

Formulations of packing problems



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Learning objectives

At the end of this chapter you will be able to:

- ▶ write the ILP model of the Bin Packing Problem and explain the link between the assignment variables and the activation variables;
- ▶ write the two LP relaxations and their duals, and prove that their optimal value is $\sum_j w_j / c$;
- ▶ round the LP bound up to obtain a lower bound on the number of bins;
- ▶ prove that the LP bound is at least one half of the optimal value, and that the factor is tight;
- ▶ write the ILP model of the Cutting Stock Problem, with integer variables for the copies of the items.

Prerequisites

- ▶ From the module *Linear Programming*, the chapter «[Duality theory](#)»: the dual of an LP problem, weak and strong duality.
- ▶ From the module *Integer linear programming*, the chapters «[Integer linear programming](#)» and «[Primal and dual bounds](#)»: ILP problems, LP relaxations and bounds.
- ▶ From this module, the chapter «[Packing problems](#)»: the problem and its assumptions.

Overview of the family

Introduction: Overview of the family

- ▶ In this chapter we write the ILP models of the packing problems: the Bin Packing Problem (BPP) and the Cutting Stock Problem (CSP)
- ▶ We first see what the models of the family have in common: the same variables, the same constraints, the same objective function
- ▶ The models are the starting point for the LP relaxations, and the relaxations give lower bounds on the number of bins

Bins, assignment variables and activation variables

- ▶ The ILP models of the packing problems use an upper bound m on the number of bins and the set $\mathcal{B} = \{1, 2, \dots, m\}$ of the bins that may be used.
- ▶ The **assignment variables** x_{jb} place the items $j \in \mathcal{I}$ into the bins $b \in \mathcal{B}$, and the binary **activation variables** y_b say whether bin b is used.
- ▶ Each bin gives one capacity constraint, which also links the two kinds of variables, and the objective function minimizes the number $\sum_{b \in \mathcal{B}} y_b$ of bins used.

Packing models are assignment models

The packing models are **assignment models**: every item, or every copy of an item, is assigned to a used bin, and the items assigned to a bin respect its capacity.

The members of the family change the domain of the assignment variables.

The members of the family

- ▶ The table lists the members of the family, with the name of their ILP model and the section that writes it.

Problem	Model	Assignment variables	Section
BPP	ILP_{BPP}	binary x_{jb}	An ILP model for the Bin Packing Problem
CSP	ILP_{CSP}	integer x_{jb} , the copies of item j in bin b	An ILP model for the Cutting Stock Problem

Summary

The common structure:

- ▶ an upper bound m on the number of bins, and the set $\mathcal{B}$ of the bins
- ▶ assignment variables x_{jb} and binary activation variables y_b
- ▶ one capacity constraint per bin, and the number of used bins as objective function

The members:

- ▶ BPP: binary assignment variables
- ▶ CSP: integer assignment variables, which count the copies

An ILP model for the Bin Packing Problem

Introduction: An ILP model for the Bin Packing Problem

- ▶ The problem of this section is the Bin Packing Problem (BPP), defined in the chapter «[Packing problems](#)»
- ▶ We introduce the variables, write the ILP_{BPP} model and describe its constraints
- ▶ We explain how the capacity constraints link the two kinds of variables, and we write the model of a small instance

The assignment variables

- ▶ Given an upper bound $m \in \mathbb{Z}_{\geq 2}$ on the number of bins used in any optimal BPP solution, we introduce the following mn **binary variables**:

$$x_{jb} = \begin{cases} 1 & \text{if item } j \text{ is packed in bin } b \\ 0 & \text{otherwise} \end{cases}$$

$$\forall j \in \underbrace{\{1, 2, \dots, n\}}_{=\mathcal{I}}, \quad \forall b \in \underbrace{\{1, 2, \dots, m\}}_{=\mathcal{B}},$$

The activation variables

- ▶ We also introduce the following m **binary variables**:

$$y_b = \begin{cases} 1 & \text{if bin } b \text{ is used} \\ 0 & \text{otherwise} \end{cases} \quad \forall b \in \underbrace{\{1, 2, \dots, m\}}_{=\mathcal{B}}.$$

- ▶ The number n of items is a valid upper bound on the minimum number of bins needed to pack all the items since, in the worst possible case, each item is packed in a different bin.
- ▶ Also the number of bins used in any feasible BPP solution is a valid upper bound.

The ILP_{BPP} model

Using these variables, an Integer Linear Programming (ILP) model for the Bin Packing Problem (BPP) reads as follows:

$$\begin{aligned} \min \quad & \sum_{b \in \mathcal{B}} y_b \\ \text{subject to} \quad & \sum_{b \in \mathcal{B}} x_{jb} = 1, & \forall j \in \mathcal{I}, \\ & - \sum_{j \in \mathcal{I}} w_j x_{jb} + c y_b \geq 0, & \forall b \in \mathcal{B}, \\ & x_{jb} \in \{0, 1\}, & \forall j \in \mathcal{I}, \forall b \in \mathcal{B}, \\ & y_b \in \{0, 1\}, & \forall b \in \mathcal{B}. \end{aligned}$$

This ILP model is referred to as the ILP_{BPP} model.

Description of the objective function and constraints

- ▶ The linear objective function minimizes the total number of used bins.
- ▶ The constraints $\sum_{b \in \mathcal{B}} x_{jb} = 1$ ensure that each item is packed in exactly one bin, so that all items are packed (n linear constraints).
- ▶ The capacity constraints link the assignment variables with the activation variables and impose the capacity restrictions: if bin b is used, the weight of its items is at most c ; if it is not used, it contains no item (m linear constraints).
- ▶ Finally, the constraints $x_{jb} \in \{0, 1\}$ and $y_b \in \{0, 1\}$ define the variables of the ILP model.

The capacity constraints do two jobs at once:
they limit the weight in each bin and they link x_{jb} to y_b .

Let us see the link between the variables in detail.

Explanation of the link between the variables (1/2)

- ▶ **If at least one item is packed in bin b , then the bin is used.** The logical implication is

$$(x_{1b} \text{ OR } x_{2b} \text{ OR } \dots \text{ OR } x_{nb}) \implies y_b.$$

Since $w_j > 0$, packing an item in bin b gives $\sum_{j \in \mathcal{J}} w_j x_{jb} > 0$, and the constraint $\sum_{j \in \mathcal{J}} w_j x_{jb} \leq c y_b$ forces $y_b = 1$.

- ▶ **If bin b is not used, then it is empty.** This is the contrapositive of the previous implication. For $y_b = 0$ the constraint reads $\sum_{j \in \mathcal{J}} w_j x_{jb} \leq 0$, and since $w_j > 0$ we obtain $x_{jb} = 0$ for every item j .

Explanation of the link between the variables (2/2)

- ▶ If bin b is used, then it contains at least one item, but only in an optimal solution.
- ▶ The constraints do not impose it.
- ▶ However, if $x_{jb} = 0$ for all j and $y_b = 1$, setting $y_b = 0$ keeps all the constraints satisfied and decreases the objective value by one.
- ▶ Hence, in every optimal solution, empty bins are not counted as used.

