



SAPIENZA
UNIVERSITÀ DI ROMA

Combinatorial Optimization

Volume 2 — Formulations and their properties

Integer linear programming models of the problems of Volume 1, their linear programming relaxations and duals, the bounds they give, and the comparison between weaker and stronger formulations.

Fabio Furini

fabiofurini.github.io

Department of Computer, Control and Management Engineering — Sapienza University of Rome

Online version and downloadable material:
fabiofurini.github.io/combinatorial-optimization/
github.com/fabiofurini/combinatorial-optimization

Academic year 2026–2027

Contents

9	Formulations of knapsack problems	3
9.1	Overview of the family	4
9.2	An ILP model for the 0/1 Knapsack Problem	5
9.3	A first Linear Programming relaxation	6
9.4	A second Linear Programming relaxation	9
9.5	0/1 Knapsack Problem upper bounds	17
9.6	An ILP model for the Subset Sum Problem	19
9.7	An ILP model for the Bounded Knapsack Problem	20
9.8	An ILP model for the Unbounded Knapsack Problem	22
9.9	An ILP model for the Multidimensional Knapsack Problem	23
9.10	An ILP model for the Multiple Knapsack Problem	24
9.11	An ILP model for the Multiple-Choice Knapsack Problem	26
9.12	An ILP model for the Knapsack Problem with Setups	27
9.13	Typical mistakes	29
9.14	In summary	29
10	Formulations of packing problems	31
10.1	Overview of the family	32
10.2	An ILP model for the Bin Packing Problem	32
10.3	A first Linear Programming relaxation	35
10.4	A second Linear Programming relaxation	39
10.5	An ILP model for the Cutting Stock Problem	46
10.6	Typical mistakes	48
10.7	In summary	49
11	Formulations of coloring problems	50
11.1	Overview of the family	51
11.2	An ILP model for the Vertex Coloring Problem	51
11.3	A first Linear Programming relaxation	54
11.4	A second Linear Programming relaxation	59
11.5	Typical mistakes	62
11.6	In summary	62

12	Formulations of clique problems	64
12.1	Overview of the family	65
12.2	An ILP model for the Maximum Clique Problem: the edge formulation	65
12.3	A first Linear Programming relaxation	67
12.4	A second Linear Programming relaxation	69
12.5	Clique inequalities: a stronger formulation	72
12.6	The relation with vertex coloring	75
12.7	Stable sets on the complement graph	77
12.8	ILP models for the weighted variants	80
12.9	Typical mistakes	83
12.10	In summary	83
13	Formulations of routing problems	85
13.1	Overview of the family	86
13.2	ILP models for the Travelling Salesman Problem	86
13.3	A model with position variables (MTZ)	87
13.4	A model with subtour elimination constraints (DFJ)	90
13.5	Linear Programming relaxations	93
13.6	ILP models for the metric and asymmetric TSP	99
13.7	Typical mistakes	101
13.8	In summary	102
14	Formulations of scheduling problems	103
14.1	Overview of the family	104
14.2	An ILP model for the Scheduling on Identical Parallel Machines Problem	104
14.3	A first Linear Programming relaxation	107
14.4	A second Linear Programming relaxation	110
14.5	Additional considerations	114
14.6	Typical mistakes	114
14.7	In summary	115
15	Formulations of lot sizing problems	116
15.1	Overview of the family	117
15.2	ILP models for the Capacitated Lot Sizing Problem	117
15.3	A formulation with a big-M link	118
15.4	The Linear Programming relaxation of the big-M formulation	121
15.5	A shortest-path formulation	123
15.6	Comparing the formulations	128
15.7	ILP models for the variants	128
15.8	Typical mistakes	144
15.9	In summary	144

Formulations of knapsack problems

Prerequisites

- From the module *Linear Programming*, [Chapter 9 «Duality theory»](#): the dual of an LP problem, weak and strong duality.
- From the module *Integer linear programming*, [Chapter 1 «Integer linear programming»](#) and [Chapter 2 «Primal and dual bounds»](#): ILP problems, LP relaxations and upper bounds.
- From this module, [Chapter 2 «Knapsack problems»](#): the 0/1 knapsack problem and its variants, their assumptions and their reference instances.

Objectives

At the end of this chapter you will be able to:

- write the ILP model of the 0/1 Knapsack Problem and explain each of its constraints;
- write the ILP models of the SSP, the BKP, the UKP, the MDKP, the MKP, the MCKP and the KPS, and write them on a numerical instance;
- write the two LP relaxations and their duals, and compute their optimal solutions with closed formulas;
- compute the split item, the residual capacity and the value $z(\text{LP}_{\text{KP}})$;
- compute the Dantzig and the Martello-Toth upper bounds, rounding them correctly;
- explain why $z(\text{LP}_{\text{KP}})$ is never larger than twice $z(\text{ILP}_{\text{KP}})$.

9.1 Overview of the family

- The ILP models of the knapsack problems use one variable x_j for each item $j \in \mathcal{I}$: a **binary variable** when the item is selected or not, an **integer variable** when it counts the selected copies of the item.
- Each capacity gives one **knapsack constraint** on the total weight of the selected items, and the objective function maximizes their total profit.
- The members of the family add an index to the variables (the knapsack k of item j in the MKP), some constraints on the classes of items (MCKP, KPS) or binary setup variables y_h (KPS).

The common pattern of the knapsack models is

$$\max \left\{ \sum_{j \in \mathcal{I}} p_j x_j : \sum_{j \in \mathcal{I}} w_j x_j \leq c, \quad x_j \in D_j \quad \forall j \in \mathcal{I} \right\},$$

where the domain D_j is $\{0, 1\}$, $\{0, 1, \dots, b_j\}$ or $\mathbb{Z}_{\geq 0}$. The members of the family change the domains and add capacities, knapsacks or classes.

- The table lists the members of the family, with the name of their ILP model and the section that writes it.

Problem	Model	Variables	Section
KP	ILP _{KP}	binary x_j	9.2
SSP	ILP _{SSP}	binary x_j	9.6
BKP	ILP _{BKP}	integer $x_j \in \{0, 1, \dots, b_j\}$	9.7
UKP	ILP _{UKP}	integer $x_j \in \mathbb{Z}_{\geq 0}$	9.8
MDKP	ILP _{MDKP}	binary x_j , one constraint per capacity	9.9
MKP	ILP _{MKP}	binary x_{jk} , item j in knapsack k	9.10
MCKP	ILP _{MCKP}	binary x_j , one item per class	9.11
KPS	ILP _{KPS}	binary x_j and setup variables y_h	9.12

9.2 An ILP model for the 0/1 Knapsack Problem

The 0/1 Knapsack Problem and its reference instance (Example 2.1) are defined in [Chapter 2](#) «Knapsack problems», [Section 2.2](#).

- We introduce the following n **binary variables**:

$$x_j = \begin{cases} 1 & \text{if item } j \text{ is selected in the knapsack} \\ 0 & \text{otherwise} \end{cases} \quad \forall j \in \underbrace{\{1, 2, \dots, n\}}_{=\mathcal{J}}. \quad (9.1)$$

Using these variables, an Integer Linear Programming (ILP) model for the 0/1 Knapsack Problem (KP) reads as follows:

$$\max \sum_{j \in \mathcal{J}} p_j x_j \quad (9.2a)$$

$$\text{subject to } \sum_{j \in \mathcal{J}} w_j x_j \leq c, \quad (9.2b)$$

$$x_j \in \{0, 1\}, \quad \forall j \in \mathcal{J}. \quad (9.2c)$$

This ILP model is referred to as the ILP_{KP} model.

- Description of the objective function and constraints of the ILP_{KP} model:
 - the linear objective function (9.2a) maximizes the total profit of the items selected in the knapsack;
 - constraint (9.2b) ensures that the total weight of the selected items does not exceed the knapsack capacity (one linear constraint);
 - finally, constraints (9.2c) define the variables of the ILP model.

Example 9.1: ILP_{KP} model

We consider a knapsack of capacity $c = 9$ and a set $\mathcal{J} = \{1, 2, 3, 4, 5, 6\}$ of 6 items ($n = 6$), with the following profits and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
p_j	6	5	8	9	6	7
w_j	2	3	6	7	5	9

This is the KP instance of Example 2.1, and we refer to it as [Example 9.1](#). The ILP_{KP} model for this KP instance is:

$$\begin{aligned} \max \quad & 6x_1 + 5x_2 + 8x_3 + 9x_4 + 6x_5 + 7x_6 \\ \text{subject to} \quad & 2x_1 + 3x_2 + 6x_3 + 7x_4 + 5x_5 + 9x_6 \leq 9 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \in \{0, 1\} \end{aligned}$$

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{KP}})$ is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 1, \quad \tilde{x}_5 = 0, \quad \tilde{x}_6 = 0$$

The optimal value $z(\text{ILP}_{\text{KP}})$ is 15. The associated optimal solution of the KP is $\tilde{S} = \{1, 4\}$, and it is the unique optimal solution.

9.3 A first Linear Programming relaxation

To obtain a Linear Programming (LP) relaxation of the ILP_{KP} model, we relax constraints (9.2c) with the following ones:

$$x_j \geq 0, \quad \forall j \in \mathcal{J}. \quad (9.3)$$

This LP model is referred to as the LP-0_{KP} model.

Example 9.2: LP-0_{KP} model

The LP-0_{KP} model for the KP instance of [Example 9.1](#) is:

$$\begin{aligned} \max \quad & 6x_1 + 5x_2 + 8x_3 + 9x_4 + 6x_5 + 7x_6 \\ \text{subject to} \quad & 2x_1 + 3x_2 + 6x_3 + 7x_4 + 5x_5 + 9x_6 \leq 9 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

An optimal solution $\tilde{\mathbf{x}}(\text{LP-0}_{\text{KP}})$ is:

$$\tilde{x}_1 = \frac{9}{2}, \quad \tilde{x}_2 = \tilde{x}_3 = \tilde{x}_4 = \tilde{x}_5 = \tilde{x}_6 = 0$$

The optimal value $z(\text{LP-0}_{\text{KP}})$ is 27.

Accordingly we have the upper bound:

$$z(\text{ILP}_{\text{KP}}) \leq 27$$

Using a non-negative dual variable π for the constraint (9.2b), the dual model of the LP-0_{KP} model is:

$$\min \quad c\pi \quad (9.4a)$$

$$\text{subject to} \quad w_j \pi \geq p_j, \quad \forall j \in \mathcal{J}, \quad (9.4b)$$

$$\pi \geq 0. \quad (9.4c)$$

This LP model is referred to as the $D(\text{LP-0}_{\text{KP}})$ model.

Example 9.3: $D(\text{LP-0}_{\text{KP}})$ model

The dual $D(\text{LP-0}_{\text{KP}})$ model for the KP instance of [Example 9.1](#) is:

$$\begin{array}{llll}
 \min & 9 & \pi & \\
 \text{subject to} & 2 & \pi & \geq 6 \\
 & 3 & \pi & \geq 5 \\
 & 6 & \pi & \geq 8 \\
 & 7 & \pi & \geq 9 \\
 & 5 & \pi & \geq 6 \\
 & 9 & \pi & \geq 7 \\
 & & \pi & \geq 0
 \end{array}$$

An optimal solution $\tilde{\pi}(D(\text{LP-0}_{\text{KP}}))$ is:

$$\tilde{\pi} = 3$$

The optimal value $z(D(\text{LP-0}_{\text{KP}}))$ is 27.

- The following proposition provides formulas to compute an optimal solution of the primal LP-0_{KP} model and of the dual $D(\text{LP-0}_{\text{KP}})$ model.

Proposition 9.1

If the first item $j = 1$ has the largest efficiency, an optimal solution of the primal LP-0_{KP} model is:

$$\tilde{x}_1 = \frac{c}{w_1} \quad \text{and} \quad \tilde{x}_j = 0, \quad \forall j \in \{2, 3, \dots, n\}, \quad (9.5)$$

and an optimal solution of the dual $D(\text{LP-0}_{\text{KP}})$ model is:

$$\tilde{\pi} = \frac{p_1}{w_1}. \quad (9.6)$$

Corollary 9.2

The optimal value of the primal LP-0_{KP} and dual $D(\text{LP-0}_{\text{KP}})$ models is:

$$z(\text{LP-0}_{\text{KP}}) = p_1 \frac{c}{w_1} = z(D(\text{LP-0}_{\text{KP}})) \quad (9.7)$$

- The proof proceeds in three steps:

1. We show that (9.5) is a feasible solution for the primal LP-0_{KP} model;
2. We show that (9.6) is a feasible solution for the dual $D(\text{LP-0}_{\text{KP}})$ model;
3. We show that both feasible solutions have the same objective function value.

Therefore, by LP Strong Duality, (9.5) and (9.6) are a pair of optimal primal and dual solutions.

Proof.

1. We have:

$$\sum_{j \in \mathcal{J}} w_j \tilde{x}_j = w_1 \frac{c}{w_1} = c$$

thus constraint (9.2b) is satisfied.

Since $\tilde{x}_j \geq 0$, for every $j \in \mathcal{J}$, constraints (9.3) are satisfied.

Therefore, (9.5) is a feasible solution for the primal LP-0_{KP} model.

2. For every $j \in \mathcal{J}$, we have:

$$w_j \tilde{\pi} = w_j \frac{p_1}{w_1} \geq p_j \quad \text{since} \quad \frac{p_1}{w_1} \geq \frac{p_j}{w_j}$$

thus constraints (9.4b) are satisfied.

We have $\tilde{\pi} \geq 0$, thus constraint (9.4c) is satisfied.

Therefore, (9.6) is a feasible solution for the dual $D(\text{LP-0}_{\text{KP}})$ model.

3. For the objective functions, we have:

$$\sum_{j \in \mathcal{J}} p_j \tilde{x}_j = p_1 \frac{c}{w_1} = c \tilde{\pi}$$

thus the primal and dual objective function values are equal.

Therefore, by LP Strong Duality, (9.5) and (9.6) are a pair of optimal primal and dual solutions. □

9.4 A second Linear Programming relaxation

To obtain an LP relaxation of the ILP_{KP} model, we relax constraints (9.2c) with the following ones:

$$x_j \leq 1, \quad \forall j \in \mathcal{J}, \quad (9.8a)$$

$$x_j \geq 0, \quad \forall j \in \mathcal{J}. \quad (9.8b)$$

This LP model is referred to as the LP_{KP} model.

Example 9.4: LP_{KP} model

The LP_{KP} model for the KP instance of [Example 9.1](#) is:

$$\begin{array}{llllllll} \max & 6 & x_1 & + & 5 & x_2 & + & 8 & x_3 & + & 9 & x_4 & + & 6 & x_5 & + & 7 & x_6 \\ \text{subject to} & 2 & x_1 & + & 3 & x_2 & + & 6 & x_3 & + & 7 & x_4 & + & 5 & x_5 & + & 9 & x_6 & \leq & 9 \\ & & x_1 & & & & & & & & & & & & & & & \leq & 1 \\ & & & & x_2 & & & & & & & & & & & & & \leq & 1 \\ & & & & & & x_3 & & & & & & & & & & & \leq & 1 \\ & & & & & & & & x_4 & & & & & & & & & \leq & 1 \\ & & & & & & & & & & x_5 & & & & & & & \leq & 1 \\ & & & & & & & & & & & & x_6 & & & & & \leq & 1 \\ & & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & & & & & \geq & 0 \end{array}$$

An optimal solution $\tilde{\mathbf{x}}(\text{LP}_{\text{KP}})$ is:

$$\tilde{x}_1 = \tilde{x}_2 = 1, \quad \tilde{x}_3 = \frac{2}{3}, \quad \tilde{x}_4 = \tilde{x}_5 = \tilde{x}_6 = 0$$

The optimal value $z(\text{LP}_{\text{KP}})$ is $\frac{49}{3}$.

Accordingly we have the upper bound:

$$z(\text{ILP}_{\text{KP}}) \leq \frac{49}{3}$$

Using a non-negative dual variable π for the constraint (9.2b) and a non-negative dual

variable μ_j for each constraint (9.8a), the dual model of the LP_{KP} model is:

$$\min \quad c\pi + \sum_{j \in \mathcal{J}} \mu_j \quad (9.9a)$$

$$\text{subject to} \quad w_j \pi + \mu_j \geq p_j, \quad \forall j \in \mathcal{J}, \quad (9.9b)$$

$$\pi \geq 0, \quad (9.9c)$$

$$\mu_j \geq 0, \quad \forall j \in \mathcal{J}. \quad (9.9d)$$

This LP model is referred to as the $D(\text{LP}_{\text{KP}})$ model.

Example 9.5: $D(\text{LP}_{\text{KP}})$ model

The dual $D(\text{LP}_{\text{KP}})$ model for the KP instance of [Example 9.1](#) is:

$$\begin{aligned} \min \quad & 9\pi + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 \\ & 2\pi + \mu_1 \geq 6 \\ & 3\pi + \mu_2 \geq 5 \\ & 6\pi + \mu_3 \geq 8 \\ & 7\pi + \mu_4 \geq 9 \\ & 5\pi + \mu_5 \geq 6 \\ & 9\pi + \mu_6 \geq 7 \\ & \pi, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6 \geq 0 \end{aligned}$$

An optimal solution $(\tilde{\pi}, \tilde{\boldsymbol{\mu}})(D(\text{LP}_{\text{KP}}))$ is:

$$\tilde{\pi} = \frac{4}{3}, \quad \tilde{\mu}_1 = \frac{10}{3}, \quad \tilde{\mu}_2 = 1, \quad \tilde{\mu}_3 = \tilde{\mu}_4 = \tilde{\mu}_5 = \tilde{\mu}_6 = 0$$

The optimal value $z(D(\text{LP}_{\text{KP}}))$ is $\frac{49}{3}$.

9.4.1 Item efficiency

Definition 9.3: efficiency

The **efficiency** of an item $j \in \mathcal{J}$ is defined as the ratio between its profit and its weight:

$$\frac{p_j}{w_j} \quad (9.10)$$

- The efficiency of an item is a measure of its profitability per unit of weight: it indicates how much profit is gained for each unit of weight.

Example 9.6: item efficiencies

Let us consider the KP instance of [Example 9.1](#). We have:

Items		$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
Profits	p_j	6	5	8	9	6	7
Weights	w_j	2	3	6	7	5	9
Efficiencies	$\frac{p_j}{w_j}$	3	$\frac{5}{3}$	$\frac{4}{3}$	$\frac{9}{7}$	$\frac{6}{5}$	$\frac{7}{9}$

The items are sorted by non-increasing efficiency, since:

$$3 \geq \frac{5}{3} \geq \frac{4}{3} \geq \frac{9}{7} \geq \frac{6}{5} \geq \frac{7}{9}$$

Sorting the items by non-increasing efficiency requires a sorting algorithm with running time $O(n \log n)$, which is a polynomial function of the number n of items.

9.4.2 Split item and residual capacity

Definition 9.4: split item and residual capacity

Given a set $\mathcal{J} = \{1, 2, \dots, n\}$ of n items ordered by non-increasing efficiency, the **split item** is the item

$$s \in \mathcal{J} \quad \text{such that} \quad \sum_{j=1}^{s-1} w_j \leq c \quad \text{and} \quad \sum_{j=1}^s w_j > c \quad (9.11)$$

and the **residual capacity** is

$$rc = c - \sum_{j=1}^{s-1} w_j \quad (9.12)$$

- The split item is the first one that does not fit entirely in the knapsack if all the previous ones are selected. It is also called **break item** or **critical item**, and it can be written as

$$s = \min \left\{ j \in \mathcal{J} : \sum_{k=1}^j w_k > c \right\}$$

- The residual capacity is the knapsack capacity remaining when all the items that precede the split item are selected. It satisfies $0 \leq rc < w_s$.

Example 9.7: split item and residual capacity

For the KP instance of [Example 9.1](#), the split item is $s = 3$ since:

$$\sum_{j=1}^2 w_j = 2 + 3 = 5 \leq 9 \quad \text{and} \quad \sum_{j=1}^3 w_j = 2 + 3 + 6 = 11 > 9$$

and the residual capacity is:

$$rc = c - \sum_{j=1}^2 w_j = 9 - 2 - 3 = 4$$

- The following proposition provides formulas to compute an optimal solution of the primal LP_{KP} model and of the dual $D(LP_{KP})$ model.

Proposition 9.5

If the items are sorted by non-increasing efficiency, an optimal solution of the primal LP_{KP} model is:

$$\tilde{x}_j = 1, \quad \forall j \in \{1, 2, \dots, s-1\}, \quad \tilde{x}_s = \frac{rc}{w_s} \quad \text{and} \quad \tilde{x}_j = 0, \quad \forall j \in \{s+1, s+2, \dots, n\}, \quad (9.13)$$

and an optimal solution of the dual $D(LP_{KP})$ model is:

$$\tilde{\pi} = \frac{p_s}{w_s}, \quad \tilde{\mu}_j = p_j - w_j \frac{p_s}{w_s}, \quad \forall j \in \{1, 2, \dots, s-1\} \quad \text{and} \quad \tilde{\mu}_j = 0, \quad \forall j \in \{s, s+1, \dots, n\}. \quad (9.14)$$

Corollary 9.6

The optimal value of the primal LP_{KP} and dual $D(LP_{KP})$ models is:

$$z(LP_{KP}) = \sum_{j=1}^{s-1} p_j + p_s \frac{rc}{w_s} = z(D(LP_{KP})) \quad (9.15)$$

- The proof proceeds in three steps:

1. We show that (9.13) is a feasible solution for the primal LP_{KP} model;
2. We show that (9.14) is a feasible solution for the dual $D(LP_{KP})$ model;
3. We show that both feasible solutions have the same objective function value.

Therefore, by LP Strong Duality, (9.13) and (9.14) are a pair of optimal primal and dual solutions.

Proof.

1. We have:

$$\sum_{j \in \mathcal{J}} w_j \tilde{x}_j = \sum_{j=1}^{s-1} w_j + w_s \frac{rc}{w_s} = \sum_{j=1}^{s-1} w_j + c - \sum_{j=1}^{s-1} w_j = c$$

thus, constraint (9.2b) is satisfied.

Since $rc < w_s$, we have $0 \leq \tilde{x}_j \leq 1$, for every $j \in \mathcal{J}$. Accordingly, constraints (9.8a) and (9.8b) are satisfied.

Therefore, (9.13) is a feasible solution for the primal LP_{KP} model. □

Proof.

2. For every $j \in \{1, 2, \dots, s-1\}$, we have:

$$w_j \tilde{\pi} + \tilde{\mu}_j = w_j \frac{p_s}{w_s} + \left(p_j - w_j \frac{p_s}{w_s} \right) = p_j$$

and, for every $j \in \{s, s+1, \dots, n\}$, we have:

$$w_j \tilde{\pi} + \tilde{\mu}_j = w_j \frac{p_s}{w_s} \geq p_j \quad \text{since} \quad \frac{p_s}{w_s} \geq \frac{p_j}{w_j}$$

thus constraints (9.9b) are satisfied.

We have $\tilde{\pi} \geq 0$. For every $j \in \{1, 2, \dots, s-1\}$, we have:

$$p_j - w_j \frac{p_s}{w_s} \geq 0 \quad \text{since} \quad \frac{p_j}{w_j} \geq \frac{p_s}{w_s}$$

accordingly, $\tilde{\mu}_j \geq 0$ for every $j \in \mathcal{J}$. Thus constraint (9.9c) and constraints (9.9d) are satisfied.

Therefore, (9.14) is a feasible solution for the dual $D(LP_{KP})$ model.

3. For the objective functions we have:

$$\begin{aligned}
 \sum_{j \in \mathcal{J}} p_j \tilde{x}_j &= \sum_{j=1}^{s-1} p_j + p_s \frac{rc}{w_s} \\
 &= \sum_{j=1}^{s-1} p_j + \frac{p_s}{w_s} \left(c - \sum_{j=1}^{s-1} w_j \right) \\
 &= c \frac{p_s}{w_s} + \sum_{j=1}^{s-1} \left(p_j - w_j \frac{p_s}{w_s} \right) \\
 &= c \tilde{\pi} + \sum_{j \in \mathcal{J}} \tilde{\mu}_j
 \end{aligned}$$

thus the primal and dual objective function values are equal.

Therefore, by LP Strong Duality, (9.13) and (9.14) are a pair of optimal primal and dual solutions. □

In the majority of KP instances the ratio between $z(\text{LP}_{\text{KP}})$ and $z(\text{ILP}_{\text{KP}})$ slightly exceeds one: $z(\text{LP}_{\text{KP}})$ typically is a tight upper bound for $z(\text{ILP}_{\text{KP}})$.

- We report an alternative proof for the formulas (9.13). We proceed by contradiction, proving that a different solution cannot be optimal.
- In this proof we assume that items with the same efficiency are merged into a single item. As we are allowed to choose any fraction of an item, this does not change the problem, and we avoid discussing symmetric solutions.

Proof. Assume $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ is an optimal solution of LP_{KP} different from $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ determined with formulas (9.13). Every optimal solution of LP_{KP} has a total weight equal to c , i.e., constraint (9.2b) is satisfied at equality (otherwise it can be improved).

Assume that no pair of items $h < k$ exists with $\bar{x}_h < 1$ and $\bar{x}_k > 0$. Then $\bar{\mathbf{x}}$ selects the items in order of efficiency, filling the knapsack item by item up to the weight c : this is exactly $\tilde{\mathbf{x}}$, a contradiction. Hence such a pair of items $h < k$ exists (the case $k = s$ is included).

Let:

$$d = \min \{ w_k \bar{x}_k, w_h (1 - \bar{x}_h) \} > 0$$

Construct a new feasible solution $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)$ which is identical to $\bar{\mathbf{x}}$ except that:

$$\hat{x}_k = \bar{x}_k - \frac{d}{w_k} \geq 0 \quad \text{and} \quad \hat{x}_h = \bar{x}_h + \frac{d}{w_h} \leq 1$$

By the definition of d , we have $\hat{x}_k \geq 0$ and $\hat{x}_h \leq 1$, and $\hat{\mathbf{x}}$ is a feasible solution since:

$$\sum_{j \in \mathcal{J}} w_j \hat{x}_j = \sum_{j \in \mathcal{J}} w_j \bar{x}_j + w_h \frac{d}{w_h} - w_k \frac{d}{w_k} = \sum_{j \in \mathcal{J}} w_j \bar{x}_j = c$$

The objective function value of $\hat{\mathbf{x}}$ is:

$$\sum_{j \in \mathcal{J}} p_j \hat{x}_j = \sum_{j \in \mathcal{J}} p_j \bar{x}_j + d \underbrace{\left(\frac{p_h}{w_h} - \frac{p_k}{w_k} \right)}_{>0 \text{ since } \frac{p_h}{w_h} > \frac{p_k}{w_k}} > \sum_{j \in \mathcal{J}} p_j \bar{x}_j$$

Then the objective function value of $\hat{\mathbf{x}}$ is larger than the objective function value of $\bar{\mathbf{x}}$, contradicting the fact that $\bar{\mathbf{x}}$ is an optimal solution.

The strict inequality $p_h/w_h > p_k/w_k$ holds for every $h < k$, since items with the same efficiency have been merged. $\square$

Example 9.8: Primal and dual LP solutions

For the KP instance of [Example 9.1](#), we have $s = 3$ and $rc = 4$.

Using formulas (9.13), we can compute the following optimal solution of the LP_{KP} model:

$$\tilde{x}_1 = \tilde{x}_2 = 1, \quad \tilde{x}_3 = \frac{2}{3}, \quad \tilde{x}_4 = \tilde{x}_5 = \tilde{x}_6 = 0$$

Using formulas (9.14), we can compute the following optimal solution of the $D(\text{LP}_{\text{KP}})$ model:

$$\tilde{\pi} = \frac{4}{3}, \quad \tilde{\mu}_1 = 6 - 2 \frac{4}{3} = \frac{10}{3}, \quad \tilde{\mu}_2 = 5 - 3 \frac{4}{3} = 1, \quad \tilde{\mu}_3 = \tilde{\mu}_4 = \tilde{\mu}_5 = \tilde{\mu}_6 = 0$$

Using formula (9.15), the optimal value of both LP models, primal and dual, is:

$$\sum_{j=1}^2 p_j + p_s \frac{rc}{w_s} = 6 + 5 + 8 \frac{2}{3} = \frac{49}{3}$$

9.4.3 Some important theoretical properties

- The following two propositions show that the ratio between $z(\text{LP}_{\text{KP}})$ and $z(\text{ILP}_{\text{KP}})$ is at most two for all KP instances. Furthermore, there exist instances for which the value two can be asymptotically achieved as the capacity $c \rightarrow +\infty$.

Proposition 9.7

For every KP instance, we have:

$$\frac{z(\text{LP}_{\text{KP}})}{z(\text{ILP}_{\text{KP}})} \leq 2 \quad (9.16)$$

Proof. We consider two feasible solutions of the ILP_{KP} model:

1. The solution selecting all items preceding the split item:

$$\bar{x}_j = 1, \quad \forall j \in \{1, 2, \dots, s-1\} \quad \text{and} \quad \bar{x}_j = 0, \quad \forall j \in \{s, s+1, \dots, n\}$$

which is feasible by definition of the split item, and provides the valid lower bound:

$$\sum_{j=1}^{s-1} p_j \leq z(\text{ILP}_{\text{KP}})$$

2. The solution selecting only the split item:

$$\bar{x}_j = 0, \quad \forall j \in \mathcal{J} \setminus \{s\} \quad \text{and} \quad \bar{x}_s = 1$$

which is feasible by assumption that $w_s \leq c$, and provides the valid lower bound:

$$p_s \leq z(\text{ILP}_{\text{KP}})$$

Adding the two inequalities yields:

$$2 z(\text{ILP}_{\text{KP}}) \geq \sum_{j=1}^{s-1} p_j + p_s \geq z(\text{LP}_{\text{KP}})$$

and thus:

$$\frac{z(\text{LP}_{\text{KP}})}{z(\text{ILP}_{\text{KP}})} \leq 2$$

□

Proposition 9.8

There are KP instances such that:

$$\frac{z(\text{LP}_{\text{KP}})}{z(\text{ILP}_{\text{KP}})} \rightarrow 2 \quad \text{as} \quad c \rightarrow +\infty \quad (9.17)$$

Proof. Given $\vartheta \in \mathbb{Z}_{\geq 1}$, consider the KP instance with knapsack capacity $c = 2\vartheta$ and $n = 2$ items having the following profits and weights:

$$p_1 = 1, \quad w_1 = \vartheta + 1 \quad \text{and} \quad p_2 = 1, \quad w_2 = \vartheta + 1$$

Since only one item can be selected, an optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{KP}})$ of the ILP_{KP} model is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 0 \quad \text{and} \quad z(\text{ILP}_{\text{KP}}) = 1$$

The split item is $s = 2$, and the residual capacity is:

$$rc = 2\vartheta - (\vartheta + 1) = \vartheta - 1$$

Accordingly, an optimal solution $\tilde{\mathbf{x}}(\text{LP}_{\text{KP}})$ of the LP_{KP} model is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = \frac{\vartheta - 1}{\vartheta + 1} \quad \text{and} \quad z(\text{LP}_{\text{KP}}) = 1 + \frac{\vartheta - 1}{\vartheta + 1} = \frac{2\vartheta}{\vartheta + 1}$$

It follows that:

$$\lim_{\vartheta \rightarrow +\infty} \frac{z(\text{LP}_{\text{KP}})}{z(\text{ILP}_{\text{KP}})} = \lim_{\vartheta \rightarrow +\infty} \frac{\frac{2\vartheta}{\vartheta+1}}{1} = \lim_{\vartheta \rightarrow +\infty} \frac{2(\vartheta+1) - 2}{\vartheta+1} = \lim_{\vartheta \rightarrow +\infty} 2 - \underbrace{\frac{2}{\vartheta+1}}_{\rightarrow 0} = 2$$

Accordingly, we conclude that:

$$\frac{z(\text{LP}_{\text{KP}})}{z(\text{ILP}_{\text{KP}})} \rightarrow 2 \quad \text{as} \quad c \rightarrow +\infty$$

□

9.5 0/1 Knapsack Problem upper bounds

- The following two propositions require that the items are sorted by non-increasing efficiency.

Proposition 9.9: Dantzig upper bound

For every KP instance, we have:

$$z(\text{ILP}_{\text{KP}}) \leq \underbrace{\sum_{j=1}^{s-1} p_j}_{\text{UB}_D} + \left\lfloor p_s \frac{rc}{w_s} \right\rfloor \quad (9.18)$$

and UB_D is called the **Dantzig upper bound** for the KP.

Proof. We have:

$$\lfloor z(\text{LP}_{\text{KP}}) \rfloor = \left\lfloor \sum_{j=1}^{s-1} p_j + p_s \frac{rc}{w_s} \right\rfloor = \sum_{j=1}^{s-1} p_j + \left\lfloor p_s \frac{rc}{w_s} \right\rfloor$$

which is a valid upper bound for $z(\text{ILP}_{\text{KP}})$ since $p_j \in \mathbb{Z}_{\geq 1}$, for every $j \in \mathcal{J}$. $\square$

- The Martello-Toth bound uses the items $s-1$ and $s+1$. Since $w_1 \leq c$ and $\sum_j w_j > c$ we have $s \geq 2$, so the item $s-1$ always exists. If $s = n$ the item $s+1$ does not exist, and we set $p_{n+1}/w_{n+1} := 0$: in this case $\text{UB}_0 = \sum_{j=1}^{s-1} p_j$.

Proposition 9.10: Martello-Toth upper bound

For every KP instance, we have:

$$z(\text{ILP}_{\text{KP}}) \leq \text{UB}_{\text{MT}} = \max \left\{ \text{UB}_0, \text{UB}_1 \right\} \quad (9.19)$$

where

$$\text{UB}_0 = \sum_{j=1}^{s-1} p_j + \left\lfloor rc \frac{p_{s+1}}{w_{s+1}} \right\rfloor$$

$$\text{UB}_1 = \sum_{j=1}^{s-1} p_j + \left\lfloor p_s - (w_s - rc) \frac{p_{s-1}}{w_{s-1}} \right\rfloor$$

and UB_{MT} is called the **Martello-Toth upper bound** for the KP.

Proof. Since x_s cannot take a fractional value, an optimal solution $\tilde{x}(\text{ILP}_{\text{KP}})$ can be obtained from an optimal solution $\tilde{x}(\text{LP}_{\text{KP}})$ either without the split item s or with the split item s and hence removing at least one of the first $s-1$ items.

1. If the split item is out, we have $z(\text{ILP}_{\text{KP}}) \leq \text{UB}_0$ since the residual capacity rc is filled with items having the largest possible efficiency $\frac{p_{s+1}}{w_{s+1}}$
2. If the split item is in, we have $z(\text{ILP}_{\text{KP}}) \leq \text{UB}_1$ since it is supposed that the item to be removed has exactly the minimum necessary weight $w_s - rc$ and the smallest possible efficiency $\frac{p_{s-1}}{w_{s-1}}$

$\square$

Example 9.9: Dantzig and Martello-Toth upper bound

For the KP instance of [Example 9.1](#), using formula (9.18), we can compute the Dantzig upper bound:

$$UB_D = \sum_{j=1}^2 p_j + \left\lfloor p_3 \frac{rc}{w_3} \right\rfloor = 6 + 5 + \left\lfloor 8 \frac{2}{3} \right\rfloor = 11 + \left\lfloor \frac{16}{3} \right\rfloor = 16$$

Using formula (9.19), we can compute the Martello-Toth upper bound:

$$UB_0 = \sum_{j=1}^2 p_j + \left\lfloor rc \frac{p_4}{w_4} \right\rfloor = 6 + 5 + \left\lfloor 4 \frac{9}{7} \right\rfloor = 11 + \left\lfloor \frac{36}{7} \right\rfloor = 16$$

$$UB_1 = \sum_{j=1}^2 p_j + \left\lfloor p_3 - (w_3 - rc) \frac{p_2}{w_2} \right\rfloor = 6 + 5 + \left\lfloor 8 - (6 - 4) \frac{5}{3} \right\rfloor = 11 + \left\lfloor \frac{14}{3} \right\rfloor = 15$$

$$UB_{MT} = \max \left\{ UB_0, UB_1 \right\} = \max \left\{ 16, 15 \right\} = 16$$

For this KP instance, the Martello-Toth upper bound is as strong as the Dantzig upper bound.

9.6 An ILP model for the Subset Sum Problem

The Subset Sum Problem and its reference instance (Example 2.2) are defined in [Chapter 2 «Knapsack problems», Section 2.3](#).

- We introduce the following n **binary variables**:

$$x_j = \begin{cases} 1 & \text{if item } j \text{ is selected in the subset} \\ 0 & \text{otherwise} \end{cases} \quad \forall j \in \mathcal{J}. \quad (9.20)$$

Using these variables, an Integer Linear Programming (ILP) model for the Subset Sum Problem (SSP) reads as follows:

$$\max \sum_{j \in \mathcal{J}} w_j x_j \quad (9.21a)$$

$$\text{subject to } \sum_{j \in \mathcal{J}} w_j x_j \leq c, \quad (9.21b)$$

$$x_j \in \{0, 1\}, \quad \forall j \in \mathcal{J}. \quad (9.21c)$$

This ILP model is referred to as the ILP_{SSP} model.

- Description of the objective function and constraints of the ILP_{SSP} model:
 - the linear objective function (9.21a) maximizes the total weight of the items selected in the subset;

- constraint (9.21b) ensures that the total weight of the selected items does not exceed the target value (one linear constraint);
- finally, constraints (9.21c) define the variables of the ILP model.
- The ILP_{SSP} model is the ILP_{KP} model with $p_j = w_j$, for every $j \in \mathcal{J}$: the answer to the SSP is **yes** if and only if $z(\text{ILP}_{\text{SSP}}) = c$.

Example 9.10: ILP_{SSP} model

We consider a target value $c = 9$ and a set $\mathcal{J} = \{1, 2, 3, 4, 5, 6\}$ of 6 items ($n = 6$), with the following weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
w_j	2	3	6	7	5	9

This is the SSP instance of Example 2.2. The ILP_{SSP} model for this SSP instance is:

$$\begin{aligned}
 \max \quad & 2x_1 + 3x_2 + 6x_3 + 7x_4 + 5x_5 + 9x_6 \\
 \text{subject to} \quad & 2x_1 + 3x_2 + 6x_3 + 7x_4 + 5x_5 + 9x_6 \leq 9 \\
 & x_1, x_2, x_3, x_4, x_5, x_6 \in \{0, 1\}
 \end{aligned}$$

An optimal solution $\tilde{x}(\text{ILP}_{\text{SSP}})$ is:

$$\tilde{x}_1 = 1, \tilde{x}_2 = 0, \tilde{x}_3 = 0, \tilde{x}_4 = 1, \tilde{x}_5 = 0, \tilde{x}_6 = 0$$

The optimal value $z(\text{ILP}_{\text{SSP}})$ is $9 = c$, so the answer to the SSP is **yes**. The associated subset is $\tilde{S} = \{1, 4\}$. The optimal solution is not unique: the subsets $\{2, 3\}$ and $\{6\}$ also reach the target value.

9.7 An ILP model for the Bounded Knapsack Problem

The Bounded Knapsack Problem and its reference instance (Example 2.3) are defined in Chapter 2 «Knapsack problems», Section 2.4.

- We introduce the following n **integer variables**:

$$x_j = \text{number of copies of item } j \text{ inserted in the knapsack} \quad \forall j \in \mathcal{J}. \quad (9.22)$$

Using these variables, an Integer Linear Programming (ILP) model for the Bounded Knapsack Problem (BKP) reads as follows:

$$\max \sum_{j \in \mathcal{J}} p_j x_j \quad (9.23a)$$

$$\text{subject to} \quad \sum_{j \in \mathcal{J}} w_j x_j \leq c, \quad (9.23b)$$

$$x_j \in \{0, 1, \dots, b_j\}, \quad \forall j \in \mathcal{J}. \quad (9.23c)$$

This ILP model is referred to as the ILP_{BKP} model.

- Description of the objective function and constraints of the ILP_{BKP} model:
 - the linear objective function (9.23a) maximizes the total profit of the item copies selected in the knapsack;
 - constraint (9.23b) ensures that the total weight of the selected item copies does not exceed the knapsack capacity (one linear constraint);
 - finally, constraints (9.23c) define the variables of the ILP model: each item j is taken at most b_j times.

Example 9.11: ILP_{BKP} model

We consider a knapsack of capacity $c = 9$ and a set $\mathcal{J} = \{1, 2, 3, 4, 5, 6\}$ of 6 items ($n = 6$), with the following profits, weights and available copies:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
p_j	6	5	8	9	6	7
w_j	2	3	6	7	5	9
b_j	2	2	1	1	1	1

This is the BKP instance of Example 2.3. The ILP_{BKP} model for this BKP instance is:

$$\begin{aligned}
 \max \quad & 6x_1 + 5x_2 + 8x_3 + 9x_4 + 6x_5 + 7x_6 \\
 \text{subject to} \quad & 2x_1 + 3x_2 + 6x_3 + 7x_4 + 5x_5 + 9x_6 \leq 9 \\
 & x_1, x_2 \in \{0, 1, 2\} \\
 & x_3, x_4, x_5, x_6 \in \{0, 1\}
 \end{aligned}$$

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{BKP}})$ is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 1, \quad \tilde{x}_6 = 0$$

The optimal value $z(\text{ILP}_{\text{BKP}})$ is 18. The associated optimal solution of the BKP takes two copies of item 1 and one copy of item 5, and it is the unique optimal solution.

