



SAPIENZA
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Combinatorial Optimization

Exercises 2 — Formulations and their properties

Worked exercises on the chapters of Volume 2, formulations and their properties:
each exercise is followed by its complete solution.

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Formulations of knapsack problems

9.1 Knapsack: Dantzig and Martello-Toth upper bounds

Upper bounds used in these exercises

The bounds of Chapter 9 «Formulations of knapsack problems», for a KP instance with the items sorted by non-increasing efficiency. The **split item** $s \in \mathcal{J}$ and the **residual capacity** rc are

$$\sum_{j=1}^{s-1} w_j \leq c \quad \text{and} \quad \sum_{j=1}^s w_j > c, \quad rc = c - \sum_{j=1}^{s-1} w_j. \quad (\text{K1})$$

The **Dantzig upper bound** for $z(\text{ILP}_{\text{KP}})$ is

$$\text{UB}_D = \sum_{j=1}^{s-1} p_j + \left\lfloor p_s \frac{rc}{w_s} \right\rfloor. \quad (\text{K2})$$

The **Martello-Toth upper bound** for $z(\text{ILP}_{\text{KP}})$ is $\text{UB}_{\text{MT}} = \max\{\text{UB}_0, \text{UB}_1\}$, where

$$\text{UB}_0 = \sum_{j=1}^{s-1} p_j + \left\lfloor rc \frac{p_{s+1}}{w_{s+1}} \right\rfloor \quad \text{and} \quad \text{UB}_1 = \sum_{j=1}^{s-1} p_j + \left\lfloor p_s - (w_s - rc) \frac{p_{s-1}}{w_{s-1}} \right\rfloor. \quad (\text{K3})$$

Exercise 9.1 — Upper bounds for the 0/1 Knapsack Problem (1)

We consider a knapsack capacity $c = 11$ and a set $\mathcal{J} = \{1, 2, 3, 4, 5\}$ of 5 items ($n = 5$) with the following profits and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
p_j	8	6	14	6	2
w_j	3	3	9	5	2

Questions

1. Check if the items are sorted by non-increasing efficiency otherwise sort them in this order.
2. Compute the Dantzig and the Martello-Toth upper bound and determine which one is stronger for this KP instance.

Solution

We use the split item and the residual capacity (K1), the Dantzig bound (K2) and the Martello-Toth bound (K3) of the box at the beginning of the chapter.

1. The items are sorted by non-increasing efficiency since we have:

$$\frac{8}{3} \geq \frac{6}{3} \geq \frac{14}{9} \geq \frac{6}{5} \geq \frac{2}{2}$$

2. The split item is $s = 3$ since:

$$\sum_{j=1}^2 w_j = 3 + 3 = 6 \leq 11 \quad \text{and} \quad \sum_{j=1}^3 w_j = 3 + 3 + 9 = 15 > 11$$

and the residual capacity is:

$$rc = c - \sum_{j=1}^2 w_j = 11 - 3 - 3 = 5$$

The Dantzig upper bound:

$$UB_D = \sum_{j=1}^2 p_j + \left\lfloor p_3 \frac{rc}{w_3} \right\rfloor = 8 + 6 + \left\lfloor 14 \frac{5}{9} \right\rfloor = 14 + \left\lfloor \frac{70}{9} \right\rfloor = 21$$

The Martello-Toth upper bound is:

$$UB_0 = \sum_{j=1}^2 p_j + \left\lfloor rc \frac{p_4}{w_4} \right\rfloor = 8 + 6 + \left\lfloor 5 \frac{6}{5} \right\rfloor = 14 + 6 = 20$$

$$UB_1 = \sum_{j=1}^2 p_j + \left\lfloor p_3 - (w_3 - rc) \frac{p_2}{w_2} \right\rfloor = 8 + 6 + \left\lfloor 14 - (9 - 5) \frac{6}{3} \right\rfloor = 14 + 6 = 20$$

$$UB_{MT} = \max \left\{ UB_0, UB_1 \right\} = \max \left\{ 20, 20 \right\} = 20$$

For this KP instance, the Martello-Toth upper bound is stronger than Dantzig upper bound since we have:

$$UB_{MT} < UB_D$$

Exercise 9.2 — Upper bounds for the 0/1 Knapsack Problem (2)

We consider a knapsack capacity $c = 8$ and a set $\mathcal{J} = \{1, 2, 3, 4\}$ of 4 items ($n = 4$) with the following profits and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
p_j	40	42	25	12
w_j	4	7	5	3

Questions

1. Check if the items are sorted by non-increasing efficiency otherwise sort them in this order.
2. Compute the Dantzig and the Martello-Toth upper bound and determine which one is stronger for this KP instance.

Solution

We use the split item and the residual capacity (K1), the Dantzig bound (K2) and the Martello-Toth bound (K3) of the box at the beginning of the chapter.

1. The items are sorted by non-increasing efficiency since we have:

$$\frac{40}{4} \geq \frac{42}{7} \geq \frac{25}{5} \geq \frac{12}{3}$$

2. The split item is $s = 2$ since:

$$w_1 = 4 \leq 8 \quad \text{and} \quad \sum_{j=1}^2 w_j = 4 + 7 = 11 > 8$$

and the residual capacity is:

$$rc = c - w_1 = 8 - 4 = 4$$

The Dantzig upper bound is:

$$UB_D = p_1 + \left\lfloor p_2 \frac{rc}{w_2} \right\rfloor = 40 + \left\lfloor 42 \frac{4}{7} \right\rfloor = 40 + \lfloor 24 \rfloor = 64$$

The Martello-Toth upper bound is:

$$UB_0 = p_1 + \left\lfloor rc \frac{p_3}{w_3} \right\rfloor = 40 + \left\lfloor 4 \frac{25}{5} \right\rfloor = 40 + \lfloor 20 \rfloor = 60$$

$$UB_1 = p_1 + \left\lfloor p_2 - (w_2 - rc) \frac{p_1}{w_1} \right\rfloor = 40 + \left\lfloor 42 - (7 - 4) \frac{40}{4} \right\rfloor = 40 + \lfloor 12 \rfloor = 52$$

$$UB_{MT} = \max \left\{ UB_0, UB_1 \right\} = \max \left\{ 60, 52 \right\} = 60$$

For this KP instance, the Martello-Toth upper bound is stronger than Dantzig upper bound since we have:

$$UB_{MT} < UB_D$$

Exercise 9.3 — Upper bounds for the 0/1 Knapsack Problem (3)

We consider a knapsack capacity $c = 15$ and a set $\mathcal{J} = \{1, 2, 3, 4\}$ of 4 items ($n = 4$) with the following profits and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
p_j	24	17	12	6
w_j	10	8	6	5

Questions

1. Check if the items are sorted by non-increasing efficiency otherwise sort them in this order.
2. Compute the Dantzig and the Martello-Toth upper bound and determine which one is stronger for this KP instance.

Solution

We use the split item and the residual capacity (K1), the Dantzig bound (K2) and the Martello-Toth bound (K3) of the box at the beginning of the chapter.

1. The items are sorted by non-increasing efficiency since we have:

$$\frac{24}{10} \geq \frac{17}{8} \geq \frac{12}{6} \geq \frac{6}{5}$$

2. The split item is $s = 2$ since:

$$w_1 = 10 \leq 15 \quad \text{and} \quad \sum_{j=1}^2 w_j = 10 + 8 = 18 > 15$$

and the residual capacity is:

$$rc = c - w_1 = 15 - 10 = 5$$

The Dantzig upper bound is:

$$UB_D = p_1 + \left\lfloor p_2 \frac{rc}{w_2} \right\rfloor = 24 + \left\lfloor 17 \frac{5}{8} \right\rfloor = 24 + \left\lfloor \frac{85}{8} \right\rfloor = 34$$

The Martello-Toth upper bound is:

$$UB_0 = p_1 + \left\lfloor rc \frac{p_3}{w_3} \right\rfloor = 24 + \left\lfloor 5 \frac{12}{6} \right\rfloor = 24 + \lfloor 10 \rfloor = 34$$

$$UB_1 = p_1 + \left\lfloor p_2 - (w_2 - rc) \frac{p_1}{w_1} \right\rfloor = 24 + \left\lfloor 17 - (8 - 5) \frac{24}{10} \right\rfloor = 24 + \left\lfloor \frac{49}{5} \right\rfloor = 33$$

$$UB_{MT} = \max \left\{ UB_0, UB_1 \right\} = \max \left\{ 34, 33 \right\} = 34$$

For this KP instance, the Martello-Toth upper bound is as strong as the Dantzig upper bound since we have:

$$UB_{MT} = UB_D$$

Exercise 9.4 — Upper bounds for the 0/1 Knapsack Problem (4)

We consider a knapsack capacity $c = 9$ and a set $\mathcal{I} = \{1, 2, 3, 4, 5, 6\}$ of 6 items ($n = 6$) with the following profits, weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
p_j	6	5	8	9	6	7
w_j	2	3	6	7	5	9

Questions

1. Check if the items are sorted by non-increasing efficiency otherwise sort them in this order.

2. Compute the Dantzig and the Martello-Toth upper bound and determine which one is stronger for this KP instance.

Solution

We use the split item and the residual capacity (K1), the Dantzig bound (K2) and the Martello-Toth bound (K3) of the box at the beginning of the chapter.