Now we write the ILP_{BPP} model of a small instance, with three items, and we look at its optimal solutions.

Example (1/4): the ILP_{BPP} model

- ▶ We consider a bin of capacity $c = 10$ and a set $\mathcal{J} = \{1, 2, 3\}$ of 3 items ($n = 3$), with the following item weights:

	$j = 1$	$j = 2$	$j = 3$
w_j	7	6	3

- ▶ In the rest of the chapter we refer to this BPP instance as the **example instance**.
- ▶ We write the ILP_{BPP} model with $m = 3$.

Example (2/4): the ILP_{BPP} model

The ILP_{BPP} model with $m = 3$ for this BPP instance is:

$$\begin{array}{llllll} \min & y_1 & + y_2 & + y_3 & & \\ \text{subject to} & x_{11} & + x_{12} & + x_{13} & & = 1 \\ & x_{21} & + x_{22} & + x_{23} & & = 1 \\ & x_{31} & + x_{32} & + x_{33} & & = 1 \\ & -7x_{11} & -6x_{21} & -3x_{31} & + 10y_1 & \geq 0 \\ & -7x_{12} & -6x_{22} & -3x_{32} & + 10y_2 & \geq 0 \\ & -7x_{13} & -6x_{23} & -3x_{33} & + 10y_3 & \geq 0 \\ & x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33} & \in & \{0, 1\} \\ & y_1, y_2, y_3 & \in & \{0, 1\} \end{array}$$

Example (3/4): the ILP_{BPP} model

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{BPP}})$ is:

$$\tilde{x}_{11} = \tilde{x}_{31} = 1, \quad \tilde{x}_{12} = \tilde{x}_{13} = \tilde{x}_{21} = 0, \quad \tilde{x}_{22} = 1,$$

$$\tilde{x}_{23} = \tilde{x}_{32} = \tilde{x}_{33} = 0, \quad \tilde{y}_1 = \tilde{y}_2 = 1, \quad \tilde{y}_3 = 0$$

Another optimal solution, obtained by permuting the indices of bins 1 and 2, is:

$$\tilde{x}_{12} = \tilde{x}_{32} = 1, \quad \tilde{x}_{11} = \tilde{x}_{13} = \tilde{x}_{22} = 0, \quad \tilde{x}_{21} = 1,$$

$$\tilde{x}_{23} = \tilde{x}_{31} = \tilde{x}_{33} = 0, \quad \tilde{y}_1 = \tilde{y}_2 = 1, \quad \tilde{y}_3 = 0.$$

Example (4/4): the ILP_{BPP} model

The optimal value is:

$$z(\text{ILP}_{\text{BPP}}) = 2$$

The associated optimal solution of the BPP is given by the packing patterns

$$\tilde{P}_1 = \{1, 3\} \quad \text{and} \quad \tilde{P}_2 = \{2\}$$

The optimal solution is not unique: also the packing patterns $\{1\}$ and $\{2, 3\}$ use two bins.

Summary (1/2)

The ILP_{BPP} model:

- ▶ mn binary assignment variables x_{jb} and m binary activation variables y_b
- ▶ each item in exactly one bin, one capacity constraint per bin
- ▶ minimize the number of used bins

Summary (2/2)

The link between the variables:

- ▶ a non-empty bin is used, an unused bin is empty
- ▶ an empty bin is not counted as used, but only in an optimal solution

The example instance: $z(\text{ILP}_{\text{BPP}}) = 2$, with patterns $\{1, 3\}$ and $\{2\}$

A first Linear Programming relaxation

Introduction: A first Linear Programming relaxation

- ▶ We relax the binary domains of the ILP_{BPP} model and obtain an LP model, the $LP-0_{BPP}$ model
- ▶ Its optimal value is a lower bound on the number of bins: we compute it on the example instance and then for every instance
- ▶ We write the dual model and prove that the optimal value is $\sum_j w_j/c$, with two different proofs

The LP-0_{BPP} model

To obtain a Linear Programming (LP) relaxation of the ILP_{BPP} model, we relax the constraints $x_{jb} \in \{0, 1\}$ and $y_b \in \{0, 1\}$ with the following ones:

$$\begin{aligned}x_{jb} &\geq 0, & \forall j \in \mathcal{J}, \forall b \in \mathcal{B}, \\ y_b &\geq 0, & \forall b \in \mathcal{B}.\end{aligned}$$

This LP model is referred to as the LP-0_{BPP} model.

Example (1/2): the LP-0_{BPP} model

The LP-0_{BPP} model for the example instance is:

$$\begin{array}{llllllll} \min & & y_1 & + y_2 & + y_3 & & & \\ \text{subject to} & x_{11} & + x_{12} & + x_{13} & & & & = 1 \\ & x_{21} & + x_{22} & + x_{23} & & & & = 1 \\ & x_{31} & + x_{32} & + x_{33} & & & & = 1 \\ & -7x_{11} & - 6x_{21} & - 3x_{31} & & + 10y_1 & & \geq 0 \\ & -7x_{12} & - 6x_{22} & - 3x_{32} & & + 10y_2 & & \geq 0 \\ & -7x_{13} & - 6x_{23} & - 3x_{33} & & + 10y_3 & & \geq 0 \\ & x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33} & & & & & & \geq 0 \\ & y_1, y_2, y_3 & & & & & & \geq 0 \end{array}$$

Example (2/2): the LP-0_{BPP} model

An optimal solution $\tilde{\mathbf{x}}$ (LP-0_{BPP}) is:

$$\tilde{x}_{11} = \tilde{x}_{12} = \tilde{x}_{13} = \tilde{x}_{21} = \tilde{x}_{22} = \tilde{x}_{23} = \tilde{x}_{31} = \tilde{x}_{32} = \tilde{x}_{33} = \frac{1}{3},$$

$$\tilde{y}_1 = \tilde{y}_2 = \tilde{y}_3 = \frac{8}{15}$$

The optimal value $z(\text{LP-0}_{\text{BPP}})$ is $\frac{8}{5}$. Accordingly, we have the lower bound:

$$\frac{8}{5} \leq z(\text{ILP}_{\text{BPP}})$$

and, since $z(\text{ILP}_{\text{BPP}})$ is an integer, $z(\text{ILP}_{\text{BPP}}) \geq \lceil \frac{8}{5} \rceil = 2$.

The dual variables of the $LP-0_{BPP}$ model

Using

- ▶ a free dual variable μ_j for each constraint $\sum_{b \in \mathcal{B}} x_{jb} = 1$
($j \in \mathcal{I}$)
- ▶ a non-negative dual variable π_b for each capacity constraint
($b \in \mathcal{B}$)

we write the dual model of the $LP-0_{BPP}$ model.