9.8 An ILP model for the Unbounded Knapsack Problem

The Unbounded Knapsack Problem and its reference instance (Example 2.4) are defined in Chapter 2 «Knapsack problems», Section 2.5.

- We introduce the following n **integer variables**:

$$x_j = \text{number of copies of item } j \text{ inserted in the knapsack} \quad \forall j \in \mathcal{J}. \quad (9.24)$$

Using these variables, an Integer Linear Programming (ILP) model for the Unbounded Knapsack Problem (UKP) reads as follows:

$$\max \sum_{j \in \mathcal{J}} p_j x_j \quad (9.25a)$$

$$\text{subject to } \sum_{j \in \mathcal{J}} w_j x_j \leq c, \quad (9.25b)$$

$$x_j \in \mathbb{Z}_{\geq 0}, \quad \forall j \in \mathcal{J}. \quad (9.25c)$$

This ILP model is referred to as the ILP_{UKP} model.

- Description of the objective function and constraints of the ILP_{UKP} model:
 - the linear objective function (9.25a) maximizes the total profit of the item copies selected in the knapsack;
 - constraint (9.25b) ensures that the total weight of the selected item copies does not exceed the knapsack capacity (one linear constraint);
 - finally, constraints (9.25c) define the variables of the ILP model; the capacity constraint already gives $x_j \leq \hat{b}_j = \lfloor c/w_j \rfloor$, for every $j \in \mathcal{J}$.

Example 9.12: ILP_{UKP} model

We consider a knapsack of capacity $c = 9$ and a set $\mathcal{J} = \{1, 2, 3, 4, 5, 6\}$ of 6 items ($n = 6$), each available in an unlimited number of copies, with the following profits and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
p_j	6	5	8	9	6	7
w_j	2	3	6	7	5	9

This is the UKP instance of Example 2.4. The ILP_{UKP} model for this UKP instance is:

$$\begin{aligned} \max \quad & 6x_1 + 5x_2 + 8x_3 + 9x_4 + 6x_5 + 7x_6 \\ \text{subject to} \quad & 2x_1 + 3x_2 + 6x_3 + 7x_4 + 5x_5 + 9x_6 \leq 9 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{UKP}})$ is:

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0, \quad \tilde{x}_6 = 0$$

The optimal value $z(\text{ILP}_{\text{UKP}})$ is 24. The associated optimal solution of the UKP takes four copies of item 1, and it is the unique optimal solution.

9.9 An ILP model for the Multidimensional Knapsack Problem

The Multidimensional Knapsack Problem and its reference instance (Example 2.5) are defined in Chapter 2 «Knapsack problems», Section 2.6.

- We introduce the following n **binary variables**:

$$x_j = \begin{cases} 1 & \text{if item } j \text{ is selected in the knapsack} \\ 0 & \text{otherwise} \end{cases} \quad \forall j \in \mathcal{J}. \quad (9.26)$$

Using these variables, an Integer Linear Programming (ILP) model for the Multidimensional Knapsack Problem (MDKP) reads as follows:

$$\max \sum_{j \in \mathcal{J}} p_j x_j \quad (9.27a)$$

$$\text{subject to } \sum_{j \in \mathcal{J}} w_{dj} x_j \leq c_d, \quad \forall d \in \mathcal{D}, \quad (9.27b)$$

$$x_j \in \{0, 1\}, \quad \forall j \in \mathcal{J}. \quad (9.27c)$$

This ILP model is referred to as the ILP_{MDKP} model.

- Description of the objective function and constraints of the ILP_{MDKP} model:
 - the linear objective function (9.27a) maximizes the total profit of the items selected in the knapsack;
 - constraints (9.27b) ensure that, in every dimension d , the total weight of the selected items does not exceed the capacity c_d (k linear constraints, one for each dimension);
 - finally, constraints (9.27c) define the variables of the ILP model.

Example 9.13: ILP_{MDKP} model

We consider a 2-dimensional knapsack ($k = 2$) with capacities $c_1 = 10$ and $c_2 = 12$, and a set $\mathcal{J} = \{1, 2, 3, 4\}$ of 4 items ($n = 4$), with the following profits and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
p_j	6	7	5	9
w_{1j}	4	5	3	7
w_{2j}	6	4	5	3

This is the MDKP instance of Example 2.5. The ILP_{MDKP} model for this MDKP instance is:

$$\begin{aligned}
 \max \quad & 6x_1 + 7x_2 + 5x_3 + 9x_4 \\
 \text{subject to} \quad & 4x_1 + 5x_2 + 3x_3 + 7x_4 \leq 10 \\
 & 6x_1 + 4x_2 + 5x_3 + 3x_4 \leq 12 \\
 & x_1, x_2, x_3, x_4 \in \{0, 1\}
 \end{aligned}$$

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{MDKP}})$ is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = 1, \quad \tilde{x}_4 = 1$$

The optimal value $z(\text{ILP}_{\text{MDKP}})$ is 14. The associated optimal solution of the MDKP is $\tilde{S} = \{3, 4\}$, and it is the unique optimal solution.

9.10 An ILP model for the Multiple Knapsack Problem

The Multiple Knapsack Problem and its reference instance (Example 2.6) are defined in [Chapter 2 «Knapsack problems», Section 2.7](#).

- We introduce the following nm **binary variables**:

$$x_{jk} = \begin{cases} 1 & \text{if item } j \text{ is inserted in knapsack } k \\ 0 & \text{otherwise} \end{cases} \quad \forall j \in \mathcal{I}, \forall k \in \mathcal{K}. \quad (9.28)$$

Using these variables, an Integer Linear Programming (ILP) model for the Multiple Knapsack Problem (MKP) reads as follows:

$$\max \sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{I}} p_j x_{jk} \quad (9.29a)$$

$$\text{subject to} \quad \sum_{j \in \mathcal{I}} w_j x_{jk} \leq c_k, \quad \forall k \in \mathcal{K}, \quad (9.29b)$$

$$\sum_{k \in \mathcal{K}} x_{jk} \leq 1, \quad \forall j \in \mathcal{I}, \quad (9.29c)$$

$$x_{jk} \in \{0, 1\}, \quad \forall j \in \mathcal{I}, \forall k \in \mathcal{K}. \quad (9.29d)$$

This ILP model is referred to as the ILP_{MKP} model.

- Description of the objective function and constraints of the ILP_{MKP} model:
 - the linear objective function (9.29a) maximizes the total profit of the items inserted in the knapsacks;
 - constraints (9.29b) ensure that, in every knapsack k , the total weight of the assigned items does not exceed the capacity c_k (m linear constraints, one for each knapsack);
 - constraints (9.29c) ensure that each item is inserted in at most one knapsack (n linear constraints, one for each item);

– finally, constraints (9.29d) define the variables of the ILP model.

Example 9.14: ILP_{MKP} model

We consider $m = 2$ knapsacks with capacities $c_1 = 10$ and $c_2 = 12$, and a set $\mathcal{J} = \{1, 2, 3, 4, 5\}$ of 5 items ($n = 5$), with the following profits and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
p_j	6	8	5	7	9
w_j	4	6	5	3	7

This is the MKP instance of Example 2.6. The model has 10 variables, so we write one constraint per line. The ILP_{MKP} model for this MKP instance is:

$$\begin{aligned}
 \max \quad & 6x_{11} + 8x_{21} + 5x_{31} + 7x_{41} + 9x_{51} \\
 & + 6x_{12} + 8x_{22} + 5x_{32} + 7x_{42} + 9x_{52} \\
 \text{subject to} \quad & 4x_{11} + 6x_{21} + 5x_{31} + 3x_{41} + 7x_{51} \leq 10 \\
 & 4x_{12} + 6x_{22} + 5x_{32} + 3x_{42} + 7x_{52} \leq 12 \\
 & x_{11} + x_{12} \leq 1 \\
 & x_{21} + x_{22} \leq 1 \\
 & x_{31} + x_{32} \leq 1 \\
 & x_{41} + x_{42} \leq 1 \\
 & x_{51} + x_{52} \leq 1 \\
 & x_{11}, x_{21}, x_{31}, x_{41}, x_{51}, x_{12}, x_{22}, x_{32}, x_{42}, x_{52} \in \{0, 1\}
 \end{aligned}$$

An optimal solution $\tilde{x}(\text{ILP}_{\text{MKP}})$ is:

$$\tilde{x}_{11} = \tilde{x}_{21} = 1, \quad \tilde{x}_{42} = \tilde{x}_{52} = 1, \quad \text{and all the other variables equal to 0.}$$

The optimal value $z(\text{ILP}_{\text{MKP}})$ is 30. The associated optimal solution of the MKP is $(\tilde{S}_1, \tilde{S}_2) = (\{1, 2\}, \{4, 5\})$. The optimal solution is not unique: $(\{2, 4\}, \{1, 5\})$ and $(\{4, 5\}, \{1, 2\})$ have the same value.

9.11 An ILP model for the Multiple-Choice Knapsack Problem

The Multiple-Choice Knapsack Problem and its reference instance (Example 2.7) are defined in Chapter 2 «Knapsack problems», Section 2.8.

- We introduce the following n **binary variables**:

$$x_j = \begin{cases} 1 & \text{if item } j \text{ is selected in the knapsack} \\ 0 & \text{otherwise} \end{cases} \quad \forall j \in \mathcal{J}. \quad (9.30)$$

Using these variables, an Integer Linear Programming (ILP) model for the Multiple-Choice Knapsack Problem (MCKP) reads as follows:

$$\max \sum_{j \in \mathcal{J}} p_j x_j \quad (9.31a)$$

$$\text{subject to } \sum_{j \in \mathcal{J}} w_j x_j \leq c, \quad (9.31b)$$

$$\sum_{j \in \mathcal{J}_h} x_j = 1, \quad \forall h \in \{1, 2, \dots, q\}, \quad (9.31c)$$

$$x_j \in \{0, 1\}, \quad \forall j \in \mathcal{J}. \quad (9.31d)$$

This ILP model is referred to as the ILP_{MCKP} model.

- Description of the objective function and constraints of the ILP_{MCKP} model:
 - the linear objective function (9.31a) maximizes the total profit of the items selected in the knapsack;
 - constraint (9.31b) ensures that the total weight of the selected items does not exceed the knapsack capacity (one linear constraint);
 - constraints (9.31c) ensure that exactly one item is selected from each class (q linear constraints, one for each class);
 - finally, constraints (9.31d) define the variables of the ILP model.

Example 9.15: ILP_{MCKP} model

We consider a knapsack of capacity $c = 15$ and a set $\mathcal{J} = \{1, 2, 3, 4, 5, 6\}$ of 6 items ($n = 6$), partitioned into $q = 3$ classes $\mathcal{J}_1 = \{1, 2\}$, $\mathcal{J}_2 = \{3, 4\}$ and $\mathcal{J}_3 = \{5, 6\}$, with the following profits and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
p_j	6	8	5	7	4	9
w_j	4	6	5	8	3	7

This is the MCKP instance of Example 2.7. The ILP_{MCKP} model for this MCKP instance is:

$$\begin{array}{llllllllll}
 \max & 6 & x_1 & + & 8 & x_2 & + & 5 & x_3 & + & 7 & x_4 & + & 4 & x_5 & + & 9 & x_6 \\
 \text{subject to} & 4 & x_1 & + & 6 & x_2 & + & 5 & x_3 & + & 8 & x_4 & + & 3 & x_5 & + & 7 & x_6 & \leq & 15 \\
 & & x_1 & & + & x_2 & & & & & & & & & & & & = & 1 \\
 & & & & & & & x_3 & & + & x_4 & & & & & & & = & 1 \\
 & & & & & & & & & & & & x_5 & & + & x_6 & & = & 1 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \in & \{0, 1\}
 \end{array}$$

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{MCKP}})$ is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 1, \quad \tilde{x}_5 = 1, \quad \tilde{x}_6 = 0$$

The optimal value $z(\text{ILP}_{\text{MCKP}})$ is 17. The associated optimal solution of the MCKP is $\tilde{S} = \{1, 4, 5\}$. The optimal solution is not unique: the subset $\{2, 3, 5\}$ has the same value.

9.12 An ILP model for the Knapsack Problem with Setups

The Knapsack Problem with Setups and its reference instance (Example 2.8) are defined in Chapter 2 «Knapsack problems», Section 2.9.

- We introduce the following n **binary variables**:

$$x_j = \begin{cases} 1 & \text{if item } j \text{ is selected in the knapsack} \\ 0 & \text{otherwise} \end{cases} \quad \forall j \in \mathcal{J}. \quad (9.32)$$

- We introduce the following q **binary variables**:

$$y_h = \begin{cases} 1 & \text{if class } h \text{ is activated (its setup is paid)} \\ 0 & \text{otherwise} \end{cases} \quad \forall h \in \{1, 2, \dots, q\}. \quad (9.33)$$

Using these variables, an Integer Linear Programming (ILP) model for the Knapsack Problem with Setups (KPS) reads as follows:

$$\max \sum_{j \in \mathcal{J}} p_j x_j - \sum_{h=1}^q f_h y_h \quad (9.34a)$$

$$\text{subject to} \quad \sum_{j \in \mathcal{J}} w_j x_j + \sum_{h=1}^q s_h y_h \leq c, \quad (9.34b)$$

$$x_j \leq y_h, \quad \forall h \in \{1, 2, \dots, q\}, \quad \forall j \in \mathcal{J}_h, \quad (9.34c)$$

$$x_j \in \{0, 1\}, \quad \forall j \in \mathcal{J}, \quad (9.34d)$$

$$y_h \in \{0, 1\}, \quad \forall h \in \{1, 2, \dots, q\}. \quad (9.34e)$$

This ILP model is referred to as the ILP_{KPS} model.

- Description of the objective function and constraints of the ILP_{KPS} model:
 - the linear objective function (9.34a) maximizes the total revenue of the selected items minus the setup costs of the activated classes;
 - constraint (9.34b) ensures that the total weight of the selected items plus the setup weights of the activated classes does not exceed the knapsack capacity (one linear constraint);
 - constraints (9.34c) ensure that an item can be selected only if its class is activated (n linear constraints, one for each item);
 - finally, constraints (9.34d) and (9.34e) define the variables of the ILP model.

Example 9.16: ILP_{KPS} model

We consider a knapsack of capacity $c = 20$ and a set $\mathcal{J} = \{1, 2, 3, 4, 5, 6\}$ of 6 items ($n = 6$), partitioned into $q = 2$ classes $\mathcal{J}_1 = \{1, 2, 3\}$ and $\mathcal{J}_2 = \{4, 5, 6\}$, with setup costs $f_1 = 3$, $f_2 = 4$, setup weights $s_1 = 2$, $s_2 = 3$, and the following revenues and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
p_j	6	8	5	7	9	4
w_j	4	5	3	6	7	2

This is the KPS instance of Example 2.8. The ILP_{KPS} model for this KPS instance is:

$$\begin{aligned}
 & \max \quad 6x_1 + 8x_2 + 5x_3 + 7x_4 + 9x_5 + 4x_6 - 3y_1 - 4y_2 \\
 \text{subject to} \quad & 4x_1 + 5x_2 + 3x_3 + 6x_4 + 7x_5 + 2x_6 + 2y_1 + 3y_2 \leq 20 \\
 & x_1 - y_1 \leq 0 \\
 & x_2 - y_1 \leq 0 \\
 & x_3 - y_1 \leq 0 \\
 & x_4 - y_2 \leq 0 \\
 & x_5 - y_2 \leq 0 \\
 & x_6 - y_2 \leq 0 \\
 & x_1, x_2, x_3, x_4, x_5, x_6 \in \{0, 1\} \\
 & y_1, y_2 \in \{0, 1\}
 \end{aligned}$$

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{KPS}})$, $\tilde{\mathbf{y}}(\text{ILP}_{\text{KPS}})$ is:

$$\tilde{x}_1 = \tilde{x}_2 = \tilde{x}_3 = 1, \quad \tilde{x}_4 = \tilde{x}_5 = 0, \quad \tilde{x}_6 = 1, \quad \tilde{y}_1 = \tilde{y}_2 = 1$$

The optimal value $z(\text{ILP}_{\text{KPS}})$ is 16. The associated optimal solution of the KPS is $\tilde{S} = \{1, 2, 3, 6\}$. The optimal solution is not unique: the subsets $\{1, 2, 3\}$ and $\{4, 5, 6\}$ have the same value.

9.13 Typical mistakes

Warning

- **Forgetting to sort the items.** The formulas (9.13), (9.14) and the bounds hold only if the items are sorted by non-increasing efficiency: with another order the vector (9.13) is feasible but not optimal.
- **Using $z(\text{LP-0}_{\text{KP}})$ as the bound.** The first relaxation has no upper bounds on the variables, and it is much weaker: in [Example 9.1](#) it gives 27, against $\frac{49}{3}$ for the second relaxation.
- **Rounding when the profits are not integer.** The bounds UB_D and UB_{MT} round down the fractional part: this is valid only because all the profits are integer.
- **Confusing the split item with the last selected item.** The split item s is the first item that does not fit: in [Example 9.1](#) it is $s = 3$, while the items 1 and 2 are selected entirely.
- **Assuming that the LP optimum is unique.** With items of equal efficiency the LP_{KP} problem has other optimal solutions; the formulas (9.13) give one of them.
- **Copying the ILP_{KP} model to the variants.** In the BKP and in the UKP the variables are integer, not binary; in the MCKP the class constraints (9.31c) are equalities; in the KPS, without the constraints (9.34c) the model selects items with $y_h = 0$ and never pays a setup.

9.14 In summary

Summary

- The ILP_{KP} model has one binary variable per item, one capacity constraint and the objective of maximizing the total profit.
- The relaxation LP_{KP} is solved by sorting the items by non-increasing efficiency: the items before the split item are selected, the split item is taken fractionally, the others are discarded.
- The dual of LP_{KP} has the optimal solution $\tilde{\pi} = p_s/w_s$, and $z(\text{LP}_{\text{KP}})$ is never larger than twice $z(\text{ILP}_{\text{KP}})$.
- Rounding down $z(\text{LP}_{\text{KP}})$ gives the Dantzig bound; the Martello-Toth bound considers separately the split item out and in.
- Each variant of [Chapter 2 «Knapsack problems»](#) changes one ingredient of the ILP_{KP} model: integer variables (BKP, UKP), one capacity constraint per dimension (MDKP), one variable per item and knapsack (MKP), one equality per class (MCKP), setup variables y_h with $x_j \leq y_h$ (KPS); the SSP is the KP with $p_j = w_j$.

Dashboard of [Example 9.1](#). Upper bounds on $z(\text{ILP}_{\text{KP}})$ computed in this chapter:

Bound	Value	From
First relaxation	27	$z(\text{LP-0KP})$, item 1 with $\frac{9}{2}$ copies
Second relaxation	$\frac{49}{3}$	$z(\text{LP}_{\text{KP}})$, split item $s = 3$
Dantzig bound	16	$\lfloor \frac{49}{3} \rfloor$
Martello-Toth bound	16	$\max\{16, 15\}$
Optimal value	15	solution $\{1, 4\}$ of the ILP_{KP} model

The gap between the best upper bound (16) and the optimal value (15) is closed by the exact algorithms of the following chapters.

Formulations of packing problems

Prerequisites

- From the module *Linear Programming*, [Chapter 9 «Duality theory»](#): the dual of an LP problem, weak and strong duality.
- From the module *Integer linear programming*, [Chapter 1 «Integer linear programming»](#) and [Chapter 2 «Primal and dual bounds»](#): ILP problems, LP relaxations and bounds.
- From this module, [Chapter 3 «Packing problems»](#): the problem and its assumptions.

Objectives

At the end of this chapter you will be able to:

- write the ILP model of the Bin Packing Problem and explain the link between the assignment variables and the activation variables;
- write the two LP relaxations and their duals, and prove that their optimal value is $\sum_j w_j/c$;
- round the LP bound up to obtain a lower bound on the number of bins;
- prove that the LP bound is at least one half of the optimal value, and that the factor is tight;
- write the ILP model of the Cutting Stock Problem, with integer variables for the copies of the items.

10.1 Overview of the family

- The ILP models of the packing problems use an upper bound m on the number of bins and the set $\mathcal{B} = \{1, 2, \dots, m\}$ of the bins that may be used.
- The **assignment variables** x_{jb} place the items $j \in \mathcal{I}$ into the bins $b \in \mathcal{B}$, and the binary **activation variables** y_b say whether bin b is used.
- Each bin gives one capacity constraint, which also links the two kinds of variables, and the objective function minimizes the number $\sum_{b \in \mathcal{B}} y_b$ of bins used.

The packing models are **assignment models**: every item, or every copy of an item, is assigned to a used bin, and the items assigned to a bin respect its capacity. The members of the family change the domain of the assignment variables.

- The table lists the members of the family, with the name of their ILP model and the section that writes it.

Problem	Model	Assignment variables	Section
BPP	ILP _{BPP}	binary x_{jb}	10.2
CSP	ILP _{CSP}	integer x_{jb} , the copies of item j in bin b	10.5

10.2 An ILP model for the Bin Packing Problem

The problem of this section is the Bin Packing Problem (BPP), defined in [Chapter 3 «Packing problems»](#).

- Given an upper bound $m \in \mathbb{Z}_{\geq 2}$ on the number of bins used in any optimal BPP solution, we introduce the following mn **binary variables**:

$$x_{jb} = \begin{cases} 1 & \text{if item } j \text{ is packed in bin } b \\ 0 & \text{otherwise} \end{cases} \quad \forall j \in \underbrace{\{1, 2, \dots, n\}}_{=\mathcal{I}}, \quad \forall b \in \underbrace{\{1, 2, \dots, m\}}_{=\mathcal{B}},$$

and the following m **binary variables**:

$$y_b = \begin{cases} 1 & \text{if bin } b \text{ is used} \\ 0 & \text{otherwise} \end{cases} \quad \forall b \in \underbrace{\{1, 2, \dots, m\}}_{=\mathcal{B}}.$$

The number n of items is a valid upper bound on the minimum number of bins needed to pack all the items since, in the worst possible case, each item is packed in a different bin. Also the number of bins used in any feasible BPP solution is a valid upper bound.

Using these variables, an Integer Linear Programming (ILP) model for the Bin Packing

Problem (BPP) reads as follows:

$$\min \sum_{b \in \mathcal{B}} y_b \quad (10.1a)$$

$$\text{subject to } \sum_{b \in \mathcal{B}} x_{jb} = 1, \quad \forall j \in \mathcal{I}, \quad (10.1b)$$

$$- \sum_{j \in \mathcal{I}} w_j x_{jb} + c y_b \geq 0, \quad \forall b \in \mathcal{B}, \quad (10.1c)$$

$$x_{jb} \in \{0, 1\}, \quad \forall j \in \mathcal{I}, \forall b \in \mathcal{B}, \quad (10.1d)$$

$$y_b \in \{0, 1\}, \quad \forall b \in \mathcal{B}. \quad (10.1e)$$

This ILP model is referred to as the ILP_{BPP} model.

- Description of the objective function and constraints of the ILP_{BPP} model:
 - the linear objective function (10.1a) minimizes the total number of used bins;
 - constraints (10.1b) ensure that each item is packed in exactly one bin, so that all items are packed (n linear constraints);
 - constraints (10.1c) link the assignment variables with the activation variables and impose the capacity restrictions: if bin b is used, the weight of its items is at most c ; if it is not used, it contains no item (m linear constraints);
 - finally, constraints (10.1d) and (10.1e) define the variables of the ILP model.

Explanation of the link between the variables

- **If at least one item is packed in bin b , then the bin is used.** The logical implication is

$$(x_{1b} \text{ OR } x_{2b} \text{ OR } \dots \text{ OR } x_{nb}) \implies y_b.$$

Since $w_j > 0$, packing an item in bin b gives $\sum_{j \in \mathcal{I}} w_j x_{jb} > 0$, and the constraint $\sum_{j \in \mathcal{I}} w_j x_{jb} \leq c y_b$ forces $y_b = 1$.

- **If bin b is not used, then it is empty.** This is the contrapositive of the previous implication. For $y_b = 0$ the constraint reads $\sum_{j \in \mathcal{I}} w_j x_{jb} \leq 0$, and since $w_j > 0$ we obtain $x_{jb} = 0$ for every item j .
- **If bin b is used, then it contains at least one item, but only in an optimal solution.** The constraints do not impose it. However, if $x_{jb} = 0$ for all j and $y_b = 1$, setting $y_b = 0$ keeps all the constraints satisfied and decreases the objective value by one. Hence, in every optimal solution, empty bins are not counted as used.

Example 10.1: ILP_{BPP} model

We consider a bin of capacity $c = 10$ and a set $\mathcal{J} = \{1, 2, 3\}$ of 3 items ($n = 3$), with the following item weights:

	$j = 1$	$j = 2$	$j = 3$
w_j	7	6	3

We refer to this BPP instance as [Example 10.1](#).

The ILP_{BPP} model with $m = 3$ for this BPP instance is:

$$\begin{aligned}
 & \min && y_1 & + & y_2 & + & y_3 \\
 \text{subject to} & & x_{11} & + & x_{12} & + & x_{13} & & & = & 1 \\
 & & & & x_{21} & + & x_{22} & + & x_{23} & & = & 1 \\
 & & & & & & x_{31} & + & x_{32} & + & x_{33} & & = & 1 \\
 & - & 7x_{11} & & - & 6x_{21} & & - & 3x_{31} & & + & 10y_1 & & \geq & 0 \\
 & & - & 7x_{12} & & - & 6x_{22} & & - & 3x_{32} & & + & 10y_2 & & \geq & 0 \\
 & & & - & 7x_{13} & & - & 6x_{23} & & - & 3x_{33} & & + & 10y_3 & & \geq & 0 \\
 & & x_{11}, & x_{12}, & x_{13}, & x_{21}, & x_{22}, & x_{23}, & x_{31}, & x_{32}, & x_{33}, & y_1, & y_2, & y_3 & \in & \{0, 1\}
 \end{aligned}$$

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{BPP}})$ is:

$$\tilde{x}_{11} = \tilde{x}_{31} = 1, \quad \tilde{x}_{12} = \tilde{x}_{13} = \tilde{x}_{21} = 0, \quad \tilde{x}_{22} = 1, \quad \tilde{x}_{23} = \tilde{x}_{32} = \tilde{x}_{33} = 0, \quad \tilde{y}_1 = \tilde{y}_2 = 1, \quad \tilde{y}_3 = 0$$

Another optimal solution, obtained by permuting the indices of bins 1 and 2, is:

$$\tilde{x}_{12} = \tilde{x}_{32} = 1, \quad \tilde{x}_{11} = \tilde{x}_{13} = \tilde{x}_{22} = 0, \quad \tilde{x}_{21} = 1, \quad \tilde{x}_{23} = \tilde{x}_{31} = \tilde{x}_{33} = 0, \quad \tilde{y}_1 = \tilde{y}_2 = 1, \quad \tilde{y}_3 = 0.$$

The optimal value is:

$$z(\text{ILP}_{\text{BPP}}) = 2$$

The associated optimal solution of the BPP is given by the packing patterns

$$\tilde{P}_1 = \{1, 3\} \quad \text{and} \quad \tilde{P}_2 = \{2\}$$

The optimal solution is not unique: also the packing patterns $\{1\}$ and $\{2, 3\}$ use two bins.

$$x_{jb} \geq 0, \quad \forall j \in \mathcal{J}, \quad \forall b \in \mathcal{B}, \quad (10.2a)$$

$$y_b \geq 0, \quad \forall b \in \mathcal{B}. \quad (10.2b)$$

Example 10.2: LP-0_{BPP} model

$$\begin{array}{llllllllllllll}
\min & & & & & & & & & & y_1 & + & y_2 & + & y_3 \\
\text{subject to} & x_{11} & + & x_{12} & + & x_{13} & & & & & & & & & = 1 \\
& & & & & & x_{21} & + & x_{22} & + & x_{23} & & & & = 1 \\
& & & & & & & & & & x_{31} & + & x_{32} & + & x_{33} & = 1 \\
& - & 7x_{11} & & & - & 6x_{21} & & & - & 3x_{31} & & + & 10y_1 & \geq 0 \\
& & - & 7x_{12} & & & - & 6x_{22} & & & - & 3x_{32} & & + & 10y_2 & \geq 0 \\
& & & & - & 7x_{13} & & & - & 6x_{23} & & & - & 3x_{33} & & + & 10y_3 \geq 0 \\
& x_{11}, & x_{12}, & x_{13}, & x_{21}, & x_{22}, & x_{23}, & x_{31}, & x_{32}, & x_{33}, & y_1, & y_2, & y_3 & \geq 0
\end{array}$$
$$\tilde{x}_{11} = \tilde{x}_{12} = \tilde{x}_{13} = \tilde{x}_{21} = \tilde{x}_{22} = \tilde{x}_{23} = \tilde{x}_{31} = \tilde{x}_{32} = \tilde{x}_{33} = \frac{1}{3}, \quad \tilde{y}_1 = \tilde{y}_2 = \tilde{y}_3 = \frac{8}{15}$$

The optimal value $z(\text{LP-0}_{\text{BPP}})$ is $\frac{8}{5}$.

$$\frac{8}{5} \leq z(\text{ILP}_{\text{BPP}})$$

and, since $z(\text{ILP}_{\text{BPP}})$ is an integer, $z(\text{ILP}_{\text{BPP}}) \geq \lceil \frac{8}{5} \rceil = 2$.

Using

- a free dual variable μ_j for each constraint (10.1b) ($j \in \mathcal{J}$)
- a non-negative dual variable π_b for each constraint (10.1c) ($b \in \mathcal{B}$)

the dual model of the LP-0_{BPP} model reads as follows:

$$\max \sum_{j \in \mathcal{J}} \mu_j \quad (10.3a)$$

$$\text{subject to } \mu_j - w_j \pi_b \leq 0, \quad \forall j \in \mathcal{J}, \forall b \in \mathcal{B}, \quad (10.3b)$$

$$c \pi_b \leq 1, \quad \forall b \in \mathcal{B}, \quad (10.3c)$$

$$\mu_j \geq 0, \quad \forall j \in \mathcal{J}, \quad (10.3d)$$

$$\pi_b \geq 0, \quad \forall b \in \mathcal{B}. \quad (10.3e)$$

This LP model is referred to as the dual $D(\text{LP-0}_{\text{BPP}})$ model.

Example 10.3: $D(\text{LP-0}_{\text{BPP}})$ model

The dual $D(\text{LP-0}_{\text{BPP}})$ model for the BPP instance of [Example 10.1](#) is:

max	μ_1	$+ \mu_2$	$+ \mu_3$				
subject to	μ_1			$- 7\pi_1$			≤ 0
	μ_1				$- 7\pi_2$		≤ 0
	μ_1					$- 7\pi_3$	≤ 0
		μ_2		$- 6\pi_1$			≤ 0
		μ_2			$- 6\pi_2$		≤ 0
		μ_2				$- 6\pi_3$	≤ 0
			μ_3	$- 3\pi_1$			≤ 0
			μ_3		$- 3\pi_2$		≤ 0
			μ_3			$- 3\pi_3$	≤ 0
				$10\pi_1$			≤ 1
					$10\pi_2$		≤ 1
						$10\pi_3$	≤ 1
	$\mu_1,$	$\mu_2,$	μ_3				≥ 0
				$\pi_1,$	$\pi_2,$	π_3	≥ 0

An optimal solution $(\tilde{\mu}, \tilde{\pi})$ of $D(\text{LP-0}_{\text{BPP}})$ is:

$$\tilde{\mu}_1 = \frac{7}{10}, \quad \tilde{\mu}_2 = \frac{3}{5}, \quad \tilde{\mu}_3 = \frac{3}{10}, \quad \tilde{\pi}_1 = \tilde{\pi}_2 = \tilde{\pi}_3 = \frac{1}{10}$$

The optimal value $z(D(\text{LP-0}_{\text{BPP}}))$ is $\frac{8}{5}$.

Proposition 10.1

An optimal solution of the primal LP-0_{BPP} model is:

$$\begin{aligned}\tilde{x}_{jb} &= \frac{1}{m}, & \forall j \in \mathcal{J}, \forall b \in \mathcal{B}, \\ \tilde{y}_b &= \frac{\sum_{j \in \mathcal{J}} w_j}{m c}, & \forall b \in \mathcal{B},\end{aligned}\tag{10.4}$$

and an optimal solution of the dual $D(\text{LP-0}_{\text{BPP}})$ model is:

$$\begin{aligned}\tilde{\mu}_j &= \frac{w_j}{c}, & \forall j \in \mathcal{J}, \\ \tilde{\pi}_b &= \frac{1}{c}, & \forall b \in \mathcal{B}.\end{aligned}\tag{10.5}$$

Corollary 10.2

The optimal value of the primal LP-0_{BPP} and dual $D(\text{LP-0}_{\text{BPP}})$ models is:

$$z(\text{LP-0}_{\text{BPP}}) = \frac{\sum_{j \in \mathcal{J}} w_j}{c} = z(D(\text{LP-0}_{\text{BPP}}))\tag{10.6}$$

- The proof proceeds in three steps:
 1. We show that (10.4) is a feasible solution for the primal LP-0_{BPP} model;
 2. We show that (10.5) is a feasible solution for the dual $D(\text{LP-0}_{\text{BPP}})$ model;
 3. We show that both feasible solutions have the same objective function value.

Therefore, by LP Weak Duality, (10.4) and (10.5) are a pair of optimal primal and dual solutions.

Proof.

1. For every $j \in \mathcal{J}$, we have:

$$\sum_{b \in \mathcal{B}} \tilde{x}_{jb} = \sum_{b \in \mathcal{B}} \frac{1}{m} = \frac{m}{m} = 1$$

thus constraints (10.1b) are satisfied.

For every $b \in \mathcal{B}$, we have:

$$-\sum_{j \in \mathcal{J}} w_j \tilde{x}_{jb} + c \tilde{y}_b = -\sum_{j \in \mathcal{J}} w_j \frac{1}{m} + c \frac{\sum_{j \in \mathcal{J}} w_j}{m c} = 0$$

thus constraints (10.1c) are satisfied.

We have $\tilde{x}_{jb} \geq 0$ and $\tilde{y}_b \geq 0$, for every $j \in \mathcal{J}$ and every $b \in \mathcal{B}$, thus constraints (10.2a) and (10.2b) are satisfied.

Therefore, (10.4) provides a feasible solution for the primal LP-0_{BPP} model.

2. For every $b \in \mathcal{B}$ and every $j \in \mathcal{J}$, we have:

$$\tilde{\mu}_j - w_j \tilde{\pi}_b = \frac{w_j}{c} - \frac{w_j}{c} = 0$$

thus constraints (10.3b) are satisfied.

For every $b \in \mathcal{B}$, we have:

$$c \tilde{\pi}_b = \frac{c}{c} = 1$$

thus constraints (10.3c) are satisfied.

We have $\tilde{\mu}_j \geq 0$ and $\tilde{\pi}_b \geq 0$, for every $b \in \mathcal{B}$ and every $j \in \mathcal{J}$, thus constraints (10.3d) and (10.3e) are satisfied.

Therefore, (10.5) provides a feasible solution for the dual $D(\text{LP-0}_{\text{BPP}})$ model.

3. For the objective functions, we have:

$$\sum_{b \in \mathcal{B}} \tilde{y}_b = \sum_{b \in \mathcal{B}} \frac{\sum_{j \in \mathcal{J}} w_j}{m c} = \frac{\sum_{j \in \mathcal{J}} w_j}{c} = \sum_{j \in \mathcal{J}} \tilde{\mu}_j$$

thus the primal and dual objective function values are equal.

Therefore, by LP Weak Duality, (10.4) and (10.5) are a pair of optimal primal and dual solutions. □

- This is an alternative proof that does not rely on the dual formulation but instead reasons directly on the minimum objective function value imposed by the constraints.

Proposition 10.3

An optimal solution of the primal LP-0_{BPP} model is:

$$\begin{aligned} \tilde{x}_{jb} &= \frac{1}{m}, & \forall j \in \mathcal{J}, \forall b \in \mathcal{B}, \\ \tilde{y}_b &= \frac{\sum_{j \in \mathcal{J}} w_j}{m c}, & \forall b \in \mathcal{B}, \end{aligned} \tag{10.7}$$

and the optimal value is

$$z(\text{LP-0}_{\text{BPP}}) = \frac{\sum_{j \in \mathcal{J}} w_j}{c}. \tag{10.8}$$

Proof. Summing the capacity constraints (10.1c) over all $b \in \mathcal{B}$ gives

$$c \sum_{b \in \mathcal{B}} y_b \geq \sum_{b \in \mathcal{B}} \sum_{j \in \mathcal{J}} w_j x_{jb} = \sum_{j \in \mathcal{J}} w_j \underbrace{\sum_{b \in \mathcal{B}} x_{jb}}_{\text{by (10.1b)=1}} = \sum_{j \in \mathcal{J}} w_j,$$

hence

$$\sum_{b \in \mathcal{B}} y_b \geq \frac{1}{c} \sum_{j \in \mathcal{J}} w_j. \tag{10.9}$$

Therefore every feasible solution of LP-0_{BPP} satisfies (10.9).

Now consider the point defined in (10.7). For every $j \in \mathcal{J}$, we have

$$\sum_{b \in \mathcal{B}} \tilde{x}_{jb} = m \frac{1}{m} = 1,$$

so constraints (10.1b) hold. Moreover, for every $b \in \mathcal{B}$, we have

$$\sum_{j \in \mathcal{J}} w_j \tilde{x}_{jb} = \sum_{j \in \mathcal{J}} w_j \frac{1}{m} = \frac{1}{m} \sum_{j \in \mathcal{J}} w_j = c \tilde{y}_b,$$

so constraints (10.1c) hold with equality. Finally, constraints (10.2a) and (10.2b) are satisfied by construction. Hence the point defined in (10.7) is a feasible solution for LP-0_{BPP} .

The objective value of (10.7) is

$$\sum_{b \in \mathcal{B}} \tilde{y}_b = m \frac{\sum_{j \in \mathcal{J}} w_j}{m c} = \frac{1}{c} \sum_{j \in \mathcal{J}} w_j,$$

which matches the lower bound (10.9). Therefore the point (10.7) is an optimal solution with objective value (10.8). $\square$

10.4 A second Linear Programming relaxation

To obtain a Linear Programming (LP) relaxation of the ILP_{BPP} model, we relax constraints (10.1d) and (10.1e) with the following ones:

$$x_{jb} \geq 0, \quad \forall j \in \mathcal{J}, \forall b \in \mathcal{B}, \quad (10.10a)$$

$$y_b \leq 1, \quad \forall b \in \mathcal{B}, \quad (10.10b)$$

$$y_b \geq 0, \quad \forall b \in \mathcal{B}. \quad (10.10c)$$

This LP model is referred to as the LP_{BPP} model.

Example 10.4: LP_{BPP} model

The LP_{BPP} model for the BPP instance of [Example 10.1](#) is:

$$\begin{aligned}
 & \min && y_1 & + y_2 & + y_3 \\
 \text{subject to} & x_{11} & + x_{12} & + x_{13} & & & = 1 \\
 & & x_{21} & + x_{22} & + x_{23} & & = 1 \\
 & & & x_{31} & + x_{32} & + x_{33} & = 1 \\
 & -7x_{11} & & -6x_{21} & & -3x_{31} & + 10y_1 & \geq 0 \\
 & & -7x_{12} & & -6x_{22} & & -3x_{32} & + 10y_2 & \geq 0 \\
 & & & -7x_{13} & & -6x_{23} & & -3x_{33} & + 10y_3 & \geq 0 \\
 & & & & & & y_1 & & \leq 1 \\
 & & & & & & & y_2 & \leq 1 \\
 & & & & & & & & y_3 & \leq 1 \\
 & x_{11}, & x_{12}, & x_{13}, & x_{21}, & x_{22}, & x_{23}, & x_{31}, & x_{32}, & x_{33}, & y_1, & y_2, & y_3 & \geq 0
 \end{aligned}$$

An optimal solution $\tilde{\mathbf{x}}(\text{LP}_{\text{BPP}})$ is:

$$\tilde{x}_{11} = \tilde{x}_{12} = \tilde{x}_{13} = \tilde{x}_{21} = \tilde{x}_{22} = \tilde{x}_{23} = \tilde{x}_{31} = \tilde{x}_{32} = \tilde{x}_{33} = \frac{1}{3}, \quad \tilde{y}_1 = \tilde{y}_2 = \tilde{y}_3 = \frac{8}{15}$$

The optimal value $z(\text{LP}_{\text{BPP}})$ is $\frac{8}{5}$. Accordingly, we have the same lower bound as before.

Using

- a free dual variable μ_j for each constraint (10.1b) ($j \in \mathcal{J}$)
- a non-negative dual variable π_b for each constraint (10.1c) ($b \in \mathcal{B}$)
- a non-positive dual variable λ_b for each constraint (10.10b) ($b \in \mathcal{B}$)

the dual model of the LP_{BPP} model reads as follows:

$$\max \sum_{j \in \mathcal{J}} \mu_j + \sum_{b \in \mathcal{B}} \lambda_b \quad (10.11a)$$

$$\text{subject to } \mu_j - w_j \pi_b \leq 0, \quad \forall j \in \mathcal{J}, \forall b \in \mathcal{B}, \quad (10.11b)$$

$$c \pi_b + \lambda_b \leq 1, \quad \forall b \in \mathcal{B}, \quad (10.11c)$$

$$\mu_j \geq 0, \quad \forall j \in \mathcal{J}, \quad (10.11d)$$

$$\pi_b \geq 0, \quad \forall b \in \mathcal{B}, \quad (10.11e)$$

$$\lambda_b \leq 0, \quad \forall b \in \mathcal{B}. \quad (10.11f)$$

This LP model is referred to as the dual $D(\text{LP}_{\text{BPP}})$ model.