1. The items are sorted by non-increasing efficiency since we have:

$$\frac{6}{2} \geq \frac{5}{3} \geq \frac{8}{6} \geq \frac{9}{7} \geq \frac{6}{5} \geq \frac{7}{9}$$

2. The split item is $s = 3$ since:

$$\sum_{j=1}^2 w_j = 2 + 3 = 5 \leq 9 \quad \text{and} \quad \sum_{j=1}^3 w_j = 2 + 3 + 6 = 11 > 9$$

and the residual capacity is:

$$rc = c - \sum_{j=1}^2 w_j = 9 - 2 - 3 = 4$$

The Dantzig upper bound is:

$$UB_D = \sum_{j=1}^2 p_j + \left\lfloor p_3 \frac{rc}{w_3} \right\rfloor = 6 + 5 + \left\lfloor 8 \frac{4}{6} \right\rfloor = 11 + \left\lfloor \frac{16}{3} \right\rfloor = 16$$

The Martello-Toth upper bound is:

$$UB_0 = \sum_{j=1}^2 p_j + \left\lfloor rc \frac{p_4}{w_4} \right\rfloor = 6 + 5 + \left\lfloor 4 \frac{9}{7} \right\rfloor = 11 + \left\lfloor \frac{36}{7} \right\rfloor = 16$$

$$UB_1 = \sum_{j=1}^2 p_j + \left\lfloor p_3 - (w_3 - rc) \frac{p_2}{w_2} \right\rfloor = 6 + 5 + \left\lfloor 8 - (6 - 4) \frac{5}{3} \right\rfloor = 11 + \left\lfloor \frac{14}{3} \right\rfloor = 15$$

$$UB_{MT} = \max \left\{ UB_0, UB_1 \right\} = \max \left\{ 16, 15 \right\} = 16$$

For this KP instance, the Martello-Toth upper bound is as strong as the Dantzig upper bound since we have:

$$UB_{MT} = UB_D$$

Exercise 9.5 — Upper bounds for the 0/1 Knapsack Problem (5)

We consider a knapsack capacity $c = 21$ and a set $\mathcal{J} = \{1, 2, 3, 4, 5, 6\}$ of 6 items ($n = 6$) with the following profits, weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
p_j	24	25	13	10	9	2
w_j	2	6	8	7	10	3

Questions

1. Check if the items are sorted by non-increasing efficiency otherwise sort them in this order.
2. Compute the Dantzig and the Martello-Toth upper bound and determine which one is stronger for this KP instance.

Solution

We use the split item and the residual capacity (K1), the Dantzig bound (K2) and the Martello-Toth bound (K3) of the box at the beginning of the chapter.

1. The items are sorted by non-increasing efficiency since we have:

$$\frac{24}{2} \geq \frac{25}{6} \geq \frac{13}{8} \geq \frac{10}{7} \geq \frac{9}{10} \geq \frac{2}{3}$$

2. The split item is $s = 4$ since:

$$\sum_{j=1}^3 w_j = 2 + 6 + 8 = 16 \leq 21 \quad \text{and} \quad \sum_{j=1}^4 w_j = 2 + 6 + 8 + 7 = 23 > 21$$

and the residual capacity is:

$$rc = c - \sum_{j=1}^3 w_j = 21 - 2 - 6 - 8 = 5$$

The Dantzig upper bound is:

$$UB_D = \sum_{j=1}^3 p_j + \left\lfloor p_4 \frac{rc}{w_4} \right\rfloor = 24 + 25 + 13 + \left\lfloor 10 \frac{5}{7} \right\rfloor = 62 + \left\lfloor \frac{50}{7} \right\rfloor = 69$$

The Martello-Toth upper bound is:

$$UB_0 = \sum_{j=1}^3 p_j + \left\lfloor rc \frac{p_5}{w_5} \right\rfloor = 24 + 25 + 13 + \left\lfloor 5 \frac{9}{10} \right\rfloor = 62 + \left\lfloor \frac{9}{2} \right\rfloor = 66$$

$$\begin{aligned} UB_1 &= \sum_{j=1}^3 p_j + \left\lfloor p_4 - (w_4 - rc) \frac{p_3}{w_3} \right\rfloor \\ &= 24 + 25 + 13 + \left\lfloor 10 - (7 - 5) \frac{13}{8} \right\rfloor = 62 + \left\lfloor \frac{27}{4} \right\rfloor = 68 \end{aligned}$$

$$UB_{MT} = \max \left\{ UB_0, UB_1 \right\} = \max \left\{ 66, 68 \right\} = 68$$

For this KP instance, the Martello-Toth upper bound is stronger than the Dantzig upper bound since we have:

$$UB_{MT} < UB_D$$

9.2 ILP models of the knapsack variants

Exercise 9.6 — ILP model of the Subset Sum Problem

The Subset Sum Problem (SSP) and its ILP_{SSP} model are in [Chapter 9 «Formulations of knapsack problems»](#). We consider a target value $c = 16$ and a set $\mathcal{J} = \{1, 2, \dots, 6\}$ of 6 items ($n = 6$), with the following weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
w_j	2	4	5	8	9	13

Questions

1. Write the ILP_{SSP} model of the chapter for this instance.
2. Solve the model with Gurobi. Give the optimal value, an optimal solution and the associated solution of the SSP, and say whether the optimal solution is unique.

Solution

1. The model has 6 binary variables and one linear constraint (besides the domains of the variables). The ILP_{SSP} model for this instance is:

$$\begin{aligned} \max \quad & 2x_1 + 4x_2 + 5x_3 + 8x_4 + 9x_5 + 13x_6 \\ \text{subject to} \quad & 2x_1 + 4x_2 + 5x_3 + 8x_4 + 9x_5 + 13x_6 \leq 16 \\ & x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad x_6 \in \{0, 1\} \end{aligned}$$

2. Gurobi returns the optimal value $z(\text{ILP}_{\text{SSP}}) = 16$. An optimal solution is

$$\tilde{x}_1 = 1, \tilde{x}_2 = 0, \tilde{x}_3 = 1, \tilde{x}_4 = 0, \tilde{x}_5 = 1, \tilde{x}_6 = 0$$

The selected items have total weight $2 + 5 + 9 = 16$. Since $z(\text{ILP}_{\text{SSP}}) = c = 16$, the answer to the SSP is **yes**, with the subset $\tilde{S} = \{1, 3, 5\}$.

Enumerating the 22 subsets of weight at most 16 shows that $\tilde{S}$ is the only subset of weight exactly 16: the optimal solution is unique.

Exercise 9.7 — ILP model of the Bounded Knapsack Problem

The Bounded Knapsack Problem (BKP) and its ILP_{BKP} model are in [Chapter 9 «Formulations of knapsack problems»](#). We consider a knapsack of capacity $c = 18$ and a set $\mathcal{J} = \{1, 2, \dots, 5\}$ of 5 items ($n = 5$), with the following profits, weights and available copies:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
p_j	5	12	11	7	4
w_j	5	7	3	3	8
b_j	3	2	3	2	2

Questions

1. Write the ILP_{BKP} model of the chapter for this instance.
2. Solve the model with Gurobi. Give the optimal value, an optimal solution and the associated solution of the BKP, and say whether the optimal solution is unique.

Solution

1. The model has 5 integer variables and one linear constraint (besides the domains of the variables). The ILP_{BKP} model for this instance is:

$$\begin{aligned}
 & \max \quad 5x_1 + 12x_2 + 11x_3 + 7x_4 + 4x_5 \\
 & \text{subject to} \quad 5x_1 + 7x_2 + 3x_3 + 3x_4 + 8x_5 \leq 18 \\
 & \quad x_1, \quad \quad \quad x_3 \quad \quad \quad \in \{0, 1, 2, 3\} \\
 & \quad \quad \quad x_2, \quad \quad \quad x_4, \quad \quad x_5 \in \{0, 1, 2\}
 \end{aligned}$$

2. Gurobi returns the optimal value $z(\text{ILP}_{\text{BKP}}) = 47$. An optimal solution is

$$\tilde{x}_1 = 0, \tilde{x}_2 = 0, \tilde{x}_3 = 3, \tilde{x}_4 = 2, \tilde{x}_5 = 0$$

It takes three copies of item 3 and two copies of item 4: the total weight is $3 \cdot 3 + 2 \cdot 3 = 15 \leq 18$ and every $\tilde{x}_j$ is at most b_j .

Enumerating the 68 feasible vectors of copies shows that the optimal solution is unique.

Exercise 9.8 — ILP model of the Unbounded Knapsack Problem

The Unbounded Knapsack Problem (UKP) and its ILP_{UKP} model are in [Chapter 9 «Formulations of knapsack problems»](#). We consider a knapsack of capacity $c = 19$ and a set $\mathcal{J} = \{1, 2, \dots, 5\}$ of 5 items ($n = 5$), each available in an unlimited number of copies, with the following profits and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
p_j	4	9	7	5	8
w_j	4	5	6	6	9

Questions

1. Write the ILP_{UKP} model of the chapter for this instance.
2. Solve the model with Gurobi. Give the optimal value, an optimal solution and the associated solution of the UKP, and say whether the optimal solution is unique.

Solution

1. The model has 5 integer variables and one linear constraint (besides the domains of the variables). The ILP_{UKP} model for this instance is:

$$\begin{aligned}
 & \max \quad 4x_1 + 9x_2 + 7x_3 + 5x_4 + 8x_5 \\
 & \text{subject to} \quad 4x_1 + 5x_2 + 6x_3 + 6x_4 + 9x_5 \leq 19 \\
 & \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5 \in \mathbb{Z}_{\geq 0}
 \end{aligned}$$

2. Gurobi returns the optimal value $z(\text{ILP}_{\text{UKP}}) = 31$. An optimal solution is

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 3, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0$$

It takes one copy of item 1 and three copies of item 2: the total weight is $1 \cdot 4 + 3 \cdot 5 = 19 \leq 19$.

The capacity constraint gives $x_j \leq \hat{b}_j = \lfloor c/w_j \rfloor$, that is $\hat{b}_1 = 4$, $\hat{b}_2 = 3$, $\hat{b}_3 = 3$, $\hat{b}_4 = 3$, $\hat{b}_5 = 2$. Enumerating the 54 feasible vectors of copies within these bounds shows that the optimal solution is unique.