The dual model $D(\text{LP-0}_{\text{BPP}})$

The dual model of the LP-0_{BPP} model reads as follows:

$$\begin{aligned} \max \quad & \sum_{j \in \mathcal{J}} \mu_j \\ \text{subject to} \quad & \mu_j - w_j \pi_b \leq 0, \quad \forall j \in \mathcal{J}, \forall b \in \mathcal{B}, \\ & c \pi_b \leq 1, \quad \forall b \in \mathcal{B}, \\ & \mu_j \geq 0, \quad \forall j \in \mathcal{J}, \\ & \pi_b \geq 0, \quad \forall b \in \mathcal{B}. \end{aligned}$$

This LP model is referred to as the dual $D(\text{LP-0}_{\text{BPP}})$ model.

Example (1/3): the $D(\text{LP-0}_{\text{BPP}})$ model

The dual $D(\text{LP-0}_{\text{BPP}})$ model for the example instance is (objective and first constraints):

$$\begin{array}{llllllll} \max & \mu_1 & + \mu_2 & + \mu_3 & & & & \\ \text{subject to} & \mu_1 & & & - 7\pi_1 & & & \leq 0 \\ & \mu_1 & & & & - 7\pi_2 & & \leq 0 \\ & \mu_1 & & & & & - 7\pi_3 & \leq 0 \\ & & \mu_2 & & - 6\pi_1 & & & \leq 0 \\ & & \mu_2 & & & - 6\pi_2 & & \leq 0 \\ & & \mu_2 & & & & - 6\pi_3 & \leq 0 \\ & & & \mu_3 & - 3\pi_1 & & & \leq 0 \\ & & & \mu_3 & & - 3\pi_2 & & \leq 0 \\ & & & \mu_3 & & & - 3\pi_3 & \leq 0 \end{array}$$

Example (2/3): the $D(\text{LP-0}_{\text{BPP}})$ model

The model continues with the remaining constraints and the domains:

$$\begin{array}{rccccccc} & & & & 10\pi_1 & & \leq & 1 \\ & & & & & 10\pi_2 & \leq & 1 \\ & & & & & & 10\pi_3 & \leq & 1 \\ \mu_1, & \mu_2, & \mu_3 & & & & & \geq & 0 \\ & & & \pi_1, & \pi_2, & \pi_3 & \geq & 0 \end{array}$$

Example (3/3): the $D(\text{LP-0}_{\text{BPP}})$ model

An optimal solution $(\tilde{\mu}, \tilde{\pi})$ of $D(\text{LP-0}_{\text{BPP}})$ is:

$$\tilde{\mu}_1 = \frac{7}{10}, \quad \tilde{\mu}_2 = \frac{3}{5}, \quad \tilde{\mu}_3 = \frac{3}{10}, \quad \tilde{\pi}_1 = \tilde{\pi}_2 = \tilde{\pi}_3 = \frac{1}{10}$$

The optimal value $z(D(\text{LP-0}_{\text{BPP}}))$ is $\frac{8}{5}$.

On the example instance the optimal value is $\frac{8}{5} = \frac{7+6+3}{10}$.

We now give optimal primal and dual solutions for **every** BPP instance.

Optimal solutions of LP-0_{BPP} and of its dual

Proposition 1

An optimal solution of the primal LP-0_{BPP} model is:

$$\tilde{x}_{jb} = \frac{1}{m}, \quad \forall j \in \mathcal{J}, \forall b \in \mathcal{B},$$

$$\tilde{y}_b = \frac{\sum_{j \in \mathcal{J}} w_j}{m c}, \quad \forall b \in \mathcal{B},$$

and an optimal solution of the dual $D(\text{LP-0}_{\text{BPP}})$ model is:

$$\tilde{\mu}_j = \frac{w_j}{c}, \quad \forall j \in \mathcal{J},$$

$$\tilde{\pi}_b = \frac{1}{c}, \quad \forall b \in \mathcal{B}.$$

The optimal value of $LP-0_{BPP}$

Corollary 1

The optimal value of the primal $LP-0_{BPP}$ and dual $D(LP-0_{BPP})$ models is:

$$z(LP-0_{BPP}) = \frac{\sum_{j \in \mathcal{J}} w_j}{c} = z(D(LP-0_{BPP}))$$

How the proof proceeds

- ▶ The proof proceeds in three steps:
 1. We show that the primal solution $\tilde{x}_{jb}, \tilde{y}_b$ of the proposition is a feasible solution for the primal $LP-0_{BPP}$ model;
 2. We show that the dual solution $\tilde{\mu}_j, \tilde{\pi}_b$ of the proposition is a feasible solution for the dual $D(LP-0_{BPP})$ model;
 3. We show that both feasible solutions have the same objective function value.

- ▶ Therefore, by LP Weak Duality, the two solutions are a pair of optimal primal and dual solutions.

Proof (1/4)

1. For every $j \in \mathcal{J}$, we have:

$$\sum_{b \in \mathcal{B}} \tilde{x}_{jb} = \sum_{b \in \mathcal{B}} \frac{1}{m} = \frac{m}{m} = 1$$

thus the constraints $\sum_{b \in \mathcal{B}} x_{jb} = 1$ are satisfied.

For every $b \in \mathcal{B}$, we have:

$$-\sum_{j \in \mathcal{J}} w_j \tilde{x}_{jb} + c \tilde{y}_b = -\sum_{j \in \mathcal{J}} w_j \frac{1}{m} + c \frac{\sum_{j \in \mathcal{J}} w_j}{m c} = 0$$

thus the capacity constraints are satisfied.

Proof (2/4)

1. We have $\tilde{x}_{jb} \geq 0$ and $\tilde{y}_b \geq 0$, for every $j \in \mathcal{J}$ and every $b \in \mathcal{B}$, thus the non-negativity constraints are satisfied.

Therefore, $\tilde{x}_{jb}$, $\tilde{y}_b$ provide a feasible solution for the primal LP-0_{BPP} model.

2. For every $b \in \mathcal{B}$ and every $j \in \mathcal{J}$, we have:

$$\tilde{\mu}_j - w_j \tilde{\pi}_b = \frac{w_j}{c} - \frac{w_j}{c} = 0$$

thus the constraints $\mu_j - w_j \pi_b \leq 0$ are satisfied.

Proof (3/4)

2. For every $b \in \mathcal{B}$, we have:

$$c \tilde{\pi}_b = \frac{c}{c} = 1$$

thus the constraints $c \pi_b \leq 1$ are satisfied.

We have $\tilde{\mu}_j \geq 0$ and $\tilde{\pi}_b \geq 0$, for every $b \in \mathcal{B}$ and every $j \in \mathcal{J}$, thus the domains of the dual variables are respected.

Therefore, $\tilde{\mu}_j, \tilde{\pi}_b$ provide a feasible solution for the dual $D(\text{LP-0}_{\text{BPP}})$ model.