Example 10.5: $D(\text{LP}_{\text{BPP}})$ model

The $D(\text{LP}_{\text{BPP}})$ model of Example 10.1 is:

$$\begin{array}{llllllll}
 \max & \mu_1 & + \mu_2 & + \mu_3 & & + \lambda_1 & + \lambda_2 & + \lambda_3 \\
 \text{subject to} & \mu_1 & & & - 7\pi_1 & & & \leq 0 \\
 & \mu_1 & & & & - 7\pi_2 & & \leq 0 \\
 & \mu_1 & & & & & - 7\pi_3 & \leq 0 \\
 & & \mu_2 & & - 6\pi_1 & & & \leq 0 \\
 & & \mu_2 & & & - 6\pi_2 & & \leq 0 \\
 & & \mu_2 & & & & - 6\pi_3 & \leq 0 \\
 & & & \mu_3 & - 3\pi_1 & & & \leq 0 \\
 & & & \mu_3 & & - 3\pi_2 & & \leq 0 \\
 & & & \mu_3 & & & - 3\pi_3 & \leq 0 \\
 & & & & 10\pi_1 & & + \lambda_1 & \leq 1 \\
 & & & & & 10\pi_2 & & + \lambda_2 & \leq 1 \\
 & & & & & & 10\pi_3 & & + \lambda_3 & \leq 1 \\
 & \mu_1, & \mu_2, & \mu_3 & & & & & & \geq 0 \\
 & & & & \pi_1, & \pi_2, & \pi_3 & & & \geq 0 \\
 & & & & & & & \lambda_1, & \lambda_2, & \lambda_3 & \leq 0
 \end{array}$$

The optimal value $z(D(\text{LP}_{\text{BPP}}))$ is $\frac{8}{5}$. An optimal solution $(\tilde{\mu}, \tilde{\pi}, \tilde{\lambda})$ of $D(\text{LP}_{\text{BPP}})$ is:

$$\tilde{\mu}_1 = \frac{7}{10}, \quad \tilde{\mu}_2 = \frac{3}{5}, \quad \tilde{\mu}_3 = \frac{3}{10}, \quad \tilde{\pi}_1 = \tilde{\pi}_2 = \tilde{\pi}_3 = \frac{1}{10}, \quad \tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{\lambda}_3 = 0$$

- The constraints:

$$x_{jb} \leq 1, \quad \forall j \in \mathcal{J}, \forall b \in \mathcal{B},$$

are also valid since, for every $j \in \mathcal{J}$ and every $b \in \mathcal{B}$, we have:

$$x_{jb} \in \{0, 1\} \quad \Rightarrow \quad x_{jb} \leq 1$$

but they are implied by constraints (10.1b).

Proposition 10.4

An optimal solution of the primal LP_{BPP} model is:

$$\begin{aligned}\tilde{x}_{jb} &= \frac{1}{m}, & \forall j \in \mathcal{J}, \forall b \in \mathcal{B}, \\ \tilde{y}_b &= \frac{\sum_{j \in \mathcal{J}} w_j}{m c}, & \forall b \in \mathcal{B},\end{aligned}\tag{10.12}$$

and an optimal solution of the dual $D(\text{LP}_{\text{BPP}})$ model is:

$$\begin{aligned}\tilde{\mu}_j &= \frac{w_j}{c}, & \forall j \in \mathcal{J}, \\ \tilde{\pi}_b &= \frac{1}{c}, & \forall b \in \mathcal{B}, \\ \tilde{\lambda}_b &= 0, & \forall b \in \mathcal{B}.\end{aligned}\tag{10.13}$$

Corollary 10.5

The optimal value of the primal LP_{BPP} and dual $D(\text{LP}_{\text{BPP}})$ models is:

$$z(\text{LP}_{\text{BPP}}) = \frac{\sum_{j \in \mathcal{J}} w_j}{c} = z(D(\text{LP}_{\text{BPP}}))\tag{10.14}$$

Proof. The proof is similar to the previous one. In addition, it is necessary to observe that

$$\tilde{y}_b = \frac{\sum_{j \in \mathcal{J}} w_j}{m c} \leq 1, \quad \forall b \in \mathcal{B},$$

since $\sum_j w_j/c \leq z(\text{ILP}_{\text{BPP}}) \leq m$, where the last inequality holds because m is an upper bound on the number of bins in any optimal BPP solution. Accordingly, constraints (10.10b) are also satisfied. □

10.4.1 Some important theoretical properties

- The following two propositions show that the ratio between $z(\text{LP}_{\text{BPP}})$ and $z(\text{ILP}_{\text{BPP}})$ is at least one half for every BPP instance (**by hypothesis** $z(\text{ILP}_{\text{BPP}}) > 1$ since not all items fit in a single bin). Furthermore, there exist instances for which the value one half can be asymptotically achieved as the capacity $c \rightarrow +\infty$.

Proposition 10.6

For every BPP instance, we have:

$$\frac{z(\text{LP}_{\text{BPP}})}{z(\text{ILP}_{\text{BPP}})} \geq \frac{1}{2} \quad (10.15)$$

Proof. Consider an optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{BPP}})$ of value $z = z(\text{ILP}_{\text{BPP}})$ (number of used bins). We can assume, without loss of generality, that bins $1, 2, \dots, z$ are the non-empty ones.

In an optimal solution, no two bins can have total item weight at most $c/2$, since otherwise their contents could be merged into a single bin without exceeding the bin capacity c (contradicting optimality).

If all non-empty bins have total item weight greater than $c/2$, then

$$\sum_{j \in \mathcal{J}} w_j = \sum_{b=1}^z \sum_{j \in \mathcal{J}} w_j \tilde{x}_{jb} > z \frac{c}{2} \implies \frac{\sum_{j \in \mathcal{J}} w_j}{z} > \frac{1}{2} \implies \frac{z(\text{LP}_{\text{BPP}})}{z(\text{ILP}_{\text{BPP}})} \geq \frac{1}{2}.$$

Otherwise, suppose that there exists a bin with total item weight at most $c/2$. Renumbering bins if necessary, let it be bin z , so that

$$\sum_{j \in \mathcal{J}} w_j \tilde{x}_{jz} \leq \frac{c}{2}.$$

Since $z > 1$, there exists another bin say bin 1; by optimality, we must have

$$\sum_{j \in \mathcal{J}} w_j \tilde{x}_{j1} + \sum_{j \in \mathcal{J}} w_j \tilde{x}_{jz} > c,$$

and, by optimality, every bin in $\{2, 3, \dots, z-1\}$ has total item weight $> c/2$. Hence,

$$\begin{aligned} \sum_{j \in \mathcal{J}} w_j &= \sum_{b=1}^z \sum_{j \in \mathcal{J}} w_j \tilde{x}_{jb} = \underbrace{\sum_{b=2}^{z-1} \sum_{j \in \mathcal{J}} w_j \tilde{x}_{jb}}_{\geq \frac{c}{2}(z-2)} + \underbrace{\sum_{j \in \mathcal{J}} w_j \tilde{x}_{j1} + \sum_{j \in \mathcal{J}} w_j \tilde{x}_{jz}}_{> c} \\ &> \frac{c}{2}(z-2) + c = z \frac{c}{2} \implies \frac{\sum_{j \in \mathcal{J}} w_j}{z} > \frac{1}{2} \implies \frac{z(\text{LP}_{\text{BPP}})}{z(\text{ILP}_{\text{BPP}})} \geq \frac{1}{2}. \end{aligned}$$

□

Proposition 10.7

There are BPP instances such that:

$$\frac{z(\text{LP}_{\text{BPP}})}{z(\text{ILP}_{\text{BPP}})} \rightarrow \frac{1}{2} \quad \text{as} \quad c \rightarrow +\infty \quad (10.16)$$

Proof. Given $\vartheta \in \mathbb{Z}_{\geq 1}$, consider the BPP instance with bin capacity $c = 2\vartheta$ and n items having the following weights:

$$w_j = \vartheta + 1, \quad \forall j \in \mathcal{I}.$$

No bin can contain more than one item, thus each item must be packed alone in its own bin. Therefore, the BPP optimal value is

$$z(\text{ILP}_{\text{BPP}}) = n.$$

For the LP relaxation we have:

$$z(\text{LP}_{\text{BPP}}) = \frac{n(\vartheta + 1)}{2\vartheta}.$$

It follows that:

$$\lim_{\vartheta \rightarrow +\infty} \frac{z(\text{LP}_{\text{BPP}})}{z(\text{ILP}_{\text{BPP}})} = \lim_{\vartheta \rightarrow +\infty} \frac{\frac{n(\vartheta+1)}{2\vartheta}}{n} = \lim_{\vartheta \rightarrow +\infty} \left(\frac{1}{2} + \underbrace{\frac{1}{2\vartheta}}_{\rightarrow 0} \right) = \frac{1}{2}.$$

Accordingly, we conclude that:

$$\frac{z(\text{LP}_{\text{BPP}})}{z(\text{ILP}_{\text{BPP}})} \rightarrow \frac{1}{2} \quad \text{as} \quad c \rightarrow +\infty.$$

□

- The following two propositions show that the ratio between $\lceil z(\text{LP}_{\text{BPP}}) \rceil$ and $z(\text{ILP}_{\text{BPP}})$ is at least one half for every BPP instance. Furthermore, there exist instances for which the value one half can be asymptotically achieved as the number of items $n \rightarrow +\infty$.

Proposition 10.8

For every BPP instance, we have:

$$\frac{\lceil z(\text{LP}_{\text{BPP}}) \rceil}{z(\text{ILP}_{\text{BPP}})} \geq \frac{1}{2} \quad (10.17)$$

Proof. Consider an optimal solution $\tilde{x}(\text{ILP}_{\text{BPP}})$ of value $z = z(\text{ILP}_{\text{BPP}})$ (number of used bins). We can assume, without loss of generality, that bins $1, 2, \dots, z$ are the non-empty

ones.

In an optimal solution, no two bins can have total item weight at most $c/2$, since otherwise their contents could be merged into a single bin without exceeding the bin capacity c (contradicting optimality).

If all non-empty bins have total item weight greater than $c/2$, then

$$\begin{aligned} \sum_{j \in \mathcal{J}} w_j &= \sum_{b=1}^z \sum_{j \in \mathcal{J}} w_j \tilde{x}_{jb} > \frac{zc}{2} \implies \frac{\sum_{j \in \mathcal{J}} w_j}{c} > \frac{z}{2} \implies \left\lceil \frac{\sum_{j \in \mathcal{J}} w_j}{c} \right\rceil \geq \left\lceil \frac{z}{2} \right\rceil \geq \frac{z}{2} \\ \implies \frac{\left\lceil \frac{\sum_{j \in \mathcal{J}} w_j}{c} \right\rceil}{z} &\geq \frac{1}{2} \implies \frac{\lceil z(\text{LP}_{\text{BPP}}) \rceil}{z(\text{ILP}_{\text{BPP}})} \geq \frac{1}{2}. \end{aligned}$$

Otherwise, suppose that there exists a bin with total item weight at most $c/2$. Renumbering bins if necessary, let it be bin z . Then we have:

$$\sum_{j \in \mathcal{J}} w_j = \sum_{b=1}^z \sum_{j \in \mathcal{J}} w_j \tilde{x}_{jb} > \sum_{b=1}^{z-1} \sum_{j \in \mathcal{J}} w_j \tilde{x}_{jb} > (z-1) \frac{c}{2} \implies z < 2 \sum_{j \in \mathcal{J}} \frac{w_j}{c} + 1.$$

Since z must be an integer, this implies:

$$z \leq \left\lceil 2 \sum_{j \in \mathcal{J}} \frac{w_j}{c} \right\rceil \leq 2 \left\lceil \sum_{j \in \mathcal{J}} \frac{w_j}{c} \right\rceil.$$

Therefore:

$$\frac{\left\lceil \sum_{j \in \mathcal{J}} \frac{w_j}{c} \right\rceil}{z} \geq \frac{1}{2} \implies \frac{\lceil z(\text{LP}_{\text{BPP}}) \rceil}{z(\text{ILP}_{\text{BPP}})} \geq \frac{1}{2}.$$

□

Proposition 10.9

There are BPP instances such that:

$$\frac{\lceil z(\text{LP}_{\text{BPP}}) \rceil}{z(\text{ILP}_{\text{BPP}})} \rightarrow \frac{1}{2} \quad \text{as } n \rightarrow +\infty \quad (10.18)$$

Proof. Given $n \in \mathbb{Z}_{\geq 2}$ (take n even), consider the BPP instance with bin capacity $c = n$ and n items having the following weights:

$$w_j = \frac{c}{2} + 1 = \frac{n}{2} + 1, \quad \forall j \in \mathcal{J}.$$

No bin can contain more than one item, since $w_j > c/2$ for every j . Thus each item must be packed alone in its own bin. Therefore, the BPP optimal value is

$$z(\text{ILP}_{\text{BPP}}) = n.$$

For the LP relaxation we have:

$$z(\text{LP}_{\text{BPP}}) = \frac{\sum_{j \in \mathcal{J}} w_j}{c} = \frac{n \left(\frac{n}{2} + 1 \right)}{n} = \frac{n}{2} + 1,$$

hence $\lceil z(\text{LP}_{\text{BPP}}) \rceil = \frac{n}{2} + 1$ (since n is even). It follows that:

$$\lim_{n \rightarrow +\infty} \frac{\lceil z(\text{LP}_{\text{BPP}}) \rceil}{z(\text{ILP}_{\text{BPP}})} = \lim_{n \rightarrow +\infty} \frac{\frac{n}{2} + 1}{n} = \lim_{n \rightarrow +\infty} \left(\frac{1}{2} + \underbrace{\frac{1}{n}}_{\rightarrow 0} \right) = \frac{1}{2}.$$

Accordingly, we conclude that:

$$\frac{\lceil z(\text{LP}_{\text{BPP}}) \rceil}{z(\text{ILP}_{\text{BPP}})} \rightarrow \frac{1}{2} \quad \text{as} \quad n \rightarrow +\infty.$$

□

10.5 An ILP model for the Cutting Stock Problem

The problem of this section is the Cutting Stock Problem (CSP), defined in [Chapter 3 «Packing problems»](#), with its reference instance.

- Given an upper bound $m \in \mathbb{Z}_{\geq 2}$ on the number of bins used in any optimal CSP solution, we introduce the following mn **integer variables**:

$$x_{jb} = \text{number of copies of item } j \text{ packed in bin } b \quad \forall j \in \mathcal{J}, \quad \forall b \in \underbrace{\{1, 2, \dots, m\}}_{=\mathcal{B}},$$

and the following m **binary variables**:

$$y_b = \begin{cases} 1 & \text{if bin } b \text{ is used} \\ 0 & \text{otherwise} \end{cases} \quad \forall b \in \mathcal{B}.$$

The total number $\sum_{j \in \mathcal{J}} d_j$ of copies is a valid value of m , and so is the number of bins of any feasible CSP solution.

Using these variables, an Integer Linear Programming (ILP) model for the Cutting Stock Problem (CSP) reads as follows:

$$\min \sum_{b \in \mathcal{B}} y_b \tag{10.19a}$$

$$\text{subject to} \quad \sum_{b \in \mathcal{B}} x_{jb} = d_j, \quad \forall j \in \mathcal{J}, \tag{10.19b}$$

$$- \sum_{j \in \mathcal{J}} w_j x_{jb} + c y_b \geq 0, \quad \forall b \in \mathcal{B}, \tag{10.19c}$$

$$x_{jb} \in \mathbb{Z}_{\geq 0}, \quad \forall j \in \mathcal{J}, \forall b \in \mathcal{B}, \quad (10.19d)$$

$$y_b \in \{0, 1\}, \quad \forall b \in \mathcal{B}. \quad (10.19e)$$

This ILP model is referred to as the ILP_{CSP} model.

- Description of the objective function and constraints of the ILP_{CSP} model:
 - the linear objective function (10.19a) minimizes the total number of used bins;
 - constraints (10.19b) ensure that exactly d_j copies of each item j are packed (n linear constraints);
 - constraints (10.19c) link the integer variables with the activation variables and impose the capacity restrictions, as constraints (10.1c) of the ILP_{BPP} model (m linear constraints);
 - finally, constraints (10.19d) and (10.19e) define the variables of the ILP model.
- With $d_j = 1$ for every item, constraints (10.19b) give $x_{jb} \in \{0, 1\}$, and the ILP_{CSP} model is the ILP_{BPP} model.
- The copies of an item are identical: a single integer variable counts them in each bin. The ILP_{BPP} model on the extended set of items would use $m \sum_j d_j$ binary variables instead of mn integer ones.

Example 10.6: ILP_{CSP} model

We consider bins of capacity $c = 10$ and a set $\mathcal{J} = \{1, 2, 3, 4\}$ of 4 items ($n = 4$), with the following weights and required copies:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
w_j	4	5	3	6
d_j	2	1	3	1

This is the CSP instance of [Chapter 3 «Packing problems»](#), where a solution with 3 bins is found by eye: hence $m = 3$ is a valid upper bound. The ILP_{CSP} model with $m = 3$ for this CSP instance is:

$$\begin{array}{llll}
\min & y_1 + y_2 + y_3 & & \\
\text{subject to} & x_{11} + x_{12} + x_{13} & = & 2 \\
& x_{21} + x_{22} + x_{23} & = & 1 \\
& x_{31} + x_{32} + x_{33} & = & 3 \\
& x_{41} + x_{42} + x_{43} & = & 1 \\
& -4x_{11} - 5x_{21} - 3x_{31} - 6x_{41} + 10y_1 & \geq & 0 \\
& -4x_{12} - 5x_{22} - 3x_{32} - 6x_{42} + 10y_2 & \geq & 0 \\
& -4x_{13} - 5x_{23} - 3x_{33} - 6x_{43} + 10y_3 & \geq & 0 \\
& x_{11}, x_{12}, x_{13}, \dots, x_{43} & \in & \mathbb{Z}_{\geq 0} \\
& y_1, y_2, y_3 & \in & \{0, 1\}
\end{array}$$

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{CSP}})$ is:

$$\tilde{x}_{11} = \tilde{x}_{41} = 1, \quad \tilde{x}_{32} = 3, \quad \tilde{x}_{13} = \tilde{x}_{23} = 1, \quad \tilde{y}_1 = \tilde{y}_2 = \tilde{y}_3 = 1,$$

with all the other variables equal to 0. The optimal value is $z(\text{ILP}_{\text{CSP}}) = 3$, as found in Chapter 3. The bins contain one copy of items 1 and 4 (weight 10), three copies of item 3 (weight 9), and one copy of items 1 and 2 (weight 9). The optimal solution is not unique: the solution of Chapter 3, with the bins $\{4, 1\}$, $\{1, 3, 3\}$ and $\{2, 3\}$, also uses 3 bins.

10.6 Typical mistakes

Warning

- **Choosing m too small.** The number m of bins must be an upper bound on the number of bins of an optimal solution: with $m < z(\text{ILP}_{\text{BPP}})$ the model has no feasible solution. The value $m = n$ is always valid.
- **Forgetting to round up.** The optimal value of the ILP_{BPP} model is an integer, so the lower bound from the relaxations is $\lceil \sum_j w_j/c \rceil$; in Example 10.1 it is $\lceil \frac{8}{5} \rceil = 2$, not $\frac{8}{5}$.
- **Believing that the model counts only non-empty bins.** The constraints allow $y_b = 1$ for an empty bin: it is the objective function that switches such bins off, in every optimal solution.
- **Ignoring the symmetry.** Permuting the indices of the bins gives another optimal solution with the same packing: the ILP_{BPP} model has many equivalent optimal solutions.
- **Expecting a better relaxation from $y_b \leq 1$.** The second relaxation has the same optimal value as the first one: the upper bounds $y_b \leq 1$ do not cut off the optimal solution of the first relaxation.
- **Declaring binary the variables of the cutting stock model.** In the ILP_{CSP}

model x_{jb} counts the copies of item j in bin b : with $x_{jb} \in \{0, 1\}$ the three copies of item 3 of the example could not share a bin.

10.7 In summary

Summary

- The ILP_{BPP} model has $m \cdot n$ assignment variables x_{jb} and m activation variables y_b ; the capacity constraints link them.
- Both LP relaxations have optimal value $z(\text{LP}_{\text{BPP}}) = \sum_j w_j/c$, with dual solution $\tilde{\mu}_j = w_j/c$ and $\tilde{\pi}_b = 1/c$.
- Since the optimal value is an integer, $\lceil z(\text{LP}_{\text{BPP}}) \rceil$ is a lower bound on the number of bins.
- The ratio between $z(\text{LP}_{\text{BPP}})$ and $z(\text{ILP}_{\text{BPP}})$ is at least $\frac{1}{2}$, and the value $\frac{1}{2}$ is asymptotically attained; the same holds for $\lceil z(\text{LP}_{\text{BPP}}) \rceil$.
- The ILP_{CSP} model has the structure of the ILP_{BPP} model, with mn integer variables x_{jb} that count the copies and the demands d_j on the right-hand side; on the CSP instance of Chapter 3 its optimal value is 3.

Dashboard of Example 10.1. Bounds on $z(\text{ILP}_{\text{BPP}})$ computed in this chapter:

Bound	Value	From
First relaxation	$\frac{8}{5}$	$z(\text{LP-0BPP})$
Second relaxation	$\frac{8}{5}$	$z(\text{LP}_{\text{BPP}})$
Lower bound	2	$\lceil \frac{8}{5} \rceil$
Optimal value	2	patterns $\{1, 3\}$ and $\{2\}$

Here the rounded bound is already equal to the optimal value.

Formulations of coloring problems

Prerequisites

- From the module *Linear Programming*, [Chapter 9 «Duality theory»](#): the dual of an LP problem, weak and strong duality.
- From the module *Integer linear programming*, [Chapter 1 «Integer linear programming»](#) and [Chapter 2 «Primal and dual bounds»](#): ILP problems, LP relaxations and bounds.
- From this module, [Chapter 4 «Coloring problems»](#): the problem, its assumptions and the notation on graphs.

Objectives

At the end of this chapter you will be able to:

- write the ILP model of the Vertex Coloring Problem and explain the role of the activation variables y_c ;
- write the two LP relaxations and their duals, and compute their optimal solutions;
- prove that the optimal value of the LP relaxations is 2 for every graph with an edge;
- explain why this bound is very weak, and why the model is not enough to color a graph.

11.1 Overview of the family

- The ILP models of the coloring problems use an upper bound k on the number of colors and the set $\mathcal{C} = \{1, 2, \dots, k\}$ of the colors that may be used.
- The binary **assignment variables** x_{vc} give a color $c \in \mathcal{C}$ to every vertex $v \in V$, and the binary **activation variables** y_c say whether color c is used.
- Each edge and each color give one constraint, which forbids the same color on the two endpoints of the edge, and the objective function minimizes the number $\sum_{c \in \mathcal{C}} y_c$ of colors used.

The coloring models are **assignment models**, as the packing models: a vertex is assigned to a color as an item is assigned to a bin, and the edge constraints replace the capacity constraints.

- The table lists the members of the family, with the name of their ILP model and the section that writes it.

Problem	Model	Variables	Section
VCP	ILP _{VCP}	binary x_{vc} and y_c	11.2

11.2 An ILP model for the Vertex Coloring Problem

The Vertex Coloring Problem and its assumptions are defined in [Chapter 4 «Coloring problems», Section 4.2](#).

- Given a graph $G = (V, E)$, for a vertex $v \in V$ we denote by $\delta(v)$ the set of the edges incident to v .
- Given an upper bound $k \in \mathbb{Z}_{\geq 2}$ on the number of colors used in any optimal VCP solution, we introduce the following $k \underbrace{|V|}_{=n}$ binary variables:

$$x_{vc} = \begin{cases} 1 & \text{if vertex } v \text{ takes color } c, \\ 0 & \text{otherwise} \end{cases} \quad \forall v \in \underbrace{\{1, 2, \dots, n\}}_{=V}, \forall c \in \underbrace{\{1, 2, \dots, k\}}_{=\mathcal{C}}, \quad (11.1)$$

and the following k binary variables:

$$y_c = \begin{cases} 1 & \text{if color } c \text{ is used,} \\ 0 & \text{otherwise} \end{cases} \quad \forall c \in \underbrace{\{1, 2, \dots, k\}}_{=\mathcal{C}}. \quad (11.2)$$

The number n of vertices is a valid upper bound on the minimum number of colors needed to color all the vertices since, in the worst possible case, each vertex takes a different color. Also the number of colors used in any feasible VCP solution is a valid upper bound.

Using these variables, an Integer Linear Programming (ILP) model for the Vertex Coloring Problem (VCP) reads as follows:

$$\min \sum_{c \in \mathcal{C}} y_c \quad (11.3a)$$

$$\text{subject to } \sum_{c \in \mathcal{C}} x_{vc} = 1, \quad \forall v \in V, \quad (11.3b)$$

$$-x_{uc} - x_{vc} + y_c \geq 0, \quad \forall c \in \mathcal{C}, \forall \{u, v\} \in E, \quad (11.3c)$$

$$x_{vc} \in \{0, 1\}, \quad \forall v \in V, \forall c \in \mathcal{C}, \quad (11.3d)$$

$$y_c \in \{0, 1\}, \quad \forall c \in \mathcal{C}. \quad (11.3e)$$

This ILP model is referred to as the ILP_{VCP} model.

- Description of the objective function and constraints of the ILP_{VCP} model:
 - the linear objective function (11.3a) minimizes the total number of used colors;
 - constraints (11.3b) ensure that each vertex takes exactly one color (n linear constraints);
 - constraints (11.3c) ensure that, for each color and each edge, the two adjacent vertices cannot both take that color, and that an unused color is taken by no vertex ($k \underbrace{|E|}_{=m}$ linear constraints);
 - finally, constraints (11.3d) and (11.3e) define the variables of the ILP model.

Explanation of the link between the variables

- **If color c is assigned to a vertex, then it is used.** The logical implication is

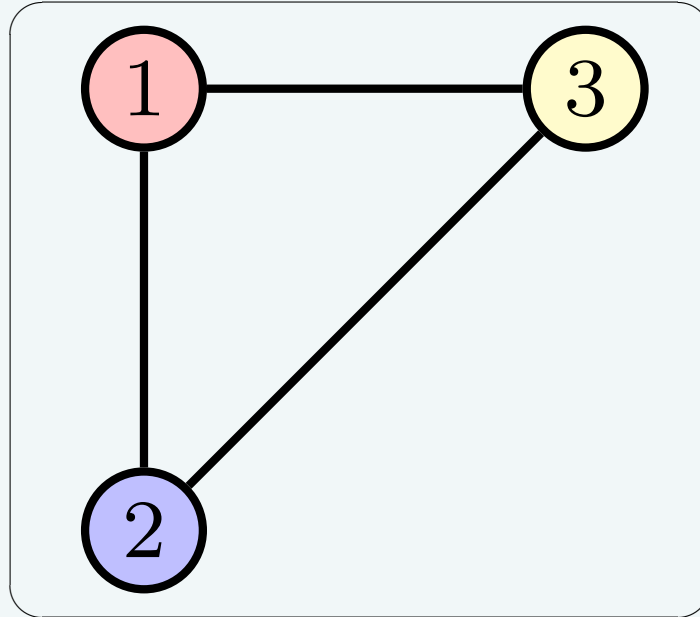
$$(x_{1c} \text{ OR } x_{2c} \text{ OR } \dots \text{ OR } x_{nc}) \implies y_c.$$

Every vertex has an incident edge $\{u, v\}$, and the constraint $x_{uc} + x_{vc} \leq y_c$ forces $y_c = 1$ as soon as $x_{vc} = 1$.

- **If color c is not used, then no vertex takes it.** This is the contrapositive of the previous implication. For $y_c = 0$ the edge constraint gives $x_{uc} = x_{vc} = 0$ for every edge $\{u, v\} \in E$.
- **If color c is used, then some vertex takes it, but only in an optimal solution.** The constraints do not impose it. However, if $x_{vc} = 0$ for all v and $y_c = 1$, setting $y_c = 0$ keeps all the constraints satisfied and decreases the objective value by one. Hence, in every optimal solution, unused colors are not counted.

Example 11.1: ILP_{VCP} model

We consider the following graph $G = (V, E)$ with 3 vertices ($n = 3$) and 3 edges ($m = 3$):



The vertex set V and the edge set E are:

$$V = \{ 1, 2, 3 \}, \quad E = \{ \{1,2\}, \{1,3\}, \{2,3\} \}.$$

We refer to this VCP instance as [Example 11.1](#).

The ILP_{VCP} model with $k = 3$ for this instance is:

$$\begin{array}{llllll}
 \min & & & & & y_1 & + & y_2 & + & y_3 \\
 \text{subject to} & x_{11} & + & x_{12} & + & x_{13} & & & & = & 1 \\
 & & & & & & x_{21} & + & x_{22} & + & x_{23} & & & = & 1 \\
 & & & & & & & & & & & x_{31} & + & x_{32} & + & x_{33} & & = & 1 \\
 & -x_{11} & & & & & -x_{21} & & & & & & + & y_1 & & & \geq & 0 \\
 & -x_{11} & & & & & & & & & -x_{31} & & + & y_1 & & & \geq & 0 \\
 & & & & & & -x_{21} & & & & -x_{31} & & + & y_1 & & & \geq & 0 \\
 & & & -x_{12} & & & & -x_{22} & & & & & & + & y_2 & & \geq & 0 \\
 & & & -x_{12} & & & & & & & & -x_{32} & & + & y_2 & & \geq & 0 \\
 & & & & & & & -x_{22} & & & & -x_{32} & & + & y_2 & & \geq & 0 \\
 & & & & & -x_{13} & & & & -x_{23} & & & & & + & y_3 & \geq & 0 \\
 & & & & & -x_{13} & & & & & & & -x_{33} & & + & y_3 & \geq & 0 \\
 & & & & & & & & & -x_{23} & & & -x_{33} & & + & y_3 & \geq & 0 \\
 & x_{11}, & x_{12}, & x_{13}, & x_{21}, & x_{22}, & x_{23}, & x_{31}, & x_{32}, & x_{33}, & y_1, & y_2, & y_3 & \in & \{0,1\}
 \end{array}$$

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{VCP}})$ is:

$$\tilde{x}_{11} = 1, \quad \tilde{x}_{12} = \tilde{x}_{13} = \tilde{x}_{21} = 0, \quad \tilde{x}_{22} = 1, \quad \tilde{x}_{23} = \tilde{x}_{31} = \tilde{x}_{32} = 0, \quad \tilde{x}_{33} = 1, \quad \tilde{y}_1 = \tilde{y}_2 = \tilde{y}_3 = 1$$

The optimal value $z(\text{ILP}_{\text{VCP}})$ is 3.

The associated optimal solution of the VCP is given by the color classes $\tilde{Q}_1 = \{1\}$, $\tilde{Q}_2 = \{2\}$ and $\tilde{Q}_3 = \{3\}$. The optimal partition is unique, but the $3!$ permutations of the colors give 6 different optimal solutions.

11.3 A first Linear Programming relaxation

To obtain a Linear Programming (LP) relaxation of the ILP_{VCP} model, we relax constraints (11.3d) and (11.3e) with the following ones:

$$x_{vc} \geq 0, \quad \forall v \in V, \forall c \in \mathcal{C}, \quad (11.4a)$$

$$y_c \geq 0, \quad \forall c \in \mathcal{C}. \quad (11.4b)$$

This LP model is referred to as the LP-0_{VCP} model.

Example 11.2: LP-0_{VCP} model

The LP-0_{VCP} model for the VCP instance of [Example 11.1](#) is:

$$\begin{array}{llllllllllll}
 \min & & & & & & & & & & y_1 & + & y_2 & + & y_3 \\
 \text{subject to} & x_{11} & + & x_{12} & + & x_{13} & & & & & & & & & = & 1 \\
 & & & & & & x_{21} & + & x_{22} & + & x_{23} & & & & & = & 1 \\
 & & & & & & & & & & x_{31} & + & x_{32} & + & x_{33} & & = & 1 \\
 & -x_{11} & & & & & -x_{21} & & & & & + & y_1 & & & \geq & 0 \\
 & -x_{11} & & & & & & & & & -x_{31} & & + & y_1 & & \geq & 0 \\
 & & & & & & -x_{21} & & & & -x_{31} & & + & y_1 & & \geq & 0 \\
 & & & -x_{12} & & & & -x_{22} & & & & & + & y_2 & & \geq & 0 \\
 & & & -x_{12} & & & & & & & -x_{32} & & + & y_2 & & \geq & 0 \\
 & & & & & & -x_{22} & & & & -x_{32} & & + & y_2 & & \geq & 0 \\
 & & & & & -x_{13} & & & -x_{23} & & & & + & y_3 & \geq & 0 \\
 & & & & & -x_{13} & & & & & -x_{33} & & + & y_3 & \geq & 0 \\
 & & & & & & & -x_{23} & & & -x_{33} & & + & y_3 & \geq & 0 \\
 & x_{11}, & x_{12}, & x_{13}, & x_{21}, & x_{22}, & x_{23}, & x_{31}, & x_{32}, & x_{33}, & y_1, & y_2, & y_3 & \geq & 0
 \end{array}$$

An optimal solution $\tilde{x}(\text{LP-0}_{\text{VCP}})$ is:

$$\tilde{x}_{11} = \tilde{x}_{12} = \tilde{x}_{21} = \tilde{x}_{22} = \tilde{x}_{31} = \tilde{x}_{32} = \frac{1}{2}, \quad \tilde{x}_{13} = \tilde{x}_{23} = \tilde{x}_{33} = 0, \quad \tilde{y}_1 = \tilde{y}_2 = 1, \quad \tilde{y}_3 = 0$$

The optimal value $z(\text{LP-0}_{\text{VCP}})$ is 2.

Accordingly we have the lower bound:

$$2 \leq z(\text{ILP}_{\text{VCP}})$$

Using

- a free dual variable μ_v for each constraint (11.3b) ($v \in V$)
- a non-negative dual variable π_{ce} for each constraint (11.3c) ($c \in \mathcal{C}$, $e = \{u, v\} \in E$)

the dual model of the LP-0_{VCP} model reads as follows:

$$\max \sum_{v \in V} \mu_v \quad (11.5a)$$

$$\text{subject to } \mu_v - \sum_{e \in \delta(v)} \pi_{ce} \leq 0, \quad \forall v \in V, \forall c \in \mathcal{C}, \quad (11.5b)$$

$$\sum_{e \in E} \pi_{ce} \leq 1, \quad \forall c \in \mathcal{C}, \quad (11.5c)$$

$$\mu_v \geq 0, \quad \forall v \in V, \quad (11.5d)$$

$$\pi_{ce} \geq 0, \quad \forall c \in \mathcal{C}, \forall e \in E. \quad (11.5e)$$

The obtained LP model is referred to as the $D(\text{LP-0}_{\text{VCP}})$ model.

Example 11.3: $D(\text{LP-0}_{\text{VCP}})$ model

The dual $D(\text{LP-0}_{\text{VCP}})$ model for the VCP instance of [Example 11.1](#) is:

$$\begin{array}{llllllllll}
 \max & \mu_1 & + \mu_2 & + \mu_3 & & & & & & \\
 \text{subject to} & \mu_1 & & & -\pi_{112} & -\pi_{113} & & & & \leq 0 \\
 & \mu_1 & & & & & -\pi_{212} & -\pi_{213} & & \leq 0 \\
 & \mu_1 & & & & & & & -\pi_{312} & -\pi_{313} \leq 0 \\
 & & \mu_2 & & -\pi_{112} & & -\pi_{123} & & & \leq 0 \\
 & & \mu_2 & & & & -\pi_{212} & & -\pi_{223} & \leq 0 \\
 & & \mu_2 & & & & & & -\pi_{312} & -\pi_{323} \leq 0 \\
 & & & \mu_3 & & -\pi_{113} & -\pi_{123} & & & \leq 0 \\
 & & & \mu_3 & & & & -\pi_{213} & -\pi_{223} & \leq 0 \\
 & & & \mu_3 & & & & & & -\pi_{313} & -\pi_{323} \leq 0 \\
 & & & & \pi_{112} & +\pi_{113} & +\pi_{123} & & & \leq 1 \\
 & & & & & & & \pi_{212} & +\pi_{213} & +\pi_{223} \leq 1 \\
 & & & & & & & & & \pi_{312} & +\pi_{313} & +\pi_{323} \leq 1 \\
 & \mu_1, & \mu_2, & \mu_3 & & & & & & & \geq 0 \\
 & & & & \pi_{112}, & \pi_{113}, & \pi_{123}, & \pi_{212}, & \pi_{213}, & \pi_{223}, & \pi_{312}, & \pi_{313}, & \pi_{323} \geq 0
 \end{array}$$

The optimal value $z(D(\text{LP-0}_{\text{VCP}}))$ is 2. An optimal solution $(\tilde{\mu}, \tilde{\pi})$ of $D(\text{LP-0}_{\text{VCP}})$ is:

$$\tilde{\mu}_1 = \tilde{\mu}_2 = 1, \quad \tilde{\mu}_3 = 0, \quad \tilde{\pi}_{112} = \tilde{\pi}_{212} = \tilde{\pi}_{312} = 1, \quad \text{all the other } \tilde{\pi}_{ce} = 0$$

Proposition 11.1

An optimal solution of the primal LP-0_{VCP} model is:

$$\tilde{x}_{v1} = \tilde{x}_{v2} = \frac{1}{2}, \quad \forall v \in V \quad \text{and} \quad \tilde{x}_{vc} = 0, \quad \forall v \in V, \forall c \in \{3, 4, \dots, k\}, \quad (11.6a)$$

$$\tilde{y}_1 = \tilde{y}_2 = 1 \quad \text{and} \quad \tilde{y}_c = 0, \quad \forall c \in \{3, 4, \dots, k\}, \quad (11.6b)$$

and an optimal solution of the dual $D(\text{LP-0}_{\text{VCP}})$ model is:

$$\tilde{\mu}_u = \tilde{\mu}_v = 1 \quad \text{and} \quad \tilde{\mu}_w = 0, \quad \forall w \in V \setminus \{u, v\}, \quad (11.7a)$$

$$\tilde{\pi}_{c\{u,v\}} = 1, \quad \forall c \in \mathcal{C} \quad \text{and} \quad \tilde{\pi}_{ce} = 0, \quad \forall c \in \mathcal{C}, e \in E \setminus \{\{u, v\}\}, \quad (11.7b)$$

where $\{u, v\}$ is any edge in E .

Corollary 11.2

The optimal value of the primal LP-0_{VCP} and dual $D(\text{LP-0}_{\text{VCP}})$ models is:

$$z(\text{LP-0}_{\text{VCP}}) = 2 = z(D(\text{LP-0}_{\text{VCP}})) \quad (11.8)$$

- The proof proceeds in three steps:

1. We show that (11.6) is a feasible solution for the primal LP-0_{VCP} model;
2. We show that (11.7) is a feasible solution for the dual $D(\text{LP-0}_{\text{VCP}})$ model;
3. We show that both feasible solutions have the same objective function value.

Therefore, by LP Weak Duality, (11.6) and (11.7) are a pair of optimal primal and dual solutions.

Proof.

1. For every $v \in V$, we have:

$$\sum_{c \in \mathcal{C}} \tilde{x}_{vc} = \frac{1}{2} + \frac{1}{2} = 1$$

Thus constraints (11.3b) are satisfied.

For $c \in \{1, 2\}$ and every $\{u, v\} \in E$, we have:

$$\tilde{y}_c - \tilde{x}_{uc} - \tilde{x}_{vc} = 1 - \frac{1}{2} - \frac{1}{2} = 0$$

and for $c \in \{3, 4, \dots, k\}$ we have $\tilde{y}_c - \tilde{x}_{uc} - \tilde{x}_{vc} = 0 - 0 - 0 = 0$. Thus constraints (11.3c) are satisfied.

We have $\tilde{x}_{vc} \geq 0$ and $\tilde{y}_c \geq 0$, for every $v \in V$ and every $c \in \mathcal{C}$, thus constraints (11.4a) and (11.4b) are satisfied.

Therefore, (11.6) provides a feasible solution for the primal LP-0_{VCP} model.

2. For every $c \in \mathcal{C}$, we have:

$$\sum_{e \in E} \tilde{\pi}_{ce} = 1$$

Thus constraints (11.5c) are satisfied.

For every $c \in \mathcal{C}$ and $w \in \{u, v\}$, we have:

$$\tilde{\mu}_w - \sum_{e \in \delta(w)} \tilde{\pi}_{ce} = 1 - 1 = 0$$

For every $c \in \mathcal{C}$ and $w \in V \setminus \{u, v\}$, we have $\tilde{\mu}_w = 0$ and $\tilde{\pi}_{ce} = 0$ for every $e \in \delta(w)$ (since $\tilde{\pi}_{ce}$ is nonzero only for $e = \{u, v\}$, which is not incident to w), so

$$\tilde{\mu}_w - \sum_{e \in \delta(w)} \tilde{\pi}_{ce} = 0 - 0 = 0.$$

Thus the constraints (11.5b) are satisfied.

We have $\tilde{\mu}_v \geq 0$, $\forall v \in V$ and $\tilde{\pi}_{ce} \geq 0$ for every $c \in \mathcal{C}$ and every $e \in E$. Thus constraints (11.5d) and (11.5e) are satisfied.

Therefore, (11.7) provides a feasible solution for the dual $D(\text{LP-0}_{\text{VCP}})$ model.