Exercise 9.9 — ILP model of the Multidimensional Knapsack Problem

The Multidimensional Knapsack Problem (MDKP) and its ILP_{MDKP} model are in [Chapter 9 «Formulations of knapsack problems»](#). We consider a 2-dimensional knapsack ($k = 2$) with capacities $c_1 = 17$ and $c_2 = 14$, and a set $\mathcal{J} = \{1, 2, \dots, 5\}$ of 5 items ($n = 5$), with the following profits and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
p_j	10	8	11	6	9
w_{1j}	9	9	3	6	3
w_{2j}	5	5	2	3	9

Questions

1. Write the ILP_{MDKP} model of the chapter for this instance.
2. Solve the model with Gurobi. Give the optimal value, an optimal solution and the associated solution of the MDKP, and say whether the optimal solution is unique.

Solution

1. The model has 5 binary variables and $k = 2$ linear constraints, one for each dimension (besides the domains of the variables). The ILP_{MDKP} model for this instance is:

$$\begin{aligned}
 & \max \quad 10x_1 + 8x_2 + 11x_3 + 6x_4 + 9x_5 \\
 & \text{subject to} \quad 9x_1 + 9x_2 + 3x_3 + 6x_4 + 3x_5 \leq 17 \\
 & \quad \quad \quad 5x_1 + 5x_2 + 2x_3 + 3x_4 + 9x_5 \leq 14 \\
 & \quad \quad \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5 \in \{0, 1\}
 \end{aligned}$$

2. Gurobi returns the optimal value $z(\text{ILP}_{\text{MDKP}}) = 26$. An optimal solution is

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = 1, \quad \tilde{x}_4 = 1, \quad \tilde{x}_5 = 1$$

The associated subset is $\tilde{S} = \{3, 4, 5\}$, with weight $12 \leq 17$ in dimension 1 and $14 \leq 14$ in dimension 2.

Enumerating the 16 feasible subsets shows that the optimal solution is unique.

Exercise 9.10 — ILP model of the Multiple Knapsack Problem

The Multiple Knapsack Problem (MKP) and its ILP_{MKP} model are in [Chapter 9 «Formulations of knapsack problems»](#). We consider $m = 2$ knapsacks with capacities $c_1 = 12$ and $c_2 = 14$, and a set $\mathcal{J} = \{1, 2, \dots, 5\}$ of 5 items ($n = 5$), with the following profits and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
p_j	10	12	6	4	9
w_j	9	4	5	7	8

Questions

1. Write the ILP_{MKP} model of the chapter for this instance.
2. Solve the model with Gurobi. Give the optimal value, an optimal solution and the

associated solution of the MKP, and say whether the optimal solution is unique.

Solution

1. The model has $nm = 10$ binary variables, $m = 2$ capacity constraints and $n = 5$ assignment constraints (besides the domains of the variables). With so many variables we write one constraint per line. The ILP_{MKP} model for this instance is:

$$\begin{aligned}
 \max \quad & 10x_{11} + 12x_{21} + 6x_{31} + 4x_{41} + 9x_{51} \\
 & + 10x_{12} + 12x_{22} + 6x_{32} + 4x_{42} + 9x_{52} \\
 \text{subject to} \quad & 9x_{11} + 4x_{21} + 5x_{31} + 7x_{41} + 8x_{51} \leq 12 \\
 & 9x_{12} + 4x_{22} + 5x_{32} + 7x_{42} + 8x_{52} \leq 14 \\
 & x_{11} + x_{12} \leq 1 \\
 & x_{21} + x_{22} \leq 1 \\
 & x_{31} + x_{32} \leq 1 \\
 & x_{41} + x_{42} \leq 1 \\
 & x_{51} + x_{52} \leq 1 \\
 & x_{11}, x_{21}, x_{31}, \dots, x_{52} \in \{0, 1\}
 \end{aligned}$$

2. Gurobi returns the optimal value $z(\text{ILP}_{\text{MKP}}) = 37$. An optimal solution is

$$\tilde{x}_{21} = 1, \tilde{x}_{51} = 1, \tilde{x}_{12} = 1, \tilde{x}_{32} = 1,$$

with all the other variables equal to 0. The associated solution of the MKP is $(\tilde{S}_1, \tilde{S}_2) = (\{2, 5\}, \{1, 3\})$: knapsack 1 has weight $4 + 8 = 12 \leq 12$, knapsack 2 has weight $9 + 5 = 14 \leq 14$, and item 4 is left out.

Enumerating the 81 feasible assignments of the items to the knapsacks shows that the optimal solution is unique.

Exercise 9.11 — ILP model of the Multiple-Choice Knapsack Problem

The Multiple-Choice Knapsack Problem (MCKP) and its ILP_{MCKP} model are in [Chapter 9 «Formulations of knapsack problems»](#). We consider a knapsack of capacity $c = 14$ and a set $\mathcal{J} = \{1, 2, \dots, 7\}$ of 7 items ($n = 7$), partitioned into $q = 3$ classes $\mathcal{J}_1 = \{1, 2\}$, $\mathcal{J}_2 = \{3, 4, 5\}$ and $\mathcal{J}_3 = \{6, 7\}$, with the following profits and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$
p_j	6	4	12	7	11	12	6
w_j	6	6	4	3	6	3	8

Questions

1. Write the ILP_{MCKP} model of the chapter for this instance.

2. Solve the model with Gurobi. Give the optimal value, an optimal solution and the associated solution of the MCKP, and say whether the optimal solution is unique.

Solution

1. The model has 7 binary variables, one capacity constraint and $q = 3$ class constraints (besides the domains of the variables). The ILP_{MCKP} model for this instance is:

$$\begin{aligned}
 & \max \quad 6x_1 + 4x_2 + 12x_3 + 7x_4 + 11x_5 + 12x_6 + 6x_7 \\
 & \text{subject to} \quad 6x_1 + 6x_2 + 4x_3 + 3x_4 + 6x_5 + 3x_6 + 8x_7 \leq 14 \\
 & \quad \quad \quad x_1 + x_2 = 1 \\
 & \quad \quad \quad \quad \quad x_3 + x_4 + x_5 = 1 \\
 & \quad \quad \quad \quad \quad \quad \quad x_6 + x_7 = 1 \\
 & \quad \quad \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad x_6, \quad x_7 \in \{0, 1\}
 \end{aligned}$$

2. Gurobi returns the optimal value $z(\text{ILP}_{\text{MCKP}}) = 30$. An optimal solution is

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = 1, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0, \quad \tilde{x}_6 = 1, \quad \tilde{x}_7 = 0$$

The associated subset is $\tilde{S} = \{1, 3, 6\}$: one item per class, with total weight $6 + 4 + 3 = 13 \leq 14$.

Only 4 of the 12 choices of one item per class fit in the knapsack; enumerating them shows that the optimal solution is unique.

Exercise 9.12 — ILP model of the Knapsack Problem with Setups

The Knapsack Problem with Setups (KPS) and its ILP_{KPS} model are in [Chapter 9 «Formulations of knapsack problems»](#). We consider a knapsack of capacity $c = 19$ and a set $\mathcal{J} = \{1, 2, \dots, 6\}$ of 6 items ($n = 6$), partitioned into $q = 2$ classes $\mathcal{J}_1 = \{1, 2, 3\}$ and $\mathcal{J}_2 = \{4, 5, 6\}$, with setup costs $f_1 = 4$, $f_2 = 2$, setup weights $s_1 = 1$, $s_2 = 4$, and the following revenues and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
p_j	4	3	8	5	4	11
w_j	7	6	3	6	7	4

Questions

1. Write the ILP_{KPS} model of the chapter for this instance.
2. Solve the model with Gurobi. Give the optimal value, an optimal solution and the associated solution of the KPS, and say whether the optimal solution is unique.

- Enumerating the 27 feasible subsets shows that the optimal solution is unique.

Formulations of packing problems

Formulations used in these exercises. The model ILP_{BPP} of the chapter [Formulations of packing problems](#), with the set of bins $\mathcal{B} = \{1, \dots, m\}$:

$$\min \sum_{b \in \mathcal{B}} y_b \quad (\text{B0})$$

$$\text{subject to } \sum_{b \in \mathcal{B}} x_{jb} = 1, \quad \forall j \in \mathcal{J}, \quad (\text{B1})$$

$$- \sum_{j \in \mathcal{J}} w_j x_{jb} + c y_b \geq 0, \quad \forall b \in \mathcal{B}, \quad (\text{B2})$$

$$x_{jb} \in \{0, 1\}, \quad \forall j \in \mathcal{J}, \forall b \in \mathcal{B}, \quad (\text{B3})$$

$$y_b \in \{0, 1\}, \quad \forall b \in \mathcal{B}. \quad (\text{B4})$$

The relaxations LP-0_{BPP} and LP_{BPP} of the chapter relax (B3) and (B4) into $x_{jb} \geq 0$ and $y_b \geq 0$, with also $y_b \leq 1$ in LP_{BPP} . For every instance and every valid m the two relaxations have the same optimal value

$$z(\text{LP-0}_{\text{BPP}}) = z(\text{LP}_{\text{BPP}}) = \frac{\sum_{j \in \mathcal{J}} w_j}{c}, \quad (\text{B5})$$

attained by $\tilde{x}_{jb} = \frac{1}{m}$ and $\tilde{y}_b = \frac{\sum_j w_j}{m c}$; an optimal dual solution is $\tilde{\mu}_j = \frac{w_j}{c}$ and $\tilde{\pi}_b = \frac{1}{c}$ (and $\tilde{\lambda}_b = 0$ for LP_{BPP}).