Proof (4/4)

3. For the objective functions, we have:

$$\sum_{b \in \mathcal{B}} \tilde{y}_b = \sum_{b \in \mathcal{B}} \frac{\sum_{j \in \mathcal{J}} w_j}{m \, c} = \frac{\sum_{j \in \mathcal{J}} w_j}{c} = \sum_{j \in \mathcal{J}} \tilde{\mu}_j$$

thus the primal and dual objective function values are equal.

Therefore, by LP Weak Duality, the two solutions are a pair of optimal primal and dual solutions. □

An alternative proof

- ▶ This is an alternative proof that does not rely on the dual formulation but instead reasons directly on the minimum objective function value imposed by the constraints.
- ▶ The idea: summing the capacity constraints gives a lower bound on $\sum_{b \in \mathcal{B}} y_b$ that holds for every feasible solution; then we find a feasible solution that attains it.

The optimal value of LP-0_{BPP} , without the dual

Proposition 2

An optimal solution of the primal LP-0_{BPP} model is:

$$\tilde{x}_{jb} = \frac{1}{m}, \quad \forall j \in \mathcal{J}, \forall b \in \mathcal{B},$$

$$\tilde{y}_b = \frac{\sum_{j \in \mathcal{J}} w_j}{m c}, \quad \forall b \in \mathcal{B},$$

and the optimal value is

$$z(\text{LP-0}_{\text{BPP}}) = \frac{\sum_{j \in \mathcal{J}} w_j}{c}.$$

Proof (1/3)

Summing the capacity constraints over all $b \in \mathcal{B}$ gives

$$c \sum_{b \in \mathcal{B}} y_b \geq \sum_{b \in \mathcal{B}} \sum_{j \in \mathcal{J}} w_j x_{jb} = \sum_{j \in \mathcal{J}} w_j \underbrace{\sum_{b \in \mathcal{B}} x_{jb}}_{=1} = \sum_{j \in \mathcal{J}} w_j,$$

hence

$$\sum_{b \in \mathcal{B}} y_b \geq \frac{1}{c} \sum_{j \in \mathcal{J}} w_j.$$

Therefore every feasible solution of LP-0_{BPP} satisfies this lower bound.

Proof (2/3)

Now consider the point of the proposition. For every $j \in \mathcal{J}$, we have

$$\sum_{b \in \mathcal{B}} \tilde{x}_{jb} = m \frac{1}{m} = 1,$$

so the constraints $\sum_{b \in \mathcal{B}} x_{jb} = 1$ hold.

Moreover, for every $b \in \mathcal{B}$, we have

$$\sum_{j \in \mathcal{J}} w_j \tilde{x}_{jb} = \sum_{j \in \mathcal{J}} w_j \frac{1}{m} = \frac{1}{m} \sum_{j \in \mathcal{J}} w_j = c \tilde{y}_b,$$

so the capacity constraints hold with equality.

Proof (3/3)

Finally, the non-negativity constraints are satisfied by construction. Hence the point of the proposition is a feasible solution for $LP-0_{BPP}$.

The objective value of this point is

$$\sum_{b \in \mathcal{B}} \tilde{y}_b = m \frac{\sum_{j \in \mathcal{J}} w_j}{m c} = \frac{1}{c} \sum_{j \in \mathcal{J}} w_j,$$

which matches the lower bound above.

Therefore the point is an optimal solution with objective value $\sum_{j \in \mathcal{J}} w_j / c$. □

Summary

The LP-0_{BPP} model:

- ▶ the binary domains are replaced by $x_{jb} \geq 0$ and $y_b \geq 0$
- ▶ on the example instance: $z(\text{LP-0}_{\text{BPP}}) = \frac{8}{5}$, hence $z(\text{ILP}_{\text{BPP}}) \geq \lceil \frac{8}{5} \rceil = 2$

The optimal value:

- ▶ primal solution $\tilde{x}_{jb} = 1/m$, dual solution $\tilde{\mu}_j = w_j/c$ and $\tilde{\pi}_b = 1/c$
- ▶ $z(\text{LP-0}_{\text{BPP}}) = \sum_j w_j/c$, by weak duality or by summing the capacity constraints

A second Linear Programming relaxation

Introduction: A second Linear Programming relaxation

- ▶ We relax the binary domains of the ILP_{BPP} model keeping the upper bounds $y_b \leq 1$: this is the LP_{BPP} model
- ▶ We write it on the example instance, write its dual, and show that its optimal value is again $\sum_j w_j/c$
- ▶ Then we measure the quality of this bound: it is at least one half of the optimal number of bins, and the factor one half cannot be improved

The LP_{BPP} model

To obtain a Linear Programming (LP) relaxation of the ILP_{BPP} model, we relax the constraints $x_{jb} \in \{0, 1\}$ and $y_b \in \{0, 1\}$ with the following ones:

$$x_{jb} \geq 0, \quad \forall j \in \mathcal{J}, \forall b \in \mathcal{B},$$

$$y_b \leq 1, \quad \forall b \in \mathcal{B},$$

$$y_b \geq 0, \quad \forall b \in \mathcal{B}.$$

This LP model is referred to as the LP_{BPP} model.

Example (1/3): the LP_{BPP} model

The LP_{BPP} model for the example instance is (objective, assignment and capacity constraints):

$$\begin{array}{llllll} \min & y_1 & + y_2 & + y_3 & & \\ \text{subject to} & x_{11} & + x_{12} & + x_{13} & & = 1 \\ & x_{21} & + x_{22} & + x_{23} & & = 1 \\ & x_{31} & + x_{32} & + x_{33} & & = 1 \\ & -7x_{11} & -6x_{21} & -3x_{31} & + 10y_1 & \geq 0 \\ & -7x_{12} & -6x_{22} & -3x_{32} & + 10y_2 & \geq 0 \\ & -7x_{13} & -6x_{23} & -3x_{33} & + 10y_3 & \geq 0 \end{array}$$

Example (2/3): the LP_{BPP} model

The model continues with the upper bounds $y_b \leq 1$ and the non-negativity constraints:

$$y_1 \leq 1$$

$$y_2 \leq 1$$

$$y_3 \leq 1$$

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33} \geq 0$$

$$y_1, y_2, y_3 \geq 0$$

Example (3/3): the LP_{BPP} model

An optimal solution $\tilde{\mathbf{x}}(\text{LP}_{\text{BPP}})$ is:

$$\tilde{x}_{11} = \tilde{x}_{12} = \tilde{x}_{13} = \tilde{x}_{21} = \tilde{x}_{22} = \tilde{x}_{23} = \tilde{x}_{31} = \tilde{x}_{32} = \tilde{x}_{33} = \frac{1}{3},$$

$$\tilde{y}_1 = \tilde{y}_2 = \tilde{y}_3 = \frac{8}{15}$$

The optimal value $z(\text{LP}_{\text{BPP}})$ is $\frac{8}{5}$.

Accordingly, we have the same lower bound as before.