3. For the objective functions we have:

$$\sum_{c \in \mathcal{C}} \tilde{y}_c = 1 + 1 = \sum_{v \in V} \tilde{\mu}_v$$

thus the primal and dual objective function values are equal.

Therefore, by LP Weak Duality, (11.6) and (11.7) are a pair of optimal primal and dual solutions. □

- This is an alternative proof that does not rely on the dual formulation but instead reasons directly on the minimum objective function value imposed by the constraints.

Proposition 11.3

An optimal solution of the primal LP-0_{VCP} model is:

$$\tilde{x}_{v1} = \tilde{x}_{v2} = \frac{1}{2}, \quad \forall v \in V \quad \text{and} \quad \tilde{x}_{vc} = 0, \quad \forall v \in V, \forall c \in \{3, 4, \dots, k\}, \quad (11.9a)$$

$$\tilde{y}_1 = \tilde{y}_2 = 1 \quad \text{and} \quad \tilde{y}_c = 0, \quad \forall c \in \{3, 4, \dots, k\}, \quad (11.9b)$$

and the optimal value is

$$z(\text{LP-0}_{\text{VCP}}) = 2. \quad (11.10)$$

Proof. Summing the edge constraints (11.3c) over all $c \in \mathcal{C}$, for any edge $\{u, v\} \in E$, gives

$$\sum_{c \in \mathcal{C}} y_c - \sum_{c \in \mathcal{C}} x_{uc} - \sum_{c \in \mathcal{C}} x_{vc} \geq 0 \implies \sum_{c \in \mathcal{C}} y_c \geq \underbrace{\sum_{c \in \mathcal{C}} x_{uc}}_{\text{by (11.3b)=1}} + \underbrace{\sum_{c \in \mathcal{C}} x_{vc}}_{\text{by (11.3b)=1}} = 2,$$

hence

$$\sum_{c \in \mathcal{C}} y_c \geq 2. \quad (11.11)$$

Therefore every feasible solution of LP-0_{VCP} satisfies (11.11).

Now consider the point defined in (11.9). For every $v \in V$, we have

$$\sum_{c \in \mathcal{C}} \tilde{x}_{vc} = \frac{1}{2} + \frac{1}{2} = 1,$$

so constraints (11.3b) hold. Moreover, for every $\{u, v\} \in E$ and $c \in \{1, 2\}$,

$$\tilde{y}_c - \tilde{x}_{uc} - \tilde{x}_{vc} = 1 - \frac{1}{2} - \frac{1}{2} = 0,$$

so constraints (11.3c) hold with equality, while for all $c \geq 3$ both sides are zero. Finally, constraints (11.4a) and (11.4b) are satisfied by construction. Hence the point defined in (11.9) is a feasible solution for LP-0_{VCP}.

The objective value of (11.9) is

$$\sum_{c \in \mathcal{C}} \tilde{y}_c = 1 + 1 = 2,$$

which matches the lower bound (11.11). Therefore the point (11.9) is an optimal solution with objective value (11.10). □

11.4 A second Linear Programming relaxation

To obtain a Linear Programming (LP) relaxation of the ILP_{VCP} model, we relax constraints (11.3d) and (11.3e) with the following ones:

$$x_{vc} \geq 0, \quad \forall v \in V, \forall c \in \mathcal{C}, \quad (11.12a)$$

$$y_c \leq 1, \quad \forall c \in \mathcal{C}, \quad (11.12b)$$

$$y_c \geq 0, \quad \forall c \in \mathcal{C}. \quad (11.12c)$$

This LP model is referred to as the LP_{VCP} model.

Example 11.4: LP_{VCP} model

The LP_{VCP} model for the VCP instance of [Example 11.1](#) is:

$$\begin{array}{llllllllllll}
 \min & & & & & & & & & y_1 & + & y_2 & + & y_3 \\
 \text{subject to} & x_{11} & + & x_{12} & + & x_{13} & & & & & & & & = & 1 \\
 & & & & & & x_{21} & + & x_{22} & + & x_{23} & & & & = & 1 \\
 & & & & & & & & & & x_{31} & + & x_{32} & + & x_{33} & = & 1 \\
 & -x_{11} & & & & & -x_{21} & & & & & + & y_1 & & & \geq & 0 \\
 & -x_{11} & & & & & & & & & -x_{31} & & + & y_1 & & \geq & 0 \\
 & & & & & & -x_{21} & & & & -x_{31} & & + & y_1 & & \geq & 0 \\
 & & & -x_{12} & & & & -x_{22} & & & & & + & y_2 & & \geq & 0 \\
 & & & -x_{12} & & & & & & & -x_{32} & & + & y_2 & & \geq & 0 \\
 & & & & & & & -x_{22} & & & -x_{32} & & + & y_2 & & \geq & 0 \\
 & & & & & -x_{13} & & & -x_{23} & & & & & + & y_3 & \geq & 0 \\
 & & & & & -x_{13} & & & & & -x_{33} & & + & y_3 & \geq & 0 \\
 & & & & & & & & -x_{23} & & -x_{33} & & + & y_3 & \geq & 0 \\
 & & & & & & & & & & & y_1 & & & \leq & 1 \\
 & & & & & & & & & & & & y_2 & & \leq & 1 \\
 & & & & & & & & & & & & & y_3 & \leq & 1 \\
 & x_{11}, & x_{12}, & x_{13}, & x_{21}, & x_{22}, & x_{23}, & x_{31}, & x_{32}, & x_{33}, & y_1, & y_2, & y_3 & \geq & 0
 \end{array}$$

An optimal solution $\tilde{\mathbf{x}}(LP_{VCP})$ is:

$$\tilde{x}_{11} = \tilde{x}_{12} = \tilde{x}_{21} = \tilde{x}_{22} = \tilde{x}_{31} = \tilde{x}_{32} = \frac{1}{2}, \quad \tilde{x}_{13} = \tilde{x}_{23} = \tilde{x}_{33} = 0, \quad \tilde{y}_1 = \tilde{y}_2 = 1, \quad \tilde{y}_3 = 0$$

The optimal value $z(LP_{VCP})$ is 2. Accordingly we have the same lower bound.

Using

- a free dual variable μ_v for each constraint (11.3b) ($v \in V$)
- a non-negative dual variable π_{ce} for each constraint (11.3c) ($c \in \mathcal{C}$, $e = \{u, v\} \in E$)
- a non-positive dual variable λ_c for each constraint (11.12b) ($\forall c \in \mathcal{C}$)

the dual model of the LP_{VCP} model reads as follows:

$$\max \sum_{v \in V} \mu_v + \sum_{c \in \mathcal{C}} \lambda_c \quad (11.13a)$$

$$\text{subject to } \mu_v - \sum_{e \in \delta(v)} \pi_{ce} \leq 0, \quad \forall v \in V, \forall c \in \mathcal{C}, \quad (11.13b)$$

$$\sum_{e \in E} \pi_{ce} + \lambda_c \leq 1, \quad \forall c \in \mathcal{C}, \quad (11.13c)$$

$$\mu_v \geq 0, \quad \forall v \in V, \quad (11.13d)$$

$$\pi_{ce} \geq 0, \quad \forall c \in \mathcal{C}, \forall e \in E, \quad (11.13e)$$

$$\lambda_c \leq 0, \quad \forall c \in \mathcal{C}. \quad (11.13f)$$

The obtained LP model is referred to as the $D(LP_{VCP})$ model.

Example 11.5: $D(LP_{VCP})$ model

The dual $D(LP_{VCP})$ model for the VCP instance of [Example 11.1](#) is:

$$\begin{array}{llllllllll} \max & \mu_1 + \mu_2 + \mu_3 & & & & & & & & +\lambda_1 + \lambda_2 + \lambda_3 \\ \text{subject to} & \mu_1 & & -\pi_{112} & -\pi_{113} & & & & & \leq 0 \\ & \mu_1 & & & & -\pi_{212} & -\pi_{213} & & & \leq 0 \\ & \mu_1 & & & & & & -\pi_{312} & -\pi_{313} & \leq 0 \\ & \mu_2 & & -\pi_{112} & & -\pi_{123} & & & & \leq 0 \\ & \mu_2 & & & & -\pi_{212} & & -\pi_{223} & & \leq 0 \\ & \mu_2 & & & & & & -\pi_{312} & & -\pi_{323} \leq 0 \\ & \mu_3 & & -\pi_{113} & -\pi_{123} & & & & & \leq 0 \\ & \mu_3 & & & & & -\pi_{213} & -\pi_{223} & & \leq 0 \\ & \mu_3 & & & & & & & -\pi_{313} & -\pi_{323} \leq 0 \\ & & \pi_{112} & +\pi_{113} & +\pi_{123} & & & & & +\lambda_1 \leq 1 \\ & & & & & \pi_{212} & +\pi_{213} & +\pi_{223} & & +\lambda_2 \leq 1 \\ & & & & & & & \pi_{312} & +\pi_{313} & +\pi_{323} +\lambda_3 \leq 1 \\ & \mu_1, \mu_2, \mu_3 & & & & & & & & \geq 0 \\ & & \pi_{112}, \pi_{113}, \pi_{123}, \pi_{212}, \pi_{213}, \pi_{223}, \pi_{312}, \pi_{313}, \pi_{323} & & & & & & & \geq 0 \\ & & & & & & & & & \lambda_1, \lambda_2, \lambda_3 \leq 0 \end{array}$$

The optimal value $z(D(\text{LP}_{\text{VCP}}))$ is 2. An optimal solution $(\tilde{\mu}, \tilde{\pi}, \tilde{\lambda})$ of $D(\text{LP}_{\text{VCP}})$ is:

$$\begin{aligned} \tilde{\mu}_1 = \tilde{\mu}_2 = 1, \quad \tilde{\mu}_3 = 0, \quad \tilde{\pi}_{112} = \tilde{\pi}_{212} = \tilde{\pi}_{312} = 1, \quad \text{all the other } \tilde{\pi}_{ce} = 0 \\ \tilde{\lambda}_1 = 0, \quad \tilde{\lambda}_2 = 0, \quad \tilde{\lambda}_3 = 0 \end{aligned}$$

- The constraints:

$$x_{vc} \leq 1, \quad \forall v \in V, \forall c \in \mathcal{C},$$

are also valid since, for every $v \in V$ and every $c \in \mathcal{C}$, we have:

$$x_{vc} \in \{0, 1\} \quad \Rightarrow \quad x_{vc} \leq 1$$

but they are implied by constraints (11.3b).

Proposition 11.4

An optimal solution of the primal LP_{VCP} model is:

$$\tilde{x}_{v1} = \tilde{x}_{v2} = \frac{1}{2}, \quad \forall v \in V \quad \text{and} \quad \tilde{x}_{vc} = 0, \quad \forall v \in V, \forall c \in \{3, 4, \dots, k\}, \quad (11.14a)$$

$$\tilde{y}_1 = \tilde{y}_2 = 1 \quad \text{and} \quad \tilde{y}_c = 0, \quad \forall c \in \{3, 4, \dots, k\}, \quad (11.14b)$$

and an optimal solution of the dual $D(\text{LP}_{\text{VCP}})$ model is:

$$\tilde{\mu}_u = \tilde{\mu}_v = 1 \quad \text{and} \quad \tilde{\mu}_w = 0, \quad \forall w \in V \setminus \{u, v\}, \quad (11.15a)$$

$$\tilde{\pi}_{c\{u,v\}} = 1, \quad \forall c \in \mathcal{C}, \quad \tilde{\pi}_{ce} = 0, \quad \forall c \in \mathcal{C}, e \in E \setminus \{\{u, v\}\}, \quad \text{and} \quad \tilde{\lambda}_c = 0, \quad \forall c \in \mathcal{C}, \quad (11.15b)$$

where $\{u, v\}$ is any edge in E .

Proof. The proof is similar to the previous one. In addition it is necessary to observe that:

$$\tilde{y}_c \leq 1, \quad \forall c \in \mathcal{C}.$$

Accordingly, also constraints (11.12b) are satisfied. $\square$

Corollary 11.5

The optimal value of the primal LP_{VCP} and dual $D(\text{LP}_{\text{VCP}})$ models is:

$$z(\text{LP}_{\text{VCP}}) = 2 = z(D(\text{LP}_{\text{VCP}})) \quad (11.16)$$

11.5 Typical mistakes

Warning

- **Choosing k too small.** The number k of colors must be at least the chromatic number of the graph: with fewer colors the model has no feasible solution. The value $k = n$ is always valid.
- **Trusting the LP bound.** The optimal value of both relaxations is 2 for every graph with at least one edge, even for the complete graph on n vertices, whose optimal value is n .
- **Believing that the model counts only the colors that are really used.** The constraints allow $y_c = 1$ for a color that no vertex takes: it is the objective function that switches such colors off.
- **Ignoring the symmetry.** Permuting the colors of an optimal solution gives another optimal solution: a coloring that uses q of the k colors has $k!/(k - q)!$ equivalent copies in the ILP_{VCP} model ($k!$ when it uses all the colors).
- **Writing the dual solution with a wrong edge.** The dual values $\tilde{\pi}_{ce}$ are non-zero only for one edge $e = \{u, v\}$, and for every color c : all the other edges have value zero.

11.6 In summary

Summary

- The ILP_{VCP} model has $k \cdot n$ variables x_{vc} (vertex v takes color c) and k variables y_c (color c is used); the edge constraints forbid equal colors on adjacent vertices.
- The two LP relaxations have the same optimal value 2 for every graph with an edge: put $\frac{1}{2}$ on two colors for all the vertices.
- The dual solution selects one edge $\{u, v\}$ and gives value 1 to μ_u , μ_v and $\pi_{c\{u,v\}}$ for every color.
- The bound 2 is very weak: a formulation that must color graphs needs additional valid inequalities.

Dashboard of Example 11.1 (triangle). Bounds on $z(\text{ILP}_{\text{VCP}})$ computed in this chapter:

Bound	Value	From
First relaxation	2	$z(\text{LP-0}_{\text{VCP}})$
Second relaxation	2	$z(\text{LP}_{\text{VCP}})$
Optimal value	3	color classes $\{1\}$, $\{2\}$, $\{3\}$

The gap between 2 and 3 is small here, but it grows with the size of the complete graph: 2 against n .

Formulations of clique problems

Prerequisites

- From this module, [Chapter 5 «Clique problems»](#): the problem, its assumptions, cliques, stable sets, the complement graph and $\omega(G) = \alpha(\overline{G})$.
- From this module, [Chapter 4 «Coloring problems»](#) and [Chapter 11 «Formulations of coloring problems»](#): the chromatic number $\chi(G)$ and an ILP model with its LP relaxations.
- From the module *Integer linear programming*, [Chapter 2 «Primal and dual bounds»](#), [Chapter 3 «Ideal formulations»](#) and [Chapter 4 «Valid inequalities»](#): LP relaxations, stronger formulations and valid inequalities.
- From the module *Linear Programming*, [Chapter 9 «Duality theory»](#): the dual of an LP problem, weak and strong duality.

Objectives

At the end of this chapter you will be able to:

- write the edge formulation of the Maximum Clique Problem on the non-edges of G , that is, on the edges of $\overline{G}$;
- explain why its LP relaxation is weak: the point $x_v = \frac{1}{2}$ is always feasible;
- write the clique formulation and compare the two LP bounds on a small graph;
- prove $\omega(G) \leq z(\text{LP}_{\text{MCP}}^{\text{CL}}) \leq \chi(G)$ through LP duality and a coloring formulation;
- write the edge and the clique formulations of the Maximum Stable Set Problem;
- write the ILP models of the vertex-weighted and of the edge-weighted maximum clique problems, linearizing the product $x_u x_v$ with the variables y_{uv} .

12.1 Overview of the family

- The ILP models of the clique problems use one **binary selection variable** x_v for each vertex $v \in V$, equal to 1 if the vertex is in the clique.
- Two vertices that are not adjacent cannot be both selected: the **conflict constraints** are written on the non-edges of G , that is, on the edges of the complement graph $\overline{G}$, or on its cliques.
- The objective function maximizes the number of selected vertices or their total weight; the edge-weighted problem adds one variable per edge.

The clique models select vertices with binary variables and forbid the selection of two non-adjacent vertices. The same models, written on the edges of G instead of $\overline{G}$, select stable sets.

- The table lists the members of the family, with the names of their ILP models and the sections that write them.

Problem	Models	Conflict constraints	Section
MCP	ILP _{MCP} (edge formulation)	on the non-edges of G	12.2
MCP	ILP _{MCP} ^{CL} (clique formulation)	on the maximal stable sets of G	12.5
MSSP	ILP _{MSSP} , ILP _{MSSP} ^{CL}	on the edges and the cliques of G	12.7
VWMCP	ILP _{MCP} ^{VW}	on the non-edges of G	12.8
EWMCP	ILP _{MCP} ^{EW}	on the non-edges of G	12.8

12.2 An ILP model for the Maximum Clique Problem: the edge formulation

The problem of this section is the Maximum Clique Problem (MCP), defined in [Chapter 5 «Clique problems», Section 5.2](#).

- Given a simple graph $G = (V, E)$, with $V = \{1, 2, \dots, n\}$ and m edges, we denote by

$$\overline{E} = \{ \{u, v\} : u, v \in V, u \neq v, \{u, v\} \notin E \}$$

the set of the **non-edges** of G , that is, the edges of the complement graph $\overline{G} = (V, \overline{E})$.

- We introduce the following n **binary variables**:

$$x_v = \begin{cases} 1 & \text{if vertex } v \text{ is selected in the clique} \\ 0 & \text{otherwise} \end{cases} \quad \forall v \in V. \quad (12.1)$$

Using these variables, an Integer Linear Programming (ILP) model for the Maximum Clique Problem (MCP) reads as follows:

$$\max \sum_{v \in V} x_v \quad (12.2a)$$

$$\text{subject to } x_u + x_v \leq 1, \quad \forall \{u, v\} \in \overline{E}, \quad (12.2b)$$

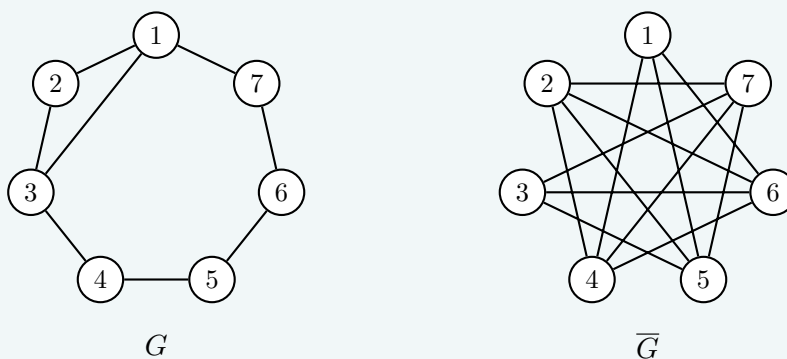
$$x_v \in \{0, 1\}, \quad \forall v \in V. \quad (12.2c)$$

This ILP model is referred to as the ILP_{MCP} model, or **edge formulation**.

- Description of the objective function and constraints of the ILP_{MCP} model:
 - the linear objective function (12.2a) maximizes the cardinality of the selected clique;
 - constraints (12.2b) ensure that no two non-adjacent vertices are both selected ($|\overline{E}| = \binom{n}{2} - m$ linear constraints, one for each edge of $\overline{G}$);
 - finally, constraints (12.2c) define the variables of the ILP model.
- For every non-edge $\{u, v\}$, the constraint $x_u + x_v \leq 1$ imposes $x_u = 1 \Rightarrow x_v = 0$. The converse is false: $x_u = x_v = 0$ is feasible.
- Read on $\overline{G}$, the same model is the **edge formulation of the MSSP**: at most one endpoint of every edge of $\overline{G}$ is selected.

Example 12.1: MCP instance

We consider the graph G with 7 vertices made of the cycle $1, 2, \dots, 7, 1$ and the chord $\{1, 3\}$ (the reference instance of Chapter 5 «Clique problems»), and its complement $\overline{G}$:



G has $m = 8$ edges, and $\overline{G}$ has the $\binom{7}{2} - 8 = 13$ non-edges of G :

$$\overline{E} = \{\{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{2, 7\}, \\ \{3, 5\}, \{3, 6\}, \{3, 7\}, \{4, 6\}, \{4, 7\}, \{5, 7\}\}.$$

We refer to this MCP instance as [Example 12.1](#).

Example 12.2: ILP_{MCP} model

The ILP_{MCP} model for the MCP instance of Example 12.1 has 7 binary variables and one constraint for each of the 13 edges of $\overline{G}$:

$$\begin{aligned}
 & \max \quad x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \\
 & \text{subject to} \quad x_1 + x_4 \leq 1 \\
 & \quad x_1 + x_5 \leq 1 \\
 & \quad x_1 + x_6 \leq 1 \\
 & \quad x_2 + x_4 \leq 1 \\
 & \quad x_2 + x_5 \leq 1 \\
 & \quad x_2 + x_6 \leq 1 \\
 & \quad x_2 + x_7 \leq 1 \\
 & \quad x_3 + x_5 \leq 1 \\
 & \quad x_3 + x_6 \leq 1 \\
 & \quad x_3 + x_7 \leq 1 \\
 & \quad x_4 + x_6 \leq 1 \\
 & \quad x_4 + x_7 \leq 1 \\
 & \quad x_5 + x_7 \leq 1 \\
 & \quad x_1, x_2, x_3, x_4, x_5, x_6, x_7 \in \{0, 1\}
 \end{aligned}$$

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{MCP}})$ is:

$$\tilde{x}_1 = \tilde{x}_2 = \tilde{x}_3 = 1, \quad \tilde{x}_4 = \tilde{x}_5 = \tilde{x}_6 = \tilde{x}_7 = 0$$

The optimal value $z(\text{ILP}_{\text{MCP}})$ is $3 = \omega(G)$. The associated optimal solution of the MCP is the clique $\tilde{K} = \{1, 2, 3\}$, and it is the unique optimal solution.

12.3 A first Linear Programming relaxation

To obtain a Linear Programming (LP) relaxation of the ILP_{MCP} model, we relax constraints (12.2c) into non-negativity constraints. The obtained LP model reads as follows:

$$\max \quad \sum_{v \in V} x_v \quad (12.3a)$$

$$\text{subject to} \quad x_u + x_v \leq 1, \quad \forall \{u, v\} \in \overline{E}, \quad (12.3b)$$

$$x_v \geq 0, \quad \forall v \in V. \quad (12.3c)$$

This LP model is referred to as the LP-0_{MCP} model.

- The upper bounds $x_v \leq 1$ are not implied in LP-0_{MCP}: if a vertex is adjacent to all the

others, it belongs to no constraint (12.3b), and LP-0_{MCP} is unbounded. In [Example 12.1](#) every vertex has a non-edge, and LP-0_{MCP} has an optimal solution.

Example 12.3: LP-0_{MCP} model

The LP-0_{MCP} model for the MCP instance of [Example 12.1](#) is:

$$\begin{aligned}
 & \max \quad x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \\
 & \text{subject to} \quad x_1 + x_4 \leq 1 \\
 & \quad x_1 + x_5 \leq 1 \\
 & \quad x_1 + x_6 \leq 1 \\
 & \quad x_2 + x_4 \leq 1 \\
 & \quad x_2 + x_5 \leq 1 \\
 & \quad x_2 + x_6 \leq 1 \\
 & \quad x_2 + x_7 \leq 1 \\
 & \quad x_3 + x_5 \leq 1 \\
 & \quad x_3 + x_6 \leq 1 \\
 & \quad x_3 + x_7 \leq 1 \\
 & \quad x_4 + x_6 \leq 1 \\
 & \quad x_4 + x_7 \leq 1 \\
 & \quad x_5 + x_7 \leq 1 \\
 & \quad x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0
 \end{aligned}$$

An optimal solution $\tilde{\mathbf{x}}(\text{LP-0}_{\text{MCP}})$ is:

$$\tilde{x}_1 = \tilde{x}_2 = \tilde{x}_3 = \tilde{x}_4 = \tilde{x}_5 = \tilde{x}_6 = \tilde{x}_7 = \frac{1}{2}$$

The optimal value $z(\text{LP-0}_{\text{MCP}})$ is $\frac{7}{2}$. Accordingly we have the upper bound:

$$z(\text{ILP}_{\text{MCP}}) \leq \frac{7}{2}$$

Using a non-negative dual variable π_{uv} for each constraint (12.3b), the dual model of the LP-0_{MCP} model is:

$$\min \sum_{\{u,v\} \in \bar{E}} \pi_{uv} \tag{12.4a}$$

$$\text{subject to} \quad \sum_{u: \{u,v\} \in \bar{E}} \pi_{uv} \geq 1, \quad \forall v \in V, \tag{12.4b}$$

$$\pi_{uv} \geq 0, \quad \forall \{u,v\} \in \bar{E}. \tag{12.4c}$$

This LP model is referred to as the $D(\text{LP-0}_{\text{MCP}})$ model.

- Constraints (12.4b) ask that every vertex is covered by a total weight of at least one on the edges of $\overline{G}$ incident to it: the $D(\text{LP-0}_{\text{MCP}})$ model is a **fractional edge cover** of the vertices with the edges of $\overline{G}$.

Example 12.4: $D(\text{LP-0}_{\text{MCP}})$ model

The dual $D(\text{LP-0}_{\text{MCP}})$ model for the MCP instance of Example 12.1 has one variable for each edge of $\overline{G}$ and one constraint for each vertex:

$$\begin{aligned}
 \min \quad & \pi_{14} + \pi_{15} + \pi_{16} + \pi_{24} + \pi_{25} + \pi_{26} + \pi_{27} \\
 & + \pi_{35} + \pi_{36} + \pi_{37} + \pi_{46} + \pi_{47} + \pi_{57} \\
 \text{subject to} \quad & \pi_{14} + \pi_{15} + \pi_{16} \geq 1 \\
 & \pi_{24} + \pi_{25} + \pi_{26} + \pi_{27} \geq 1 \\
 & \pi_{35} + \pi_{36} + \pi_{37} \geq 1 \\
 & \pi_{14} + \pi_{24} + \pi_{46} + \pi_{47} \geq 1 \\
 & \pi_{15} + \pi_{25} + \pi_{35} + \pi_{57} \geq 1 \\
 & \pi_{16} + \pi_{26} + \pi_{36} + \pi_{46} \geq 1 \\
 & \pi_{27} + \pi_{37} + \pi_{47} + \pi_{57} \geq 1 \\
 & \pi_{14}, \pi_{15}, \pi_{16}, \dots, \pi_{57} \geq 0
 \end{aligned}$$

An optimal solution $\tilde{\pi}(D(\text{LP-0}_{\text{MCP}}))$ gives value $\frac{1}{2}$ to the 7 edges of the cycle 1, 4, 7, 2, 5, 3, 6, 1 of $\overline{G}$:

$$\begin{aligned}
 \tilde{\pi}_{14} = \tilde{\pi}_{47} = \tilde{\pi}_{27} = \tilde{\pi}_{25} = \tilde{\pi}_{35} = \tilde{\pi}_{36} = \tilde{\pi}_{16} &= \frac{1}{2}, \\
 \tilde{\pi}_{15} = \tilde{\pi}_{24} = \tilde{\pi}_{26} = \tilde{\pi}_{37} = \tilde{\pi}_{46} = \tilde{\pi}_{57} &= 0
 \end{aligned}$$

Every vertex lies on two edges of the cycle and receives $\frac{1}{2} + \frac{1}{2} = 1$. The optimal value $z(D(\text{LP-0}_{\text{MCP}}))$ is $\frac{7}{2}$.

12.4 A second Linear Programming relaxation

To obtain an LP relaxation of the ILP_{MCP} model, we relax constraints (12.2c) into the bounds $0 \leq x_v \leq 1$. The obtained LP model reads as follows:

$$\max \sum_{v \in V} x_v \tag{12.5a}$$

$$\text{subject to } x_u + x_v \leq 1, \quad \forall \{u, v\} \in \overline{E}, \tag{12.5b}$$

$$x_v \leq 1, \quad \forall v \in V, \tag{12.5c}$$

$$x_v \geq 0, \quad \forall v \in V. \tag{12.5d}$$

This LP model is referred to as the LP_{MCP} model.

Example 12.5: LP_{MCP} model

The LP_{MCP} model for the MCP instance of [Example 12.1](#) is:

$$\begin{aligned}
 \max \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \\
 \text{subject to} \quad & x_1 + x_4 \leq 1 \\
 & x_1 + x_5 \leq 1 \\
 & x_1 + x_6 \leq 1 \\
 & x_2 + x_4 \leq 1 \\
 & x_2 + x_5 \leq 1 \\
 & x_2 + x_6 \leq 1 \\
 & x_2 + x_7 \leq 1 \\
 & x_3 + x_5 \leq 1 \\
 & x_3 + x_6 \leq 1 \\
 & x_3 + x_7 \leq 1 \\
 & x_4 + x_6 \leq 1 \\
 & x_4 + x_7 \leq 1 \\
 & x_5 + x_7 \leq 1 \\
 & x_1 \leq 1 \\
 & x_2 \leq 1 \\
 & x_3 \leq 1 \\
 & x_4 \leq 1 \\
 & x_5 \leq 1 \\
 & x_6 \leq 1 \\
 & x_7 \leq 1 \\
 & x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0
 \end{aligned}$$

An optimal solution $\tilde{\mathbf{x}}(\text{LP}_{\text{MCP}})$ is:

$$\tilde{x}_1 = \tilde{x}_2 = \tilde{x}_3 = \tilde{x}_4 = \tilde{x}_5 = \tilde{x}_6 = \tilde{x}_7 = \frac{1}{2}$$

The optimal value $z(\text{LP}_{\text{MCP}})$ is $\frac{7}{2}$. Accordingly we have the upper bound:

$$z(\text{ILP}_{\text{MCP}}) \leq \frac{7}{2}$$

Using a non-negative dual variable π_{uv} for each constraint (12.5b) and a non-negative dual variable λ_v for each constraint (12.5c), the dual model of the LP_{MCP} model is:

$$\min \sum_{\{u,v\} \in \bar{E}} \pi_{uv} + \sum_{v \in V} \lambda_v \quad (12.6a)$$

$$\text{subject to } \sum_{u: \{u,v\} \in \bar{E}} \pi_{uv} + \lambda_v \geq 1, \quad \forall v \in V, \quad (12.6b)$$

$$\pi_{uv} \geq 0, \quad \forall \{u, v\} \in \bar{E}, \quad (12.6c)$$

$$\lambda_v \geq 0, \quad \forall v \in V. \quad (12.6d)$$

This LP model is referred to as the $D(\text{LP}_{\text{MCP}})$ model.

Example 12.6: $D(\text{LP}_{\text{MCP}})$ model

The dual $D(\text{LP}_{\text{MCP}})$ model for the MCP instance of [Example 12.1](#) is:

$$\begin{aligned} \min \quad & \pi_{14} + \pi_{15} + \pi_{16} + \pi_{24} + \pi_{25} + \pi_{26} + \pi_{27} \\ & + \pi_{35} + \pi_{36} + \pi_{37} + \pi_{46} + \pi_{47} + \pi_{57} + \lambda_1 \\ & + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 \\ \text{subject to} \quad & \pi_{14} + \pi_{15} + \pi_{16} + \lambda_1 \geq 1 \\ & \pi_{24} + \pi_{25} + \pi_{26} + \pi_{27} + \lambda_2 \geq 1 \\ & \pi_{35} + \pi_{36} + \pi_{37} + \lambda_3 \geq 1 \\ & \pi_{14} + \pi_{24} + \pi_{46} + \pi_{47} + \lambda_4 \geq 1 \\ & \pi_{15} + \pi_{25} + \pi_{35} + \pi_{57} + \lambda_5 \geq 1 \\ & \pi_{16} + \pi_{26} + \pi_{36} + \pi_{46} + \lambda_6 \geq 1 \\ & \pi_{27} + \pi_{37} + \pi_{47} + \pi_{57} + \lambda_7 \geq 1 \\ & \pi_{14}, \pi_{15}, \pi_{16}, \dots, \pi_{57} \geq 0 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \geq 0 \end{aligned}$$

An optimal solution $(\tilde{\pi}, \tilde{\lambda})(D(\text{LP}_{\text{MCP}}))$ is the solution of [Example 12.4](#) with $\tilde{\lambda}_v = 0$ for every $v \in V$:

$$\begin{aligned} \tilde{\pi}_{14} = \tilde{\pi}_{47} = \tilde{\pi}_{27} = \tilde{\pi}_{25} = \tilde{\pi}_{35} = \tilde{\pi}_{36} = \tilde{\pi}_{16} &= \frac{1}{2}, \\ \text{all the other } \tilde{\pi}_{uv} &= 0, \quad \tilde{\lambda}_1 = \tilde{\lambda}_2 = \dots = \tilde{\lambda}_7 = 0 \end{aligned}$$

The optimal value $z(D(\text{LP}_{\text{MCP}}))$ is $\frac{7}{2}$.

Proposition 12.1: The LP bound can be arbitrarily weak

For every graph G with n vertices, $z(\text{LP}_{\text{MCP}}) \geq n/2$. Consequently, the ratio $z(\text{LP}_{\text{MCP}})/z(\text{ILP}_{\text{MCP}})$ is not bounded over all graphs.

Proof. Set $x_v = \frac{1}{2}$ for every vertex. Each constraint (12.5b) becomes $\frac{1}{2} + \frac{1}{2} = 1$, and the bounds (12.5c) and (12.5d) hold, so this is a feasible solution of LP_{MCP} of value $n/2$. Graphs without triangles and with an edge have $\omega(G) = 2$ for every n , and for them the

ratio is at least $n/4$. □

- The data are integer and the optimal value of the ILP is integer, so $\lfloor z(\text{LP}_{\text{MCP}}) \rfloor$ is also a dual bound. On [Example 12.1](#) it gives $\lfloor \frac{7}{2} \rfloor = 3 = \omega(G)$ by luck; on a graph without triangles it gives $\lfloor n/2 \rfloor$, against $\omega(G) = 2$.
- On the cycle with 5 vertices C_5 , where $\omega(C_5) = 2$, the point $x_v = \frac{1}{2}$ gives $z(\text{LP}_{\text{MCP}}) = \frac{5}{2}$.

12.5 Clique inequalities: a stronger formulation

Proposition 12.2: clique inequalities

Let $Q \subseteq V$ be a clique of $\overline{G}$, that is, a stable set of G . Then

$$\sum_{v \in Q} x_v \leq 1 \quad (12.7)$$

is a **valid inequality** for ILP_{MCP} .

Proof. A clique of G and a stable set Q of G share at most one vertex: two common vertices would be both adjacent and non-adjacent. So every feasible solution of ILP_{MCP} selects at most one vertex of Q . □

- Inequalities (12.7) are called **clique inequalities**, after the clique Q of $\overline{G}$; read on G , they are **stable set inequalities**. The non-edge constraints are the case $|Q| = 2$.
- If $Q \subseteq Q'$, the inequality of Q' implies the one of Q , since all the variables are non-negative. It is enough to write the inequalities of the **maximal** cliques of $\overline{G}$, whose family we denote by $\mathcal{Q}$.

Using the variables (12.1), a second ILP model for the Maximum Clique Problem (MCP) reads as follows:

$$\max \sum_{v \in V} x_v \quad (12.8a)$$

$$\text{subject to } \sum_{v \in Q} x_v \leq 1, \quad \forall Q \in \mathcal{Q}, \quad (12.8b)$$

$$x_v \in \{0, 1\}, \quad \forall v \in V. \quad (12.8c)$$

This ILP model is referred to as the $\text{ILP}_{\text{MCP}}^{\text{CL}}$ model, or **clique formulation**.

- Description of the objective function and constraints of the $\text{ILP}_{\text{MCP}}^{\text{CL}}$ model:
 - the linear objective function (12.8a) maximizes the cardinality of the selected clique;
 - constraints (12.8b) ensure that at most one vertex of every maximal clique of $\overline{G}$ is selected ($|\mathcal{Q}|$ linear constraints);

- finally, constraints (12.8c) define the variables of the ILP model.
- Every edge of $\overline{G}$ lies in a maximal clique, so each constraint (12.2b) is implied by a constraint (12.8b): the two models have the same feasible solutions.
- The family $\mathcal{Q}$ can be very large: Moon and Moser (1965) showed graphs with $3^{n/3}$ maximal cliques. In practice the inequalities are added only when they are violated, as in [Chapter 4 «Valid inequalities»](#) of the module *Integer linear programming*.

Example 12.7: $\text{ILP}_{\text{MCP}}^{\text{CL}}$ model

The maximal cliques of $\overline{G}$ in [Example 12.1](#) are the 5 triangles and the 2 edges

$$\mathcal{Q} = \{\{1, 4, 6\}, \{2, 4, 6\}, \{2, 4, 7\}, \{2, 5, 7\}, \{3, 5, 7\}, \{1, 5\}, \{3, 6\}\},$$

and they give 7 constraints instead of 13. The $\text{ILP}_{\text{MCP}}^{\text{CL}}$ model for this MCP instance is:

$$\begin{aligned} \max \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \\ \text{subject to} \quad & x_1 + x_4 + x_6 \leq 1 \\ & x_2 + x_4 + x_6 \leq 1 \\ & x_2 + x_4 + x_7 \leq 1 \\ & x_2 + x_5 + x_7 \leq 1 \\ & x_3 + x_5 + x_7 \leq 1 \\ & x_1 + x_5 \leq 1 \\ & x_3 + x_6 \leq 1 \\ & x_1, x_2, x_3, x_4, x_5, x_6, x_7 \in \{0, 1\} \end{aligned}$$

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{MCP}}^{\text{CL}})$ is:

$$\tilde{x}_1 = \tilde{x}_2 = \tilde{x}_3 = 1, \quad \tilde{x}_4 = \tilde{x}_5 = \tilde{x}_6 = \tilde{x}_7 = 0$$

The optimal value $z(\text{ILP}_{\text{MCP}}^{\text{CL}})$ is $3 = \omega(G)$.

To obtain an LP relaxation of the $\text{ILP}_{\text{MCP}}^{\text{CL}}$ model, we relax constraints (12.8c) into non-negativity constraints. The obtained LP model reads as follows:

$$\max \sum_{v \in V} x_v \tag{12.9a}$$

$$\text{subject to} \quad \sum_{v \in Q} x_v \leq 1, \quad \forall Q \in \mathcal{Q}, \tag{12.9b}$$

$$x_v \geq 0, \quad \forall v \in V. \tag{12.9c}$$

This LP model is referred to as the $\text{LP}_{\text{MCP}}^{\text{CL}}$ model.

- Every vertex lies in some maximal clique of $\overline{G}$, so constraints (12.9b) imply $x_v \leq 1$: the bounds (12.5c) are not needed.

- Each constraint (12.5b) is implied by a constraint (12.9b). Hence the feasible region of $\text{LP}_{\text{MCP}}^{\text{CL}}$ is contained in that of LP_{MCP} , and

$$\omega(G) \leq z(\text{LP}_{\text{MCP}}^{\text{CL}}) \leq z(\text{LP}_{\text{MCP}}).$$

Example 12.8: $\text{LP}_{\text{MCP}}^{\text{CL}}$ model

The $\text{LP}_{\text{MCP}}^{\text{CL}}$ model for the MCP instance of Example 12.1 is:

$$\begin{aligned} \max \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \\ \text{subject to} \quad & x_1 + x_4 + x_6 \leq 1 \\ & x_2 + x_4 + x_6 \leq 1 \\ & x_2 + x_4 + x_7 \leq 1 \\ & x_2 + x_5 + x_7 \leq 1 \\ & x_3 + x_5 + x_7 \leq 1 \\ & x_1 + x_5 \leq 1 \\ & x_3 + x_6 \leq 1 \\ & x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0 \end{aligned}$$

The point $\bar{x}_v = \frac{1}{2}$ is no longer feasible: $\bar{x}_1 + \bar{x}_4 + \bar{x}_6 = \frac{3}{2} > 1$. Summing the three constraints of $\{1, 4, 6\}$, $\{2, 5, 7\}$ and $\{3, 5, 7\}$ gives

$$\sum_{v=1}^7 x_v + x_5 + x_7 \leq 3 \implies \sum_{v=1}^7 x_v \leq 3.$$

An optimal solution $\tilde{x}(\text{LP}_{\text{MCP}}^{\text{CL}})$ is:

$$\tilde{x}_1 = \tilde{x}_2 = \tilde{x}_3 = 1, \quad \tilde{x}_4 = \tilde{x}_5 = \tilde{x}_6 = \tilde{x}_7 = 0$$

The optimal value $z(\text{LP}_{\text{MCP}}^{\text{CL}})$ is $3 = \omega(G)$: the LP bound drops from $\frac{7}{2}$ to 3 and closes the gap.

- Clique inequalities do not always close the gap. On C_5 the complement is again a cycle with 5 vertices, whose maximal cliques are its edges: the two formulations coincide and $z(\text{LP}_{\text{MCP}}^{\text{CL}}) = \frac{5}{2} > 2 = \omega(C_5)$.
- If C is a cycle of $\bar{G}$ with an odd number of vertices and no chord, then $\sum_{v \in C} x_v \leq (|C| - 1)/2$ is a valid inequality (an **odd cycle inequality**): a stable set of $\bar{G}$ takes at most $(|C| - 1)/2$ vertices of C . For C_5 it reads $x_1 + \dots + x_5 \leq 2$ and closes the gap.

12.6 The relation with vertex coloring

- In a coloring of G the vertices of a clique take pairwise different colors, hence (Chapter 4 «Coloring problems»)

$$\omega(G) \leq \chi(G).$$

Here we place the LP bound of the clique formulation between the two numbers.