Exercise 10.1 — ILP model of the Bin Packing Problem (1)

We consider a bin capacity $c = 10$ and a set $\mathcal{J} = \{1, 2, \dots, 5\}$ of 5 items ($n = 5$), with the following item weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
w_j	6	5	4	3	2

The packing patterns $\{1, 4\}$, $\{2, 3\}$ and $\{5\}$ are a feasible solution.

Questions

1. Explain why $m = 3$ is a valid upper bound on the number of bins used in any optimal solution.
2. Count the variables and the constraints (besides the domains of the variables) of the ILP_{BPP} model with $m = 3$.
3. Write the ILP_{BPP} model with $m = 3$ for this instance.
4. Solve the model with Gurobi. Give the optimal value, an optimal solution and the associated packing patterns, and say whether the optimal solution is unique.

Solution

1. The given packing patterns have weights 9, 9 and 2, all at most $c = 10$: they are a feasible solution with 3 bins. Hence $z(\text{ILP}_{\text{BPP}}) \leq 3$, and $m = 3$ is a valid upper bound.
2. There are $m n = 3 \cdot 5 = 15$ assignment variables x_{jb} and $m = 3$ activation variables y_b , so 18 binary variables. There are $n = 5$ assignment constraints and $m = 3$ capacity constraints, so 8 linear constraints.
3. With $\mathcal{B} = \{1, 2, 3\}$ the ILP_{BPP} model is:

$$\begin{aligned}
 \min \quad & y_1 + y_2 + y_3 \\
 \text{subject to} \quad & x_{11} + x_{12} + x_{13} = 1 \\
 & x_{21} + x_{22} + x_{23} = 1 \\
 & x_{31} + x_{32} + x_{33} = 1 \\
 & x_{41} + x_{42} + x_{43} = 1 \\
 & x_{51} + x_{52} + x_{53} = 1 \\
 & -6x_{11} - 5x_{21} - 4x_{31} - 3x_{41} - 2x_{51} + 10y_1 \geq 0 \\
 & -6x_{12} - 5x_{22} - 4x_{32} - 3x_{42} - 2x_{52} + 10y_2 \geq 0 \\
 & -6x_{13} - 5x_{23} - 4x_{33} - 3x_{43} - 2x_{53} + 10y_3 \geq 0 \\
 & x_{11}, x_{12}, x_{13}, \dots, x_{53} \in \{0, 1\} \\
 & y_1, y_2, y_3 \in \{0, 1\}
 \end{aligned}$$

4. Gurobi returns the optimal value $z(\text{ILP}_{\text{BPP}}) = 2$. The lower bound $\lceil \frac{6+5+4+3+2}{10} \rceil = 2$ confirms it. An optimal solution is

$$\tilde{x}_{11} = 1, \tilde{x}_{31} = 1, \tilde{x}_{22} = 1, \tilde{x}_{42} = 1, \tilde{x}_{52} = 1, \tilde{y}_1 = \tilde{y}_2 = 1, \tilde{y}_3 = 0,$$

with all the other variables equal to 0. The packing patterns are $\tilde{P}_1 = \{1, 3\}$ (weight 10) and $\tilde{P}_2 = \{2, 4, 5\}$ (weight 10).

The partition of the items is unique: item 1 can share a bin only with weight at most 4, and only $\{1, 3\}$ leaves a second bin of weight at most 10. The optimal solution of the model is not unique: the two patterns can go into any two of the 3 bins, which gives $3 \cdot 2 = 6$ optimal solutions.

Exercise 10.2 — LP relaxations of the BPP model (1)

We consider a bin capacity $c = 12$ and a set $\mathcal{J} = \{1, 2, \dots, 5\}$ of 5 items ($n = 5$), with the following item weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
w_j	7	6	5	4	3

We use $m = 3$ bins: the packing patterns $\{1, 3\}$, $\{2, 4\}$ and $\{5\}$ are feasible.

Questions

1. Solve the two LP relaxations LP-0BPP and LP_{BPP} with Gurobi and compare their optimal values with formula (B5).
2. Write the primal solution and the dual solution given by the chapter for this instance, and verify that they are feasible and have the same objective value.
3. Derive the lower bound on the number of bins, and decide whether it is equal to the optimal value.

Solution

We use the value (B5) of the relaxations and the primal and dual solutions given with it in the box at the beginning of the chapter.

1. Gurobi returns $z(\text{LP-0BPP}) = \frac{25}{12}$ and $z(\text{LP}_{\text{BPP}}) = \frac{25}{12}$. The formula gives the same value:

$$\frac{\sum_{j \in \mathcal{J}} w_j}{c} = \frac{7 + 6 + 5 + 4 + 3}{12} = \frac{25}{12}.$$

The upper bounds $y_b \leq 1$ do not change the optimal value.

2. The primal solution is $\tilde{x}_{jb} = \frac{1}{3}$ for every j and b , and $\tilde{y}_b = \frac{25}{3 \cdot 12} = \frac{25}{36}$ for every b .
 - Assignment: $\sum_b \tilde{x}_{jb} = 3 \cdot \frac{1}{3} = 1$ for every item j .
 - Capacity: $-\sum_j w_j \tilde{x}_{jb} + 12 \tilde{y}_b = -\frac{25}{3} + 12 \cdot \frac{25}{36} = 0$ for every bin b .
 - Bounds: $0 \leq \frac{25}{36} \leq 1$, so $\tilde{y}_b \leq 1$ holds as well.

The dual solution is $\tilde{\pi}_b = \frac{1}{12}$ for every bin and $\tilde{\mu}_j = \frac{w_j}{12}$:

$$\tilde{\mu}_1 = \frac{7}{12}, \quad \tilde{\mu}_2 = \frac{1}{2}, \quad \tilde{\mu}_3 = \frac{5}{12}, \quad \tilde{\mu}_4 = \frac{1}{3}, \quad \tilde{\mu}_5 = \frac{1}{4}.$$

Every constraint $\mu_j - w_j \pi_b \leq 0$ holds with equality, since $\frac{w_j}{12} - w_j \frac{1}{12} = 0$, and every constraint $c \pi_b \leq 1$ holds with equality, since $12 \cdot \frac{1}{12} = 1$.

Primal value: $\sum_b \tilde{y}_b = 3 \cdot \frac{25}{36} = \frac{25}{12}$. Dual value: $\sum_j \tilde{\mu}_j = \frac{25}{12}$. The two values are equal, so by LP Weak Duality both solutions are optimal.

3. Since $z(\text{ILP}_{\text{BPP}})$ is an integer, $z(\text{ILP}_{\text{BPP}}) \geq \lceil \frac{25}{12} \rceil = 3$. The feasible solution of the statement uses 3 bins, so $z(\text{ILP}_{\text{BPP}}) = 3$ (Gurobi gives the same value): the rounded bound is equal to the optimal value.

Exercise 10.3 — Feasible solutions of the BPP model (1)

We consider a bin capacity $c = 10$ and a set $\mathcal{J} = \{1, 2, \dots, 6\}$ of 6 items ($n = 6$), with the following item weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
w_j	5	4	4	3	2	2

We use the ILP_{BPP} model with $m = 3$ bins. The following candidate solutions give the items in each bin (all the variables not listed are equal to 0):

- (a) bin 1: $\{1, 3\}$, bin 2: $\{2, 4, 5\}$, bin 3: $\{6\}$, and $y_1 = y_2 = y_3 = 1$;
- (b) bin 1: $\{1, 2, 4\}$, bin 2: $\{3, 5\}$, bin 3: $\{6\}$, and $y_1 = y_2 = y_3 = 1$;
- (c) bin 1: $\{1, 4, 5\}$, bin 2: $\{2, 3, 6\}$, bin 3 empty, and $y_1 = y_2 = y_3 = 1$;
- (d) bin 1: $\{1, 4, 5\}$, bin 2: $\{2, 3, 6\}$, bin 3: $\{6\}$, and $y_1 = y_2 = y_3 = 1$;
- (e) $x_{jb} = \frac{1}{3}$ for every item j and every bin b , and $y_1 = y_2 = y_3 = \frac{2}{3}$.

Questions

- For each candidate, check the assignment constraints and the capacity constraints of the model.
- Decide which candidates are feasible for the ILP_{BPP} model and which are feasible for its relaxation LP_{BPP} , and give the objective value of the feasible ones.
- Use the candidates to find the optimal value of the instance.

Solution

- Every item is in exactly one bin. The loads are 9, 9, 2 against the capacities 10, 10, 10. All the capacity constraints hold.
 - Every item is in exactly one bin. The loads are 12, 6, 2 against the capacities 10, 10, 10. The capacity constraint of bin 1 is violated.
 - Every item is in exactly one bin. The loads are 10, 10, 0 against the capacities 10, 10, 10. All the capacity constraints hold.
 - Item 6 is in 2 bins: the assignment constraint of item 6 is violated. The loads are 10, 10, 2 against the capacities 10, 10, 10. All the capacity constraints hold.
 - Every item is in exactly one bin. The loads are $\frac{20}{3}$, $\frac{20}{3}$, $\frac{20}{3}$ against the capacities $\frac{20}{3}$, $\frac{20}{3}$, $\frac{20}{3}$. All the capacity constraints hold.
- Feasible for ILP_{BPP} , hence for LP_{BPP} , of value 3.
 - Infeasible for both models: bin 1 contains weight $12 > 10$.
 - Feasible for both models, of value 3. Bin 3 is empty but counted: setting $y_3 = 0$ keeps all the constraints satisfied and gives value 2.