The dual variables of the LP_{BPP} model

Using

- ▶ a free dual variable μ_j for each constraint $\sum_{b \in \mathcal{B}} x_{jb} = 1$ ($j \in \mathcal{I}$)
- ▶ a non-negative dual variable π_b for each capacity constraint ($b \in \mathcal{B}$)
- ▶ a non-positive dual variable λ_b for each constraint $y_b \leq 1$ ($b \in \mathcal{B}$)

we write the dual model of the LP_{BPP} model.

The dual model $D(\text{LP}_{\text{BPP}})$

The dual model of the LP_{BPP} model reads as follows:

$$\max \sum_{j \in \mathcal{I}} \mu_j + \sum_{b \in \mathcal{B}} \lambda_b$$

$$\text{subject to } \mu_j - w_j \pi_b \leq 0, \quad \forall j \in \mathcal{I}, \forall b \in \mathcal{B},$$

$$c \pi_b + \lambda_b \leq 1, \quad \forall b \in \mathcal{B},$$

$$\mu_j \geq 0, \quad \forall j \in \mathcal{I},$$

$$\pi_b \geq 0, \quad \forall b \in \mathcal{B},$$

$$\lambda_b \leq 0, \quad \forall b \in \mathcal{B}.$$

This LP model is referred to as the dual $D(\text{LP}_{\text{BPP}})$ model.

Example (1/3): the $D(\text{LP}_{\text{BPP}})$ model

The $D(\text{LP}_{\text{BPP}})$ model of the example instance is (objective and first constraints):

$$\begin{array}{llll} \max & \mu_1 + \mu_2 + \mu_3 & & + \lambda_1 + \lambda_2 + \lambda_3 \\ \text{subject to} & \mu_1 & - 7\pi_1 & \leq 0 \\ & \mu_1 & - 7\pi_2 & \leq 0 \\ & \mu_1 & - 7\pi_3 & \leq 0 \\ & \mu_2 & - 6\pi_1 & \leq 0 \\ & \mu_2 & - 6\pi_2 & \leq 0 \\ & \mu_2 & - 6\pi_3 & \leq 0 \\ & \mu_3 & - 3\pi_1 & \leq 0 \\ & \mu_3 & - 3\pi_2 & \leq 0 \\ & \mu_3 & - 3\pi_3 & \leq 0 \end{array}$$

Example (2/3): the $D(\text{LP}_{\text{BPP}})$ model

The model continues with the remaining constraints and the domains:

$$\begin{array}{rcll} 10\pi_1 & & + \lambda_1 & \leq 1 \\ & 10\pi_2 & & + \lambda_2 \leq 1 \\ & & 10\pi_3 & + \lambda_3 \leq 1 \\ \mu_1, \mu_2, \mu_3 & & & \geq 0 \\ \pi_1, \pi_2, \pi_3 & & & \geq 0 \\ & \lambda_1, \lambda_2, \lambda_3 & \leq 0 \end{array}$$

Example (3/3): the $D(\text{LP}_{\text{BPP}})$ model

The optimal value $z(D(\text{LP}_{\text{BPP}}))$ is $\frac{8}{5}$.

An optimal solution $(\tilde{\mu}, \tilde{\pi}, \tilde{\lambda})$ of $D(\text{LP}_{\text{BPP}})$ is:

$$\begin{aligned}\tilde{\mu}_1 &= \frac{7}{10}, & \tilde{\mu}_2 &= \frac{3}{5}, & \tilde{\mu}_3 &= \frac{3}{10}, \\ \tilde{\pi}_1 &= \tilde{\pi}_2 = \tilde{\pi}_3 = \frac{1}{10}, & \tilde{\lambda}_1 &= \tilde{\lambda}_2 = \tilde{\lambda}_3 = 0\end{aligned}$$

Valid constraints implied by the model

- The constraints:

$$x_{jb} \leq 1, \quad \forall j \in \mathcal{J}, \forall b \in \mathcal{B},$$

are also valid since, for every $j \in \mathcal{J}$ and every $b \in \mathcal{B}$, we have:

$$x_{jb} \in \{0, 1\} \quad \Rightarrow \quad x_{jb} \leq 1$$

but they are implied by the constraints $\sum_{b \in \mathcal{B}} x_{jb} = 1$.

Optimal solutions of LP_{BPP} and of its dual

Proposition 3

An optimal solution of the primal LP_{BPP} model is:

$$\tilde{x}_{jb} = \frac{1}{m}, \quad \forall j \in \mathcal{J}, \forall b \in \mathcal{B},$$

$$\tilde{y}_b = \frac{\sum_{j \in \mathcal{J}} w_j}{m c}, \quad \forall b \in \mathcal{B},$$

and an optimal solution of the dual $D(LP_{BPP})$ model is:

$$\tilde{\mu}_j = \frac{w_j}{c}, \quad \forall j \in \mathcal{J},$$

$$\tilde{\pi}_b = \frac{1}{c}, \quad \forall b \in \mathcal{B},$$

$$\tilde{\lambda}_b = 0, \quad \forall b \in \mathcal{B}.$$

The optimal value of LP_{BPP}

Corollary 2

The optimal value of the primal LP_{BPP} and dual $D(LP_{BPP})$ models is:

$$z(LP_{BPP}) = \frac{\sum_{j \in \mathcal{J}} w_j}{c} = z(D(LP_{BPP}))$$

Proof

The proof is similar to the previous one. In addition, it is necessary to observe that

$$\tilde{y}_b = \frac{\sum_{j \in \mathcal{J}} w_j}{m c} \leq 1, \quad \forall b \in \mathcal{B},$$

since

$$\frac{\sum_j w_j}{c} \leq z(\text{ILP}_{\text{BPP}}) \leq m,$$

where the last inequality holds because m is an upper bound on the number of bins in any optimal BPP solution.

Accordingly, the constraints $y_b \leq 1$ are also satisfied. □

Some important theoretical properties

- ▶ The following two propositions show that the ratio between $z(LP_{BPP})$ and $z(ILP_{BPP})$ is at least one half for every BPP instance.
- ▶ **By hypothesis** $z(ILP_{BPP}) > 1$, since not all items fit in a single bin.
- ▶ Furthermore, there exist instances for which the value one half can be asymptotically achieved as the capacity $c \rightarrow +\infty$.

The LP bound is at least one half

Proposition 4

For every BPP instance, we have:

$$\frac{z(\text{LP}_{\text{BPP}})}{z(\text{ILP}_{\text{BPP}})} \geq \frac{1}{2}$$

Proof (1/4)

- ▶ Consider an optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{BPP}})$ of value $z = z(\text{ILP}_{\text{BPP}})$ (number of used bins).
- ▶ We can assume, without loss of generality, that bins $1, 2, \dots, z$ are the non-empty ones.
- ▶ In an optimal solution, no two bins can have total item weight at most $c/2$, since otherwise their contents could be merged into a single bin without exceeding the bin capacity c (contradicting optimality).