- The maximal cliques of $\overline{G}$ are the **maximal stable sets** of G , and a color class is a stable set. We introduce the following $|\mathcal{Q}|$ **binary variables**:

$$y_Q = \begin{cases} 1 & \text{if the stable set } Q \text{ is a color class} \\ 0 & \text{otherwise} \end{cases} \quad \forall Q \in \mathcal{Q}. \quad (12.10)$$

Using these variables, an ILP model for the Vertex Coloring Problem (VCP) reads as follows:

$$\min \sum_{Q \in \mathcal{Q}} y_Q \quad (12.11a)$$

$$\text{subject to } \sum_{Q \in \mathcal{Q}: v \in Q} y_Q \geq 1, \quad \forall v \in V, \quad (12.11b)$$

$$y_Q \in \{0, 1\}, \quad \forall Q \in \mathcal{Q}. \quad (12.11c)$$

This ILP model is referred to as the $\text{ILP}_{\text{VCP}}^{\text{ST}}$ model, or **stable set formulation** of the VCP.

- Description of the objective function and constraints of the $\text{ILP}_{\text{VCP}}^{\text{ST}}$ model:
 - the linear objective function (12.11a) minimizes the number of color classes;
 - constraints (12.11b) ensure that every vertex belongs to at least one chosen stable set (n linear constraints);
 - finally, constraints (12.11c) define the variables of the ILP model.
- Its optimal value is $\chi(G)$: a vertex covered by two chosen sets is kept in one of them, and every subset of a stable set is stable, so the chosen sets give a coloring with the same number of colors.

Using a non-negative dual variable π_Q for each constraint (12.9b), the dual model of the $\text{LP}_{\text{MCP}}^{\text{CL}}$ model is:

$$\min \sum_{Q \in \mathcal{Q}} \pi_Q \quad (12.12a)$$

$$\text{subject to } \sum_{Q \in \mathcal{Q}: v \in Q} \pi_Q \geq 1, \quad \forall v \in V, \quad (12.12b)$$

$$\pi_Q \geq 0, \quad \forall Q \in \mathcal{Q}. \quad (12.12c)$$

This LP model is referred to as the $D(\text{LP}_{\text{MCP}}^{\text{CL}})$ model.

Proposition 12.3: clique bound from LP duality

The dual $D(\text{LP}_{\text{MCP}}^{\text{CL}})$ model is the LP relaxation of the $\text{ILP}_{\text{VCP}}^{\text{ST}}$ model, with the variables π_Q in place of y_Q . Hence

$$\omega(G) \leq z(\text{LP}_{\text{MCP}}^{\text{CL}}) \leq \chi(G).$$

Proof. Relaxing constraints (12.11c) into $y_Q \geq 0$ gives exactly the model (12.12), with y_Q in place of π_Q . By strong duality $z(\text{LP}_{\text{MCP}}^{\text{CL}}) = z(D(\text{LP}_{\text{MCP}}^{\text{CL}}))$. The first inequality holds because $\text{LP}_{\text{MCP}}^{\text{CL}}$ is a relaxation of the MCP. The second one holds because $D(\text{LP}_{\text{MCP}}^{\text{CL}})$ is a relaxation of the $\text{ILP}_{\text{VCP}}^{\text{ST}}$ model, a minimization problem with optimal value $\chi(G)$. $\square$

- The common value $z(\text{LP}_{\text{MCP}}^{\text{CL}})$ is called the **fractional chromatic number** of G . The clique formulation of the MCP and the stable set formulation of the VCP are two faces of the same LP.

Example 12.9: $D(\text{LP}_{\text{MCP}}^{\text{CL}})$ model

The dual $D(\text{LP}_{\text{MCP}}^{\text{CL}})$ model for the MCP instance of [Example 12.1](#) has one variable for each maximal stable set of G (π_{146} for $Q = \{1, 4, 6\}$, and so on):

$$\begin{aligned} \min \quad & \pi_{146} + \pi_{246} + \pi_{247} + \pi_{257} + \pi_{357} + \pi_{15} + \pi_{36} \\ \text{subject to} \quad & \pi_{146} + \pi_{15} \geq 1 \\ & \pi_{246} + \pi_{247} + \pi_{257} \geq 1 \\ & \pi_{357} + \pi_{36} \geq 1 \\ & \pi_{146} + \pi_{246} + \pi_{247} \geq 1 \\ & \pi_{257} + \pi_{357} + \pi_{15} \geq 1 \\ & \pi_{146} + \pi_{246} + \pi_{36} \geq 1 \\ & \pi_{247} + \pi_{257} + \pi_{357} \geq 1 \\ & \pi_{146}, \pi_{246}, \pi_{247}, \pi_{257}, \pi_{357}, \pi_{15}, \pi_{36} \geq 0 \end{aligned}$$

An optimal solution $\tilde{\pi}(D(\text{LP}_{\text{MCP}}^{\text{CL}}))$ gives value 1 to the multipliers used in [Example 12.8](#):

$$\tilde{\pi}_{146} = \tilde{\pi}_{257} = \tilde{\pi}_{357} = 1, \quad \tilde{\pi}_{246} = \tilde{\pi}_{247} = \tilde{\pi}_{15} = \tilde{\pi}_{36} = 0$$

The optimal value $z(D(\text{LP}_{\text{MCP}}^{\text{CL}}))$ is 3.

Example 12.10: the chain $\omega \leq z(\text{LP}_{\text{MCP}}^{\text{CL}}) \leq \chi$

- On [Example 12.1](#), the $\text{ILP}_{\text{VCP}}^{\text{ST}}$ model is:

$$\begin{aligned}
 \min \quad & y_{146} + y_{246} + y_{247} + y_{257} + y_{357} + y_{15} + y_{36} \\
 \text{subject to} \quad & y_{146} + y_{15} \geq 1 \\
 & y_{246} + y_{247} + y_{257} \geq 1 \\
 & y_{357} + y_{36} \geq 1 \\
 & y_{146} + y_{246} + y_{247} \geq 1 \\
 & y_{257} + y_{357} + y_{15} \geq 1 \\
 & y_{146} + y_{246} + y_{36} \geq 1 \\
 & y_{247} + y_{257} + y_{357} \geq 1 \\
 & y_{146}, y_{246}, y_{247}, y_{257}, y_{357}, y_{15}, y_{36} \in \{0, 1\}
 \end{aligned}$$

The integer point $\tilde{y}_{146} = \tilde{y}_{257} = \tilde{y}_{357} = 1$, all the other variables equal to 0, is the solution of [Example 12.9](#): it is a coloring with 3 colors, the classes $\{1, 4, 6\}$, $\{2, 5, 7\}$ and $\{3\}$. Hence $\chi(G) = 3$, and the chain reads $3 \leq 3 \leq 3$.

- On C_5 , with vertices $1, \dots, 5$, the maximal stable sets are the 5 non-edges. Setting $\pi_Q = \frac{1}{2}$ on each of them gives every vertex a total of 1, so the dual has value $\frac{5}{2}$, as the primal. The chain reads

$$\omega(C_5) = 2 < z(\text{LP}_{\text{MCP}}^{\text{CL}}) = \frac{5}{2} < \chi(C_5) = 3.$$

12.7 Stable sets on the complement graph

The problem of this section is the Maximum Stable Set Problem (MSSP), defined in [Chapter 5](#) «Clique problems», [Section 5.2](#).

- Given a simple graph $G = (V, E)$, with $V = \{1, 2, \dots, n\}$ and m edges, we introduce the following n **binary variables**:

$$x_v = \begin{cases} 1 & \text{if vertex } v \text{ is selected in the stable set} \\ 0 & \text{otherwise} \end{cases} \quad \forall v \in V. \quad (12.13)$$

Using these variables, an Integer Linear Programming (ILP) model for the Maximum Stable Set Problem (MSSP) reads as follows:

$$\max \sum_{v \in V} x_v \quad (12.14a)$$

$$\text{subject to } x_u + x_v \leq 1, \quad \forall \{u, v\} \in E, \quad (12.14b)$$

$$x_v \in \{0, 1\}, \quad \forall v \in V. \quad (12.14c)$$

This ILP model is referred to as the ILP_{MSSP} model, or **edge formulation** of the MSSP.

- Description of the objective function and constraints of the ILP_{MSSP} model:

- the linear objective function (12.14a) maximizes the cardinality of the selected stable set;
 - constraints (12.14b) ensure that no two adjacent vertices are both selected (m linear constraints, one for each edge of G);
 - finally, constraints (12.14c) define the variables of the ILP model.
- The ILP_{MSSP} model on G is the ILP_{MCP} model on $\overline{G}$: the non-edges of $\overline{G}$ are the edges of G . This is the identity $\alpha(G) = \omega(\overline{G})$ written as a model.

Example 12.11: ILP_{MSSP} model

We consider the graph G of Example 12.1, the cycle $1, 2, \dots, 7, 1$ with the chord $\{1, 3\}$, and we look for a maximum stable set of G . The ILP_{MSSP} model has 7 binary variables and one constraint for each of the 8 edges of G :

$$\begin{aligned}
 & \max \quad x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \\
 & \text{subject to} \quad x_1 + x_2 \leq 1 \\
 & \quad \quad \quad x_1 + x_3 \leq 1 \\
 & \quad \quad \quad x_1 + x_7 \leq 1 \\
 & \quad \quad \quad x_2 + x_3 \leq 1 \\
 & \quad \quad \quad x_3 + x_4 \leq 1 \\
 & \quad \quad \quad x_4 + x_5 \leq 1 \\
 & \quad \quad \quad x_5 + x_6 \leq 1 \\
 & \quad \quad \quad x_6 + x_7 \leq 1 \\
 & \quad \quad \quad x_1, x_2, x_3, x_4, x_5, x_6, x_7 \in \{0, 1\}
 \end{aligned}$$

An optimal solution $\tilde{x}(\text{ILP}_{\text{MSSP}})$ is:

$$\tilde{x}_1 = \tilde{x}_4 = \tilde{x}_6 = 1, \quad \tilde{x}_2 = \tilde{x}_3 = \tilde{x}_5 = \tilde{x}_7 = 0$$

The optimal value $z(\text{ILP}_{\text{MSSP}})$ is $3 = \alpha(G)$. The optimal solution is not unique: the maximum stable sets of G are $\{1, 4, 6\}$, $\{2, 4, 6\}$, $\{2, 4, 7\}$, $\{2, 5, 7\}$ and $\{3, 5, 7\}$, the five triangles of $\overline{G}$.

- A stable set and a clique of G share at most one vertex, so for every clique K of G the **clique inequality** $\sum_{v \in K} x_v \leq 1$ is valid, and it implies the constraints (12.14b) of the edges of K . As for the MCP, it is enough to write the inequalities of the family $\mathcal{K}$ of the **maximal** cliques of G .

Using the variables (12.13), a second ILP model for the Maximum Stable Set Problem (MSSP) reads as follows:

$$\max \sum_{v \in V} x_v \tag{12.15a}$$

$$\text{subject to } \sum_{v \in K} x_v \leq 1, \quad \forall K \in \mathcal{K}, \quad (12.15b)$$

$$x_v \in \{0, 1\}, \quad \forall v \in V. \quad (12.15c)$$

This ILP model is referred to as the $\text{ILP}_{\text{MSSP}}^{\text{CL}}$ model, or **clique formulation** of the MSSP.

- Description of the objective function and constraints of the $\text{ILP}_{\text{MSSP}}^{\text{CL}}$ model:
 - the linear objective function (12.15a) maximizes the cardinality of the selected stable set;
 - constraints (12.15b) ensure that at most one vertex of every maximal clique of G is selected ($|\mathcal{K}|$ linear constraints);
 - finally, constraints (12.15c) define the variables of the ILP model.
- The $\text{ILP}_{\text{MSSP}}^{\text{CL}}$ model on G is the $\text{ILP}_{\text{MCP}}^{\text{CL}}$ model on $\overline{G}$, and its LP relaxation is at least as strong as that of the ILP_{MSSP} model.

Example 12.12: $\text{ILP}_{\text{MSSP}}^{\text{CL}}$ model

The maximal cliques of the graph G of Example 12.1 are the triangle $\{1, 2, 3\}$ and the five edges $\{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 7\}, \{1, 7\}$. The $\text{ILP}_{\text{MSSP}}^{\text{CL}}$ model replaces the three constraints of the triangle with one constraint, and has 6 constraints:

$$\begin{aligned} \max \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \leq 1 \\ & x_3 + x_4 \leq 1 \\ & x_4 + x_5 \leq 1 \\ & x_5 + x_6 \leq 1 \\ & x_6 + x_7 \leq 1 \\ & x_1 + x_7 \leq 1 \\ & x_1, x_2, x_3, x_4, x_5, x_6, x_7 \in \{0, 1\} \end{aligned}$$

The optimal solution $\tilde{x}_1 = \tilde{x}_4 = \tilde{x}_6 = 1$ of Example 12.11 is optimal here too, with value $z(\text{ILP}_{\text{MSSP}}^{\text{CL}}) = 3$. The LP relaxations differ (Gurobi): the edge formulation, with $0 \leq x_v \leq 1$, has optimal value $\frac{7}{2}$ at $x_v = \frac{1}{2}$, while the clique formulation has optimal value 3, attained by $x_2 = x_5 = x_7 = 1$.

12.8 ILP models for the weighted variants

The problems of this section are the vertex-weighted (VWMCP) and the edge-weighted (EWMCP) maximum clique problems, defined in Chapter 5 «Clique problems», Section 5.5, with their reference instances.

12.8.1 The vertex-weighted maximum clique problem

- Given a simple graph $G = (V, E)$ and a vertex-weight vector $\mathbf{w} \in \mathbb{Z}_{\geq 1}^n$, we use the n **binary variables** (12.1) of the edge formulation:

$$x_v = \begin{cases} 1 & \text{if vertex } v \text{ is selected in the clique} \\ 0 & \text{otherwise} \end{cases} \quad \forall v \in V.$$

Using these variables, an Integer Linear Programming (ILP) model for the vertex-weighted maximum clique problem (VWMCP) reads as follows:

$$\max \sum_{v \in V} w_v x_v \quad (12.16a)$$

$$\text{subject to } x_u + x_v \leq 1, \quad \forall \{u, v\} \in \overline{E}, \quad (12.16b)$$

$$x_v \in \{0, 1\}, \quad \forall v \in V. \quad (12.16c)$$

This ILP model is referred to as the $\text{ILP}_{\text{MCP}}^{\text{VW}}$ model.

- Description of the objective function and constraints of the $\text{ILP}_{\text{MCP}}^{\text{VW}}$ model:
 - the linear objective function (12.16a) maximizes the total weight $w(K)$ of the selected clique;
 - constraints (12.16b) ensure that no two non-adjacent vertices are both selected ($|\overline{E}| = \binom{n}{2} - m$ linear constraints);
 - finally, constraints (12.16c) define the variables of the ILP model.
- Only the objective function changes: the feasible solutions are the cliques of G , as in the ILP_{MCP} model, and with $w_v = 1$ for every vertex the two models coincide. The clique inequalities of $\overline{G}$ are valid here too.

Example 12.13: $\text{ILP}_{\text{MCP}}^{\text{VW}}$ model

We consider the graph G with 8 vertices on a circle, where every vertex is adjacent to all the others except the opposite one: G has 24 edges, and $\overline{G}$ has the 4 edges $\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}$. The vertex weights are:

	$v = 1$	$v = 2$	$v = 3$	$v = 4$	$v = 5$	$v = 6$	$v = 7$	$v = 8$
w_v	5	8	9	5	6	5	5	9

This is the instance of the VWMCP of Chapter 5 «Clique problems». The $\text{ILP}_{\text{MCP}}^{\text{VW}}$ model has 8 binary variables and one constraint for each of the 4 edges of $\overline{G}$:

$$\begin{aligned}
 & \max \quad 5x_1 + 8x_2 + 9x_3 + 5x_4 + 6x_5 + 5x_6 + 5x_7 + 9x_8 \\
 & \text{subject to} \quad x_1 + x_5 \leq 1 \\
 & \quad \quad \quad x_2 + x_6 \leq 1 \\
 & \quad \quad \quad x_3 + x_7 \leq 1 \\
 & \quad \quad \quad x_4 + x_8 \leq 1 \\
 & \quad \quad \quad x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \in \{0, 1\}
 \end{aligned}$$

An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{MCP}}^{\text{VW}})$ is:

$$\tilde{x}_2 = \tilde{x}_3 = \tilde{x}_5 = \tilde{x}_8 = 1, \quad \tilde{x}_1 = \tilde{x}_4 = \tilde{x}_6 = \tilde{x}_7 = 0$$

The optimal value $z(\text{ILP}_{\text{MCP}}^{\text{VW}})$ is $8+9+6+9 = 32$, as stated in Chapter 5 «Clique problems». The associated optimal solution is the clique $\tilde{K} = \{2, 3, 5, 8\}$, and it is the unique optimal solution: each constraint keeps the heavier vertex of its pair.

12.8.2 The edge-weighted maximum clique problem

- Given a simple graph $G = (V, E)$ and an edge-weight vector $\mathbf{r} \in \mathbb{Z}_{\geq 1}^m$, the weight of an edge is collected only if both its endpoints are selected: the product $x_u x_v$ is not linear. We introduce the n **binary variables** (12.1) and the following m **binary variables**:

$$y_{uv} = \begin{cases} 1 & \text{if both endpoints of edge } \{u, v\} \text{ are selected in the clique} \\ 0 & \text{otherwise} \end{cases} \quad \forall \{u, v\} \in E. \quad (12.17)$$

Using these variables, an Integer Linear Programming (ILP) model for the edge-weighted maximum clique problem (EWMCP) reads as follows:

$$\max \quad \sum_{\{u,v\} \in E} r_{uv} y_{uv} \quad (12.18a)$$

$$\text{subject to} \quad y_{uv} \leq x_u, \quad \forall \{u, v\} \in E, \quad (12.18b)$$

$$y_{uv} \leq x_v, \quad \forall \{u, v\} \in E, \quad (12.18c)$$

$$x_u + x_v \leq 1, \quad \forall \{u, v\} \in \overline{E}, \quad (12.18d)$$

$$x_v \in \{0, 1\}, \quad \forall v \in V, \quad (12.18e)$$

$$y_{uv} \in \{0, 1\}, \quad \forall \{u, v\} \in E. \quad (12.18f)$$

This ILP model is referred to as the $\text{ILP}_{\text{MCP}}^{\text{EW}}$ model.

- Description of the objective function and constraints of the $\text{ILP}_{\text{MCP}}^{\text{EW}}$ model:
 - the linear objective function (12.18a) maximizes the total weight $r(K)$ of the edges with both endpoints in the selected clique;

- constraints (12.18b) and (12.18c) allow $y_{uv} = 1$ only if both endpoints are selected ($2m$ linear constraints);
 - constraints (12.18d) ensure that the selected vertices form a clique ($\binom{n}{2} - m$ linear constraints);
 - finally, constraints (12.18e) and (12.18f) define the variables of the ILP model.
- Constraints (12.18b) and (12.18c) model the implication $y_{uv} = 1 \Rightarrow x_u = x_v = 1$. The converse implication $x_u = x_v = 1 \Rightarrow y_{uv} = 1$, that is $y_{uv} \geq x_u + x_v - 1$, is not needed: since $r_{uv} > 0$, in every optimal solution y_{uv} takes the largest value allowed, $\min\{x_u, x_v\}$.

Example 12.14: ILP_{MCP}^{EW} model

We consider the graph G of Example 12.13, with the edge weights of the instance of the EWMCP of Chapter 5 «Clique problems»; they are the coefficients of the objective function below. The ILP_{MCP}^{EW} model has $8 + 24 = 32$ binary variables, 48 constraints $y_{uv} \leq x_u$, $y_{uv} \leq x_v$ and 4 non-edge constraints:

$$\begin{aligned}
 \max \quad & y_{12} + 4y_{13} + 5y_{14} + 4y_{16} + 3y_{17} + 3y_{18} \\
 & + 2y_{23} + 5y_{24} + 6y_{25} + 9y_{27} + 4y_{28} + 3y_{34} \\
 & + 6y_{35} + y_{36} + 6y_{38} + 4y_{45} + 7y_{46} + 2y_{47} \\
 & + 5y_{56} + 7y_{57} + 3y_{58} + y_{67} + 2y_{68} + 8y_{78} \\
 \text{subject to} \quad & y_{12} \leq x_1 & y_{12} \leq x_2 & y_{13} \leq x_1 & y_{13} \leq x_3 \\
 & y_{14} \leq x_1 & y_{14} \leq x_4 & y_{16} \leq x_1 & y_{16} \leq x_6 \\
 & y_{17} \leq x_1 & y_{17} \leq x_7 & y_{18} \leq x_1 & y_{18} \leq x_8 \\
 & y_{23} \leq x_2 & y_{23} \leq x_3 & y_{24} \leq x_2 & y_{24} \leq x_4 \\
 & y_{25} \leq x_2 & y_{25} \leq x_5 & y_{27} \leq x_2 & y_{27} \leq x_7 \\
 & y_{28} \leq x_2 & y_{28} \leq x_8 & y_{34} \leq x_3 & y_{34} \leq x_4 \\
 & y_{35} \leq x_3 & y_{35} \leq x_5 & y_{36} \leq x_3 & y_{36} \leq x_6 \\
 & y_{38} \leq x_3 & y_{38} \leq x_8 & y_{45} \leq x_4 & y_{45} \leq x_5 \\
 & y_{46} \leq x_4 & y_{46} \leq x_6 & y_{47} \leq x_4 & y_{47} \leq x_7 \\
 & y_{56} \leq x_5 & y_{56} \leq x_6 & y_{57} \leq x_5 & y_{57} \leq x_7 \\
 & y_{58} \leq x_5 & y_{58} \leq x_8 & y_{67} \leq x_6 & y_{67} \leq x_7 \\
 & y_{68} \leq x_6 & y_{68} \leq x_8 & y_{78} \leq x_7 & y_{78} \leq x_8 \\
 & x_1 + x_5 \leq 1 & x_2 + x_6 \leq 1 & x_3 + x_7 \leq 1 & x_4 + x_8 \leq 1 \\
 & x_1, x_2, \dots, x_8 \in \{0, 1\} \\
 & y_{12}, y_{13}, \dots, y_{78} \in \{0, 1\}
 \end{aligned}$$

An optimal solution $(\tilde{x}, \tilde{y})(\text{ILP}_{\text{MCP}}^{\text{EW}})$ is:

$$\begin{aligned}
 \tilde{x}_2 = \tilde{x}_5 = \tilde{x}_7 = \tilde{x}_8 = 1, \quad \tilde{x}_1 = \tilde{x}_3 = \tilde{x}_4 = \tilde{x}_6 = 0, \\
 \tilde{y}_{25} = \tilde{y}_{27} = \tilde{y}_{28} = \tilde{y}_{57} = \tilde{y}_{58} = \tilde{y}_{78} = 1, \quad \text{all the other } \tilde{y}_{uv} = 0
 \end{aligned}$$

The optimal value $z(\text{ILP}_{\text{MCP}}^{\text{EW}})$ is $6 + 9 + 4 + 7 + 3 + 8 = 37$, as stated in Chapter 5 «Clique

problems». The associated optimal solution is the clique $\tilde{K} = \{2, 5, 7, 8\}$, and it is the unique optimal solution.

12.9 Typical mistakes

Warning

- **Writing the constraints on the edges of G .** The edge formulation of the MCP has one constraint for each *non-edge* of G ; on the edges of G the same constraints describe the stable sets of G .
- **Trusting the LP bound of the edge formulation.** The point $x_v = \frac{1}{2}$ is always feasible, so $z(\text{LP}_{\text{MCP}}) \geq n/2$ even when $\omega(G) = 2$.
- **Forgetting the bounds $x_v \leq 1$ in the edge formulation.** A vertex adjacent to all the others appears in no constraint, and LP-0_{MCP} becomes unbounded. In the clique formulation the bounds are implied.
- **Believing that clique inequalities close every gap.** On C_5 they coincide with the non-edge constraints, and the odd cycle inequality is needed.
- **Using non-maximal cliques of $\overline{G}$.** Their inequalities are valid but dominated: the maximal cliques give the strongest constraints of this family.
- **Mixing up the graphs of the MCP and of the MSSP.** The ILP_{MSSP} model has one constraint per edge of G , the ILP_{MCP} model one per edge of $\overline{G}$.
- **Writing $y_{uv} = x_u x_v$ in the edge-weighted model.** The product is not linear: the constraints $y_{uv} \leq x_u$ and $y_{uv} \leq x_v$ replace it, and the positive weights make $y_{uv} = 1$ whenever both endpoints are selected.

12.10 In summary

Summary

- The edge formulation ILP_{MCP} has one binary variable per vertex and one constraint $x_u + x_v \leq 1$ per edge of $\overline{G}$: it is the edge formulation of the MSSP on $\overline{G}$.
- Its LP relaxation is weak: $x_v = \frac{1}{2}$ is feasible and $z(\text{LP}_{\text{MCP}}) \geq n/2$.
- The clique inequalities $\sum_{v \in Q} x_v \leq 1$, for the maximal cliques Q of $\overline{G}$, give a stronger formulation with $\omega(G) \leq z(\text{LP}_{\text{MCP}}^{\text{CL}}) \leq z(\text{LP}_{\text{MCP}})$.
- The dual of $\text{LP}_{\text{MCP}}^{\text{CL}}$ is the LP relaxation of a coloring formulation, hence $\omega(G) \leq z(\text{LP}_{\text{MCP}}^{\text{CL}}) \leq \chi(G)$.
- The ILP_{MSSP} model has one constraint $x_u + x_v \leq 1$ per edge of G ; with the maximal cliques of G it becomes the clique formulation. On [Example 12.1](#) the optimal value is $\alpha(G) = 3$.

- The vertex-weighted model changes only the objective function; the edge-weighted model adds a variable y_{uv} per edge, with $y_{uv} \leq x_u$ and $y_{uv} \leq x_v$. On the instances of Chapter 5 «Clique problems» their optimal values are 32 and 37.

Dashboard (maximization). Bounds on $\omega(G)$ computed in this chapter:

instance	$z(\text{LP}_{\text{MCP}})$	$z(\text{LP}_{\text{MCP}}^{\text{CL}})$	$\omega(G)$
Example 12.1 (cycle with a chord)	$\frac{7}{2}$	3	3
C_5	$\frac{5}{2}$	$\frac{5}{2}$	2 (odd cycle inequality)

Formulations of routing problems

Prerequisites

- From the module *Integer linear programming*, [Chapter 1 «Integer linear programming»](#) and [Chapter 2 «Primal and dual bounds»](#): ILP problems, LP relaxations and bounds.
- From this module, [Chapter 6 «Routing problems»](#): the problem, its assumptions, tours, the distance matrix and the instance of Example 6.1.

Objectives

At the end of this chapter you will be able to:

- write the ILP model of the Travelling Salesman Problem with the position variables of Miller-Tucker-Zemlin (MTZ) and explain how they eliminate subtours;
- write the ILP model with the subtour elimination constraints of Dantzig-Fulkerson-Johnson (DFJ), and prove that it is correct;
- count the variables and the constraints of the two models;
- compare the LP relaxations of the two models on the same instance;
- write the ILP model of the asymmetric TSP, and explain why the metric TSP has the same models as the TSP.

13.1 Overview of the family

- Given the set $\mathcal{V} = \{1, 2, \dots, n\}$ of cities, the models of this chapter use the set of directed arcs

$$A = \{(i, j) : i, j \in \mathcal{V}, i \neq j\}.$$

Each edge $\{i, j\}$ of the symmetric problem is represented by the two opposite arcs (i, j) and (j, i) , which have the same cost $d_{ij} = d_{ji}$.

- The ILP models of the routing problems use one binary **arc variable** x_{ij} for each arc $(i, j) \in A$, equal to 1 if the tour goes from city i to city j .
- The **degree constraints** impose that exactly one arc leaves and exactly one arc enters each city; the models differ in the constraints that eliminate the subtours.

The routing models are assignment models on the arcs, with the degree constraints, plus constraints that forbid the subtours. The objective function minimizes the total distance $\sum_{(i,j) \in A} d_{ij} x_{ij}$.

- The table lists the members of the family, with the names of their ILP models and the sections that write them.

Problem	Model	Subtour elimination	Section
TSP	ILP _{TSP} ^{MTZ}	position variables u_i	13.3
TSP	ILP _{TSP} ^{DFJ}	one constraint per subset of cities	13.4
metric TSP	ILP _{TSP} ^{MTZ} , ILP _{TSP} ^{DFJ}	the models of the TSP	13.6
ATSP	ILP _{ATSP}	the constraints of the TSP models	13.6

13.2 ILP models for the Travelling Salesman Problem

The Travelling Salesman Problem and its reference instance (Example 6.1) are defined in [Chapter 6 «Routing problems», Section 6.2](#).

- The TSP has two classical ILP models on the arc variables x_{ij} : the model of Miller, Tucker and Zemlin (MTZ) of Section 13.3, with position variables, and the model of Dantzig, Fulkerson and Johnson (DFJ) of Section 13.4, with subtour elimination constraints.
- Section 13.5 compares the Linear Programming relaxations of the two models.

13.3 A model with position variables (MTZ)

- We introduce the following $n(n-1)$ **binary variables**:

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i, j) \text{ is traversed by the tour} \\ 0 & \text{otherwise} \end{cases} \quad \forall (i, j) \in A,$$

and the following $n-1$ **continuous variables**:

$$u_i = \text{position of city } i \text{ after leaving city } 1, \quad \forall i \in \mathcal{V} \setminus \{1\}.$$

The city 1 is used only as a reference city: a tour has no preferred starting city.

Using these variables, an Integer Linear Programming (ILP) model for the Travelling Salesman Problem (TSP) reads as follows:

$$\min \sum_{i \in \mathcal{V}} \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} d_{ij} x_{ij} \quad (13.1a)$$

$$\text{subject to } \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} x_{ij} = 1, \quad \forall i \in \mathcal{V}, \quad (13.1b)$$

$$\sum_{\substack{i \in \mathcal{V} \\ i \neq j}} x_{ij} = 1, \quad \forall j \in \mathcal{V}, \quad (13.1c)$$

$$u_i - u_j + (n-1)x_{ij} \leq n-2, \quad \forall i, j \in \mathcal{V} \setminus \{1\}, i \neq j, \quad (13.1d)$$

$$x_{ij} \in \{0, 1\}, \quad \forall (i, j) \in A, \quad (13.1e)$$

$$1 \leq u_i \leq n-1, \quad \forall i \in \mathcal{V} \setminus \{1\}. \quad (13.1f)$$

This ILP model is referred to as the $\text{ILP}_{\text{TSP}}^{\text{MTZ}}$ model, or **MTZ model**.

- Description of the objective function and constraints of the $\text{ILP}_{\text{TSP}}^{\text{MTZ}}$ model:
 - the linear objective function (13.1a) minimizes the total distance travelled by the tour;
 - constraints (13.1b) ensure that exactly one arc leaves each city (n linear constraints);
 - constraints (13.1c) ensure that exactly one arc enters each city (n linear constraints);
 - constraints (13.1d) link the arc variables with the position variables and eliminate the subtours that do not contain city 1 ($(n-1)(n-2)$ linear constraints);
 - finally, constraints (13.1e) and (13.1f) define the variables of the ILP model.

Explanation of the link between the variables

- For each pair of cities i, j different from city 1, if arc (i, j) is traversed, then the position of city j is at least one unit greater than the position of city i . This logical relation is the implication

$$x_{ij} = 1 \implies u_j \geq u_i + 1.$$

- Its contrapositive is

$$u_j < u_i + 1 \implies x_{ij} = 0.$$

In particular, an arc from a city to a city in the same or in an earlier position cannot be traversed.

- Both implications are imposed by the linear constraints (13.1d). If $x_{ij} = 1$, then the constraint gives $u_i - u_j + (n - 1) \leq n - 2$, that is $u_j \geq u_i + 1$.
- Conversely, if $u_j < u_i + 1$, then the constraint gives $(n - 1)x_{ij} \leq n - 2 + u_j - u_i < n - 1$. Since x_{ij} is binary, this forces $x_{ij} = 0$.
- When $x_{ij} = 0$, the constraint becomes $u_i - u_j \leq n - 2$, which is always satisfied because both positions belong to $[1, n - 1]$. Accordingly, $n - 1$ is the smallest possible big-M coefficient for these bounds.

Proposition 13.1: MTZ constraints eliminate subtours

The constraints (13.1b)–(13.1f) are satisfied by an integer solution if and only if its selected arcs form one tour through all the cities.

Proof. Constraints (13.1b) and (13.1c) make the selected arcs a collection of vertex-disjoint directed cycles covering all cities. Suppose that one such cycle does not contain city 1 and visits, in order, $i_1, i_2, \dots, i_k$. Since every arc (i_r, i_{r+1}) is selected, the implication above gives

$$u_{i_{r+1}} \geq u_{i_r} + 1 \quad r = 1, \dots, k,$$

where $i_{k+1} = i_1$. Summing the inequalities gives $u_{i_1} \geq u_{i_1} + k$, a contradiction. Thus no cycle can avoid city 1, and the collection is one tour.

Conversely, consider any tour and start it from city 1. For every city $i \neq 1$, set u_i equal to the position at which city i is visited after leaving city 1. The bounds (13.1f) hold. If a selected arc (i, j) has $i, j \neq 1$, then city j immediately follows city i , so that $u_j = u_i + 1$ and (13.1d) holds. If the arc is not selected, constraint (13.1d) holds by the bounds on the positions. Hence the tour defines a feasible integer solution. $\square$

Example 13.1: TSP instance

We consider the set $\mathcal{V} = \{1, 2, 3, 4, 5\}$ of 5 cities ($n = 5$), with the distance matrix

d_{ij}	1	2	3	4	5
1	—	5	2	2	9
2	5	—	4	3	4
3	2	4	—	4	7
4	2	3	4	—	7
5	9	4	7	7	—

This is the instance of Example 6.1 of Chapter 6 «Routing problems». We refer to this TSP instance as [Example 13.1](#).

Example 13.2: $\text{ILP}_{\text{TSP}}^{\text{MTZ}}$ model

The $\text{ILP}_{\text{TSP}}^{\text{MTZ}}$ model for the TSP instance of [Example 13.1](#) has 20 arc variables, 4 position variables, 10 degree constraints and 12 constraints (13.1d):

$$\begin{aligned}
 \min \quad & 5x_{12} + 2x_{13} + 2x_{14} + 9x_{15} + 5x_{21} + 4x_{23} + 3x_{24} \\
 & + 4x_{25} + 2x_{31} + 4x_{32} + 4x_{34} + 7x_{35} + 2x_{41} + 3x_{42} \\
 & + 4x_{43} + 7x_{45} + 9x_{51} + 4x_{52} + 7x_{53} + 7x_{54} \\
 \text{subject to} \quad & x_{12} + x_{13} + x_{14} + x_{15} = 1 \\
 & x_{21} + x_{23} + x_{24} + x_{25} = 1 \\
 & x_{31} + x_{32} + x_{34} + x_{35} = 1 \\
 & x_{41} + x_{42} + x_{43} + x_{45} = 1 \\
 & x_{51} + x_{52} + x_{53} + x_{54} = 1 \\
 & x_{21} + x_{31} + x_{41} + x_{51} = 1 \\
 & x_{12} + x_{32} + x_{42} + x_{52} = 1 \\
 & x_{13} + x_{23} + x_{43} + x_{53} = 1 \\
 & x_{14} + x_{24} + x_{34} + x_{54} = 1 \\
 & x_{15} + x_{25} + x_{35} + x_{45} = 1 \\
 & u_2 - u_3 + 4x_{23} \leq 3 \\
 & u_2 - u_4 + 4x_{24} \leq 3 \\
 & u_2 - u_5 + 4x_{25} \leq 3 \\
 & u_3 - u_2 + 4x_{32} \leq 3 \\
 & u_3 - u_4 + 4x_{34} \leq 3 \\
 & u_3 - u_5 + 4x_{35} \leq 3 \\
 & u_4 - u_2 + 4x_{42} \leq 3 \\
 & u_4 - u_3 + 4x_{43} \leq 3 \\
 & u_4 - u_5 + 4x_{45} \leq 3 \\
 & u_5 - u_2 + 4x_{52} \leq 3 \\
 & u_5 - u_3 + 4x_{53} \leq 3 \\
 & u_5 - u_4 + 4x_{54} \leq 3 \\
 & x_{12}, x_{13}, x_{14}, \dots, x_{54} \in \{0, 1\} \\
 & u_2, u_3, u_4, u_5 \geq 1 \\
 & u_2, u_3, u_4, u_5 \leq 4
 \end{aligned}$$

An optimal solution $(\tilde{x}, \tilde{u})(\text{ILP}_{\text{TSP}}^{\text{MTZ}})$ is:

$$\begin{aligned} \tilde{x}_{13} = \tilde{x}_{35} = \tilde{x}_{52} = \tilde{x}_{24} = \tilde{x}_{41} = 1, \quad \text{all the other } \tilde{x}_{ij} = 0, \\ \tilde{u}_3 = 1, \quad \tilde{u}_5 = 2, \quad \tilde{u}_2 = 3, \quad \tilde{u}_4 = 4 \end{aligned}$$

The optimal value $z(\text{ILP}_{\text{TSP}}^{\text{MTZ}})$ is 18. The associated optimal solution of the TSP is the tour $\tilde{\sigma} = (1, 3, 5, 2, 4)$, of length $L(\tilde{\sigma}) = 2 + 7 + 4 + 3 + 2 = 18$; it is the unique optimal tour, up to the direction.

13.4 A model with subtour elimination constraints (DFJ)

- The second model uses only the $n(n-1)$ arc variables x_{ij} of the MTZ model: there are no position variables. The subtours are forbidden directly, with one constraint for every subset of cities.

Using these variables, a second Integer Linear Programming (ILP) model for the Travelling Salesman Problem (TSP) reads as follows:

$$\min \sum_{i \in \mathcal{V}} \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} d_{ij} x_{ij} \quad (13.2a)$$

$$\text{subject to } \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} x_{ij} = 1, \quad \forall i \in \mathcal{V}, \quad (13.2b)$$

$$\sum_{\substack{i \in \mathcal{V} \\ i \neq j}} x_{ij} = 1, \quad \forall j \in \mathcal{V}, \quad (13.2c)$$

$$\sum_{i \in S} \sum_{\substack{j \in S \\ j \neq i}} x_{ij} \leq |S| - 1, \quad \forall S \subseteq \mathcal{V} \setminus \{1\}, \quad |S| \geq 2, \quad (13.2d)$$

$$x_{ij} \in \{0, 1\}, \quad \forall (i, j) \in A. \quad (13.2e)$$

This ILP model is referred to as the $\text{ILP}_{\text{TSP}}^{\text{DFJ}}$ model, or **DFJ model**.

- Description of the objective function and constraints of the $\text{ILP}_{\text{TSP}}^{\text{DFJ}}$ model:
 - the objective function (13.2a) and the degree constraints (13.2b) and (13.2c) are those of the MTZ model;
 - constraints (13.2d) are called **subtour elimination constraints**: a tour cannot contain all the $|S|$ arcs of a cycle on the cities of S , so it uses at most $|S| - 1$ arcs inside S ($2^{n-1} - n$ linear constraints);
 - finally, constraints (13.2e) define the variables of the ILP model.
- The number $2^{n-1} - n$ of constraints (13.2d) is exponential in n : for $n = 5$ they are 11, for $n = 20$ they are 524 268.

Proposition 13.2: Subtour elimination constraints

The constraints (13.2b)–(13.2e) are satisfied by an integer solution if and only if its selected arcs form one tour through all the cities.

Proof. Let a tour be given and let $S \subseteq \mathcal{V} \setminus \{1\}$ with $|S| \geq 2$. If the tour used $|S|$ arcs inside S , these arcs would form a cycle on the cities of S , which is not the whole tour since $1 \notin S$: a contradiction. Hence (13.2d) holds, and the degree constraints hold by definition of a tour.