- (d) Infeasible for both models: item 6 is packed twice.
- (e) Feasible for LP_{BPP} , of value $3 \cdot \frac{2}{3} = 2$, but not for ILP_{BPP} , since the variables are fractional. Every capacity constraint holds with equality: $\frac{20}{3} = 10 \cdot \frac{2}{3}$.
3. Candidate (c) with $y_3 = 0$ uses 2 bins, with patterns $\{1, 4, 5\}$ and $\{2, 3, 6\}$, both of weight 10. The lower bound $\lceil \frac{5+4+4+3+2+2}{10} \rceil = 2$ shows that no solution uses fewer bins: $z(\text{ILP}_{\text{BPP}}) = 2$. The value 2 of candidate (e) is the value $\frac{\sum_j w_j}{c} = 2$ of the LP relaxations.

Exercise 10.4 — The bounds $y_b \leq 1$ and the choice of m (1)

We consider a bin capacity $c = 10$ and a set $\mathcal{J} = \{1, 2, \dots, 3\}$ of 3 items ($n = 3$), with the following item weights:

	$j = 1$	$j = 2$	$j = 3$
w_j	6	6	6

The two relaxations differ only in the upper bounds $y_b \leq 1$.

Questions

1. Solve LP-0_{BPP} , LP_{BPP} and ILP_{BPP} with $m = 3$ using Gurobi, and explain why the bounds $y_b \leq 1$ do not change the value of the relaxation.
2. Show that the upper bounds $x_{jb} \leq 1$ are valid but implied by the other constraints.
3. Repeat the computation with $m = 2$. Explain the result.
4. Repeat the computation with $m = 1$, and compare the two relaxations.

Solution

1. Gurobi gives $z(\text{LP-0}_{\text{BPP}}) = z(\text{LP}_{\text{BPP}}) = \frac{9}{5}$ and $z(\text{ILP}_{\text{BPP}}) = 3$. The optimal solution of the chapter has $\tilde{y}_b = \frac{18}{3 \cdot 10} = \frac{3}{5} \leq 1$: it satisfies the bounds $y_b \leq 1$, so adding them cuts off no optimal solution of LP-0_{BPP} . In general $\frac{\sum_j w_j}{m c} \leq 1$ whenever $\frac{\sum_j w_j}{c} \leq z(\text{ILP}_{\text{BPP}}) \leq m$.
2. For every item j and bin b , the assignment constraint and $x_{jk} \geq 0$ give

$$x_{jb} \leq \sum_{k \in \mathcal{B}} x_{jk} = 1.$$

Every point that satisfies the assignment constraints and the non-negativity constraints already satisfies $x_{jb} \leq 1$.

3. With $m = 2$ the ILP_{BPP} model is infeasible (Gurobi): every item has weight $6 > \frac{c}{2} = 5$, so no two items share a bin and 3 bins are needed. The relaxations are feasible, with value $\frac{9}{5}$ and $\tilde{y}_1 = \tilde{y}_2 = \frac{18}{2 \cdot 10} = \frac{9}{10} \leq 1$. A value $m < z(\text{ILP}_{\text{BPP}})$ makes the model wrong, and the relaxation does not reveal it.

4. With $m = 1$, LP-0BPP is still feasible, with $\tilde{y}_1 = \frac{9}{5}$ and value $\frac{9}{5}$. Instead LP_{BPP} is infeasible (Gurobi): the capacity constraint and the assignment constraints give $10y_1 \geq 18$, that is $y_1 \geq \frac{9}{5} > 1$, against $y_1 \leq 1$. Here the bounds $y_b \leq 1$ do make a difference: they detect that one bin cannot hold all the items.

Exercise 10.5 — Integrality gap of the BPP model (1)

Given $\vartheta \in \mathbb{Z}_{\geq 1}$, the chapter considers the BPP instances with capacity $c = 2\vartheta$ and n items of weight $w_j = \vartheta + 1$. We take $\vartheta = 4$ and $n = 4$, that is $c = 8$ and the weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
w_j	5	5	5	5

Questions

1. Use $m = 4$ and Gurobi to compute $z(\text{LP}_{\text{BPP}})$ and $z(\text{ILP}_{\text{BPP}})$, and explain the value of $z(\text{ILP}_{\text{BPP}})$.
2. Compute the absolute gap $z(\text{ILP}_{\text{BPP}}) - z(\text{LP}_{\text{BPP}})$ and the ratios $\frac{z(\text{LP}_{\text{BPP}})}{z(\text{ILP}_{\text{BPP}})}$ and $\frac{\lceil z(\text{LP}_{\text{BPP}}) \rceil}{z(\text{ILP}_{\text{BPP}})}$, and compare them with the value $\frac{1}{2}$ of the chapter.
3. Find the smallest ϑ for which the ratio $\frac{z(\text{LP}_{\text{BPP}})}{z(\text{ILP}_{\text{BPP}})}$ is at most $\frac{11}{20}$.
4. Consider the second family of the chapter with $n = 6$: capacity $c = 6$ and 6 items of weight $\frac{c}{2} + 1 = 4$. Compute the ratio $\frac{\lceil z(\text{LP}_{\text{BPP}}) \rceil}{z(\text{ILP}_{\text{BPP}})}$ with Gurobi.

Solution

We use the value (B5) of the relaxations and the primal and dual solutions given with it in the box at the beginning of the chapter.

1. Gurobi gives $z(\text{LP}_{\text{BPP}}) = \frac{4 \cdot 5}{8} = \frac{5}{2}$ and $z(\text{ILP}_{\text{BPP}}) = 4$. Every item has weight $5 > \frac{c}{2} = 4$, so two items never fit in the same bin, and each item needs its own bin.
2. The absolute gap is $4 - \frac{5}{2} = \frac{3}{2}$. The ratios are

$$\frac{z(\text{LP}_{\text{BPP}})}{z(\text{ILP}_{\text{BPP}})} = \frac{\frac{5}{2}}{4} = \frac{5}{8} \quad \text{and} \quad \frac{\lceil z(\text{LP}_{\text{BPP}}) \rceil}{z(\text{ILP}_{\text{BPP}})} = \frac{3}{4} = \frac{3}{4}.$$

Both are at least $\frac{1}{2}$, as the chapter proves for every instance.

3. For this family the ratio is

$$\frac{z(\text{LP}_{\text{BPP}})}{z(\text{ILP}_{\text{BPP}})} = \frac{\frac{n(\vartheta+1)}{2\vartheta}}{n} = \frac{\vartheta+1}{2\vartheta}.$$

The condition $\frac{\vartheta+1}{2\vartheta} \leq \frac{11}{20}$ is $20\vartheta + 20 \leq 22\vartheta$, that is $\vartheta \geq 10$. The smallest value is $\vartheta = 10$, with $c = 20$ and items of weight 11.

4. Gurobi gives $z(\text{LP}_{\text{BPP}}) = \frac{6.4}{6} = 4$ and $z(\text{ILP}_{\text{BPP}}) = 6$, since again no two items share a bin. The ratio is $4/6 = \frac{2}{3}$. In general it is $\frac{n/2+1}{n} = \frac{1}{2} + \frac{1}{n}$, which tends to $\frac{1}{2}$.

Exercise 10.6 — ILP model of the Cutting Stock Problem (CSP)

We consider a Cutting Stock Problem with bins of capacity $c = 10$ and a set $\mathcal{J} = \{1, 2, 3\}$ of 3 items, with the following weights w_j and required copies d_j :

	$j = 1$	$j = 2$	$j = 3$
w_j	6	5	3
d_j	3	1	2

We use the ILP_{CSP} model of the chapter, with integer variables x_{jb} for the number of copies of item j packed in bin b and binary variables y_b for the used bins.

Questions

1. Find a feasible solution with 4 bins, and count the variables and the constraints (besides the domains) of the ILP_{CSP} model with $m = 4$.
2. Write the ILP_{CSP} model with $m = 4$ for this instance.
3. Solve the model with Gurobi: give the optimal value and the content of the used bins.
4. Compute the weight bound $\lceil \sum_j d_j w_j / c \rceil$ and the optimal value of the LP relaxation (Gurobi), and explain the gap with the optimal value.
5. How many binary variables would the ILP_{BPP} model need on the extended set of items, with the same m ?

Solution

1. Each copy of item 1 in its own bin, with a copy of item 3 in two of them, and item 2 alone: the bins $\{1, 3\}$, $\{1, 3\}$, $\{1\}$, $\{2\}$ have weights 9, 9, 6, 5, all at most 10. Hence $m = 4$ is a valid upper bound. The model has $mn = 4 \cdot 3 = 12$ integer variables x_{jb} and $m = 4$ binary variables y_b , so 16 variables, and $n + m = 3 + 4 = 7$ linear constraints.
2. The ILP_{CSP} model is:

$$\begin{aligned}
 \min \quad & y_1 + y_2 + y_3 + y_4 \\
 \text{subject to} \quad & x_{11} + x_{12} + x_{13} + x_{14} = 3 \\
 & x_{21} + x_{22} + x_{23} + x_{24} = 1 \\
 & x_{31} + x_{32} + x_{33} + x_{34} = 2 \\
 & -6x_{11} - 5x_{21} - 3x_{31} + 10y_1 \geq 0 \\
 & -6x_{12} - 5x_{22} - 3x_{32} + 10y_2 \geq 0 \\
 & -6x_{13} - 5x_{23} - 3x_{33} + 10y_3 \geq 0 \\
 & -6x_{14} - 5x_{24} - 3x_{34} + 10y_4 \geq 0 \\
 & x_{11}, x_{12}, x_{13}, \dots, x_{34} \in \mathbb{Z}_{\geq 0} \\
 & y_1, y_2, y_3, y_4 \in \{0, 1\}
 \end{aligned}$$

3. Gurobi returns $z(\text{ILP}_{\text{CSP}}) = 4$. The used bins of the solution found contain the copies $\{2\}$ (weight 5); $\{1\}$ (weight 6); $\{1, 3\}$ (weight 9); $\{1, 3\}$ (weight 9).
4. The total weight is $\sum_j d_j w_j = 3 \cdot 6 + 1 \cdot 5 + 2 \cdot 3 = 29$, so the weight bound is $\lceil \frac{29}{10} \rceil = 3$, and Gurobi gives the optimal value $\frac{29}{10}$ for the LP relaxation, with $x_{jb} \geq 0$ and $0 \leq y_b \leq 1$. Both are smaller than the optimal value 4: two copies of item 1 never share a bin ($6 + 6 > 10$), and item 2 does not fit with a copy of item 1 ($6 + 5 > 10$), so the three copies of item 1 and item 2 need four different bins.
5. The extended set of items has $\sum_j d_j = 6$ copies, so the ILP_{BPP} model would have $4 \cdot 6 = 24$ binary variables x_{jb} and 4 variables y_b , against the 12 integer variables of the ILP_{CSP} model; it would also have many equivalent solutions, obtained by exchanging copies of the same item.