Proof (2/4)

If all non-empty bins have total item weight greater than $c/2$, then

$$\sum_{j \in \mathcal{J}} w_j = \sum_{b=1}^z \sum_{j \in \mathcal{J}} w_j \tilde{x}_{jb} > z \frac{c}{2}$$

$$\implies \frac{\frac{\sum_{j \in \mathcal{J}} w_j}{c}}{z} > \frac{1}{2} \implies \frac{z(\text{LP}_{\text{BPP}})}{z(\text{ILP}_{\text{BPP}})} \geq \frac{1}{2}.$$

Proof (3/4)

Otherwise, suppose that there exists a bin with total item weight at most $c/2$. Renumbering bins if necessary, let it be bin z , so that

$$\sum_{j \in \mathcal{J}} w_j \tilde{x}_{jz} \leq \frac{c}{2}.$$

Since $z > 1$, there exists another bin say bin 1; by optimality, we must have

$$\sum_{j \in \mathcal{J}} w_j \tilde{x}_{j1} + \sum_{j \in \mathcal{J}} w_j \tilde{x}_{jz} > c,$$

and, by optimality, every bin in $\{2, 3, \dots, z-1\}$ has total item weight $> c/2$.

Proof (4/4)

Hence,

$$\begin{aligned}\sum_{j \in \mathcal{J}} w_j &= \sum_{b=1}^z \sum_{j \in \mathcal{J}} w_j \tilde{x}_{jb} = \underbrace{\sum_{b=2}^{z-1} \sum_{j \in \mathcal{J}} w_j \tilde{x}_{jb}}_{\geq \frac{c}{2}(z-2)} + \underbrace{\sum_{j \in \mathcal{J}} w_j \tilde{x}_{j1} + \sum_{j \in \mathcal{J}} w_j \tilde{x}_{jz}}_{> c} \\ &> \frac{c}{2}(z-2) + c = z \frac{c}{2} \\ \Rightarrow \quad \frac{\frac{\sum_{j \in \mathcal{J}} w_j}{c}}{z} &> \frac{1}{2} \quad \Rightarrow \quad \frac{z(\text{LP}_{\text{BPP}})}{z(\text{ILP}_{\text{BPP}})} \geq \frac{1}{2}.\end{aligned}$$

□

The factor one half cannot be improved:
we now build instances whose ratio tends to $\frac{1}{2}$.

The factor one half is tight as $c \rightarrow +\infty$

Proposition 5

There are BPP instances such that:

$$\frac{z(\text{LP}_{\text{BPP}})}{z(\text{ILP}_{\text{BPP}})} \rightarrow \frac{1}{2} \quad \text{as} \quad c \rightarrow +\infty$$

Proof (1/2)

Given $\vartheta \in \mathbb{Z}_{\geq 1}$, consider the BPP instance with bin capacity $c = 2\vartheta$ and n items having the following weights:

$$w_j = \vartheta + 1, \quad \forall j \in \mathcal{I}.$$

No bin can contain more than one item, thus each item must be packed alone in its own bin. Therefore, the BPP optimal value is

$$z(\text{ILP}_{\text{BPP}}) = n.$$

For the LP relaxation we have:

$$z(\text{LP}_{\text{BPP}}) = \frac{n(\vartheta + 1)}{2\vartheta}.$$

Proof (2/2)

It follows that:

$$\lim_{\vartheta \rightarrow +\infty} \frac{z(\text{LP}_{\text{BPP}})}{z(\text{ILP}_{\text{BPP}})} = \lim_{\vartheta \rightarrow +\infty} \frac{\frac{n(\vartheta+1)}{2^\vartheta}}{n} = \lim_{\vartheta \rightarrow +\infty} \left(\frac{1}{2} + \underbrace{\frac{1}{2^\vartheta}}_{\rightarrow 0} \right) = \frac{1}{2}.$$

Accordingly, we conclude that:

$$\frac{z(\text{LP}_{\text{BPP}})}{z(\text{ILP}_{\text{BPP}})} \rightarrow \frac{1}{2} \quad \text{as} \quad c \rightarrow +\infty.$$



The rounded LP bound

- ▶ The following two propositions show that the ratio between $\lceil z(\text{LP}_{\text{BPP}}) \rceil$ and $z(\text{ILP}_{\text{BPP}})$ is at least one half for every BPP instance.
- ▶ Furthermore, there exist instances for which the value one half can be asymptotically achieved as the number of items $n \rightarrow +\infty$.

The rounded LP bound is at least one half

Proposition 6

For every BPP instance, we have:

$$\frac{\lceil z(\text{LP}_{\text{BPP}}) \rceil}{z(\text{ILP}_{\text{BPP}})} \geq \frac{1}{2}$$

Proof (1/4)

- ▶ Consider an optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{BPP}})$ of value $z = z(\text{ILP}_{\text{BPP}})$ (number of used bins).
- ▶ We can assume, without loss of generality, that bins $1, 2, \dots, z$ are the non-empty ones.
- ▶ In an optimal solution, no two bins can have total item weight at most $c/2$, since otherwise their contents could be merged into a single bin without exceeding the bin capacity c (contradicting optimality).

Proof (2/4)

If all non-empty bins have total item weight greater than $c/2$, then

$$\sum_{j \in \mathcal{J}} w_j = \sum_{b=1}^z \sum_{j \in \mathcal{J}} w_j \tilde{x}_{jb} > \frac{zc}{2} \quad \Rightarrow \quad \frac{\sum_{j \in \mathcal{J}} w_j}{c} > \frac{z}{2}$$

$$\Rightarrow \quad \left\lceil \frac{\sum_{j \in \mathcal{J}} w_j}{c} \right\rceil \geq \left\lceil \frac{z}{2} \right\rceil \geq \frac{z}{2}$$

$$\Rightarrow \quad \frac{\left\lceil \frac{\sum_{j \in \mathcal{J}} w_j}{c} \right\rceil}{z} \geq \frac{1}{2} \quad \Rightarrow \quad \frac{\lceil z(\text{LP}_{\text{BPP}}) \rceil}{z(\text{ILP}_{\text{BPP}})} \geq \frac{1}{2}.$$

Proof (3/4)

Otherwise, suppose that there exists a bin with total item weight at most $c/2$. Renumbering bins if necessary, let it be bin z . Then we have:

$$\begin{aligned}\sum_{j \in \mathcal{J}} w_j &= \sum_{b=1}^z \sum_{j \in \mathcal{J}} w_j \tilde{x}_{jb} > \sum_{b=1}^{z-1} \sum_{j \in \mathcal{J}} w_j \tilde{x}_{jb} > (z-1) \frac{c}{2} \\ \implies z &< 2 \sum_{j \in \mathcal{J}} \frac{w_j}{c} + 1.\end{aligned}$$

Proof (4/4)

Since z must be an integer, this implies:

$$z \leq \left\lceil 2 \sum_{j \in \mathcal{J}} \frac{w_j}{c} \right\rceil \leq 2 \left\lceil \sum_{j \in \mathcal{J}} \frac{w_j}{c} \right\rceil.$$

