \\ & + 5y_{56} + 7y_{57} + 3y_{58} + y_{67} + 2y_{68} + 8y_{78} \end{aligned}$$

Example: the ILP^{EW}_{MCP} model (2/4)

The model continues with the constraints $y_{uv} \leq x_u$ and $y_{uv} \leq x_v$:

$$\begin{array}{llll} \text{subject to} & y_{12} \leq x_1 & y_{12} \leq x_2 & y_{13} \leq x_1 & y_{13} \leq x_3 \\ & y_{14} \leq x_1 & y_{14} \leq x_4 & y_{16} \leq x_1 & y_{16} \leq x_6 \\ & y_{17} \leq x_1 & y_{17} \leq x_7 & y_{18} \leq x_1 & y_{18} \leq x_8 \\ & y_{23} \leq x_2 & y_{23} \leq x_3 & y_{24} \leq x_2 & y_{24} \leq x_4 \\ & y_{25} \leq x_2 & y_{25} \leq x_5 & y_{27} \leq x_2 & y_{27} \leq x_7 \\ & y_{28} \leq x_2 & y_{28} \leq x_8 & y_{34} \leq x_3 & y_{34} \leq x_4 \\ & y_{35} \leq x_3 & y_{35} \leq x_5 & y_{36} \leq x_3 & y_{36} \leq x_6 \end{array}$$

Example: the ILP^{EW}_{MCP} model (3/4)

The model ends with the other constraints $y_{uv} \leq x_u$, $y_{uv} \leq x_v$, the 4 non-edge constraints and the domains of the variables:

$$y_{38} \leq x_3 \quad y_{38} \leq x_8 \quad y_{45} \leq x_4 \quad y_{45} \leq x_5$$

$$y_{46} \leq x_4 \quad y_{46} \leq x_6 \quad y_{47} \leq x_4 \quad y_{47} \leq x_7$$

$$y_{56} \leq x_5 \quad y_{56} \leq x_6 \quad y_{57} \leq x_5 \quad y_{57} \leq x_7$$

$$y_{58} \leq x_5 \quad y_{58} \leq x_8 \quad y_{67} \leq x_6 \quad y_{67} \leq x_7$$

$$y_{68} \leq x_6 \quad y_{68} \leq x_8 \quad y_{78} \leq x_7 \quad y_{78} \leq x_8$$

$$x_1 + x_5 \leq 1 \quad x_2 + x_6 \leq 1 \quad x_3 + x_7 \leq 1 \quad x_4 + x_8 \leq 1$$

$$x_1, x_2, \dots, x_8 \in \{0, 1\}$$

$$y_{12}, y_{13}, \dots, y_{78} \in \{0, 1\}$$

Example: the $\text{ILP}_{\text{MCP}}^{\text{EW}}$ model (4/4)

- ▶ An optimal solution $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})(\text{ILP}_{\text{MCP}}^{\text{EW}})$ is:

$$\tilde{x}_2 = \tilde{x}_5 = \tilde{x}_7 = \tilde{x}_8 = 1, \quad \tilde{x}_1 = \tilde{x}_3 = \tilde{x}_4 = \tilde{x}_6 = 0,$$

$$\tilde{y}_{25} = \tilde{y}_{27} = \tilde{y}_{28} = \tilde{y}_{57} = \tilde{y}_{58} = \tilde{y}_{78} = 1, \quad \text{all the other } \tilde{y}_{uv} = 0$$

- ▶ The optimal value $z(\text{ILP}_{\text{MCP}}^{\text{EW}})$ is

$$6 + 9 + 4 + 7 + 3 + 8 = 37,$$

as stated in the chapter «Clique problems».

- ▶ The associated optimal solution is the clique $\tilde{K} = \{2, 5, 7, 8\}$, and it is the unique optimal solution.