Formulations of coloring problems

Formulations used in these exercises. The model ILP_{VCP} of the chapter [Formulations of coloring problems](#), with the set of colors $\mathcal{C} = \{1, \dots, k\}$:

$$\min \sum_{c \in \mathcal{C}} y_c \quad (\text{C0})$$

$$\text{subject to } \sum_{c \in \mathcal{C}} x_{vc} = 1, \quad \forall v \in V, \quad (\text{C1})$$

$$-x_{uc} - x_{vc} + y_c \geq 0, \quad \forall c \in \mathcal{C}, \forall \{u, v\} \in E, \quad (\text{C2})$$

$$x_{vc} \in \{0, 1\}, \quad \forall v \in V, \forall c \in \mathcal{C}, \quad (\text{C3})$$

$$y_c \in \{0, 1\}, \quad \forall c \in \mathcal{C}. \quad (\text{C4})$$

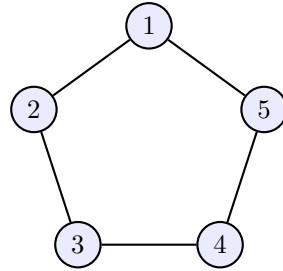
The relaxations LP-0_{VCP} and LP_{VCP} of the chapter relax (C3) and (C4) into $x_{vc} \geq 0$ and $y_c \geq 0$, with also $y_c \leq 1$ in LP_{VCP} . For every graph with at least one edge and every valid k , an optimal solution of both relaxations is

$$\tilde{x}_{v1} = \tilde{x}_{v2} = \frac{1}{2} \quad (v \in V), \quad \tilde{y}_1 = \tilde{y}_2 = 1, \quad (\text{C5})$$

with all the other variables equal to 0, and $z(\text{LP-0}_{\text{VCP}}) = z(\text{LP}_{\text{VCP}}) = 2$. An optimal dual solution chooses one edge $\{u, v\} \in E$ and sets $\tilde{\mu}_u = \tilde{\mu}_v = 1$, $\tilde{\pi}_{c\{u,v\}} = 1$ for every color c , and all the other dual variables equal to 0.

Exercise 11.1 — ILP model of the Vertex Coloring Problem (1)

We consider the following graph $G = (V, E)$ with 5 vertices ($n = 5$) and 5 edges ($m = 5$):



$$V = \{1, \dots, 5\},$$

$$E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{1, 5\}\}.$$

The color classes $\{1, 3\}$, $\{2, 4\}$ and $\{5\}$ are a feasible coloring.

Questions

1. Explain why $k = 3$ is a valid upper bound on the number of colors of an optimal solution.
2. Count the variables and the constraints (besides the domains) of the ILP_{VCP} model with $k = 3$.
3. Write the ILP_{VCP} model with $k = 3$ for this graph (write the edge constraints for all the colors c at once).
4. Solve the model with Gurobi. Give the optimal value and an optimal coloring, and count the optimal partitions of the vertices into color classes.

Solution

1. The given classes contain no edge: they are a feasible coloring with 3 colors. Hence $z(\text{ILP}_{\text{VCP}}) \leq 3$ and $k = 3$ is valid.
2. There are $k n = 3 \cdot 5 = 15$ variables x_{vc} and $k = 3$ variables y_c , so 18 binary variables. There are $n = 5$ assignment constraints and $k m = 3 \cdot 5 = 15$ edge constraints, so 20 linear constraints.
3. With $\mathcal{C} = \{1, 2, 3\}$ the ILP_{VCP} model is:

$$\begin{aligned}
& \min && y_1 + y_2 + y_3 \\
& \text{subject to} && x_{11} + x_{12} + x_{13} &= 1 \\
& && x_{21} + x_{22} + x_{23} &= 1 \\
& && x_{31} + x_{32} + x_{33} &= 1 \\
& && x_{41} + x_{42} + x_{43} &= 1 \\
& && x_{51} + x_{52} + x_{53} &= 1 \\
& && -x_{11} - x_{21} + y_1 &\geq 0 \\
& && -x_{21} - x_{31} + y_1 &\geq 0 \\
& && -x_{31} - x_{41} + y_1 &\geq 0 \\
& && -x_{41} - x_{51} + y_1 &\geq 0 \\
& && -x_{11} - x_{51} + y_1 &\geq 0 \\
& && -x_{12} - x_{22} + y_2 &\geq 0 \\
& && -x_{22} - x_{32} + y_2 &\geq 0 \\
& && -x_{32} - x_{42} + y_2 &\geq 0 \\
& && -x_{42} - x_{52} + y_2 &\geq 0 \\
& && -x_{12} - x_{52} + y_2 &\geq 0 \\
& && -x_{13} - x_{23} + y_3 &\geq 0 \\
& && -x_{23} - x_{33} + y_3 &\geq 0 \\
& && -x_{33} - x_{43} + y_3 &\geq 0 \\
& && -x_{43} - x_{53} + y_3 &\geq 0 \\
& && -x_{13} - x_{53} + y_3 &\geq 0 \\
& && x_{11}, x_{12}, x_{13}, \dots, x_{53} &\in \{0, 1\} \\
& && y_1, y_2, y_3 &\in \{0, 1\}
\end{aligned}$$

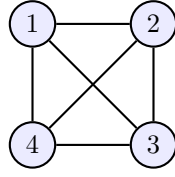
4. Gurobi returns $z(\text{ILP}_{\text{VCP}}) = 3$. Two colors are not enough: the cycle has an odd number of vertices, and colors that alternate along it would give the same color to vertices 1 and 5. An optimal coloring is $\tilde{Q}_1 = \{1, 3\}$, $\tilde{Q}_2 = \{2, 4\}$, $\tilde{Q}_3 = \{5\}$, that is

$$\tilde{x}_{11} = 1, \tilde{x}_{31} = 1, \tilde{x}_{22} = 1, \tilde{x}_{42} = 1, \tilde{x}_{53} = 1, \tilde{y}_1 = \tilde{y}_2 = \tilde{y}_3 = 1.$$

By enumeration there are 5 optimal partitions: one class is a single vertex, and the other four vertices form two pairs of non-adjacent vertices in a unique way. With the $3! = 6$ permutations of the colors, the model has 30 optimal solutions.

Exercise 11.2 — LP relaxations of the VCP model (1)

We consider the following graph $G = (V, E)$ with 4 vertices ($n = 4$) and 6 edges ($m = 6$):



$$V = \{1, \dots, 4\},$$

$$E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}.$$

The graph is complete, and we use $k = 4$ colors.

Questions

1. Solve LP-0_{VCP} , LP_{VCP} and ILP_{VCP} with Gurobi.
2. Write the primal solution (C5) for this graph and verify that it is feasible for LP_{VCP} .
3. Write the dual solution of the chapter built on the edge $\{1, 2\}$ and verify that it is feasible for $D(\text{LP-0}_{\text{VCP}})$, with the same objective value.
4. Compute the absolute gap and the ratio between the LP bound and the optimal value. What happens on the complete graph with n vertices?

Solution

We use the primal solution (C5) and the dual solution built on one edge, given in the box at the beginning of the chapter.

1. Gurobi gives $z(\text{LP-0}_{\text{VCP}}) = z(\text{LP}_{\text{VCP}}) = 2$ and $z(\text{ILP}_{\text{VCP}}) = 4$: all the vertices are pairwise adjacent, so they need 4 different colors.
2. The solution is $\tilde{x}_{v1} = \tilde{x}_{v2} = \frac{1}{2}$ for $v = 1, \dots, 4$, $\tilde{x}_{v3} = \tilde{x}_{v4} = 0$, $\tilde{y}_1 = \tilde{y}_2 = 1$ and $\tilde{y}_3 = \tilde{y}_4 = 0$.
 - Assignment: $\frac{1}{2} + \frac{1}{2} = 1$ for every vertex.
 - Edges, colors 1 and 2: $\tilde{y}_c - \tilde{x}_{uc} - \tilde{x}_{vc} = 1 - \frac{1}{2} - \frac{1}{2} = 0$. Colors 3 and 4: $0 - 0 - 0 = 0$.
 - Bounds: $0 \leq \tilde{y}_c \leq 1$ for every color.

Its value is $\tilde{y}_1 + \tilde{y}_2 = 2$.

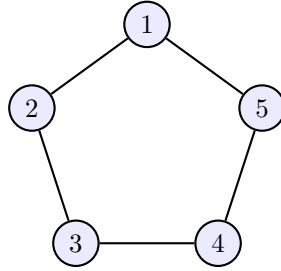
3. The dual solution is $\tilde{\mu}_1 = \tilde{\mu}_2 = 1$, $\tilde{\mu}_3 = \tilde{\mu}_4 = 0$, $\tilde{\pi}_{c\{1,2\}} = 1$ for $c = 1, \dots, 4$, and all the other $\tilde{\pi}_{ce} = 0$.
 - Vertices 1 and 2: the edge $\{1, 2\}$ is incident to both, so $\tilde{\mu}_v - \sum_{e \in \delta(v)} \tilde{\pi}_{ce} = 1 - 1 = 0$ for every color.
 - Vertices 3 and 4: $0 - 0 = 0$, since no edge incident to them has a non-zero value.
 - Colors: $\sum_{e \in E} \tilde{\pi}_{ce} = 1 \leq 1$ for every color.