Therefore:

$$\frac{\left\lceil \sum_{j \in \mathcal{J}} \frac{w_j}{c} \right\rceil}{z} \geq \frac{1}{2} \quad \Rightarrow \quad \frac{\lceil z(\text{LP}_{\text{BPP}}) \rceil}{z(\text{ILP}_{\text{BPP}})} \geq \frac{1}{2}.$$



The factor one half is tight as $n \rightarrow +\infty$

Proposition 7

There are BPP instances such that:

$$\frac{\lceil z(\text{LP}_{\text{BPP}}) \rceil}{z(\text{ILP}_{\text{BPP}})} \rightarrow \frac{1}{2} \quad \text{as} \quad n \rightarrow +\infty$$

Proof (1/2)

Given $n \in \mathbb{Z}_{\geq 2}$ (take n even), consider the BPP instance with bin capacity $c = n$ and n items having the following weights:

$$w_j = \frac{c}{2} + 1 = \frac{n}{2} + 1, \quad \forall j \in \mathcal{I}.$$

No bin can contain more than one item, since $w_j > c/2$ for every j . Thus each item must be packed alone in its own bin. Therefore, the BPP optimal value is

$$z(\text{ILP}_{\text{BPP}}) = n.$$

Proof (2/2)

For the LP relaxation we have:

$$z(\text{LP}_{\text{BPP}}) = \frac{\sum_{j \in \mathcal{J}} w_j}{c} = \frac{n \left(\frac{n}{2} + 1 \right)}{n} = \frac{n}{2} + 1,$$

hence $\lceil z(\text{LP}_{\text{BPP}}) \rceil = \frac{n}{2} + 1$ (since n is even). It follows that:

$$\lim_{n \rightarrow +\infty} \frac{\lceil z(\text{LP}_{\text{BPP}}) \rceil}{z(\text{ILP}_{\text{BPP}})} = \lim_{n \rightarrow +\infty} \frac{\frac{n}{2} + 1}{n} = \lim_{n \rightarrow +\infty} \left(\frac{1}{2} + \underbrace{\frac{1}{n}}_{\rightarrow 0} \right) = \frac{1}{2}.$$

Accordingly, we conclude that:

$$\frac{\lceil z(\text{LP}_{\text{BPP}}) \rceil}{z(\text{ILP}_{\text{BPP}})} \rightarrow \frac{1}{2} \quad \text{as} \quad n \rightarrow +\infty.$$



Summary (1/2)

The LP_{BPP} model:

- ▶ the binary domains are replaced by $x_{jb} \geq 0$, $y_b \leq 1$ and $y_b \geq 0$
- ▶ the constraints $x_{jb} \leq 1$ are valid but implied by $\sum_{b \in \mathcal{B}} x_{jb} = 1$
- ▶ on the example instance: $z(\text{LP}_{\text{BPP}}) = \frac{8}{5}$, the same lower bound as before

The optimal value:

- ▶ the same solutions as for LP-0_{BPP} , with $\tilde{\lambda}_b = 0$
- ▶ $z(\text{LP}_{\text{BPP}}) = \sum_j w_j / c$

Summary (2/2)

The quality of the bound:

- ▶ $z(\text{LP}_{\text{BPP}})/z(\text{ILP}_{\text{BPP}}) \geq \frac{1}{2}$ for every instance
- ▶ the value $\frac{1}{2}$ is attained asymptotically as $c \rightarrow +\infty$

The rounded bound:

- ▶ $\lceil z(\text{LP}_{\text{BPP}}) \rceil / z(\text{ILP}_{\text{BPP}}) \geq \frac{1}{2}$ for every instance
- ▶ the value $\frac{1}{2}$ is attained asymptotically as $n \rightarrow +\infty$

An ILP model for the Cutting Stock Problem

Introduction: An ILP model for the Cutting Stock Problem

- ▶ The problem of this section is the Cutting Stock Problem (CSP), defined in the chapter «[Packing problems](#)», with its reference instance
- ▶ Each item j must be packed in d_j identical copies
- ▶ We write the ILP_{CSP} model: the same structure as the ILP_{BPP} model, with integer variables that count the copies

The integer variables

- ▶ Given an upper bound $m \in \mathbb{Z}_{\geq 2}$ on the number of bins used in any optimal CSP solution, we introduce the following mn **integer variables**:

x_{jb} = number of copies of item j packed in bin b

$$\forall j \in \mathcal{J}, \quad \forall b \in \underbrace{\{1, 2, \dots, m\}}_{=\mathcal{B}},$$

The activation variables

- ▶ We also introduce the following m **binary variables**:

$$y_b = \begin{cases} 1 & \text{if bin } b \text{ is used} \\ 0 & \text{otherwise} \end{cases} \quad \forall b \in \mathcal{B}.$$

- ▶ The total number $\sum_{j \in \mathcal{J}} d_j$ of copies is a valid value of m , and so is the number of bins of any feasible CSP solution.

The ILP_{CSP} model

Using these variables, an Integer Linear Programming (ILP) model for the Cutting Stock Problem (CSP) reads as follows:

$$\begin{aligned} \min \quad & \sum_{b \in \mathcal{B}} y_b \\ \text{subject to} \quad & \sum_{b \in \mathcal{B}} x_{jb} = d_j, & \forall j \in \mathcal{J}, \\ & - \sum_{j \in \mathcal{J}} w_j x_{jb} + c y_b \geq 0, & \forall b \in \mathcal{B}, \\ & x_{jb} \in \mathbb{Z}_{\geq 0}, & \forall j \in \mathcal{J}, \forall b \in \mathcal{B}, \\ & y_b \in \{0, 1\}, & \forall b \in \mathcal{B}. \end{aligned}$$

This ILP model is referred to as the ILP_{CSP} model.

Description of the objective function and constraints

- ▶ The linear objective function minimizes the total number of used bins.
- ▶ The constraints $\sum_{b \in \mathcal{B}} x_{jb} = d_j$ ensure that exactly d_j copies of each item j are packed (n linear constraints).
- ▶ The capacity constraints link the integer variables with the activation variables and impose the capacity restrictions, as the capacity constraints of the ILP_{BPP} model (m linear constraints).
- ▶ Finally, the constraints $x_{jb} \in \mathbb{Z}_{\geq 0}$ and $y_b \in \{0, 1\}$ define the variables of the ILP model.

The ILP_{CSP} model and the ILP_{BPP} model

- ▶ With $d_j = 1$ for every item, the constraints $\sum_{b \in \mathcal{B}} x_{jb} = d_j$ give $x_{jb} \in \{0, 1\}$, and the ILP_{CSP} model is the ILP_{BPP} model.
- ▶ The copies of an item are identical: a single integer variable counts them in each bin.
- ▶ The ILP_{BPP} model on the extended set of items would use $m \sum_j d_j$ binary variables instead of mn integer ones.