Conversely, constraints (13.2b) and (13.2c) make the selected arcs a collection of vertex-disjoint directed cycles. If there were more than one cycle, one of them would not contain city 1. Let S be its set of cities: then $|S| \geq 2$ and the cycle uses $|S|$ arcs inside S , which violates (13.2d). Hence there is one cycle, a tour. $\square$

Example 13.3: ILP^{DFJ}_{TSP} model

The ILP^{DFJ}_{TSP} model for the TSP instance of [Example 13.1](#) has 20 arc variables, 10 degree constraints and 11 constraints (13.2d), one for every subset $S \subseteq \{2, 3, 4, 5\}$ with at least two cities:

$$\begin{aligned}
 \min \quad & 5x_{12} + 2x_{13} + 2x_{14} + 9x_{15} + 5x_{21} + 4x_{23} + 3x_{24} \\
 & + 4x_{25} + 2x_{31} + 4x_{32} + 4x_{34} + 7x_{35} + 2x_{41} + 3x_{42} \\
 & + 4x_{43} + 7x_{45} + 9x_{51} + 4x_{52} + 7x_{53} + 7x_{54} \\
 \text{subject to} \quad & x_{12} + x_{13} + x_{14} + x_{15} = 1 \\
 & x_{21} + x_{23} + x_{24} + x_{25} = 1 \\
 & x_{31} + x_{32} + x_{34} + x_{35} = 1 \\
 & x_{41} + x_{42} + x_{43} + x_{45} = 1 \\
 & x_{51} + x_{52} + x_{53} + x_{54} = 1 \\
 & x_{21} + x_{31} + x_{41} + x_{51} = 1 \\
 & x_{12} + x_{32} + x_{42} + x_{52} = 1 \\
 & x_{13} + x_{23} + x_{43} + x_{53} = 1 \\
 & x_{14} + x_{24} + x_{34} + x_{54} = 1 \\
 & x_{15} + x_{25} + x_{35} + x_{45} = 1 \\
 & x_{23} + x_{32} \leq 1 \\
 & x_{24} + x_{42} \leq 1 \\
 & x_{25} + x_{52} \leq 1 \\
 & x_{34} + x_{43} \leq 1 \\
 & x_{35} + x_{53} \leq 1 \\
 & x_{45} + x_{54} \leq 1 \\
 & x_{23} + x_{24} + x_{32} + x_{34} + x_{42} + x_{43} \leq 2 \\
 & x_{23} + x_{25} + x_{32} + x_{35} + x_{52} + x_{53} \leq 2 \\
 & x_{24} + x_{25} + x_{42} + x_{45} + x_{52} + x_{54} \leq 2 \\
 & x_{34} + x_{35} + x_{43} + x_{45} + x_{53} + x_{54} \leq 2 \\
 & x_{23} + x_{24} + x_{25} + x_{32} + x_{34} + x_{35} + x_{42} + x_{43} + x_{45} + x_{52} + x_{53} + x_{54} \leq 3 \\
 & x_{12}, x_{13}, x_{14}, \dots, x_{54} \in \{0, 1\}
 \end{aligned}$$

The subtour $2 \rightarrow 5 \rightarrow 2$ together with the subtour $1 \rightarrow 3 \rightarrow 4 \rightarrow 1$ satisfies the degree constraints but violates the constraint $x_{25} + x_{52} \leq 1$ of $S = \{2, 5\}$. An optimal solution $\tilde{\mathbf{x}}(\text{ILP}_{\text{TSP}}^{\text{DFJ}})$ is the tour $\tilde{\sigma} = (1, 3, 5, 2, 4)$:

$$\tilde{x}_{13} = \tilde{x}_{35} = \tilde{x}_{52} = \tilde{x}_{24} = \tilde{x}_{41} = 1, \quad \text{all the other } \tilde{x}_{ij} = 0$$

The optimal value $z(\text{ILP}_{\text{TSP}}^{\text{DFJ}})$ is 18.

13.5 Linear Programming relaxations

13.5.1 The assignment relaxation

Keeping only the degree constraints of the two models, and relaxing the binary constraints into non-negativity constraints, we obtain the following LP model:

$$\min \sum_{i \in \mathcal{V}} \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} d_{ij} x_{ij} \quad (13.3a)$$

$$\text{subject to } \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} x_{ij} = 1, \quad \forall i \in \mathcal{V}, \quad (13.3b)$$

$$\sum_{\substack{i \in \mathcal{V} \\ i \neq j}} x_{ij} = 1, \quad \forall j \in \mathcal{V}, \quad (13.3c)$$

$$x_{ij} \geq 0, \quad \forall (i, j) \in A. \quad (13.3d)$$

This LP model is referred to as the LP-0_{TSP} model, or **assignment relaxation**.

- Adding the upper bounds $x_{ij} \leq 1$ does not change this relaxation, nor the two relaxations below: for every arc (i, j) , non-negativity and the degree equation (13.3b) give

$$x_{ij} \leq \sum_{\substack{k \in \mathcal{V} \\ k \neq i}} x_{ik} = 1.$$

Example 13.4: LP-0_{TSP} model

The LP-0_{TSP} model for the TSP instance of [Example 13.1](#) is:

$$\begin{aligned}
 \min \quad & 5x_{12} + 2x_{13} + 2x_{14} + 9x_{15} + 5x_{21} + 4x_{23} + 3x_{24} \\
 & + 4x_{25} + 2x_{31} + 4x_{32} + 4x_{34} + 7x_{35} + 2x_{41} + 3x_{42} \\
 & + 4x_{43} + 7x_{45} + 9x_{51} + 4x_{52} + 7x_{53} + 7x_{54} \\
 \text{subject to} \quad & x_{12} + x_{13} + x_{14} + x_{15} = 1 \\
 & x_{21} + x_{23} + x_{24} + x_{25} = 1 \\
 & x_{31} + x_{32} + x_{34} + x_{35} = 1 \\
 & x_{41} + x_{42} + x_{43} + x_{45} = 1 \\
 & x_{51} + x_{52} + x_{53} + x_{54} = 1 \\
 & x_{21} + x_{31} + x_{41} + x_{51} = 1 \\
 & x_{12} + x_{32} + x_{42} + x_{52} = 1 \\
 & x_{13} + x_{23} + x_{43} + x_{53} = 1 \\
 & x_{14} + x_{24} + x_{34} + x_{54} = 1 \\
 & x_{15} + x_{25} + x_{35} + x_{45} = 1 \\
 & x_{12}, x_{13}, x_{14}, \dots, x_{54} \geq 0
 \end{aligned}$$

An optimal solution $\tilde{\mathbf{x}}(\text{LP-0}_{\text{TSP}})$ is:

$$\tilde{x}_{13} = \tilde{x}_{34} = \tilde{x}_{41} = \tilde{x}_{25} = \tilde{x}_{52} = 1, \text{ all the other } \tilde{x}_{ij} = 0$$

It is integer, but it is made of the two subtours $1 \rightarrow 3 \rightarrow 4 \rightarrow 1$ and $2 \rightarrow 5 \rightarrow 2$. The optimal value $z(\text{LP-0}_{\text{TSP}})$ is 16. Accordingly we have the lower bound:

$$16 \leq z(\text{ILP}_{\text{TSP}}^{\text{MTZ}}) = z(\text{ILP}_{\text{TSP}}^{\text{DFJ}})$$

13.5.2 The relaxation of the MTZ model

To obtain an LP relaxation of the $\text{ILP}_{\text{TSP}}^{\text{MTZ}}$ model, we relax constraints (13.1e) into non-negativity constraints; the position variables keep their bounds. The obtained LP model reads as follows:

$$\min \sum_{i \in \mathcal{V}} \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} d_{ij} x_{ij} \quad (13.4a)$$

$$\text{subject to} \quad \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} x_{ij} = 1, \quad \forall i \in \mathcal{V}, \quad (13.4b)$$

$$\sum_{\substack{i \in \mathcal{V} \\ i \neq j}} x_{ij} = 1, \quad \forall j \in \mathcal{V}, \quad (13.4c)$$

$$u_i - u_j + (n-1)x_{ij} \leq n-2, \quad \forall i, j \in \mathcal{V} \setminus \{1\}, i \neq j, \quad (13.4d)$$

$$x_{ij} \geq 0, \quad \forall (i, j) \in A, \quad (13.4e)$$

$$1 \leq u_i \leq n-1, \quad \forall i \in \mathcal{V} \setminus \{1\}. \quad (13.4f)$$

This LP model is referred to as the $\text{LP}_{\text{TSP}}^{\text{MTZ}}$ model.

Proposition 13.3: A weak region of the MTZ relaxation

Let (x, u) satisfy the degree equations (13.4b)–(13.4c), with $x_{ij} \geq 0$. If

$$x_{ij} \leq \frac{n-2}{n-1} \quad \forall i, j \in \mathcal{V} \setminus \{1\}, i \neq j,$$

then (x, u) is feasible for $\text{LP}_{\text{TSP}}^{\text{MTZ}}$ after setting $u_i = 1$ for every $i \neq 1$.

Proof. With $u_i = u_j = 1$, each constraint (13.4d) becomes

$$(n-1)x_{ij} \leq n-2,$$

which is exactly the assumed upper bound. The bounds (13.4f) on the position variables also hold. $\square$

- The proposition shows why the compact MTZ formulation can have a weak lower bound: many fractional solutions of the assignment relaxation already satisfy the ordering constraints.

Example 13.5: $\text{LP}_{\text{TSP}}^{\text{MTZ}}$ model

The $\text{LP}_{\text{TSP}}^{\text{MTZ}}$ model for the TSP instance of [Example 13.1](#) is:

$$\begin{aligned}
 \min \quad & 5x_{12} + 2x_{13} + 2x_{14} + 9x_{15} + 5x_{21} + 4x_{23} + 3x_{24} \\
 & + 4x_{25} + 2x_{31} + 4x_{32} + 4x_{34} + 7x_{35} + 2x_{41} + 3x_{42} \\
 & + 4x_{43} + 7x_{45} + 9x_{51} + 4x_{52} + 7x_{53} + 7x_{54} \\
 \text{subject to} \quad & x_{12} + x_{13} + x_{14} + x_{15} = 1 \\
 & x_{21} + x_{23} + x_{24} + x_{25} = 1 \\
 & x_{31} + x_{32} + x_{34} + x_{35} = 1 \\
 & x_{41} + x_{42} + x_{43} + x_{45} = 1 \\
 & x_{51} + x_{52} + x_{53} + x_{54} = 1 \\
 & x_{21} + x_{31} + x_{41} + x_{51} = 1 \\
 & x_{12} + x_{32} + x_{42} + x_{52} = 1 \\
 & x_{13} + x_{23} + x_{43} + x_{53} = 1 \\
 & x_{14} + x_{24} + x_{34} + x_{54} = 1 \\
 & x_{15} + x_{25} + x_{35} + x_{45} = 1 \\
 & u_2 - u_3 + 4x_{23} \leq 3 \\
 & u_2 - u_4 + 4x_{24} \leq 3 \\
 & u_2 - u_5 + 4x_{25} \leq 3 \\
 & u_3 - u_2 + 4x_{32} \leq 3 \\
 & u_3 - u_4 + 4x_{34} \leq 3 \\
 & u_3 - u_5 + 4x_{35} \leq 3 \\
 & u_4 - u_2 + 4x_{42} \leq 3 \\
 & u_4 - u_3 + 4x_{43} \leq 3 \\
 & u_4 - u_5 + 4x_{45} \leq 3 \\
 & u_5 - u_2 + 4x_{52} \leq 3 \\
 & u_5 - u_3 + 4x_{53} \leq 3 \\
 & u_5 - u_4 + 4x_{54} \leq 3 \\
 & x_{12}, x_{13}, x_{14}, \dots, x_{54} \geq 0 \\
 & u_2, u_3, u_4, u_5 \geq 1 \\
 & u_2, u_3, u_4, u_5 \leq 4
 \end{aligned}$$

An optimal solution $(\tilde{\mathbf{x}}, \tilde{\mathbf{u}})(\text{LP}_{\text{TSP}}^{\text{MTZ}})$ is:

$$\begin{aligned}
 \tilde{x}_{13} = 1, \quad \tilde{x}_{31} = \tilde{x}_{34} = \tilde{x}_{41} = \frac{1}{2}, \quad \tilde{x}_{25} = \tilde{x}_{52} = \frac{3}{4}, \\
 \tilde{x}_{24} = \tilde{x}_{42} = \tilde{x}_{45} = \tilde{x}_{54} = \frac{1}{4}, \quad \text{all the other } \tilde{x}_{ij} = 0, \\
 \tilde{u}_2 = \tilde{u}_3 = \tilde{u}_4 = \tilde{u}_5 = 1
 \end{aligned}$$

All the arcs between the cities 2, 3, 4, 5 have value at most $\frac{3}{4} = \frac{n-2}{n-1}$, as in Proposition 13.3.

The optimal value $z(\text{LP}_{\text{TSP}}^{\text{MTZ}})$ is 17. Accordingly we have the lower bound:

$$17 \leq z(\text{ILP}_{\text{TSP}}^{\text{MTZ}})$$

13.5.3 The relaxation of the DFJ model

To obtain an LP relaxation of the $\text{ILP}_{\text{TSP}}^{\text{DFJ}}$ model, we relax constraints (13.2e) into non-negativity constraints. The obtained LP model reads as follows:

$$\min \sum_{i \in \mathcal{V}} \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} d_{ij} x_{ij} \quad (13.5a)$$

$$\text{subject to } \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} x_{ij} = 1, \quad \forall i \in \mathcal{V}, \quad (13.5b)$$

$$\sum_{\substack{i \in \mathcal{V} \\ i \neq j}} x_{ij} = 1, \quad \forall j \in \mathcal{V}, \quad (13.5c)$$

$$\sum_{i \in S} \sum_{\substack{j \in S \\ j \neq i}} x_{ij} \leq |S| - 1, \quad \forall S \subseteq \mathcal{V} \setminus \{1\}, |S| \geq 2, \quad (13.5d)$$

$$x_{ij} \geq 0, \quad \forall (i, j) \in A. \quad (13.5e)$$

This LP model is referred to as the $\text{LP}_{\text{TSP}}^{\text{DFJ}}$ model.

- The MTZ model is valuable because it has only $O(n^2)$ variables and constraints. The DFJ model has a stronger relaxation, but exponentially many constraints: in practice, the subtour elimination constraints are added only when they are violated by the current solution of the relaxation.

Example 13.6: $\text{LP}_{\text{TSP}}^{\text{DFJ}}$ model

The $\text{LP}_{\text{TSP}}^{\text{DFJ}}$ model for the TSP instance of [Example 13.1](#) is:

$$\begin{aligned}
 \min \quad & 5x_{12} + 2x_{13} + 2x_{14} + 9x_{15} + 5x_{21} + 4x_{23} + 3x_{24} \\
 & + 4x_{25} + 2x_{31} + 4x_{32} + 4x_{34} + 7x_{35} + 2x_{41} + 3x_{42} \\
 & + 4x_{43} + 7x_{45} + 9x_{51} + 4x_{52} + 7x_{53} + 7x_{54} \\
 \text{subject to} \quad & x_{12} + x_{13} + x_{14} + x_{15} = 1 \\
 & x_{21} + x_{23} + x_{24} + x_{25} = 1 \\
 & x_{31} + x_{32} + x_{34} + x_{35} = 1 \\
 & x_{41} + x_{42} + x_{43} + x_{45} = 1 \\
 & x_{51} + x_{52} + x_{53} + x_{54} = 1 \\
 & x_{21} + x_{31} + x_{41} + x_{51} = 1 \\
 & x_{12} + x_{32} + x_{42} + x_{52} = 1 \\
 & x_{13} + x_{23} + x_{43} + x_{53} = 1 \\
 & x_{14} + x_{24} + x_{34} + x_{54} = 1 \\
 & x_{15} + x_{25} + x_{35} + x_{45} = 1 \\
 & x_{23} + x_{32} \leq 1 \\
 & x_{24} + x_{42} \leq 1 \\
 & x_{25} + x_{52} \leq 1 \\
 & x_{34} + x_{43} \leq 1 \\
 & x_{35} + x_{53} \leq 1 \\
 & x_{45} + x_{54} \leq 1 \\
 & x_{23} + x_{24} + x_{32} + x_{34} + x_{42} + x_{43} \leq 2 \\
 & x_{23} + x_{25} + x_{32} + x_{35} + x_{52} + x_{53} \leq 2 \\
 & x_{24} + x_{25} + x_{42} + x_{45} + x_{52} + x_{54} \leq 2 \\
 & x_{34} + x_{35} + x_{43} + x_{45} + x_{53} + x_{54} \leq 2 \\
 & x_{23} + x_{24} + x_{25} + x_{32} + x_{34} + x_{35} + x_{42} + x_{43} + x_{45} + x_{52} + x_{53} + x_{54} \leq 3 \\
 & x_{12}, x_{13}, x_{14}, \dots, x_{54} \geq 0
 \end{aligned}$$

An optimal solution $\tilde{x}(\text{LP}_{\text{TSP}}^{\text{DFJ}})$ is the tour $\tilde{\sigma} = (1, 3, 5, 2, 4)$:

$$\tilde{x}_{13} = \tilde{x}_{35} = \tilde{x}_{52} = \tilde{x}_{24} = \tilde{x}_{41} = 1, \quad \text{all the other } \tilde{x}_{ij} = 0$$

The optimal value $z(\text{LP}_{\text{TSP}}^{\text{DFJ}})$ is 18: on this instance the relaxation closes the gap.

Example 13.7: comparing the relaxations

For the TSP instance of [Example 13.1](#), the three relaxations give the following lower bounds on the optimal value 18:

Relaxation	Optimal value
LP-0 _{TSP} (assignment relaxation)	16
LP _{TSP} ^{MTZ}	17
LP _{TSP} ^{DFJ}	18

The ordering constraints raise the bound from 16 to 17; the subtour elimination constraints close the gap on this instance.

13.6 ILP models for the metric and asymmetric TSP

The problems of this section are the asymmetric travelling salesman problem (ATSP) and the metric TSP, defined in [Chapter 6 «Routing problems», Section 6.6](#).

- Given the set $\mathcal{V} = \{1, 2, \dots, n\}$ of cities and the set A of directed arcs, each arc (i, j) has its own distance d_{ij} , which may differ from d_{ji} . We introduce the following $n(n-1)$ **binary variables**:

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i, j) \text{ is traversed by the tour} \\ 0 & \text{otherwise} \end{cases} \quad \forall (i, j) \in A.$$

Using these variables, an Integer Linear Programming (ILP) model for the asymmetric travelling salesman problem (ATSP) reads as follows:

$$\min \sum_{i \in \mathcal{V}} \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} d_{ij} x_{ij} \quad (13.6a)$$

$$\text{subject to } \sum_{\substack{j \in \mathcal{V} \\ j \neq i}} x_{ij} = 1, \quad \forall i \in \mathcal{V}, \quad (13.6b)$$

$$\sum_{\substack{i \in \mathcal{V} \\ i \neq j}} x_{ij} = 1, \quad \forall j \in \mathcal{V}, \quad (13.6c)$$

$$\sum_{i \in S} \sum_{\substack{j \in S \\ j \neq i}} x_{ij} \leq |S| - 1, \quad \forall S \subseteq \mathcal{V} \setminus \{1\}, |S| \geq 2, \quad (13.6d)$$

$$x_{ij} \in \{0, 1\}, \quad \forall (i, j) \in A. \quad (13.6e)$$

This ILP model is referred to as the ILP_{ATSP} model.

- Description of the objective function and constraints of the ILP_{ATSP} model:
 - the linear objective function (13.6a) minimizes the total distance travelled along the arcs of the tour, each arc with its own distance;

- constraints (13.6b) and (13.6c) ensure that exactly one arc leaves and exactly one arc enters each city ($2n$ linear constraints);
 - constraints (13.6d) are the subtour elimination constraints ($2^{n-1} - n$ linear constraints);
 - finally, constraints (13.6e) define the variables of the ILP model.
- The ILP_{ATSP} model is the $\text{ILP}_{\text{TSP}}^{\text{DFJ}}$ model with the distances of the arcs: its proof of correctness, Proposition 13.2, never uses $d_{ij} = d_{ji}$. For the same reason the $\text{ILP}_{\text{TSP}}^{\text{MTZ}}$ model is also an ILP model of the ATSP.
 - A tour and the same tour travelled backwards use opposite arcs: in the ATSP they are different solutions, with different lengths in general.
 - The metric TSP, defined in Chapter 6 «Routing problems», has the same ILP models as the TSP: the triangle inequality is a property of the distances, not a constraint of the model. The instance of Example 13.1 is metric, so the $\text{ILP}_{\text{TSP}}^{\text{MTZ}}$ and $\text{ILP}_{\text{TSP}}^{\text{DFJ}}$ models written for it are already models of a metric TSP instance.

Example 13.8: ILP_{ATSP} model

We consider the set $\mathcal{V} = \{1, 2, 3, 4\}$ of 4 cities ($n = 4$), with the asymmetric distance matrix

d_{ij}	1	2	3	4
1	–	2	7	7
2	2	–	4	2
3	9	7	–	1
4	2	4	1	–

For instance, $d_{13} = 7$ and $d_{31} = 9$. The ILP_{ATSP} model for this ATSP instance has 12 arc variables, 8 degree constraints and 4 subtour elimination constraints, one for every subset $S \subseteq \{2, 3, 4\}$ with at least two cities:

$$\begin{aligned}
 \min \quad & 2x_{12} + 7x_{13} + 7x_{14} + 2x_{21} + 4x_{23} + 2x_{24} \\
 & + 9x_{31} + 7x_{32} + x_{34} + 2x_{41} + 4x_{42} + x_{43} \\
 \text{subject to} \quad & x_{12} + x_{13} + x_{14} = 1 \\
 & x_{21} + x_{23} + x_{24} = 1 \\
 & x_{31} + x_{32} + x_{34} = 1 \\
 & x_{41} + x_{42} + x_{43} = 1 \\
 & x_{21} + x_{31} + x_{41} = 1 \\
 & x_{12} + x_{32} + x_{42} = 1 \\
 & x_{13} + x_{23} + x_{43} = 1 \\
 & x_{14} + x_{24} + x_{34} = 1 \\
 & x_{23} + x_{32} \leq 1 \\
 & x_{24} + x_{42} \leq 1 \\
 & x_{34} + x_{43} \leq 1 \\
 & x_{23} + x_{24} + x_{32} + x_{34} + x_{42} + x_{43} \leq 2 \\
 & x_{12}, x_{13}, x_{14}, \dots, x_{43} \in \{0, 1\}
 \end{aligned}$$

The two subtours $1 \rightarrow 2 \rightarrow 1$ and $3 \rightarrow 4 \rightarrow 3$, of total length $2 + 2 + 1 + 1 = 6$, satisfy the degree constraints but violate the constraint $x_{34} + x_{43} \leq 1$ of $S = \{3, 4\}$. An optimal solution $\tilde{x}(\text{ILP}_{\text{ATSP}})$ is the tour $\tilde{\sigma} = (1, 2, 3, 4)$:

$$\tilde{x}_{12} = \tilde{x}_{23} = \tilde{x}_{34} = \tilde{x}_{41} = 1, \text{ all the other } \tilde{x}_{ij} = 0$$

The optimal value $z(\text{ILP}_{\text{ATSP}})$ is $2 + 4 + 1 + 2 = 9$, and the optimal tour is unique. The same tour travelled backwards, $(1, 4, 3, 2)$, has length $7 + 1 + 7 + 2 = 17$.

13.7 Typical mistakes

Warning

- **Writing the MTZ constraints for the city 1.** The constraints (13.1d) are written only for the pairs of cities different from city 1: the city 1 has no position variable.
- **Using a big-M smaller than $n-1$.** Written as $u_i - u_j + Mx_{ij} \leq M-1$, the constraint with $M < n-1$ cuts off feasible tours: when $x_{ij} = 0$ it requires $u_i - u_j \leq M-1 < n-2$, while two positions can differ by $n-2$; $M = n-1$ is the smallest valid value.
- **Taking $S = \mathcal{V}$ in the subtour elimination constraints.** A tour uses $|\mathcal{V}|$ arcs inside $\mathcal{V}$: the constraint would cut off every tour. The sets S are proper subsets of the cities, with at least two cities.
- **Trusting the MTZ bound.** By Proposition 13.3, many fractional solutions of the assignment relaxation are feasible for the MTZ relaxation: the compact formulation is not a strong one.

- **Listing all the subtour elimination constraints.** Their number is $2^{n-1} - n$: they are not written all at once, but added when violated.
- **Identifying a tour with its reverse in the ATSP.** With asymmetric distances the two directions of a tour are different solutions: in [Example 13.8](#) they have lengths 9 and 17.
- **Adding the triangle inequality as a constraint.** In the metric TSP the triangle inequality holds for the data d_{ij} ; the variables of the model are the same as in the TSP.

13.8 In summary

Summary

- The two models have the same arc variables x_{ij} and the same degree constraints; they differ in how they forbid the subtours.
- The MTZ model adds $n - 1$ position variables and $(n - 1)(n - 2)$ constraints: it is compact, but its relaxation is weak.
- The DFJ model adds $2^{n-1} - n$ subtour elimination constraints: the relaxation is stronger, and the constraints are generated when needed.
- Both models are correct: an integer solution is feasible if and only if its arcs form one tour.
- Both models are written on arcs, so they are also models of the ATSP: the ILP_{ATSP} model is the DFJ model with $d_{ij} \neq d_{ji}$ allowed. The metric TSP has the same models as the TSP.
- From the module *Integer linear programming*, [Chapter 8 «Branch-and-cut»](#) shows how the subtour elimination constraints are added only when they are violated.

Dashboard of the instance of the chapter. Lower bounds on the optimal value 18 computed in this chapter:

Bound	Value	From
Assignment relaxation	16	$z(\text{LP-0}_{\text{TSP}})$
MTZ relaxation	17	$z(\text{LP}_{\text{TSP}}^{\text{MTZ}})$
DFJ relaxation	18	$z(\text{LP}_{\text{TSP}}^{\text{DFJ}})$
Optimal value	18	tour $\sigma = (1, 3, 5, 2, 4)$

Formulations of scheduling problems

Prerequisites

- From the module *Linear Programming*, [Chapter 9 «Duality theory»](#): the dual of an LP problem, weak and strong duality.
- From the module *Integer linear programming*, [Chapter 1 «Integer linear programming»](#) and [Chapter 2 «Primal and dual bounds»](#): ILP problems, LP relaxations and bounds.
- From this module, [Chapter 7 «Scheduling problems»](#): the problem, its assumption, the makespan and the instance of Example 7.1.

Objectives

At the end of this chapter you will be able to:

- write the ILP model of $P||C_{\max}$ and explain how the makespan variable is linked to the machine loads;
- write the LP relaxations and their duals, and prove that their optimal value is $\sum_j t_j/m$;
- prove that the LP bound is at least $1/m$ of the optimal value, and that the factor is tight;
- use the valid inequalities and the symmetry-breaking constraints of the model.

14.1 Overview of the family

- The ILP models of the parallel machine scheduling problems use the set $\mathcal{H} = \{1, 2, \dots, m\}$ of the machines and one binary **assignment variable** x_{jh} for each job $j \in \mathcal{J}$ and each machine $h \in \mathcal{H}$.
- One continuous variable $C_{\max}$ represents the makespan: one constraint for each machine imposes that $C_{\max}$ is at least the load of the machine.
- The objective function minimizes $C_{\max}$, so at the optimum it is equal to the largest load.

The scheduling models are assignment models of the jobs to the machines, as the packing models assign items to bins; the min-max objective is linearized with the makespan variable $C_{\max}$.

- The table lists the members of the family, with the name of their ILP model and the section that writes it.

Problem	Model	Variables	Section
$P C_{\max}$	$ILP_{P C_{\max}}$	binary x_{jh} and continuous $C_{\max}$	14.2

14.2 An ILP model for the Scheduling on Identical Parallel Machines Problem

The problem of this section is the scheduling on identical parallel machines ($P||C_{\max}$), defined in [Chapter 7 «Scheduling problems»](#).

- Given m identical machines and a set $\mathcal{J} = \{1, 2, \dots, n\}$ of n jobs, with processing times t_j , we denote by $\mathcal{H} = \{1, 2, \dots, m\}$ the set of machines.
- We introduce the following nm **binary variables**:

$$x_{jh} = \begin{cases} 1 & \text{if job } j \text{ is assigned to machine } h \\ 0 & \text{otherwise} \end{cases} \quad \forall j \in \mathcal{J}, \forall h \in \mathcal{H},$$

and one **non-negative continuous variable**:

$$C_{\max} = \text{makespan of the schedule.}$$

Using these variables, an Integer Linear Programming (ILP) model for the Scheduling on Identical Parallel Machines Problem ($P||C_{\max}$) reads as follows:

$$\min \quad C_{\max} \quad (14.1a)$$

$$\text{subject to} \quad \sum_{h \in \mathcal{H}} x_{jh} = 1, \quad \forall j \in \mathcal{J}, \quad (14.1b)$$

$$- \sum_{j \in \mathcal{J}} t_j x_{jh} + C_{\max} \geq 0, \quad \forall h \in \mathcal{H}, \quad (14.1c)$$

$$x_{jh} \in \{0, 1\}, \quad \forall j \in \mathcal{J}, \forall h \in \mathcal{H}, \quad (14.1d)$$

$$C_{\max} \geq 0. \quad (14.1e)$$

This ILP model is referred to as the $\text{ILP}_{P\|C_{\max}}$ model.

- Description of the objective function and constraints of the $\text{ILP}_{P\|C_{\max}}$ model:
 - the linear objective function (14.1a) minimizes the makespan;
 - constraints (14.1b) ensure that each job is assigned to exactly one machine (n linear constraints);
 - constraints (14.1c) link the assignment variables with the makespan variable. In particular, they ensure that the makespan is greater than or equal to the load of every machine (m linear constraints);
 - finally, constraints (14.1d) and (14.1e) define the variables of the ILP model.

Explanation of the link between the variables

- For every machine h , the makespan and its load must satisfy the direct relation

$$C_{\max} \geq \sum_{j \in \mathcal{J}} t_j x_{jh},$$

which is imposed by constraint (14.1c).

- Since the constraint holds for every machine, it imposes

$$C_{\max} \geq \max_{h \in \mathcal{H}} \left\{ \sum_{j \in \mathcal{J}} t_j x_{jh} \right\}.$$

- In every optimal solution equality holds. Indeed, if $C_{\max}$ were larger than the maximum load, replacing $C_{\max}$ with the maximum load would keep all the constraints (14.1c) satisfied and would decrease the objective value by a positive amount, which contradicts optimality.
- When no job is assigned to a machine, its load is zero.

Example 14.1: $\text{ILP}_{P\|C_{\max}}$ model

We consider $m = 3$ machines ($\mathcal{H} = \{1, 2, 3\}$) and a set $\mathcal{J} = \{1, 2, \dots, 7\}$ of 7 jobs ($n = 7$), with the following processing times:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$
t_j	5	5	4	4	3	3	3

This is the instance of Example 7.1 of [Chapter 7 «Scheduling problems»](#); we refer to it as [Example 14.1](#). The $\text{ILP}_{P\|C_{\max}}$ model for this instance is:

$$\begin{aligned}
 \min \quad & C_{\max} \\
 \text{subject to} \quad & x_{11} + x_{12} + x_{13} = 1 \\
 & x_{21} + x_{22} + x_{23} = 1 \\
 & x_{31} + x_{32} + x_{33} = 1 \\
 & x_{41} + x_{42} + x_{43} = 1 \\
 & x_{51} + x_{52} + x_{53} = 1 \\
 & x_{61} + x_{62} + x_{63} = 1 \\
 & x_{71} + x_{72} + x_{73} = 1 \\
 & -5x_{11} - 5x_{21} - 4x_{31} - 4x_{41} - 3x_{51} - 3x_{61} - 3x_{71} + C_{\max} \geq 0 \\
 & -5x_{12} - 5x_{22} - 4x_{32} - 4x_{42} - 3x_{52} - 3x_{62} - 3x_{72} + C_{\max} \geq 0 \\
 & -5x_{13} - 5x_{23} - 4x_{33} - 4x_{43} - 3x_{53} - 3x_{63} - 3x_{73} + C_{\max} \geq 0 \\
 & x_{11}, x_{12}, x_{13}, \dots, x_{73} \in \{0, 1\} \\
 & C_{\max} \geq 0
 \end{aligned}$$

An optimal solution $(\tilde{x}, \tilde{C}_{\max})(\text{ILP}_{P\|C_{\max}})$ is:

$$\tilde{x}_{11} = \tilde{x}_{31} = \tilde{x}_{22} = \tilde{x}_{42} = \tilde{x}_{53} = \tilde{x}_{63} = \tilde{x}_{73} = 1, \quad \text{all the other } \tilde{x}_{jh} = 0, \quad \tilde{C}_{\max} = 9$$

It assigns the jobs $\{1, 3\}$ to machine 1, $\{2, 4\}$ to machine 2 and $\{5, 6, 7\}$ to machine 3, with loads 9, 9 and 9. The optimal value $z(\text{ILP}_{P\|C_{\max}})$ is 9, since it attains the lower bound

$$\max \left\{ \max_{j \in \mathcal{J}} t_j, \left\lceil \frac{\sum_{j \in \mathcal{J}} t_j}{m} \right\rceil \right\} = \max\{5, \lceil 27/3 \rceil\} = 9.$$

14.3 A first Linear Programming relaxation

To obtain a Linear Programming (LP) relaxation of the $\text{ILP}_{P\|C_{\max}}$ model, we relax constraints (14.1d) into non-negativity constraints. The obtained LP model reads as follows:

$$\min \quad C_{\max} \quad (14.2a)$$

$$\text{subject to} \quad \sum_{h \in \mathcal{H}} x_{jh} = 1, \quad \forall j \in \mathcal{J}, \quad (14.2b)$$

$$-\sum_{j \in \mathcal{J}} t_j x_{jh} + C_{\max} \geq 0, \quad \forall h \in \mathcal{H}, \quad (14.2c)$$

$$x_{jh} \geq 0, \quad \forall j \in \mathcal{J}, \forall h \in \mathcal{H}, \quad (14.2d)$$

$$C_{\max} \geq 0. \quad (14.2e)$$

This LP model is referred to as the $\text{LP-0}_{P\|C_{\max}}$ model.

Example 14.2: $\text{LP-0}_{P\|C_{\max}}$ model

The $\text{LP-0}_{P\|C_{\max}}$ model for the instance of [Example 14.1](#) is:

$$\begin{aligned} \min \quad & C_{\max} \\ \text{subject to} \quad & x_{11} + x_{12} + x_{13} = 1 \\ & x_{21} + x_{22} + x_{23} = 1 \\ & x_{31} + x_{32} + x_{33} = 1 \\ & x_{41} + x_{42} + x_{43} = 1 \\ & x_{51} + x_{52} + x_{53} = 1 \\ & x_{61} + x_{62} + x_{63} = 1 \\ & x_{71} + x_{72} + x_{73} = 1 \\ & -5x_{11} - 5x_{21} - 4x_{31} - 4x_{41} - 3x_{51} - 3x_{61} - 3x_{71} + C_{\max} \geq 0 \\ & -5x_{12} - 5x_{22} - 4x_{32} - 4x_{42} - 3x_{52} - 3x_{62} - 3x_{72} + C_{\max} \geq 0 \\ & -5x_{13} - 5x_{23} - 4x_{33} - 4x_{43} - 3x_{53} - 3x_{63} - 3x_{73} + C_{\max} \geq 0 \\ & x_{11}, x_{12}, x_{13}, \dots, x_{73} \geq 0 \\ & C_{\max} \geq 0 \end{aligned}$$

An optimal solution $(\tilde{x}, \tilde{C}_{\max})(\text{LP-0}_{P\|C_{\max}})$ is:

$$\tilde{x}_{jh} = \frac{1}{3}, \quad \forall j \in \mathcal{J}, \forall h \in \mathcal{H}, \quad \tilde{C}_{\max} = 9$$

The optimal value $z(\text{LP-0}_{P\|C_{\max}})$ is 9. Accordingly we have the lower bound:

$$9 \leq z(\text{ILP}_{P\|C_{\max}})$$

Using a free dual variable μ_j for each constraint (14.2b) and a non-negative dual variable π_h for each constraint (14.2c), the dual model of the $\text{LP-0}_{P\|C_{\max}}$ model is:

$$\max \sum_{j \in \mathcal{J}} \mu_j \quad (14.3a)$$

$$\text{subject to } \mu_j - t_j \pi_h \leq 0, \quad \forall j \in \mathcal{J}, \forall h \in \mathcal{H}, \quad (14.3b)$$

$$\sum_{h \in \mathcal{H}} \pi_h \leq 1, \quad (14.3c)$$

$$\mu_j \geq 0, \quad \forall j \in \mathcal{J}, \quad (14.3d)$$

$$\pi_h \geq 0, \quad \forall h \in \mathcal{H}. \quad (14.3e)$$

This LP model is referred to as the $D(\text{LP-0}_{P\|C_{\max}})$ model.

Example 14.3: $D(\text{LP-0}_{P\|C_{\max}})$ model

The dual $D(\text{LP-0}_{P\|C_{\max}})$ model for the instance of [Example 14.1](#) is:

$$\begin{array}{llll}
 \max & \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 & & \\
 \text{subject to} & \mu_1 - 5\pi_1 & \leq & 0 \\
 & \mu_1 - 5\pi_2 & \leq & 0 \\
 & \mu_1 - 5\pi_3 & \leq & 0 \\
 & \mu_2 - 5\pi_1 & \leq & 0 \\
 & \mu_2 - 5\pi_2 & \leq & 0 \\
 & \mu_2 - 5\pi_3 & \leq & 0 \\
 & \mu_3 - 4\pi_1 & \leq & 0 \\
 & \mu_3 - 4\pi_2 & \leq & 0 \\
 & \mu_3 - 4\pi_3 & \leq & 0 \\
 & \mu_4 - 4\pi_1 & \leq & 0 \\
 & \mu_4 - 4\pi_2 & \leq & 0 \\
 & \mu_4 - 4\pi_3 & \leq & 0 \\
 & \mu_5 - 3\pi_1 & \leq & 0 \\
 & \mu_5 - 3\pi_2 & \leq & 0 \\
 & \mu_5 - 3\pi_3 & \leq & 0 \\
 & \mu_6 - 3\pi_1 & \leq & 0 \\
 & \mu_6 - 3\pi_2 & \leq & 0 \\
 & \mu_6 - 3\pi_3 & \leq & 0 \\
 & \mu_7 - 3\pi_1 & \leq & 0 \\
 & \mu_7 - 3\pi_2 & \leq & 0 \\
 & \mu_7 - 3\pi_3 & \leq & 0 \\
 & \pi_1 + \pi_2 + \pi_3 & \leq & 1 \\
 & \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7 & \geq & 0 \\
 & \pi_1, \pi_2, \pi_3 & \geq & 0
 \end{array}$$

An optimal solution $(\tilde{\mu}, \tilde{\pi})(D(\text{LP-}0_{P\|C_{\max}}))$ is:

$$\tilde{\mu}_1 = \tilde{\mu}_2 = \frac{5}{3}, \quad \tilde{\mu}_3 = \tilde{\mu}_4 = \frac{4}{3}, \quad \tilde{\mu}_5 = \tilde{\mu}_6 = \tilde{\mu}_7 = 1, \quad \tilde{\pi}_1 = \tilde{\pi}_2 = \tilde{\pi}_3 = \frac{1}{3}$$

The optimal value $z(D(\text{LP-}0_{P\|C_{\max}}))$ is 9.

Proposition 14.1: Optimal value of $\text{LP-}0_{P\|C_{\max}}$

For every $P\|C_{\max}$ instance,

$$z(\text{LP-}0_{P\|C_{\max}}) = \frac{\sum_{j \in \mathcal{J}} t_j}{m}.$$

Proof. Set $x_{jh} = 1/m$ for every job and every machine, and set

$$C_{\max} = \frac{\sum_{j \in \mathcal{J}} t_j}{m}.$$

Each constraint (14.2b) is satisfied and every machine has the displayed load, hence this is a feasible primal solution. In the dual, set

$$\pi_h = \frac{1}{m} \quad \forall h \in \mathcal{H}, \quad \mu_j = \frac{t_j}{m} \quad \forall j \in \mathcal{J}.$$

Constraints (14.3b) hold at equality, $\sum_h \pi_h = 1$, and the dual objective value is

$$\sum_{j \in \mathcal{J}} \mu_j = \frac{\sum_{j \in \mathcal{J}} t_j}{m}.$$

By LP Weak Duality, the two solutions are optimal, and the optimal value is the common value of the objective functions. $\square$

14.4 A second Linear Programming relaxation

To obtain an LP relaxation of the $\text{ILP}_{P\|C_{\max}}$ model, we relax constraints (14.1d) into the bounds $0 \leq x_{jh} \leq 1$. The obtained LP model reads as follows:

$$\min \quad C_{\max} \tag{14.4a}$$

$$\text{subject to} \quad \sum_{h \in \mathcal{H}} x_{jh} = 1, \quad \forall j \in \mathcal{J}, \tag{14.4b}$$

$$- \sum_{j \in \mathcal{J}} t_j x_{jh} + C_{\max} \geq 0, \quad \forall h \in \mathcal{H}, \tag{14.4c}$$

$$x_{jh} \leq 1, \quad \forall j \in \mathcal{J}, \forall h \in \mathcal{H}, \tag{14.4d}$$

$$x_{jh} \geq 0, \quad \forall j \in \mathcal{J}, \forall h \in \mathcal{H}, \tag{14.4e}$$

$$C_{\max} \geq 0. \tag{14.4f}$$

This LP model is referred to as the $\text{LP}_{P\|C_{\max}}$ model.

- Constraints (14.4d) do not change the relaxation: by (14.4b) and (14.4e), each x_{jh} is at most one. Thus

$$z(\text{LP}_{P\|C_{\max}}) = z(\text{LP-0}_{P\|C_{\max}}) = \frac{\sum_{j \in \mathcal{J}} t_j}{m}.$$

Example 14.4: $\text{LP}_{P\|C_{\max}}$ model

The $\text{LP}_{P\|C_{\max}}$ model for the instance of [Example 14.1](#) is:

$$\begin{aligned}
 & \min \quad C_{\max} \\
 & \text{subject to} \quad x_{11} + x_{12} + x_{13} = 1 \\
 & \quad \quad \quad x_{21} + x_{22} + x_{23} = 1 \\
 & \quad \quad \quad x_{31} + x_{32} + x_{33} = 1 \\
 & \quad \quad \quad x_{41} + x_{42} + x_{43} = 1 \\
 & \quad \quad \quad x_{51} + x_{52} + x_{53} = 1 \\
 & \quad \quad \quad x_{61} + x_{62} + x_{63} = 1 \\
 & \quad \quad \quad x_{71} + x_{72} + x_{73} = 1 \\
 & \quad \quad \quad -5x_{11} - 5x_{21} - 4x_{31} - 4x_{41} - 3x_{51} - 3x_{61} - 3x_{71} + C_{\max} \geq 0 \\
 & \quad \quad \quad -5x_{12} - 5x_{22} - 4x_{32} - 4x_{42} - 3x_{52} - 3x_{62} - 3x_{72} + C_{\max} \geq 0 \\
 & \quad \quad \quad -5x_{13} - 5x_{23} - 4x_{33} - 4x_{43} - 3x_{53} - 3x_{63} - 3x_{73} + C_{\max} \geq 0 \\
 & \quad \quad \quad x_{11}, x_{12}, x_{13}, \dots, x_{73} \leq 1 \\
 & \quad \quad \quad x_{11}, x_{12}, x_{13}, \dots, x_{73} \geq 0 \\
 & \quad \quad \quad C_{\max} \geq 0
 \end{aligned}$$

An optimal solution $(\tilde{x}, \tilde{C}_{\max})(\text{LP}_{P\|C_{\max}})$ is the solution of [Example 14.2](#):

$$\tilde{x}_{jh} = \frac{1}{3}, \quad \forall j \in \mathcal{J}, \quad \forall h \in \mathcal{H}, \quad \tilde{C}_{\max} = 9$$

The optimal value $z(\text{LP}_{P\|C_{\max}})$ is 9. Accordingly we have the same lower bound.