Its value is $\tilde{\mu}_1 + \tilde{\mu}_2 = 2$, equal to the primal value: by LP Weak Duality both solutions are optimal.

4. The absolute gap is $4 - 2 = 2$ and the ratio is $\frac{z(\text{LP}_{\text{VCP}})}{z(\text{ILP}_{\text{VCP}})} = \frac{1}{2}$. On the complete graph with n vertices the optimal value is n and the LP bound is still 2: the gap $n - 2$ grows without limit and the ratio $\frac{2}{n}$ tends to 0.

Exercise 11.3 — Feasible solutions of the VCP model (1)

We consider the following graph $G = (V, E)$ with 5 vertices ($n = 5$) and 5 edges ($m = 5$):



$$V = \{1, \dots, 5\},$$

$$E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{1, 5\}\}.$$

We use the ILP_{VCP} model with $k = 3$ colors. The candidate solutions are (variables not listed are 0):

- (a) color 1: $\{1, 3\}$, color 2: $\{2, 4\}$, color 3: $\{5\}$, and $y_1 = y_2 = y_3 = 1$;
- (b) color 1: $\{1, 3\}$, color 2: $\{2, 4, 5\}$, and $y_1 = y_2 = 1, y_3 = 0$;
- (c) color 1: $\{1, 3\}$, color 2: $\{2, 4\}$, color 3: $\{5\}$, and $y_1 = y_2 = 1, y_3 = 0$;
- (d) $x_{v1} = x_{v2} = \frac{1}{2}$ for every vertex v , and $y_1 = y_2 = 1, y_3 = 0$;
- (e) $x_{vc} = \frac{1}{3}$ for every vertex v and color c , and $y_1 = y_2 = y_3 = \frac{2}{3}$.

Questions

1. For each candidate, check the assignment constraints and the edge constraints, and name a violated constraint when there is one.
2. Decide which candidates are feasible for ILP_{VCP} and which for LP_{VCP} , and give the value of the feasible ones.
3. Which candidates are optimal for LP_{VCP} ? Use Gurobi to confirm the optimal value of the relaxation.

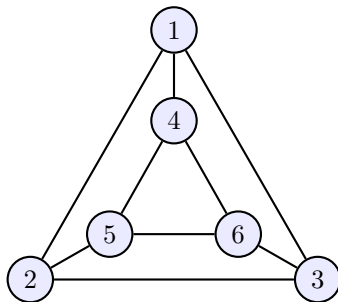
Solution

1. (a) Every vertex has one color, and no edge has both endpoints in the same class: all the constraints hold.
- (b) The edge $\{4, 5\}$ has both endpoints with color 2: the constraint $-x_{42} - x_{52} + y_2 \geq 0$ reads $-1 - 1 + 1 = -1 < 0$.

- (c) Vertex 5 takes color 3 but $y_3 = 0$: the constraint $-x_{43} - x_{53} + y_3 \geq 0$ reads $0 - 1 + 0 = -1 < 0$.
- (d) Each vertex has $\frac{1}{2} + \frac{1}{2} = 1$; for colors 1 and 2 every edge gives $1 - \frac{1}{2} - \frac{1}{2} = 0$, for color 3 it gives 0: all the constraints hold.
- (e) Each vertex has $3 \cdot \frac{1}{3} = 1$; every edge and color give $\frac{2}{3} - \frac{1}{3} - \frac{1}{3} = 0$: all the constraints hold.
2. (a) Feasible for ILP_{VCP} and for LP_{VCP} , of value 3.
 (b) Infeasible for both models.
 (c) Infeasible for both models.
 (d) Feasible for LP_{VCP} only (fractional variables), of value 2.
 (e) Feasible for LP_{VCP} only, of value $3 \cdot \frac{2}{3} = 2$.
3. The value of every feasible solution of LP_{VCP} is at least 2, and Gurobi gives $z(\text{LP}_{\text{VCP}}) = 2$: candidates (d) and (e) are both optimal for the relaxation. Candidate (d) is the solution of the chapter; (e) shows that the optimal solution of the relaxation is not unique. Candidate (a) is optimal for ILP_{VCP} , whose value is 3.

Exercise 11.4 — Dual solutions of the VCP relaxations (1)

We consider the following graph $G = (V, E)$ with 6 vertices ($n = 6$) and 9 edges ($m = 9$):



$$V = \{1, \dots, 6\},$$

$$E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{1, 4\}, \{2, 5\}, \{3, 6\}\}.$$

We use $k = 3$ colors. Every vertex has degree 3.

Questions

- Write the constraints of the dual $D(\text{LP-0}_{\text{VCP}})$ associated with the variable x_{1c} and with the variable y_c , for a color c .
- Verify that the dual solutions of the chapter built on the edge $\{1, 4\}$ and on the edge $\{2, 3\}$ are feasible, and give their value.
- Verify that $\tilde{\mu}_v = \frac{1}{3} = \frac{\deg(v)}{|E|}$ for every vertex and $\tilde{\pi}_{ce} = \frac{1}{9}$ for every color and edge is also an optimal dual solution.

4. Add the bounds $y_c \leq 1$, with dual variables $\lambda_c \leq 0$. Give an optimal solution of $D(\text{LP}_{\text{VCP}})$, and compare the two relaxations with the optimal value (use Gurobi).

Solution

We use the primal solution (C5) and the dual solution built on one edge, given in the box at the beginning of the chapter.

1. The edges incident to vertex 1 are $\{1, 2\}$, $\{1, 3\}$ and $\{1, 4\}$. The constraint of x_{1c} and the constraint of y_c are

$$\mu_1 - \pi_{c\{1,2\}} - \pi_{c\{1,3\}} - \pi_{c\{1,4\}} \leq 0, \quad \sum_{e \in E} \pi_{ce} \leq 1.$$

2. For the edge $\{1, 4\}$: $\tilde{\mu}_1 = \tilde{\mu}_4 = 1$ and $\tilde{\pi}_{c\{1,4\}} = 1$ for $c = 1, 2, 3$. Vertices 1 and 4 give $1 - 1 = 0$, the other vertices give $0 - 0 = 0$, and every color gives $\sum_e \tilde{\pi}_{ce} = 1$. The value is 2.

For the edge $\{2, 3\}$ the check is the same with vertices 2 and 3: also this solution is feasible, of value 2. Both are optimal, since $z(\text{LP-0}_{\text{VCP}}) = 2$.

3. Every vertex has 3 incident edges, so $\tilde{\mu}_v - \sum_{e \in \delta(v)} \tilde{\pi}_{ce} = \frac{1}{3} - 3 \cdot \frac{1}{9} = 0$. Every color gives $\sum_{e \in E} \tilde{\pi}_{ce} = 9 \cdot \frac{1}{9} = 1$. The value is

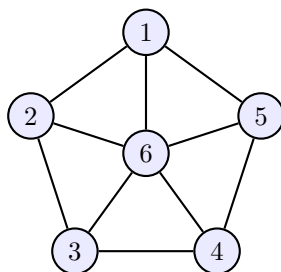
$$\sum_{v \in V} \tilde{\mu}_v = 6 \cdot \frac{1}{3} = 2,$$

the optimal value: this dual solution spreads the weight on all the edges, instead of choosing one.

4. The previous dual solutions with $\tilde{\lambda}_c = 0$ for every color satisfy $\sum_e \tilde{\pi}_{ce} + \tilde{\lambda}_c \leq 1$ and have value 2. Gurobi gives $z(\text{LP-0}_{\text{VCP}}) = z(\text{LP}_{\text{VCP}}) = 2$ and $z(\text{ILP}_{\text{VCP}}) = 3$: the bounds $y_c \leq 1$ do not strengthen the relaxation, which stays at 2 while the graph needs 3 colors (it contains the triangle $\{1, 2, 3\}$).

Exercise 11.5 — Integrality gap of the VCP model (1)

We consider the following graph $G = (V, E)$ with 6 vertices ($n = 6$) and 10 edges ($m = 10$):



$$V = \{1, \dots, 6\},$$

$$E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{1, 5\}, \\ \{1, 6\}, \{2, 6\}, \{3, 6\}, \{4, 6\}, \{5, 6\}\}.$$

The graph is a cycle on the vertices $1, \dots, 5$ with vertex 6 adjacent to all of them. The classes $\{1, 3\}, \{2, 4\}, \{5\}, \{6\}$ are a feasible coloring, so we use $k = 4$.

Questions

1. Compute $z(\text{LP}_{\text{VCP}})$ and $z(\text{ILP}_{\text{VCP}})$ with Gurobi, and explain the optimal value.
2. Compute the absolute gap and the ratio $\frac{z(\text{LP}_{\text{VCP}})}{z(\text{ILP}_{\text{VCP}})}$.
3. Solve the relaxation again with $k = 6$. Does a larger k change the bound?
4. On the complete graph with n vertices, find the smallest n for which the ratio is at most $\frac{1}{5}$.

Solution

We use the primal solution (C5) and the dual solution built on one edge, given in the box at the beginning of the chapter.