Example (1/3): the ILP_{CSP} model

- ▶ We consider bins of capacity $c = 10$ and a set $\mathcal{J} = \{1, 2, 3, 4\}$ of 4 items ($n = 4$), with the following weights and required copies:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
w_j	4	5	3	6
d_j	2	1	3	1

- ▶ This is the CSP instance of the chapter «[Packing problems](#)», where a solution with 3 bins is found by eye: hence $m = 3$ is a valid upper bound.

Example (2/3): the ILP_{CSP} model

The ILP_{CSP} model with $m = 3$ for this CSP instance is:

$$\begin{array}{llll} \min & y_1 + y_2 + y_3 & & \\ \text{subject to} & x_{11} + x_{12} + x_{13} & = & 2 \\ & x_{21} + x_{22} + x_{23} & = & 1 \\ & x_{31} + x_{32} + x_{33} & = & 3 \\ & x_{41} + x_{42} + x_{43} & = & 1 \\ & -4x_{11} - 5x_{21} - 3x_{31} - 6x_{41} + 10y_1 & \geq & 0 \\ & -4x_{12} - 5x_{22} - 3x_{32} - 6x_{42} + 10y_2 & \geq & 0 \\ & -4x_{13} - 5x_{23} - 3x_{33} - 6x_{43} + 10y_3 & \geq & 0 \\ & x_{11}, x_{12}, x_{13}, \dots, x_{43} & \in & \mathbb{Z}_{\geq 0} \\ & y_1, y_2, y_3 & \in & \{0, 1\} \end{array}$$

Example (3/3): the ILP_{CSP} model

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{CSP}})$ is:

$$\tilde{x}_{11} = \tilde{x}_{41} = 1, \quad \tilde{x}_{32} = 3, \quad \tilde{x}_{13} = \tilde{x}_{23} = 1, \quad \tilde{y}_1 = \tilde{y}_2 = \tilde{y}_3 = 1,$$

with all the other variables equal to 0.

- ▶ The optimal value is $z(\text{ILP}_{\text{CSP}}) = 3$, as found in the chapter «Packing problems».
- ▶ The bins contain one copy of items 1 and 4 (weight 10), three copies of item 3 (weight 9), and one copy of items 1 and 2 (weight 9).
- ▶ The optimal solution is not unique: the solution of the chapter «Packing problems», with the bins $\{4, 1\}$, $\{1, 3, 3\}$ and $\{2, 3\}$, also uses 3 bins.

Summary

The ILP_{CSP} model:

- ▶ mn integer variables x_{jb} , the copies of item j in bin b , and m binary variables y_b
- ▶ the demands d_j on the right-hand side of the constraints $\sum_b x_{jb} = d_j$, the same capacity constraints as the ILP_{BPP} model
- ▶ with $d_j = 1$ for every item it is the ILP_{BPP} model

The reference CSP instance:

- ▶ $z(\text{ILP}_{\text{CSP}}) = 3$, with more than one optimal solution

Typical mistakes

Introduction: Typical mistakes

- ▶ We collect the most frequent mistakes on the packing models
- ▶ They concern the choice of m , the use of the LP bound, the meaning of the variables and the symmetry of the solutions
- ▶ Recognizing them helps to write correct models and correct bounds

Typical mistakes (1/3)

Observation 1: Typical mistakes

- ▶ **Choosing m too small.** The number m of bins must be an upper bound on the number of bins of an optimal solution: with $m < z(\text{ILP}_{\text{BPP}})$ the model has no feasible solution. The value $m = n$ is always valid.
- ▶ **Forgetting to round up.** The optimal value of the ILP_{BPP} model is an integer, so the lower bound from the relaxations is $\lceil \sum_j w_j / c \rceil$; in the example instance it is $\lceil \frac{8}{5} \rceil = 2$, not $\frac{8}{5}$.

Typical mistakes (2/3)

Observation: Typical mistakes

- ▶ ***Believing that the model counts only non-empty bins.*** The constraints allow $y_b = 1$ for an empty bin: it is the objective function that switches such bins off, in every optimal solution.
- ▶ ***Ignoring the symmetry.*** Permuting the indices of the bins gives another optimal solution with the same packing: the ILP_{BPP} model has many equivalent optimal solutions.

Typical mistakes (3/3)

Observation: Typical mistakes

- ▶ ***Expecting a better relaxation from $y_b \leq 1$.*** The second relaxation has the same optimal value as the first one: the upper bounds $y_b \leq 1$ do not cut off the optimal solution of the first relaxation.
- ▶ ***Declaring binary the variables of the cutting stock model.*** In the ILP_{CSP} model x_{jb} counts the copies of item j in bin b : with $x_{jb} \in \{0, 1\}$ the three copies of item 3 of the example could not share a bin.

Summary

The model:

- ▶ m must be an upper bound on the optimal number of bins: $m = n$ is always valid
- ▶ empty bins are switched off by the objective function, not by the constraints
- ▶ the indices of the bins can be permuted: many equivalent optimal solutions

The bounds and the CSP:

- ▶ round the LP bound up; $y_b \leq 1$ does not improve it
- ▶ in the ILP_{CSP} model the variables x_{jb} are integer, not binary

In summary

Introduction: In summary

- ▶ We collect the key ideas of the chapter
- ▶ The two ILP models, the two LP relaxations and the quality of their bound
- ▶ A dashboard with the bounds computed on the example instance

Summary (1/3)

- ▶ The ILP_{BPP} model has $m \cdot n$ assignment variables x_{jb} and m activation variables y_b ; the capacity constraints link them.
- ▶ Both LP relaxations have optimal value $z(\text{LP}_{\text{BPP}}) = \sum_j w_j/c$, with dual solution $\tilde{\mu}_j = w_j/c$ and $\tilde{\pi}_b = 1/c$.
- ▶ Since the optimal value is an integer, $\lceil z(\text{LP}_{\text{BPP}}) \rceil$ is a lower bound on the number of bins.

Summary (2/3)

- ▶ The ratio between $z(\text{LP}_{\text{BPP}})$ and $z(\text{ILP}_{\text{BPP}})$ is at least $\frac{1}{2}$, and the value $\frac{1}{2}$ is asymptotically attained; the same holds for $\lceil z(\text{LP}_{\text{BPP}}) \rceil$.
- ▶ The ILP_{CSP} model has the structure of the ILP_{BPP} model, with mn integer variables x_{jb} that count the copies and the demands d_j on the right-hand side; on the CSP instance of the chapter «Packing problems» its optimal value is 3.

Summary (3/3)

Dashboard of the example instance. Bounds on $z(\text{ILP}_{\text{BPP}})$ computed in this chapter:

Bound	Value	From
First relaxation	$\frac{8}{5}$	$z(\text{LP-0}_{\text{BPP}})$
Second relaxation	$\frac{8}{5}$	$z(\text{LP}_{\text{BPP}})$
Lower bound	2	$\lceil \frac{8}{5} \rceil$
Optimal value	2	patterns $\{1, 3\}$ and $\{2\}$

Here the rounded bound is already equal to the optimal value.