Using a free dual variable μ_j for each constraint (14.4b), a non-negative dual variable π_h for each constraint (14.4c) and a non-positive dual variable λ_{jh} for each constraint (14.4d), the dual model of the $\text{LP}_{P\|C_{\max}}$ model is:

$$\max \quad \sum_{j \in \mathcal{J}} \mu_j + \sum_{j \in \mathcal{J}} \sum_{h \in \mathcal{H}} \lambda_{jh} \quad (14.5a)$$

$$\text{subject to} \quad \mu_j - t_j \pi_h + \lambda_{jh} \leq 0, \quad \forall j \in \mathcal{J}, \quad \forall h \in \mathcal{H}, \quad (14.5b)$$

$$\sum_{h \in \mathcal{H}} \pi_h \leq 1, \quad (14.5c)$$

$$\mu_j \geq 0, \quad \forall j \in \mathcal{J}, \quad (14.5d)$$

$$\pi_h \geq 0, \quad \forall h \in \mathcal{H}, \quad (14.5e)$$

$$\lambda_{jh} \leq 0, \quad \forall j \in \mathcal{J}, \quad \forall h \in \mathcal{H}. \quad (14.5f)$$

This LP model is referred to as the $D(\text{LP}_{P\|C_{\max}})$ model.

Example 14.5: $D(\text{LP}_{P\|C_{\max}})$ model

The dual $D(\text{LP}_{P\|C_{\max}})$ model for the instance of [Example 14.1](#) is:

$$\begin{aligned}
\max \quad & \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 + \lambda_{11} \\
& + \lambda_{12} + \lambda_{13} + \lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{31} + \lambda_{32} + \lambda_{33} \\
& + \lambda_{41} + \lambda_{42} + \lambda_{43} + \lambda_{51} + \lambda_{52} + \lambda_{53} + \lambda_{61} + \lambda_{62} \\
& + \lambda_{63} + \lambda_{71} + \lambda_{72} + \lambda_{73} \\
\text{subject to} \quad & \mu_1 - 5\pi_1 + \lambda_{11} \leq 0 \\
& \mu_1 - 5\pi_2 + \lambda_{12} \leq 0 \\
& \mu_1 - 5\pi_3 + \lambda_{13} \leq 0 \\
& \mu_2 - 5\pi_1 + \lambda_{21} \leq 0 \\
& \mu_2 - 5\pi_2 + \lambda_{22} \leq 0 \\
& \mu_2 - 5\pi_3 + \lambda_{23} \leq 0 \\
& \mu_3 - 4\pi_1 + \lambda_{31} \leq 0 \\
& \mu_3 - 4\pi_2 + \lambda_{32} \leq 0 \\
& \mu_3 - 4\pi_3 + \lambda_{33} \leq 0 \\
& \mu_4 - 4\pi_1 + \lambda_{41} \leq 0 \\
& \mu_4 - 4\pi_2 + \lambda_{42} \leq 0 \\
& \mu_4 - 4\pi_3 + \lambda_{43} \leq 0 \\
& \mu_5 - 3\pi_1 + \lambda_{51} \leq 0 \\
& \mu_5 - 3\pi_2 + \lambda_{52} \leq 0 \\
& \mu_5 - 3\pi_3 + \lambda_{53} \leq 0 \\
& \mu_6 - 3\pi_1 + \lambda_{61} \leq 0 \\
& \mu_6 - 3\pi_2 + \lambda_{62} \leq 0 \\
& \mu_6 - 3\pi_3 + \lambda_{63} \leq 0 \\
& \mu_7 - 3\pi_1 + \lambda_{71} \leq 0 \\
& \mu_7 - 3\pi_2 + \lambda_{72} \leq 0 \\
& \mu_7 - 3\pi_3 + \lambda_{73} \leq 0 \\
& \pi_1 + \pi_2 + \pi_3 \leq 1 \\
& \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7 \geq 0 \\
& \pi_1, \pi_2, \pi_3 \geq 0 \\
& \lambda_{11}, \lambda_{12}, \lambda_{13}, \dots, \lambda_{73} \leq 0
\end{aligned}$$

An optimal solution $(\tilde{\mu}, \tilde{\pi}, \tilde{\lambda})(D(\text{LP}_{P\|C_{\max}}))$ is the solution of [Example 14.3](#) with $\tilde{\lambda}_{jh} = 0$ for every job and every machine:

$$\tilde{\mu}_1 = \tilde{\mu}_2 = \frac{5}{3}, \quad \tilde{\mu}_3 = \tilde{\mu}_4 = \frac{4}{3}, \quad \tilde{\mu}_5 = \tilde{\mu}_6 = \tilde{\mu}_7 = 1, \quad \tilde{\pi}_1 = \tilde{\pi}_2 = \tilde{\pi}_3 = \frac{1}{3}, \quad \tilde{\lambda}_{jh} = 0$$

The optimal value $z(D(\text{LP}_{P\|C_{\max}}))$ is 9.

14.4.1 Some important theoretical properties

Proposition 14.2: Worst-case ratio of the LP bound

For every $P||C_{\max}$ instance,

$$\frac{z(\text{LP}_{P||C_{\max}})}{z(\text{ILP}_{P||C_{\max}})} \geq \frac{1}{m}.$$

Proof. Assigning every job to one machine gives a feasible schedule of makespan $\sum_j t_j$. Hence $z(\text{ILP}_{P||C_{\max}}) \leq \sum_j t_j$. By Proposition 14.1,

$$\frac{z(\text{LP}_{P||C_{\max}})}{z(\text{ILP}_{P||C_{\max}})} \geq \frac{\sum_j t_j}{\sum_j t_j} = \frac{1}{m}.$$

□

Proposition 14.3: The ratio $1/m$ is asymptotically attained

For a fixed number $m \geq 2$ of machines, there are instances for which

$$\frac{z(\text{LP}_{P||C_{\max}})}{z(\text{ILP}_{P||C_{\max}})} \rightarrow \frac{1}{m}.$$

Proof. For $M \in \mathbb{Z}_{\geq 2}$, consider one job of processing time M and m jobs of processing time one. The long job alone forces $z(\text{ILP}_{P||C_{\max}}) \geq M$. The schedule that assigns the long job alone to machine 1 and the m unit jobs to the other $m - 1$ machines has loads at most $2 \leq M$, hence the makespan M is feasible. Thus $z(\text{ILP}_{P||C_{\max}}) = M$. Proposition 14.1 gives

$$z(\text{LP}_{P||C_{\max}}) = \frac{M + m}{m}.$$

Therefore,

$$\lim_{M \rightarrow +\infty} \frac{z(\text{LP}_{P||C_{\max}})}{z(\text{ILP}_{P||C_{\max}})} = \lim_{M \rightarrow +\infty} \left(\frac{1}{m} + \frac{1}{M} \right) = \frac{1}{m}.$$

□

Example 14.6: an integrality gap

Consider two machines and three jobs with processing times $(3, 3, 2)$. Proposition 14.1 gives $z(\text{LP}_{P||C_{\max}}) = 8/2 = 4$. In every integer schedule one machine processes at least two of the three jobs, so its load is at least $3 + 2 = 5$; the schedule $\{3, 2\}, \{3\}$ reaches it, so $z(\text{ILP}_{P||C_{\max}}) = 5$. The fractional assignment

$$x_{j1} = x_{j2} = \frac{1}{2} \quad \forall j \in \{1, 2, 3\}$$

has load four on both machines, whereas the integer optimum has loads five and three.

14.5 Additional considerations

- For every job j , the inequality

$$C_{\max} \geq t_j \quad (14.6)$$

is a **valid inequality** for model (14.1). In every integer schedule, job j is assigned to one machine, whose load and therefore makespan are at least t_j (n linear constraints);

- The inequality

$$C_{\max} \geq \left\lceil \frac{\sum_{j \in \mathcal{J}} t_j}{m} \right\rceil \quad (14.7)$$

is also a **valid inequality** for model (14.1): the m machine loads sum to $\sum_j t_j$, so their maximum is at least their average and is an integer in every integer schedule (one linear constraint);

- The machines are identical. The constraints

$$\sum_{j \in \mathcal{J}} t_j x_{jh} \geq \sum_{j \in \mathcal{J}} t_j x_{j,h+1} \quad \forall h \in \{1, \dots, m-1\} \quad (14.8)$$

are **optimality-preserving constraints**: they are not valid for every feasible assignment, but every schedule can be relabelled so that its machine loads are non-increasing, without changing the makespan.

- The constraints (14.8) remove the equivalent relabellings of the machines only when the loads are distinct. With equal loads they remove none: in the instance of the chapter all the optimal schedules have loads 9, 9, 9, and all the 12 optimal solutions of the model satisfy them.

14.6 Typical mistakes

Warning

- **Using the LP bound alone.** The value $\sum_j t_j/m$ ignores the longest job: with one very long job it is much smaller than the optimal value. The valid inequality $C_{\max} \geq t_j$ must be added.
- **Forgetting to round up.** All the processing times are integer, so $C_{\max}$ is an integer in every integer schedule: the lower bound is $\lceil \sum_j t_j/m \rceil$.
- **Believing that the symmetry constraints remove all the equivalent solutions.** They remove the relabellings of the machines only when the loads are distinct; with equal loads they remove none.
- **Forgetting the direction of the load constraints.** The constraints (14.1c) bound $C_{\max}$ from below: it is the minimization that makes $C_{\max}$ equal to the largest load.
- **Mixing the indices.** The variable x_{jh} has the job j first and the machine h second.

14.7 In summary

Summary

- The $\text{ILP}_{P\|C_{\max}}$ model has $n \cdot m$ assignment variables x_{jh} and the makespan variable $C_{\max}$, linked by one load constraint for each machine.
- The LP relaxation has optimal value $z(\text{LP}_{P\|C_{\max}}) = \sum_j t_j/m$, attained by assigning every job with value $1/m$ to every machine; the dual solution is $\tilde{\pi}_h = 1/m$ and $\tilde{\mu}_j = t_j/m$.
- The ratio between $z(\text{LP}_{P\|C_{\max}})$ and $z(\text{ILP}_{P\|C_{\max}})$ is at least $1/m$, and $1/m$ is asymptotically attained.
- The valid inequalities $C_{\max} \geq t_j$ and $C_{\max} \geq \lceil \sum_j t_j/m \rceil$ strengthen the model.

Dashboard of the instance of the chapter ($m = 3$, $t = (5, 5, 4, 4, 3, 3, 3)$). Lower bounds on the optimal value 9:

Bound	Value	From
LP relaxation	9	$\sum_j t_j/m = 27/3$
Valid inequalities	9	$\max\{5, \lceil 27/3 \rceil\}$
Optimal value	9	loads 9, 9, 9

On the instance of the integrality gap ($m = 2$, $t = (3, 3, 2)$) the LP bound is 4 and the optimal value is 5.

Formulations of lot sizing problems

Prerequisites

- From the module *Integer linear programming*, [Chapter 1 «Integer linear programming»](#) and [Chapter 2 «Primal and dual bounds»](#): ILP problems, LP relaxations and lower bounds.
- From this module, [Chapter 8 «Lot sizing problems»](#): the problem, its assumptions, the plans and the reference instance.

Objectives

At the end of this chapter you will be able to:

- write the big-M formulation of the capacitated lot sizing problem and compute a weak and a strong big-M value;
- write the shortest-path formulation and interpret its variables as the arcs of a network;
- represent a production plan as a combination of paths of the network;
- compare the LP bounds of the formulations on the reference instance;
- write the ILP model of the uncapacitated lot sizing problem;
- extend the big-M formulation to several products with setup times, to the initial stock, to overtime and to unsatisfied demand, and compute the strong big-M values of each model.

15.1 Overview of the family

- The big- M models of the lot sizing problems use, for every period $t \in T$, an integer **production variable** x_t , an integer **inventory variable** s_t and a binary **setup variable** y_t .
- The **flow balancing constraints** link the inventories of consecutive periods to the production and to the demand; the setup variables are linked to the production by big- M constraints.
- The objective function minimizes the total cost of setups, production and holding. The shortest-path formulation of the CLS replaces the production and inventory variables with the arcs of a network.

The lot sizing models are flow models over the periods, with binary setup variables that switch the production on. The members of the family add products, setup times, an initial stock, overtime or unsatisfied demand to the same balance and big- M constraints.

- The table lists the members of the family, with the names of their models and the sections that write them.

Problem	Models	Formulation	Section
CLS	ILP _{CLSP}	big- M formulation	15.3
CLS	MIP _{CLSP} ^{SP}	shortest-path formulation	15.5
ULS	ILP _{ULS}	big- M , no capacity	15.7
MLS	ILP _{CLSP} ^P	big- M , setup times	15.7
LS-SI	ILP _{CLSP} ^{SI}	big- M , initial stock variables	15.7
LS-OT	ILP _{CLSP} ^{OT}	big- M , overtime variables	15.7
LS-UD	ILP _{CLSP} ^{UD}	big- M , unsatisfied demand variables	15.7

15.2 ILP models for the Capacitated Lot Sizing Problem

The Capacitated Lot Sizing Problem and its reference instance (Example 8.1) are defined in Chapter 8 «Lot sizing problems», Section 8.2.

- The CLS has two formulations: the big- M formulation ILP_{CLSP} of Section 15.3 and the shortest-path formulation MIP_{CLSP}^{SP} of Section 15.5.
- Section 15.6 compares the bounds of their Linear Programming relaxations on the reference instance.

15.3 A formulation with a big-M link

- Given a set $T = \{1, 2, \dots, m\}$ of periods with demands d_t , for $j \leq t$ we write $D_{j,t} = \sum_{k=j}^t d_k$ for the total demand of the periods $j, j+1, \dots, t$.
- We introduce the following **integer variables** and **binary variables**, for every period $t \in T$:

$$x_t \in \mathbb{Z}_{\geq 0} \text{ (production),} \quad s_t \in \mathbb{Z}_{\geq 0} \text{ (inventory at the end of the period),}$$

$$y_t = \begin{cases} 1 & \text{if there is a setup in period } t \\ 0 & \text{otherwise} \end{cases}$$

The inventory at the beginning of the horizon is the constant s_0 .

- The setup variables are linked to the quantities by the logical implication

$$x_t > 0 \implies y_t = 1,$$

modeled by the **big-M constraint** $x_t \leq M_t y_t$, where M_t is an upper bound on the production of period t .

- If the setups are ignored, the best plan produces in every period exactly its demand: it has no inventory and, in the reference instance, cost 0. This is a very weak lower bound: the setup costs must be modeled.

Using these variables, an Integer Linear Programming (ILP) model for the capacitated lot sizing problem (CLS) reads as follows:

$$\min \sum_{t \in T} (sc_t y_t + vc_t x_t + hc_t s_t) \quad (15.1a)$$

$$\text{subject to } s_{t-1} + x_t = d_t + s_t, \quad \forall t \in T, \quad (15.1b)$$

$$x_t \leq M_t y_t, \quad \forall t \in T, \quad (15.1c)$$

$$x_t \leq cap_t, \quad \forall t \in T, \quad (15.1d)$$

$$s_0 = s_m = 0, \quad (15.1e)$$

$$x_t \in \mathbb{Z}_{\geq 0}, \quad \forall t \in T, \quad (15.1f)$$

$$s_t \in \mathbb{Z}_{\geq 0}, \quad \forall t \in T, \quad (15.1g)$$

$$y_t \in \{0, 1\}, \quad \forall t \in T. \quad (15.1h)$$

This ILP model is referred to as the ILP_{CLSP} model, or **big-M formulation**.

- Description of the objective function and constraints of the ILP_{CLSP} model:
 - the linear objective function (15.1a) minimizes the cost of the setups, of the production and of the holding;

- constraints (15.1b) are the **flow balancing constraints**: the demand of period t is satisfied by the production of the period and by the inventory of the previous period, and the surplus becomes the inventory at the end of period t (m linear constraints);
- constraints (15.1c) link the quantities with the setups, and constraints (15.1d) are the capacity constraints (m linear constraints each);
- constraint (15.1e) imposes an empty inventory at the beginning and at the end of the horizon;
- finally, constraints (15.1f), (15.1g) and (15.1h) define the variables of the ILP model.

15.3.1 Weak and strong big-M values

- **A weak value.** Since $s_m = 0$ and the inventories are non-negative, the production of period t cannot exceed the demand of the periods $t, \dots, m$: $M_t^{\text{dem}} = D_{t,m}$ is a valid value.
- **A second value.** By the capacity constraints (15.1d), $M_t^{\text{cap}} = \text{cap}_t$ is also valid.
- **A strong value.** The best value is the smallest of the two:

$$M_t = \min \{ \text{cap}_t, D_{t,m} \}.$$

The smaller the big-M value, the stronger the LP relaxation: the constraint $x_t \leq M_t y_t$ cuts off the fractional values of y_t below x_t/M_t . A value that is too small, however, cuts off feasible plans.

Example 15.1: the reference instance

We consider $m = 5$ periods, with the following data:

period t	1	2	3	4	5
demand d_t	2	1	3	4	1
capacity cap_t	4	4	4	4	4
setup cost sc_t	50	50	50	50	50
production cost vc_t	0	0	0	0	0
holding cost hc_t	10	10	10	10	10

This is the instance of Example 8.1 of [Chapter 8 «Lot sizing problems»](#). The big-M values are:

period t	1	2	3	4	5
$M_t^{\text{dem}} = D_{t,5}$	11	9	8	5	1
$M_t^{\text{cap}} = \text{cap}_t$	4	4	4	4	4
M_t	4	4	4	4	1

We refer to this CLS instance as [Example 15.1](#).

Example 15.2: ILP_{CLSP} model

The ILP_{CLSP} model with the strong big-M values for the CLS instance of [Example 15.1](#) is ($s_0 = 0$ is a constant, and it does not appear):

$$\begin{aligned}
 \min \quad & 50 y_1 + 50 y_2 + 50 y_3 + 50 y_4 + 50 y_5 + 10 s_1 + 10 s_2 \\
 & + 10 s_3 + 10 s_4 + 10 s_5 \\
 \text{subject to} \quad & x_1 - s_1 = 2 \\
 & s_1 + x_2 - s_2 = 1 \\
 & s_2 + x_3 - s_3 = 3 \\
 & s_3 + x_4 - s_4 = 4 \\
 & s_4 + x_5 - s_5 = 1 \\
 & x_1 - 4 y_1 \leq 0 \\
 & x_2 - 4 y_2 \leq 0 \\
 & x_3 - 4 y_3 \leq 0 \\
 & x_4 - 4 y_4 \leq 0 \\
 & x_5 - y_5 \leq 0 \\
 & x_1 \leq 4 \\
 & x_2 \leq 4 \\
 & x_3 \leq 4 \\
 & x_4 \leq 4 \\
 & x_5 \leq 4 \\
 & s_5 = 0 \\
 & x_1, x_2, x_3, x_4, x_5 \in \mathbb{Z}_{\geq 0} \\
 & s_1, s_2, s_3, s_4, s_5 \in \mathbb{Z}_{\geq 0} \\
 & y_1, y_2, y_3, y_4, y_5 \in \{0, 1\}
 \end{aligned}$$

An optimal solution $(\tilde{\mathbf{y}}, \tilde{\mathbf{x}}, \tilde{\mathbf{s}})$ (ILP_{CLSP}) is:

$$\tilde{\mathbf{y}} = (1, 0, 1, 1, 0), \quad \tilde{\mathbf{x}} = (3, 0, 4, 4, 0), \quad \tilde{\mathbf{s}} = (1, 0, 1, 1, 0)$$

The optimal value $z(\text{ILP}_{\text{CLSP}})$ is $3 \cdot 50 + 3 \cdot 10 = 180$, the cost of the optimal plan found by enumeration in [Chapter 8 «Lot sizing problems»](#). The weak big-M values give the same optimal solution.

15.4 The Linear Programming relaxation of the big-M formulation

To obtain a Linear Programming (LP) relaxation of the ILP_{CLSP} model, we relax constraints (15.1f)–(15.1h) into non-negativity constraints and the bounds $y_t \leq 1$. The obtained LP model reads as follows:

$$\min \sum_{t \in T} (sc_t y_t + vc_t x_t + hc_t s_t) \quad (15.2a)$$

$$\text{subject to } s_{t-1} + x_t = d_t + s_t, \quad \forall t \in T, \quad (15.2b)$$

$$x_t \leq M_t y_t, \quad \forall t \in T, \quad (15.2c)$$

$$x_t \leq cap_t, \quad \forall t \in T, \quad (15.2d)$$

$$s_0 = s_m = 0, \quad (15.2e)$$

$$y_t \leq 1, \quad \forall t \in T, \quad (15.2f)$$

$$x_t \geq 0, \quad \forall t \in T, \quad (15.2g)$$

$$s_t \geq 0, \quad \forall t \in T, \quad (15.2h)$$

$$y_t \geq 0, \quad \forall t \in T. \quad (15.2i)$$

This LP model is referred to as the $\text{LP}_{\text{CLSP}}^{\text{dem}}$, $\text{LP}_{\text{CLSP}}^{\text{cap}}$ or $\text{LP}_{\text{CLSP}}^{\text{strong}}$ model, when the big-M values are M_t^{dem} , M_t^{cap} or the strong values M_t .

Example 15.3: big-M values and LP bounds

The LP model (15.2) for the CLS instance of [Example 15.1](#) is:

$$\begin{aligned}
 \min \quad & 50 y_1 + 50 y_2 + 50 y_3 + 50 y_4 + 50 y_5 + 10 s_1 + 10 s_2 \\
 & + 10 s_3 + 10 s_4 + 10 s_5 \\
 \text{subject to} \quad & x_1 - s_1 = 2 \\
 & s_1 + x_2 - s_2 = 1 \\
 & s_2 + x_3 - s_3 = 3 \\
 & s_3 + x_4 - s_4 = 4 \\
 & s_4 + x_5 - s_5 = 1 \\
 & x_1 - M_1 y_1 \leq 0 \\
 & x_2 - M_2 y_2 \leq 0 \\
 & x_3 - M_3 y_3 \leq 0 \\
 & x_4 - M_4 y_4 \leq 0 \\
 & x_5 - M_5 y_5 \leq 0 \\
 & x_1 \leq 4 \\
 & x_2 \leq 4 \\
 & x_3 \leq 4 \\
 & x_4 \leq 4 \\
 & x_5 \leq 4 \\
 & s_5 = 0 \\
 & y_1 \leq 1 \\
 & y_2 \leq 1 \\
 & y_3 \leq 1 \\
 & y_4 \leq 1 \\
 & y_5 \leq 1 \\
 & x_1, x_2, x_3, x_4, x_5 \geq 0 \\
 & s_1, s_2, s_3, s_4, s_5 \geq 0 \\
 & y_1, y_2, y_3, y_4, y_5 \geq 0
 \end{aligned}$$

with the values $M_1, \dots, M_5$ of the table of [Example 15.1](#). The three choices give the following optimal solutions $(\tilde{y}, \tilde{x}, \tilde{s})$, with $\tilde{s}_5 = 0$:

Model	$\tilde{y}$	$\tilde{x}$	$(\tilde{s}_1, \dots, \tilde{s}_4)$	optimal value
$\text{LP}_{\text{CLSP}}^{\text{dem}}$	$(\frac{2}{11}, \frac{1}{9}, \frac{1}{2}, \frac{4}{5}, 0)$	$(2, 1, 4, 4, 0)$	$(0, 0, 1, 1)$	$\frac{9865}{99} \approx 99.65$
$\text{LP}_{\text{CLSP}}^{\text{cap}}$	$(\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, 1, \frac{1}{4})$	$(2, 1, 3, 4, 1)$	$(0, 0, 0, 0)$	$\frac{275}{2} = 137.5$
$\text{LP}_{\text{CLSP}}^{\text{strong}}$	$(\frac{1}{2}, \frac{1}{4}, 1, 1, 0)$	$(2, 1, 4, 4, 0)$	$(0, 0, 1, 1)$	$\frac{315}{2} = 157.5$

In every solution $\tilde{y}_t = \tilde{x}_t / M_t$: the smaller M_t , the larger the cost of the setups paid by the LP. The integer optimal solution of [Example 15.2](#) has value $z(\text{ILP}_{\text{CLSP}}) = 180$.

15.5 A shortest-path formulation

- The second formulation does not decide the quantities directly. For every $t, k \in T$, with $k \geq t$, we introduce the following **continuous variable**: $zv_{tk} \geq 0$, the fraction of the demand $D_{t,k}$ of the periods $t, \dots, k$ that is produced in period t .
- Producing in period t the demand of the periods $t, \dots, k$ costs the production of $D_{t,k}$ units and the holding of the units that wait for their period:

$$cv_{tk} = vc_t D_{t,k} + \sum_{s=t+1}^k \sum_{u=t}^{s-1} hc_u d_s.$$

Using these variables and the binary variables y_t , a Mixed Integer Linear Programming (MIP) model for the CLS reads as follows:

$$\min \sum_{t \in T} sc_t y_t + \sum_{t \in T} \sum_{k=t}^m cv_{tk} zv_{tk} \quad (15.3a)$$

$$\text{subject to } \sum_{k=1}^m zv_{1k} = 1, \quad (15.3b)$$

$$\sum_{u=1}^{t-1} zv_{u,t-1} - \sum_{k=t}^m zv_{tk} = 0, \quad \forall t \in T \setminus \{1\}, \quad (15.3c)$$

$$\sum_{k=t}^m zv_{tk} \leq y_t, \quad \forall t \in T, \quad (15.3d)$$

$$\sum_{k=t}^m D_{t,k} zv_{tk} \leq cap_t, \quad \forall t \in T, \quad (15.3e)$$

$$zv_{tk} \geq 0, \quad \forall t, k \in T, k \geq t, \quad (15.3f)$$

$$y_t \in \{0, 1\}, \quad \forall t \in T. \quad (15.3g)$$

This MIP model is referred to as the $\text{MIP}_{\text{CLSP}}^{\text{SP}}$ model, or **shortest-path formulation**.

- Description of the objective function and constraints of the $\text{MIP}_{\text{CLSP}}^{\text{SP}}$ model:
 - the linear objective function (15.3a) minimizes the cost of the setups and of the arcs of the plan;
 - constraints (15.3b) and (15.3c) are the flow constraints of a network (m linear constraints);
 - constraints (15.3d) force the setup if the product is produced in period t , and constraints (15.3e) impose the capacities (m linear constraints each);
 - finally, constraints (15.3f) and (15.3g) define the variables of the MIP model.
- The network is drawn in Figure 15.1: the nodes are $0, 1, \dots, m$, and the variable zv_{tk} is the arc from node $t - 1$ to node k , with cost cv_{tk} . A unit of flow leaves node 0 (constraint (15.3b)), and the flow is conserved at the other nodes (constraints (15.3c)).

- A path from node 0 to node m is a plan that produces only when the stock is empty: each arc $(t-1, k)$ is a production in period t for the periods $t, \dots, k$. This is why the formulation is called a shortest-path formulation: with the capacities relaxed, the optimal plan is a shortest path.
- Constraints (15.3d) link the arcs with the setups with the coefficient 1 instead of a big-M value. In constraints (15.3e) the quantity produced in period t is $x_t = \sum_k D_{t,k} zv_{tk}$.
- The paths are the building blocks, and a plan that is not a path is a **combination of paths**: the quantities are integer, but the variables $z v_{tk}$ may be fractional.

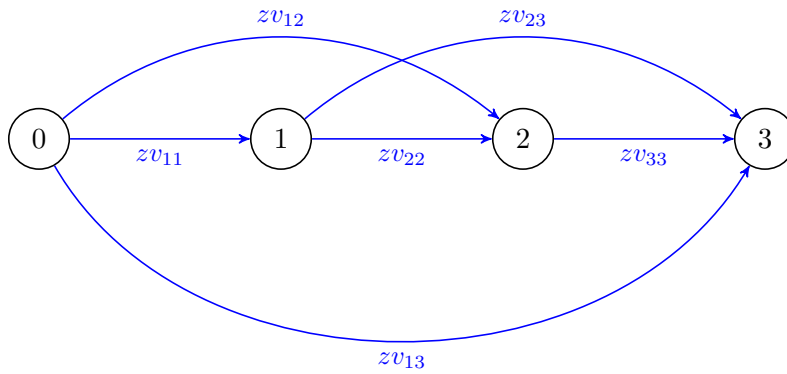


Figure 15.1: Network of the variables $z v_{tk}$ for $m = 3$: the variable $z v_{tk}$ is the arc from node $t-1$ to node k .

Example 15.4: $\text{MIP}_{\text{CLSP}}^{\text{SP}}$ model and combinations of paths

For the CLS instance of Example 15.1 the demands $D_{t,k}$ and the costs cv_{tk} are (the production cost is zero, so cv_{tk} is only holding cost):

$t \backslash k$	$D_{t,k}$					cv_{tk}				
	1	2	3	4	5	1	2	3	4	5
1	2	3	6	10	11	0	10	70	190	230
2		1	4	8	9		0	30	110	140
3			3	7	8			0	40	60
4				4	5				0	10
5					1					0

The $\text{MIP}_{\text{CLSP}}^{\text{SP}}$ model for this instance is:

$$\begin{aligned}
 \min \quad & 50 y_1 + 50 y_2 + 50 y_3 + 50 y_4 + 50 y_5 + 10 z v_{12} + 70 z v_{13} \\
 & + 190 z v_{14} + 230 z v_{15} + 30 z v_{23} + 110 z v_{24} + 140 z v_{25} + 40 z v_{34} \\
 & + 60 z v_{35} + 10 z v_{45} \\
 \text{subject to} \quad & z v_{11} + z v_{12} + z v_{13} + z v_{14} + z v_{15} = 1 \\
 & z v_{11} - z v_{22} - z v_{23} - z v_{24} - z v_{25} = 0 \\
 & z v_{12} + z v_{22} - z v_{33} - z v_{34} - z v_{35} = 0 \\
 & z v_{13} + z v_{23} + z v_{33} - z v_{44} - z v_{45} = 0 \\
 & z v_{14} + z v_{24} + z v_{34} + z v_{44} - z v_{55} = 0 \\
 & z v_{11} + z v_{12} + z v_{13} + z v_{14} + z v_{15} - y_1 \leq 0 \\
 & z v_{22} + z v_{23} + z v_{24} + z v_{25} - y_2 \leq 0 \\
 & z v_{33} + z v_{34} + z v_{35} - y_3 \leq 0 \\
 & z v_{44} + z v_{45} - y_4 \leq 0 \\
 & z v_{55} - y_5 \leq 0 \\
 & 2 z v_{11} + 3 z v_{12} + 6 z v_{13} + 10 z v_{14} + 11 z v_{15} \leq 4 \\
 & z v_{22} + 4 z v_{23} + 8 z v_{24} + 9 z v_{25} \leq 4 \\
 & 3 z v_{33} + 7 z v_{34} + 8 z v_{35} \leq 4 \\
 & 4 z v_{44} + 5 z v_{45} \leq 4 \\
 & z v_{55} \leq 4 \\
 & z v_{11}, z v_{12}, z v_{13}, \dots, z v_{55} \geq 0 \\
 & y_1, y_2, y_3, y_4, y_5 \in \{0, 1\}
 \end{aligned}$$

An optimal solution $(\tilde{z}v, \tilde{y})(\text{MIP}_{\text{CLSP}}^{\text{SP}})$ is:

$$\tilde{z}v_{12} = 1, \quad \tilde{z}v_{33} = \frac{4}{5}, \quad \tilde{z}v_{35} = \frac{1}{5}, \quad \tilde{z}v_{45} = \frac{4}{5}, \quad \tilde{y} = (1, 0, 1, 1, 0),$$

and all the other variables $z v_{tk}$ are zero. It is the optimal plan $\tilde{x} = (3, 0, 4, 4, 0)$, which is not a path: in period 3 it produces 4 units, more than the demand 3 of period 3, and less than the demand 7 of the periods 3 and 4. It is the combination, with weights $\frac{4}{5}$ and $\frac{1}{5}$, of the paths $0 \rightarrow 2 \rightarrow 3 \rightarrow 5$ (plan $(3, 0, 3, 5, 0)$) and $0 \rightarrow 2 \rightarrow 5$ (plan $(3, 0, 8, 0, 0)$). The quantities are $x_1 = D_{1,2} z v_{12} = 3$, $x_3 = 3 \cdot \frac{4}{5} + 8 \cdot \frac{1}{5} = 4$ and $x_4 = 5 \cdot \frac{4}{5} = 4$. The optimal value $z(\text{MIP}_{\text{CLSP}}^{\text{SP}})$ is

$$\underbrace{3 \cdot 50}_{\text{setups}} + \underbrace{10 \cdot 1 + 0 \cdot \frac{4}{5} + 60 \cdot \frac{1}{5} + 10 \cdot \frac{4}{5}}_{=30} = 180.$$

To obtain an LP relaxation of the $\text{MIP}_{\text{CLSP}}^{\text{SP}}$ model, we relax constraints (15.3g) into the bounds $0 \leq y_t \leq 1$. The obtained LP model reads as follows:

$$\min \sum_{t \in T} s c_t y_t + \sum_{t \in T} \sum_{k=t}^m c v_{tk} z v_{tk} \quad (15.4a)$$

$$\text{subject to } \sum_{k=1}^m z v_{1k} = 1, \quad (15.4b)$$

$$\sum_{u=1}^{t-1} z v_{u,t-1} - \sum_{k=t}^m z v_{tk} = 0, \quad \forall t \in T \setminus \{1\}, \quad (15.4c)$$

$$\sum_{k=t}^m z v_{tk} \leq y_t, \quad \forall t \in T, \quad (15.4d)$$

$$\sum_{k=t}^m D_{t,k} z v_{tk} \leq \text{cap}_t, \quad \forall t \in T, \quad (15.4e)$$

$$y_t \leq 1, \quad \forall t \in T, \quad (15.4f)$$

$$z v_{tk} \geq 0, \quad \forall t, k \in T, \ k \geq t, \quad (15.4g)$$

$$y_t \geq 0, \quad \forall t \in T. \quad (15.4h)$$

This LP model is referred to as the $\text{LP}_{\text{CLSP}}^{\text{SP}}$ model.

Example 15.5: $\text{LP}_{\text{CLSP}}^{\text{SP}}$ model

The $\text{LP}_{\text{CLSP}}^{\text{SP}}$ model for the CLS instance of [Example 15.1](#) is:

$$\begin{aligned}
 \min \quad & 50 y_1 + 50 y_2 + 50 y_3 + 50 y_4 + 50 y_5 + 10 z v_{12} + 70 z v_{13} \\
 & + 190 z v_{14} + 230 z v_{15} + 30 z v_{23} + 110 z v_{24} + 140 z v_{25} + 40 z v_{34} \\
 & + 60 z v_{35} + 10 z v_{45} \\
 \text{subject to} \quad & z v_{11} + z v_{12} + z v_{13} + z v_{14} + z v_{15} = 1 \\
 & z v_{11} - z v_{22} - z v_{23} - z v_{24} - z v_{25} = 0 \\
 & z v_{12} + z v_{22} - z v_{33} - z v_{34} - z v_{35} = 0 \\
 & z v_{13} + z v_{23} + z v_{33} - z v_{44} - z v_{45} = 0 \\
 & z v_{14} + z v_{24} + z v_{34} + z v_{44} - z v_{55} = 0 \\
 & z v_{11} + z v_{12} + z v_{13} + z v_{14} + z v_{15} - y_1 \leq 0 \\
 & z v_{22} + z v_{23} + z v_{24} + z v_{25} - y_2 \leq 0 \\
 & z v_{33} + z v_{34} + z v_{35} - y_3 \leq 0 \\
 & z v_{44} + z v_{45} - y_4 \leq 0 \\
 & z v_{55} - y_5 \leq 0 \\
 & 2 z v_{11} + 3 z v_{12} + 6 z v_{13} + 10 z v_{14} + 11 z v_{15} \leq 4 \\
 & z v_{22} + 4 z v_{23} + 8 z v_{24} + 9 z v_{25} \leq 4 \\
 & 3 z v_{33} + 7 z v_{34} + 8 z v_{35} \leq 4 \\
 & 4 z v_{44} + 5 z v_{45} \leq 4 \\
 & z v_{55} \leq 4 \\
 & y_1 \leq 1 \\
 & y_2 \leq 1 \\
 & y_3 \leq 1 \\
 & y_4 \leq 1 \\
 & y_5 \leq 1 \\
 & z v_{11}, z v_{12}, z v_{13}, \dots, z v_{55} \geq 0 \\
 & y_1, y_2, y_3, y_4, y_5 \geq 0
 \end{aligned}$$

An optimal solution $(\tilde{z}\mathbf{v}, \tilde{\mathbf{y}})(\text{LP}_{\text{CLSP}}^{\text{SP}})$ is:

$$\tilde{z}v_{12} = 1, \quad \tilde{z}v_{33} = \frac{4}{5}, \quad \tilde{z}v_{35} = \frac{1}{5}, \quad \tilde{z}v_{45} = \frac{4}{5}, \quad \tilde{\mathbf{y}} = (1, 0, 1, \frac{4}{5}, 0),$$

and all the other variables $z v_{tk}$ are zero: the arcs of the optimal plan, with the setup of period 4 paid only for the fraction $\frac{4}{5}$. The optimal value $z(\text{LP}_{\text{CLSP}}^{\text{SP}})$ is $180 - \frac{1}{5} \cdot 50 = 170$.

15.6 Comparing the formulations

- The formulations have the same optimal value $z(\text{ILP}_{\text{CLSP}}) = z(\text{MIP}_{\text{CLSP}}^{\text{SP}}) = 180$, but their LP relaxations have different optimal values. On the reference instance the lower bounds, computed with the exact simplex algorithm and checked with Gurobi, are the following.
- The last column is the gap of Chapter 1 «Combinatorial optimization problems» between the bound and the optimal value; it is undefined for the bound 0.

Model	Lower bound	Gap $\frac{180-\text{LB}}{\text{LB}}$
Without setups	0	undefined
$\text{LP}_{\text{CLSP}}^{\text{dem}} (M_t = D_{t,m})$	99.65	80.6%
$\text{LP}_{\text{CLSP}}^{\text{cap}} (M_t = \text{cap}_t)$	137.50	30.9%
$\text{LP}_{\text{CLSP}}^{\text{strong}} (M_t = \min\{\text{cap}_t, D_{t,m}\})$	157.50	14.3%
$\text{LP}_{\text{CLSP}}^{\text{SP}}$ (shortest path)	170	5.9%

- The bound 170 of the shortest-path formulation is the optimal value of the problem without capacity (plan $(3, 0, 8, 0, 0)$, of cost 170): the capacity constraints (15.4e) act only on the quantities, and a combination of paths can satisfy them.
- The price of the shortest-path formulation is its size: there are $O(m^2)$ variables zv_{tk} , against the $O(m)$ variables of the big-M formulation.

15.7 ILP models for the variants

The problems of this section are the variants of the capacitated lot sizing problem defined in Chapter 8 «Lot sizing problems», Section 8.6. Each subsection writes the ILP model of one variant, in the order of that chapter.

15.7.1 The uncapacitated lot sizing problem

- Given a set $T = \{1, 2, \dots, m\}$ of periods with demands d_t and no binding capacity, we introduce, for every period $t \in T$, the following **integer variables** and **binary variables**:

$$x_t \in \mathbb{Z}_{\geq 0} \text{ (production),} \quad s_t \in \mathbb{Z}_{\geq 0} \text{ (inventory at the end of the period),}$$

$$y_t = \begin{cases} 1 & \text{if there is a setup in period } t \\ 0 & \text{otherwise} \end{cases}$$

- Since $s_m = 0$, the production of period t is at most the demand $D_{t,m}$ of the periods $t, \dots, m$: this is the big-M value of the link between x_t and y_t .

Using these variables, an Integer Linear Programming (ILP) model for the uncapacitated

lot sizing problem (ULS) reads as follows:

$$\min \sum_{t \in T} (sc_t y_t + vc_t x_t + hc_t s_t) \quad (15.5a)$$

$$\text{subject to } s_{t-1} + x_t = d_t + s_t, \quad \forall t \in T, \quad (15.5b)$$

$$x_t \leq D_{t,m} y_t, \quad \forall t \in T, \quad (15.5c)$$

$$s_0 = s_m = 0, \quad (15.5d)$$

$$x_t \in \mathbb{Z}_{\geq 0}, \quad \forall t \in T, \quad (15.5e)$$

$$s_t \in \mathbb{Z}_{\geq 0}, \quad \forall t \in T, \quad (15.5f)$$

$$y_t \in \{0, 1\}, \quad \forall t \in T. \quad (15.5g)$$

This ILP model is referred to as the ILP_{ULS} model.