Summary: ILP models for the weighted variants

Vertex weights:

- ▶ The ILP_{MCP}^{VW} model changes only the objective function:
$$\max \sum_{v \in V} w_v x_v$$
- ▶ On the instance of the chapter «Clique problems»:
 $\tilde{K} = \{2, 3, 5, 8\}$, value 32

Edge weights:

- ▶ The ILP_{MCP}^{EW} model adds a variable y_{uv} per edge, with
 $y_{uv} \leq x_u$ and $y_{uv} \leq x_v$
- ▶ The converse constraint $y_{uv} \geq x_u + x_v - 1$ is not needed, since
 $r_{uv} > 0$
- ▶ On the instance of the chapter «Clique problems»:
 $\tilde{K} = \{2, 5, 7, 8\}$, value 37

Typical mistakes

Introduction: Typical mistakes

- ▶ Seven frequent mistakes in writing and using the clique models
- ▶ They concern the graph on which the constraints are written, the strength of the LP bounds, and the linearization of the edge-weighted model
- ▶ Checking them is a quick way to avoid wrong models

Observation 1: typical mistakes (1/3)

- ▶ *Writing the constraints on the edges of G . The edge formulation of the MCP has one constraint for each non-edge of G ; on the edges of G the same constraints describe the stable sets of G .*
- ▶ *Trusting the LP bound of the edge formulation. The point $x_v = \frac{1}{2}$ is always feasible, so $z(\text{LP}_{\text{MCP}}) \geq n/2$ even when $\omega(G) = 2$.*
- ▶ *Forgetting the bounds $x_v \leq 1$ in the edge formulation. A vertex adjacent to all the others appears in no constraint, and LP-0_{MCP} becomes unbounded. In the clique formulation the bounds are implied.*

Observation: typical mistakes (2/3)

- ▶ *Believing that clique inequalities close every gap. On C_5 they coincide with the non-edge constraints, and the odd cycle inequality is needed.*
- ▶ *Using non-maximal cliques of $\overline{G}$. Their inequalities are valid but dominated: the maximal cliques give the strongest constraints of this family.*

Observation: typical mistakes (3/3)

- ▶ *Mixing up the graphs of the MCP and of the MSSP. The ILP_{MSSP} model has one constraint per edge of G , the ILP_{MCP} model one per edge of $\overline{G}$.*
- ▶ *Writing $y_{uv} = x_u x_v$ in the edge-weighted model. The product is not linear: the constraints $y_{uv} \leq x_u$ and $y_{uv} \leq x_v$ replace it, and the positive weights make $y_{uv} = 1$ whenever both endpoints are selected.*

Summary: Typical mistakes

- ▶ MCP: constraints on the edges of $\overline{G}$; MSSP: constraints on the edges of G
- ▶ The LP bound of the edge formulation is weak, and without $x_v \leq 1$ it can be unbounded
- ▶ Clique inequalities do not close every gap, and only the maximal cliques give the strongest ones
- ▶ The product $x_u x_v$ is replaced by $y_{uv} \leq x_u$ and $y_{uv} \leq x_v$

Summary (1/3)

- ▶ The edge formulation ILP_{MCP} has one binary variable per vertex and one constraint $x_u + x_v \leq 1$ per edge of $\overline{G}$: it is the edge formulation of the MSSP on $\overline{G}$.
- ▶ Its LP relaxation is weak: $x_v = \frac{1}{2}$ is feasible and $z(\text{LP}_{\text{MCP}}) \geq n/2$.
- ▶ The clique inequalities $\sum_{v \in Q} x_v \leq 1$, for the maximal cliques Q of $\overline{G}$, give a stronger formulation with $\omega(G) \leq z(\text{LP}_{\text{MCP}}^{\text{CL}}) \leq z(\text{LP}_{\text{MCP}})$.

Summary (2/3)

- ▶ The dual of LP_{MCP}^{CL} is the LP relaxation of a coloring formulation, hence $\omega(G) \leq z(LP_{MCP}^{CL}) \leq \chi(G)$.
- ▶ The ILP_{MSSP} model has one constraint $x_u + x_v \leq 1$ per edge of G ; with the maximal cliques of G it becomes the clique formulation. On the cycle with a chord the optimal value is $\alpha(G) = 3$.
- ▶ The vertex-weighted model changes only the objective function; the edge-weighted model adds a variable y_{uv} per edge, with $y_{uv} \leq x_u$ and $y_{uv} \leq x_v$. On the instances of the chapter «Clique problems» their optimal values are 32 and 37.

Summary (3/3)

Dashboard (maximization). Bounds on $\omega(G)$ computed in this chapter:

instance	$z(\text{LP}_{\text{MCP}})$	$z(\text{LP}_{\text{MCP}}^{\text{CL}})$	$\omega(G)$
cycle with a chord	$\frac{7}{2}$	3	3
C_5	$\frac{5}{2}$	$\frac{5}{2}$	2 (odd cycle inequality)