- Description of the objective function and constraints of the ILP_{ULS} model:
 - the linear objective function (15.5a) minimizes the cost of the setups, of the production and of the holding;
 - constraints (15.5b) are the flow balancing constraints (m linear constraints);
 - constraints (15.5c) force a setup in every period with positive production (m linear constraints);
 - constraint (15.5d) imposes an empty inventory at the beginning and at the end of the horizon;
 - finally, constraints (15.5e)–(15.5g) define the variables of the ILP model.
- The ILP_{ULS} model is the ILP_{CLSP} model without the capacity constraints (15.1d) and with the big-M values $M_t = D_{t,m}$. The DP algorithm of Wagner and Whitin of [Chapter 25](#) «DP for the uncapacitated lot sizing problem» solves the ULS in polynomial time.

Example 15.6: ILP_{ULS} model

We consider the CLS instance of [Example 15.1](#) without the capacities: the demands are $\mathbf{d} = (2, 1, 3, 4, 1)$, the setup costs $sc_t = 50$, the holding costs $hc_t = 10$, and $vc_t = 0$. The big-M values are $D_{t,5} = (11, 9, 8, 5, 1)$, and the ILP_{ULS} model for this ULS instance is:

$$\begin{aligned}
 \min \quad & 50 y_1 + 50 y_2 + 50 y_3 + 50 y_4 + 50 y_5 + 10 s_1 \\
 & + 10 s_2 + 10 s_3 + 10 s_4 + 10 s_5 \\
 \text{subject to} \quad & x_1 - s_1 = 2 \\
 & s_1 + x_2 - s_2 = 1 \\
 & s_2 + x_3 - s_3 = 3 \\
 & s_3 + x_4 - s_4 = 4 \\
 & s_4 + x_5 - s_5 = 1 \\
 & x_1 - 11 y_1 \leq 0 \\
 & x_2 - 9 y_2 \leq 0 \\
 & x_3 - 8 y_3 \leq 0 \\
 & x_4 - 5 y_4 \leq 0 \\
 & x_5 - y_5 \leq 0 \\
 & s_5 = 0 \\
 & x_1, x_2, x_3, x_4, x_5 \in \mathbb{Z}_{\geq 0} \\
 & s_1, s_2, s_3, s_4, s_5 \in \mathbb{Z}_{\geq 0} \\
 & y_1, y_2, y_3, y_4, y_5 \in \{0, 1\}
 \end{aligned}$$

An optimal solution $(\tilde{y}, \tilde{x}, \tilde{s})(\text{ILP}_{\text{ULS}})$ is:

$$\tilde{y} = (1, 0, 1, 0, 0), \quad \tilde{x} = (3, 0, 8, 0, 0), \quad \tilde{s} = (1, 0, 5, 1, 0)$$

The optimal value $z(\text{ILP}_{\text{ULS}})$ is $2 \cdot 50 + (1 + 5 + 1) \cdot 10 = 170$, the bound of the shortest-path formulation of the CLS. The optimal solution is not unique: the plan $(3, 0, 3, 5, 0)$, with three setups and the stocks $(1, 0, 0, 1, 0)$, also costs $3 \cdot 50 + 2 \cdot 10 = 170$.

15.7.2 Several products and setup times

- A set P of products shares the resource of each period. We write d_{pt} , sc_{pt} , vc_{pt} and hc_{pt} for the data of product $p \in P$ in period $t \in T$. Producing a unit of p takes the time $vt_{pt} \in \mathbb{Z}_{\geq 1}$, a setup takes the time $st_{pt} \in \mathbb{Z}_{\geq 0}$, and the capacity cap_t is a time. For $j \leq k$ we write $sd_{pjk} = \sum_{u=j}^k d_{pu}$ for the demand of product p in the periods $j, \dots, k$.
- We introduce, for every product $p \in P$ and every period $t \in T$, the following **integer variables** and **binary variables**:

$$x_{pt} \in \mathbb{Z}_{\geq 0} \text{ (production)}, \quad s_{pt} \in \mathbb{Z}_{\geq 0} \text{ (inventory at the end of the period)},$$

$$y_{pt} = \begin{cases} 1 & \text{if there is a setup of product } p \text{ in period } t \\ 0 & \text{otherwise} \end{cases}$$

The inventories at the beginning of the horizon are the constants $s_{p0} = 0$.

Using these variables, an Integer Linear Programming (ILP) model for the lot sizing problem with several products and setup times (MLS) reads as follows:

$$\min \sum_{p \in P} \sum_{t \in T} (sc_{pt} y_{pt} + vc_{pt} x_{pt} + hc_{pt} s_{pt}) \quad (15.6a)$$

$$\text{subject to } s_{p,t-1} + x_{pt} = d_{pt} + s_{pt}, \quad \forall p \in P, \forall t \in T, \quad (15.6b)$$

$$\sum_{p \in P} (st_{pt} y_{pt} + vt_{pt} x_{pt}) \leq cap_t, \quad \forall t \in T, \quad (15.6c)$$

$$x_{pt} \leq M_{pt} y_{pt}, \quad \forall p \in P, \forall t \in T, \quad (15.6d)$$

$$s_{p0} = s_{pm} = 0, \quad \forall p \in P, \quad (15.6e)$$

$$x_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T, \quad (15.6f)$$

$$s_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T, \quad (15.6g)$$

$$y_{pt} \in \{0, 1\}, \quad \forall p \in P, \forall t \in T. \quad (15.6h)$$

This ILP model is referred to as the ILP_{CLSP}^P model.

- Description of the objective function and constraints of the ILP_{CLSP}^P model:
 - the linear objective function (15.6a) minimizes the cost of the setups, of the production and of the holding of all the products;
 - constraints (15.6b) are the flow balancing constraints of every product ($|P| m$ linear constraints);
 - constraints (15.6c) impose that the setup times and the production times of each period do not exceed its capacity (m linear constraints);
 - constraints (15.6d) link the quantities with the setups ($|P| m$ linear constraints);
 - constraints (15.6e) impose an empty inventory at the beginning and at the end of the horizon;
 - finally, constraints (15.6f)–(15.6h) define the variables of the ILP model.
- A setup of p in period t leaves the time $cap_t - st_{pt}$ for the production, and $s_{pm} = 0$ bounds the production by the remaining demand. The strong big-M value is

$$M_{pt} = \min \left\{ \left\lfloor \frac{cap_t - st_{pt}}{vt_{pt}} \right\rfloor, s_{ptm} \right\}.$$

- The products interact only through the capacity constraints (15.6c): without them, the model splits into $|P|$ independent single-product models.

Example 15.7: ILP_{CLSP}^P model

We consider $|P| = 2$ products and $m = 3$ periods, with capacities $cap_t = 9$, production times $vt_{pt} = 1$, setup times $st_{pt} = 1$, holding costs $hc_{pt} = 5$ and production costs $vc_{pt} = 0$.

The setup costs are $sc_{1t} = 30$ and $sc_{2t} = 40$, and the demands are:

period t	product 1			product 2		
	1	2	3	1	2	3
d_{pt}	3	2	4	2	4	1
sd_{ptm}	9	6	4	7	5	1
M_{pt}	8	6	4	7	5	1

The $\text{ILP}_{\text{CLSP}}^P$ model with the strong big-M values for this instance is:

$$\begin{aligned}
\min \quad & 30 y_{11} + 30 y_{12} + 30 y_{13} + 40 y_{21} + 40 y_{22} + 40 y_{23} \\
& + 5 s_{11} + 5 s_{12} + 5 s_{13} + 5 s_{21} + 5 s_{22} + 5 s_{23} \\
\text{subject to} \quad & x_{11} - s_{11} = 3 \\
& s_{11} + x_{12} - s_{12} = 2 \\
& s_{12} + x_{13} - s_{13} = 4 \\
& x_{21} - s_{21} = 2 \\
& s_{21} + x_{22} - s_{22} = 4 \\
& s_{22} + x_{23} - s_{23} = 1 \\
& x_{11} + x_{21} + y_{11} + y_{21} \leq 9 \\
& x_{12} + x_{22} + y_{12} + y_{22} \leq 9 \\
& x_{13} + x_{23} + y_{13} + y_{23} \leq 9 \\
& x_{11} - 8 y_{11} \leq 0 \\
& x_{12} - 6 y_{12} \leq 0 \\
& x_{13} - 4 y_{13} \leq 0 \\
& x_{21} - 7 y_{21} \leq 0 \\
& x_{22} - 5 y_{22} \leq 0 \\
& x_{23} - y_{23} \leq 0 \\
& s_{13} = 0 \\
& s_{23} = 0 \\
& x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \in \mathbb{Z}_{\geq 0} \\
& s_{11}, s_{12}, s_{13}, s_{21}, s_{22}, s_{23} \in \mathbb{Z}_{\geq 0} \\
& y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23} \in \{0, 1\}
\end{aligned}$$

An optimal solution $(\tilde{y}, \tilde{x}, \tilde{s})(\text{ILP}_{\text{CLSP}}^P)$ is:

$$\begin{aligned}
& \tilde{y}_{11} = \tilde{y}_{13} = \tilde{y}_{21} = \tilde{y}_{22} = 1, \quad \tilde{y}_{12} = \tilde{y}_{23} = 0, \\
& \tilde{x}_{11} = 5, \quad \tilde{x}_{13} = 4, \quad \tilde{x}_{21} = 2, \quad \tilde{x}_{22} = 5, \quad \tilde{x}_{12} = \tilde{x}_{23} = 0, \\
& \tilde{s}_{11} = 2, \quad \tilde{s}_{22} = 1, \quad \text{all the other } \tilde{s}_{pt} = 0
\end{aligned}$$

The optimal value $z(\text{ILP}_{\text{CLSP}}^P)$ is $2 \cdot 30 + 2 \cdot 40 + (2 + 1) \cdot 5 = 155$, and the optimal solution is unique (Gurobi). Period 1 uses its whole capacity, $5 + 2 + 1 + 1 = 9$. Without the capacity

constraints product 2 would be produced only in period 1, with value 140: this plan needs the time $5 + 7 + 1 + 1 = 14 > 9$.

15.7.3 Initial stock

- The data are those of the lot sizing problem with several products and setup times. In addition, the initial inventory of product p is a decision, bought at the cost $f c_p$ per unit.
- We introduce the integer variables x_{pt} , s_{pt} and the binary variables y_{pt} of the $\text{ILP}_{\text{CLSP}}^P$ model, and, for every product $p \in P$, the following **integer variable**:

$si_p \in \mathbb{Z}_{\geq 0}$ (inventory of product p at the beginning of the horizon).

Using these variables, an Integer Linear Programming (ILP) model for the lot sizing problem with initial stock (LS-SI) reads as follows:

$$\begin{aligned} \min \quad & \sum_{p \in P} f c_p si_p + \\ & \sum_{p \in P} \sum_{t \in T} (s c_{pt} y_{pt} + v c_{pt} x_{pt} + h c_{pt} s_{pt}) \end{aligned} \quad (15.7a)$$

$$\text{subject to} \quad si_p + x_{p1} = d_{p1} + s_{p1}, \quad \forall p \in P, \quad (15.7b)$$

$$s_{p,t-1} + x_{pt} = d_{pt} + s_{pt}, \quad \forall p \in P, \forall t \in T \setminus \{1\}, \quad (15.7c)$$

$$\sum_{p \in P} (s t_{pt} y_{pt} + v t_{pt} x_{pt}) \leq cap_t, \quad \forall t \in T, \quad (15.7d)$$

$$x_{pt} \leq M_{pt} y_{pt}, \quad \forall p \in P, \forall t \in T, \quad (15.7e)$$

$$s_{pm} = 0, \quad \forall p \in P, \quad (15.7f)$$

$$x_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T, \quad (15.7g)$$

$$s_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T, \quad (15.7h)$$

$$si_p \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \quad (15.7i)$$

$$y_{pt} \in \{0, 1\}, \quad \forall p \in P, \forall t \in T. \quad (15.7j)$$

This ILP model is referred to as the $\text{ILP}_{\text{CLSP}}^{\text{SI}}$ model.

- Description of the objective function and constraints of the $\text{ILP}_{\text{CLSP}}^{\text{SI}}$ model:
 - the linear objective function (15.7a) minimizes the cost of the initial inventory, of the setups, of the production and of the holding;
 - constraints (15.7b) and (15.7c) are the flow balancing constraints of every product, with the initial inventory si_p in the first period ($|P|m$ linear constraints);
 - constraints (15.7d) impose that the setup times and the production times of each period do not exceed its capacity (m linear constraints); constraints (15.7e) link the quantities with the setups ($|P|m$ linear constraints);

- constraints (15.7f) impose an empty inventory at the end of the horizon ($|P|$ linear constraints);
- finally, constraints (15.7g)–(15.7j) define the variables of the ILP model.
- The $\text{ILP}_{\text{CLSP}}^{\text{SI}}$ model is the $\text{ILP}_{\text{CLSP}}^P$ model with the variables si_p in place of the constants $s_{p0} = 0$, and with their cost in the objective function; the strong big-M values M_{pt} are the same.

Example 15.8: $\text{ILP}_{\text{CLSP}}^{\text{SI}}$ model

We consider $|P| = 2$ products and $m = 3$ periods, with capacities $cap_1 = 2500$, $cap_2 = 2100$, $cap_3 = 2500$, the initial inventory costs $fc_1 = fc_2 = 3000$ and the setup costs $sc_{pt} = 10000$. The other data are:

period t	product 1			product 2		
	1	2	3	1	2	3
d_{pt}	100	200	10	10	100	150
vt_{pt}	20	10	10	10	5	5
st_{pt}	100	100	100	110	110	110
hc_{pt}	110	100	100	105	105	105
vc_{pt}	1100	1000	1000	900	800	800

The big-M values are:

period t	1	2	3	1	2	3
sd_{ptm}	310	210	10	260	250	150
$\lfloor (cap_t - st_{pt})/vt_{pt} \rfloor$	120	200	240	239	398	478
M_{pt}	120	200	10	239	250	150

The $\text{ILP}_{\text{CLSP}}^{\text{SI}}$ model with the strong big-M values for this instance is:

$$\begin{aligned}
\min \quad & 3000 si_1 + 3000 si_2 + 10000 y_{11} + 10000 y_{12} + 10000 y_{13} + 10000 y_{21} \\
& + 10000 y_{22} + 10000 y_{23} + 1100 x_{11} + 1000 x_{12} + 1000 x_{13} + 900 x_{21} \\
& + 800 x_{22} + 800 x_{23} + 110 s_{11} + 100 s_{12} + 100 s_{13} + 105 s_{21} \\
& + 105 s_{22} + 105 s_{23} \\
\text{subject to} \quad & si_1 + x_{11} - s_{11} = 100 \\
& s_{11} + x_{12} - s_{12} = 200 \\
& s_{12} + x_{13} - s_{13} = 10 \\
& si_2 + x_{21} - s_{21} = 10 \\
& s_{21} + x_{22} - s_{22} = 100 \\
& s_{22} + x_{23} - s_{23} = 150 \\
& 20 x_{11} + 10 x_{21} + 100 y_{11} + 110 y_{21} \leq 2500 \\
& 10 x_{12} + 5 x_{22} + 100 y_{12} + 110 y_{22} \leq 2100 \\
& 10 x_{13} + 5 x_{23} + 100 y_{13} + 110 y_{23} \leq 2500 \\
& x_{11} - 120 y_{11} \leq 0 \\
& x_{12} - 200 y_{12} \leq 0 \\
& x_{13} - 10 y_{13} \leq 0 \\
& x_{21} - 239 y_{21} \leq 0 \\
& x_{22} - 250 y_{22} \leq 0 \\
& x_{23} - 150 y_{23} \leq 0 \\
& s_{13} = 0 \\
& s_{23} = 0 \\
& si_1, si_2 \in \mathbb{Z}_{\geq 0} \\
& x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \in \mathbb{Z}_{\geq 0} \\
& s_{11}, s_{12}, s_{13}, s_{21}, s_{22}, s_{23} \in \mathbb{Z}_{\geq 0} \\
& y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23} \in \{0, 1\}
\end{aligned}$$

An optimal solution of the $\text{ILP}_{\text{CLSP}}^{\text{SI}}$ model is:

$$\begin{aligned}
& \tilde{x}_{11} = 59, \quad \tilde{x}_{12} = 200, \quad \tilde{x}_{13} = 10, \quad \tilde{x}_{21} = 110, \quad \tilde{x}_{22} = 0, \quad \tilde{x}_{23} = 150, \\
& \tilde{si}_1 = 41, \quad \tilde{si}_2 = 0, \quad \tilde{s}_{21} = 100, \quad \text{all the other } \tilde{s}_{pt} = 0,
\end{aligned}$$

with a setup in every period except for product 2 in period 2. The optimal value $z(\text{ILP}_{\text{CLSP}}^{\text{SI}})$ is 677 400, and the optimal solution is unique (Gurobi). Product 1 starts with 41 units bought at the beginning of the horizon, because period 1 has no time to produce all its demand.

To obtain an LP relaxation of the $\text{ILP}_{\text{CLSP}}^{\text{SI}}$ model, we relax constraints (15.7g)–(15.7j) into non-negativity constraints and the bounds $y_{pt} \leq 1$. The obtained LP model reads as follows:

$$\min \sum_{p \in P} f c_p s i_p + \sum_{p \in P} \sum_{t \in T} (s c_{pt} y_{pt} + v c_{pt} x_{pt} + h c_{pt} s_{pt}) \quad (15.8a)$$

$$\text{subject to } s i_p + x_{p1} = d_{p1} + s_{p1}, \quad \forall p \in P, \quad (15.8b)$$

$$s_{p,t-1} + x_{pt} = d_{pt} + s_{pt}, \quad \forall p \in P, \forall t \in T \setminus \{1\}, \quad (15.8c)$$

$$\sum_{p \in P} (s t_{pt} y_{pt} + v t_{pt} x_{pt}) \leq \text{cap}_t, \quad \forall t \in T, \quad (15.8d)$$

$$x_{pt} \leq M_{pt} y_{pt}, \quad \forall p \in P, \forall t \in T, \quad (15.8e)$$

$$s_{pm} = 0, \quad \forall p \in P, \quad (15.8f)$$

$$y_{pt} \leq 1, \quad \forall p \in P, \forall t \in T, \quad (15.8g)$$

$$x_{pt} \geq 0, \quad \forall p \in P, \forall t \in T, \quad (15.8h)$$

$$s_{pt} \geq 0, \quad \forall p \in P, \forall t \in T, \quad (15.8i)$$

$$y_{pt} \geq 0, \quad \forall p \in P, \forall t \in T, \quad (15.8j)$$

$$s i_p \geq 0, \quad \forall p \in P. \quad (15.8k)$$

This LP model is referred to as the $\text{LP}_{\text{CLSP}}^{\text{SI}}$ model.

- In the shortest-path formulation each product has its own network, with an additional continuous variable $w_{pt} \geq 0$, the arc from node 0 to node t , for the fraction of the initial inventory that covers the demands of the first t periods. The costs of the arcs are

$$c v_{ptk} = v c_{pt} s d_{ptk} + \sum_{s=t+1}^k \sum_{u=t}^{s-1} h c_{pu} d_{ps}, \quad c i_{pt} = f c_p s d_{p1t} + \sum_{s=2}^t \sum_{u=1}^{s-1} h c_{pu} d_{ps}.$$

Using these variables, a Mixed Integer Linear Programming (MIP) model for the lot sizing problem with initial stock (LS-SI) reads as follows:

$$\min \sum_{p \in P} \sum_{t \in T} (s c_{pt} y_{pt} + c i_{pt} w_{pt}) + \sum_{p \in P} \sum_{t \in T} \sum_{k=t}^m c v_{ptk} z v_{ptk} \quad (15.9a)$$

$$\text{subject to } \sum_{k=1}^m (w_{pk} + z v_{p1k}) = 1, \quad \forall p \in P, \quad (15.9b)$$

$$w_{p,t-1} + \sum_{u=1}^{t-1} z v_{p,u,t-1} - \sum_{k=t}^m z v_{ptk} = 0, \quad \forall p \in P, \forall t \in T \setminus \{1\}, \quad (15.9c)$$

$$\sum_{k=t}^m z v_{ptk} \leq y_{pt}, \quad \forall p \in P, \forall t \in T, \quad (15.9d)$$

$$\sum_{p \in P} s t_{pt} y_{pt} + \sum_{p \in P} \sum_{k=t}^m v t_{pt} s d_{ptk} z v_{ptk} \leq cap_t, \quad \forall t \in T, \quad (15.9e)$$

$$\sum_{k=t}^m s d_{ptk} z v_{ptk} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T, \quad (15.9f)$$

$$\sum_{t \in T} s d_{p1t} w_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \quad (15.9g)$$

$$z v_{ptk} \geq 0, \quad \forall p \in P, \forall t, k \in T, k \geq t, \quad (15.9h)$$

$$w_{pt} \geq 0, \quad \forall p \in P, \forall t \in T, \quad (15.9i)$$

$$y_{pt} \in \{0, 1\}, \quad \forall p \in P, \forall t \in T. \quad (15.9j)$$

This MIP model is referred to as the $\text{MIP}_{\text{CLSP}}^{\text{SP,SI}}$ model. Relaxing constraints (15.9j) into $0 \leq y_{pt} \leq 1$ and removing constraints (15.9f) and (15.9g) gives the LP model referred to as the $\text{LP}_{\text{CLSP}}^{\text{SP,SI}}$ model.

- Description of the objective function and constraints of the $\text{MIP}_{\text{CLSP}}^{\text{SP,SI}}$ model:
 - the linear objective function (15.9a) minimizes the cost of the setups, of the initial inventory and of the arcs of the plans of all the products;
 - constraints (15.9b) and (15.9c) are the flow constraints of the network of every product, with the arcs w_{pt} of the initial inventory leaving node 0 ($|P| m$ linear constraints);
 - constraints (15.9d) force the setup of product p in period t if it is produced ($|P| m$ linear constraints), and constraints (15.9e) impose the capacities, setup times included (m linear constraints);
 - constraints (15.9f) and (15.9g) keep integer the quantities and the initial inventories;
 - finally, constraints (15.9h)–(15.9j) define the variables of the MIP model.
- The quantity produced is $x_{pt} = \sum_{k=t}^m s d_{ptk} z v_{ptk}$, and the initial inventory is $si_p = \sum_{t \in T} s d_{p1t} w_{pt}$: constraints (15.9f) and (15.9g) keep them integer, as in the $\text{ILP}_{\text{CLSP}}^{\text{SI}}$ model.
- With one product and $v t_t = 1$ these constraints are not needed: once the setups are fixed, what remains is a minimum cost flow problem, which has an integer optimal solution.
- With several products and production times $v t_{pt} > 1$, a fractional quantity can use the capacity better. Without constraints (15.9f) and (15.9g) the model is only a relaxation of the problem, and its optimal value can be smaller than $z(\text{ILP}_{\text{CLSP}}^{\text{SI}})$ (Example 15.9).

Example 15.9: lower bounds with two products

For the instance of [Example 15.8](#), ignoring the setups (in the costs and in the capacities), the optimal value is 575400, attained by the integer solution $x_{11} = 120$, $x_{12} = 160$, $x_{13} = 10$, $x_{21} = 10$, $x_{22} = 100$, $x_{23} = 150$, $s_{11} = 40$, $si_1 = 20$. The lower bounds and the optimal value of the formulations are:

Model	Value
Without setups	575 400.00
LP_{CLSP}^{SI} with $M_{pt} = sd_{ptm}$	638 882.20
LP_{CLSP}^{SI} with the strong M_{pt}	651 267.37
$LP_{CLSP}^{SP,SI}$	668 928.57
$MIP_{CLSP}^{SP,SI}$ without (15.9f) and (15.9g)	676 450.00
ILP_{CLSP}^{SI} and $MIP_{CLSP}^{SP,SI}$ (integer optimum)	677 400

Without constraints (15.9f) and (15.9g) the shortest-path model reaches 676 450 with the same setups and the fractional quantities $x_{11} = 59.5$ and $si_1 = 40.5$: period 1 then uses its whole capacity, $100 + 20 \cdot 59.5 + 110 + 10 \cdot 110 = 2500$.

15.7.4 Overtime

- The data are those of the lot sizing problem with several products and setup times, but the time of each period is not a fixed capacity: in period t the working time in normal hours costs cw_t per unit, up to cap_t^N , and the overtime costs co_t per unit, up to cap_t^O .
- We introduce the integer variables x_{pt} , s_{pt} and the binary variables y_{pt} of the ILP_{CLSP}^P model, and, for every period $t \in T$, the following **integer variables**:

$$\mu_t \in \mathbb{Z}_{\geq 0} \text{ (working time in normal hours),} \quad \nu_t \in \mathbb{Z}_{\geq 0} \text{ (overtime).}$$

Since the times vt_{pt} and st_{pt} are integer, the working times can be taken integer.

Using these variables, an Integer Linear Programming (ILP) model for the lot sizing problem with overtime (LS-OT) reads as follows:

$$\min \sum_{p \in P} \sum_{t \in T} (sc_{pt} y_{pt} + vc_{pt} x_{pt} + hc_{pt} s_{pt}) + \sum_{t \in T} (cw_t \mu_t + co_t \nu_t) \quad (15.10a)$$

$$\text{subject to} \quad s_{p,t-1} + x_{pt} = d_{pt} + s_{pt}, \quad \forall p \in P, \forall t \in T, \quad (15.10b)$$

$$\sum_{p \in P} (st_{pt} y_{pt} + vt_{pt} x_{pt}) \leq \mu_t + \nu_t, \quad \forall t \in T, \quad (15.10c)$$

$$\mu_t \leq cap_t^N, \quad \forall t \in T, \quad (15.10d)$$

$$\nu_t \leq cap_t^O, \quad \forall t \in T, \quad (15.10e)$$

$$x_{pt} \leq M_{pt} y_{pt}, \quad \forall p \in P, \forall t \in T, \quad (15.10f)$$

$$s_{p0} = s_{pm} = 0, \quad \forall p \in P, \quad (15.10g)$$

$$x_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T, \quad (15.10h)$$

$$s_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T, \quad (15.10i)$$

$$\mu_t \in \mathbb{Z}_{\geq 0}, \quad \forall t \in T, \quad (15.10j)$$

$$\nu_t \in \mathbb{Z}_{\geq 0}, \quad \forall t \in T, \quad (15.10k)$$

$$y_{pt} \in \{0, 1\}, \quad \forall p \in P, \forall t \in T. \quad (15.10l)$$

This ILP model is referred to as the $\text{ILP}_{\text{CLSP}}^{\text{OT}}$ model.

- Description of the objective function and constraints of the $\text{ILP}_{\text{CLSP}}^{\text{OT}}$ model:
 - the linear objective function (15.10a) adds to the costs of the $\text{ILP}_{\text{CLSP}}^P$ model the cost of the working time in normal hours and in overtime;
 - constraints (15.10b) are the flow balancing constraints of every product ($|P| m$ linear constraints);
 - constraints (15.10c) impose that the setup times and the production times of each period are covered by the working time, in normal hours and in overtime (m linear constraints);
 - constraints (15.10d) and (15.10e) bound the two working times (m linear constraints each);
 - constraints (15.10f) link the quantities with the setups ($|P| m$ linear constraints);
 - constraints (15.10g) impose an empty inventory at the beginning and at the end of the horizon;
 - finally, constraints (15.10h)–(15.10l) define the variables of the ILP model.
- The time available in period t is at most $\text{cap}_t^N + \text{cap}_t^O$, so the strong big-M value is

$$M_{pt} = \min \left\{ \left\lfloor \frac{\text{cap}_t^N + \text{cap}_t^O - st_{pt}}{vt_{pt}} \right\rfloor, sd_{ptm} \right\}.$$

- With $co_t > cw_t$ an optimal solution uses overtime only when the normal hours of the period are exhausted.

Example 15.10: $\text{ILP}_{\text{CLSP}}^{\text{OT}}$ model

We consider the CLS instance of [Example 15.1](#), with one product ($|P| = 1$, and we drop the index p), production times $vt_t = 1$ and setup times $st_t = 0$. The capacity $\text{cap}_t = 4$ becomes the normal hours $\text{cap}_t^N = 4$, at the cost $cw_t = 1$, and one unit of overtime is available in every period: $\text{cap}_t^O = 1$, at the cost $co_t = 6$. The big-M values are $M_t = \min\{5, D_{t,5}\}$, that is $(5, 5, 5, 5, 1)$. The $\text{ILP}_{\text{CLSP}}^{\text{OT}}$ model for this instance is:

$$\begin{array}{llll}
\min & 50 y_1 + 50 y_2 + 50 y_3 + 50 y_4 + 50 y_5 + 10 s_1 \\
& + 10 s_2 + 10 s_3 + 10 s_4 + 10 s_5 + \mu_1 + \mu_2 \\
& + \mu_3 + \mu_4 + \mu_5 + 6 \nu_1 + 6 \nu_2 + 6 \nu_3 \\
& + 6 \nu_4 + 6 \nu_5 \\
\text{subject to} & x_1 - s_1 & = & 2 \\
& s_1 + x_2 - s_2 & = & 1 \\
& s_2 + x_3 - s_3 & = & 3 \\
& s_3 + x_4 - s_4 & = & 4 \\
& s_4 + x_5 - s_5 & = & 1 \\
& x_1 - \mu_1 - \nu_1 & \leq & 0 \\
& x_2 - \mu_2 - \nu_2 & \leq & 0 \\
& x_3 - \mu_3 - \nu_3 & \leq & 0 \\
& x_4 - \mu_4 - \nu_4 & \leq & 0 \\
& x_5 - \mu_5 - \nu_5 & \leq & 0 \\
& x_1 - 5 y_1 & \leq & 0 \\
& x_2 - 5 y_2 & \leq & 0 \\
& x_3 - 5 y_3 & \leq & 0 \\
& x_4 - 5 y_4 & \leq & 0 \\
& x_5 - y_5 & \leq & 0 \\
& \mu_1 & \leq & 4 \\
& \mu_2 & \leq & 4 \\
& \mu_3 & \leq & 4 \\
& \mu_4 & \leq & 4 \\
& \mu_5 & \leq & 4 \\
& \nu_1 & \leq & 1 \\
& \nu_2 & \leq & 1 \\
& \nu_3 & \leq & 1 \\
& \nu_4 & \leq & 1 \\
& \nu_5 & \leq & 1 \\
& s_5 & = & 0 \\
& x_1, x_2, x_3, x_4, x_5 & \in & \mathbb{Z}_{\geq 0} \\
& s_1, s_2, s_3, s_4, s_5 & \in & \mathbb{Z}_{\geq 0} \\
& \mu_1, \mu_2, \mu_3, \dots, \mu_5 & \in & \mathbb{Z}_{\geq 0} \\
& y_1, y_2, y_3, y_4, y_5 & \in & \{0, 1\}
\end{array}$$

An optimal solution of the $\text{ILP}_{\text{CLSP}}^{\text{OT}}$ model is:

$$\begin{aligned}\tilde{\mathbf{y}} &= (1, 0, 1, 1, 0), & \tilde{\mathbf{x}} &= (3, 0, 3, 5, 0), & \tilde{\mathbf{s}} &= (1, 0, 0, 1, 0), \\ \tilde{\boldsymbol{\mu}} &= (3, 0, 3, 4, 0), & \tilde{\boldsymbol{\nu}} &= (0, 0, 0, 1, 0)\end{aligned}$$

The optimal value $z(\text{ILP}_{\text{CLSP}}^{\text{OT}})$ is $3 \cdot 50 + 2 \cdot 10 + 10 \cdot 1 + 1 \cdot 6 = 186$, and the optimal solution is unique (Gurobi). One unit of overtime in period 4, at the cost 6, replaces one unit stored from period 3 to period 4, at the cost 10: the optimal plan $(3, 0, 4, 4, 0)$ of the CLS costs 191 here.

15.7.5 Unsatisfied demand

- The data are those of the lot sizing problem with several products and setup times. In addition, a part of the demand d_{pt} may be left unsatisfied, at the penalty cu_{pt} per unit.
- We introduce the integer variables x_{pt} , s_{pt} and the binary variables y_{pt} of the $\text{ILP}_{\text{CLSP}}^P$ model, and, for every product $p \in P$ and every period $t \in T$, the following **integer variable**:

$$\varepsilon_{pt} \in \mathbb{Z}_{\geq 0} \text{ (demand of product } p \text{ in period } t \text{ left unsatisfied).}$$

Using these variables, an Integer Linear Programming (ILP) model for the lot sizing problem with unsatisfied demand (LS-UD) reads as follows:

$$\begin{aligned}\min \quad & \sum_{p \in P} \sum_{t \in T} (sc_{pt} y_{pt} + vc_{pt} x_{pt} + hc_{pt} s_{pt}) + \\ & \sum_{p \in P} \sum_{t \in T} cu_{pt} \varepsilon_{pt}\end{aligned}\tag{15.11a}$$

$$\text{subject to } s_{p,t-1} + x_{pt} = d_{pt} - \varepsilon_{pt} + s_{pt}, \quad \forall p \in P, \forall t \in T,\tag{15.11b}$$

$$\sum_{p \in P} (st_{pt} y_{pt} + vt_{pt} x_{pt}) \leq cap_t, \quad \forall t \in T,\tag{15.11c}$$

$$x_{pt} \leq M_{pt} y_{pt}, \quad \forall p \in P, \forall t \in T,\tag{15.11d}$$

$$\varepsilon_{pt} \leq d_{pt}, \quad \forall p \in P, \forall t \in T,\tag{15.11e}$$

$$s_{p0} = s_{pm} = 0, \quad \forall p \in P,\tag{15.11f}$$

$$x_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T,\tag{15.11g}$$

$$s_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T,\tag{15.11h}$$

$$\varepsilon_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T, \quad (15.11i)$$

$$y_{pt} \in \{0, 1\}, \quad \forall p \in P, \forall t \in T. \quad (15.11j)$$

This ILP model is referred to as the $\text{ILP}_{\text{CLSP}}^{\text{UD}}$ model.

- Description of the objective function and constraints of the $\text{ILP}_{\text{CLSP}}^{\text{UD}}$ model:
 - the linear objective function (15.11a) adds to the costs of the $\text{ILP}_{\text{CLSP}}^P$ model the penalty of the unsatisfied demand;
 - constraints (15.11b) are the flow balancing constraints, with the demand reduced by its unsatisfied part ($|P|m$ linear constraints);
 - constraints (15.11c) are the capacity constraints and constraints (15.11d) link the quantities with the setups (m and $|P|m$ linear constraints);
 - constraints (15.11e) ensure that the unsatisfied demand does not exceed the demand ($|P|m$ linear constraints);
 - constraints (15.11f) impose an empty inventory at the beginning and at the end of the horizon;
 - finally, constraints (15.11g)–(15.11j) define the variables of the ILP model.
- The big-M values of the $\text{ILP}_{\text{CLSP}}^P$ model remain valid, since the production can only decrease. With $\varepsilon_{pt} = 0$ for every product and period the model is the $\text{ILP}_{\text{CLSP}}^P$ model.

Example 15.11: $\text{ILP}_{\text{CLSP}}^{\text{UD}}$ model

We consider the CLS instance of [Example 15.1](#), with one product ($|P| = 1$, and we drop the index p), production times $vt_t = 1$, setup times $st_t = 0$ and the strong big-M values $M_t = (4, 4, 4, 4, 1)$. Each unit of unsatisfied demand costs $cu_t = 18$. The $\text{ILP}_{\text{CLSP}}^{\text{UD}}$ model for this instance is:

$$\begin{aligned}
\min \quad & 50 y_1 + 50 y_2 + 50 y_3 + 50 y_4 + 50 y_5 + 10 s_1 \\
& + 10 s_2 + 10 s_3 + 10 s_4 + 10 s_5 + 18 \varepsilon_1 + 18 \varepsilon_2 \\
& + 18 \varepsilon_3 + 18 \varepsilon_4 + 18 \varepsilon_5 \\
\text{subject to} \quad & x_1 - s_1 + \varepsilon_1 = 2 \\
& s_1 + x_2 - s_2 + \varepsilon_2 = 1 \\
& s_2 + x_3 - s_3 + \varepsilon_3 = 3 \\
& s_3 + x_4 - s_4 + \varepsilon_4 = 4 \\
& s_4 + x_5 - s_5 + \varepsilon_5 = 1 \\
& x_1 \leq 4 \\
& x_2 \leq 4 \\
& x_3 \leq 4 \\
& x_4 \leq 4 \\
& x_5 \leq 4 \\
& x_1 - 4 y_1 \leq 0 \\
& x_2 - 4 y_2 \leq 0 \\
& x_3 - 4 y_3 \leq 0 \\
& x_4 - 4 y_4 \leq 0 \\
& x_5 - y_5 \leq 0 \\
& \varepsilon_1 \leq 2 \\
& \varepsilon_2 \leq 1 \\
& \varepsilon_3 \leq 3 \\
& \varepsilon_4 \leq 4 \\
& \varepsilon_5 \leq 1 \\
& s_5 = 0 \\
& x_1, x_2, x_3, x_4, x_5 \in \mathbb{Z}_{\geq 0} \\
& s_1, s_2, s_3, s_4, s_5 \in \mathbb{Z}_{\geq 0} \\
& \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5 \in \mathbb{Z}_{\geq 0} \\
& y_1, y_2, y_3, y_4, y_5 \in \{0, 1\}
\end{aligned}$$

An optimal solution of the $\text{ILP}_{\text{CLSP}}^{\text{UD}}$ model is:

$$\tilde{\mathbf{y}} = (0, 0, 1, 1, 0), \quad \tilde{\mathbf{x}} = (0, 0, 3, 4, 0), \quad \tilde{\mathbf{s}} = (0, 0, 0, 0, 0), \quad \tilde{\boldsymbol{\varepsilon}} = (2, 1, 0, 0, 1)$$

The optimal value $z(\text{ILP}_{\text{CLSP}}^{\text{UD}})$ is $2 \cdot 50 + 4 \cdot 18 = 172$, and the optimal solution is unique (Gurobi). The demands of the periods 1, 2 and 5 are not satisfied: their 4 units cost 72, and the plan saves one setup and all the holding costs of the optimal plan $(3, 0, 4, 4, 0)$ of the CLS, of cost 180.

15.8 Typical mistakes

Warning

- **Taking a big-M value that is too small.** A value smaller than the production of a feasible plan cuts off that plan: the values $D_{t,m}$ and cap_t are both valid.
- **Taking a big-M value that is too large.** Every valid value gives a correct model, but a large value gives a weak relaxation: the setup variables become almost zero in the LP solutions, as in the bound 99.65 of the reference instance.
- **Forgetting the setup times in the capacity constraints.** With several products the constraints contain $st_{pt} y_{pt}$ besides $vt_{pt} x_{pt}$.
- **Counting the holding cost of the demand of the production period.** In cv_{tk} the units for period t are not stored: the holding sum starts from the demand of period $t + 1$.
- **Believing that the shortest-path formulation needs binary paths.** The variables zv_{tk} are continuous: a feasible plan that is not a path is a combination of paths.
- **Forgetting the integrality of the quantities with several products.** With production times $vt_{pt} > 1$ the continuous variables zv_{ptk} can give fractional quantities: the shortest-path model needs the integrality constraints (15.9f) and (15.9g), otherwise it is only a relaxation ($676\,450 < 677\,400$).
- **Keeping the fixed capacity in the model with overtime.** The time of period t is bounded by the working time $\mu_t + \nu_t$, and the big-M values use $cap_t^N + cap_t^O$.
- **Leaving the unsatisfied demand unbounded.** Without $\varepsilon_{pt} \leq d_{pt}$ the balance constraints would allow a negative net demand, that is, inventory created from nothing at the price of the penalty.

15.9 In summary

Summary

- Without the setups the CLS is a simple flow problem; the setups need the binary variables y_t and a link with the quantities.
- In the big-M formulation the link $x_t \leq M_t y_t$ is correct for every valid M_t , but the LP bound improves when M_t decreases: the strong value is $\min\{cap_t, D_{t,m}\}$.
- The shortest-path formulation writes the plans as paths in a network, with the link $\sum_k zv_{tk} \leq y_t$ and no big-M value: its relaxation is the strongest.
- The two formulations have the same optimal value; they differ in the lower bound and in the size of the model.
- The ULS model is the big-M model without capacities and with $M_t = D_{t,m}$: on the

reference instance its optimal value is 170.

- The variants keep the structure of the big-M model: several products share the capacity with their setup times, the initial stock adds the variables si_p , overtime adds the working times μ_t, ν_t , and the unsatisfied demand adds the variables ε_{pt} . On the reference instance, overtime gives 186 and unsatisfied demand 172.

Dashboard of the reference instance. Lower bounds on the optimal value computed in this chapter:

Bound	Value	From
Without setups	0	lot-for-lot plan
Big-M, $M_t = D_{t,m}$	99.65	$z(\text{LP}_{\text{CLSP}}^{\text{dem}}) = \frac{9865}{99}$
Big-M, $M_t = cap_t$	137.5	$z(\text{LP}_{\text{CLSP}}^{\text{cap}})$
Strong big-M	157.5	$z(\text{LP}_{\text{CLSP}}^{\text{strong}})$
Shortest path	170	$z(\text{LP}_{\text{CLSP}}^{\text{SP}})$
Optimal value	180	plan (3, 0, 4, 4, 0)