1. Gurobi gives $z(\text{LP}_{\text{VCP}}) = 2$ and $z(\text{ILP}_{\text{VCP}}) = 4$. The cycle $1, \dots, 5$ needs 3 colors, and vertex 6, adjacent to all its vertices, needs a fourth one.
2. The absolute gap is $4 - 2 = 2$ and the ratio is $\frac{z(\text{LP}_{\text{VCP}})}{z(\text{ILP}_{\text{VCP}})} = \frac{1}{2}$.
3. With $k = 6$ Gurobi gives again $z(\text{LP}_{\text{VCP}}) = 2$. The value 2 does not depend on k : the solution of the chapter uses only colors 1 and 2, whatever the number of colors.
4. On the complete graph the ratio is $\frac{2}{n}$, and $\frac{2}{n} \leq \frac{1}{5}$ means $n \geq 10$. The smallest value is $n = 10$.

Formulations of clique problems

Formulations used in these exercises. The models of the chapter [Formulations of clique problems](#), where $\bar{E}$ is the set of non-edges of G and $\mathcal{Q}$ the set of the maximal stable sets of G (the maximal cliques of the complement graph). The edge formulation ILP_{MCP} :

$$\max \sum_{v \in V} x_v \quad (\text{M0})$$

$$\text{subject to } x_u + x_v \leq 1, \quad \forall \{u, v\} \in \bar{E}, \quad (\text{M1})$$

$$x_v \in \{0, 1\}, \quad \forall v \in V. \quad (\text{M2})$$

The clique formulation $\text{ILP}_{\text{MCP}}^{\text{CL}}$:

$$\max \sum_{v \in V} x_v \quad (\text{M3})$$

$$\text{subject to } \sum_{v \in Q} x_v \leq 1, \quad \forall Q \in \mathcal{Q}, \quad (\text{M4})$$

$$x_v \in \{0, 1\}, \quad \forall v \in V. \quad (\text{M5})$$

The relaxation LP_{MCP} relaxes (M2) into $0 \leq x_v \leq 1$; the relaxation $\text{LP}_{\text{MCP}}^{\text{CL}}$ relaxes (M5) into $x_v \geq 0$, since (M4) gives $x_v \leq 1$. The models ILP_{MSSP} and $\text{ILP}_{\text{MSSP}}^{\text{CL}}$ of the MSSP are (M0)–(M2) with (M1) written on the edges of G , and (M3)–(M5) with (M4) written on the maximal cliques of G . The model $\text{ILP}_{\text{MCP}}^{\text{VW}}$ of the VWMCP is (M0)–(M2) with the objective function $\max \sum_{v \in V} w_v x_v$. The model $\text{ILP}_{\text{MCP}}^{\text{EW}}$ of the EWMCP adds a binary variable y_{uv} for each edge $\{u, v\} \in E$:

$$\max \sum_{\{u, v\} \in E} r_{uv} y_{uv} \quad (\text{M6})$$

$$\text{subject to } y_{uv} \leq x_u, \quad \forall \{u, v\} \in E, \quad (\text{M7})$$

$$y_{uv} \leq x_v, \quad \forall \{u, v\} \in E, \quad (\text{M8})$$

$$x_u + x_v \leq 1, \quad \forall \{u, v\} \in \bar{E}, \quad (\text{M9})$$

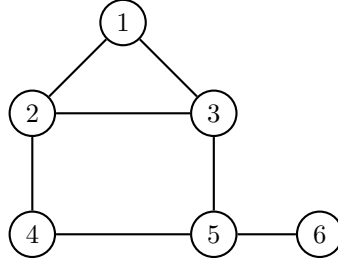
$$x_v \in \{0, 1\}, \quad \forall v \in V, \quad (\text{M10})$$

$$y_{uv} \in \{0, 1\}, \quad \forall \{u, v\} \in E. \quad (\text{M11})$$

Exercise 12.1 — ILP model of the Maximum Clique Problem (MCP)

We consider the graph $G = (V, E)$ with $V = \{1, \dots, 6\}$ and the 7 edges

$$E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 5\}, \{4, 5\}, \{5, 6\}\}.$$

**Questions**

1. Write the non-edges of G . How many variables and constraints does the ILP_{MCP} model have?
2. Write the ILP_{MCP} model of G .
3. Solve the model with Gurobi: give $z(\text{ILP}_{\text{MCP}})$ and the optimal solution. Is it unique?
4. Which ILP model of the Maximum Stable Set Problem on $\overline{G}$ do you obtain? Explain.

Solution

1. G has $\binom{6}{2} - 7 = 8$ non-edges:

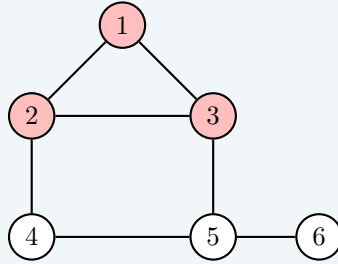
$$\overline{E} = \{\{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 4\}, \{3, 6\}, \{4, 6\}\}.$$

The model has $n = 6$ binary variables and one constraint per non-edge: 8 linear constraints.

2. The ILP_{MCP} model is:

$$\begin{aligned}
 & \max \quad x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\
 & \text{subject to} \quad x_1 + x_4 \leq 1 \\
 & \quad x_1 + x_5 \leq 1 \\
 & \quad x_1 + x_6 \leq 1 \\
 & \quad x_2 + x_5 \leq 1 \\
 & \quad x_2 + x_6 \leq 1 \\
 & \quad x_3 + x_4 \leq 1 \\
 & \quad x_3 + x_6 \leq 1 \\
 & \quad x_4 + x_6 \leq 1 \\
 & \quad x_1, x_2, x_3, x_4, x_5, x_6 \in \{0, 1\}
 \end{aligned}$$

3. Gurobi returns $z(\text{ILP}_{\text{MCP}}) = 3$ with $\tilde{x}_1 = \tilde{x}_2 = \tilde{x}_3 = 1$ and $\tilde{x}_4 = \tilde{x}_5 = \tilde{x}_6 = 0$: the clique $\{1, 2, 3\}$. The enumeration of the cliques shows that it is the only triangle, and no clique has 4 vertices (only vertices 2, 3 and 5 have degree 3). The optimal solution is unique.

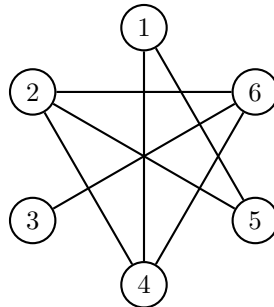


4. The same model: its constraints are written on the edges of $\overline{G}$, whose edge set is $\overline{E}$, and $x_u + x_v \leq 1$ says that at most one endpoint of every edge of $\overline{G}$ is selected. It is the edge formulation of the MSSP on $\overline{G}$, and $\alpha(\overline{G}) = \omega(G) = 3$.

Exercise 12.2 — The complement graph and the clique formulation (MCP)

We consider the graph $G = (V, E)$ with $V = \{1, \dots, 6\}$ and the 7 edges

$$E = \{\{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{3, 6\}, \{4, 6\}\}.$$



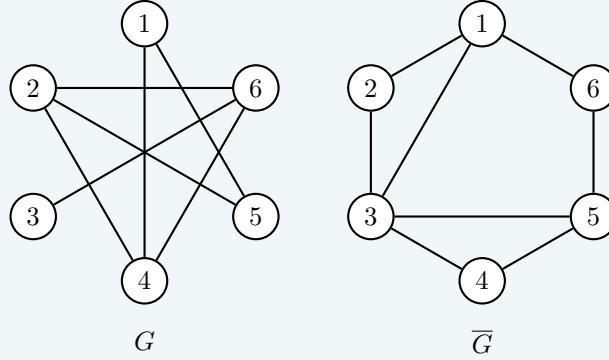
Questions

1. Write the edges of the complement graph $\overline{G}$ and draw it next to G .
2. List the maximal cliques of $\overline{G}$.
3. Write the clique formulation $\text{ILP}_{\text{MCP}}^{\text{CL}}$ and compare its number of constraints with the ILP_{MCP} model.
4. Show that every constraint of ILP_{MCP} is implied by a constraint of $\text{ILP}_{\text{MCP}}^{\text{CL}}$.
5. Solve both models with Gurobi and give a maximum clique of G .

Solution

1. $\overline{G}$ has the $\binom{6}{2} - 7 = 8$ non-edges of G :

$$\overline{E} = \{\{1, 2\}, \{1, 3\}, \{1, 6\}, \{2, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{5, 6\}\}.$$



2. The maximal cliques of $\overline{G}$ are the two triangles $\{1, 2, 3\}$ and $\{3, 4, 5\}$, which share vertex 3, and the two edges $\{1, 6\}$ and $\{5, 6\}$: vertex 6 is adjacent in $\overline{G}$ only to 1 and 5, which are not adjacent there.
3. The clique formulation has one constraint per maximal clique of $\overline{G}$:

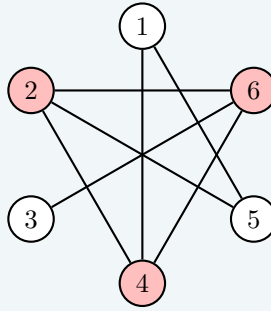
$$\begin{aligned} \max \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \leq 1 \\ & x_3 + x_4 + x_5 \leq 1 \\ & x_1 + x_6 \leq 1 \\ & x_5 + x_6 \leq 1 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \in \{0, 1\} \end{aligned}$$

It has 4 constraints, against the 8 constraints of ILP_{MCP} (one per edge of $\overline{G}$).

4. Every edge of $\overline{G}$ lies in a maximal clique, and the variables are non-negative:

constraint of ILP_{MCP}	implied by
$x_1 + x_2 \leq 1$	$x_1 + x_2 + x_3 \leq 1$
$x_1 + x_3 \leq 1$	$x_1 + x_2 + x_3 \leq 1$
$x_1 + x_6 \leq 1$	$x_1 + x_6 \leq 1$
$x_2 + x_3 \leq 1$	$x_1 + x_2 + x_3 \leq 1$
$x_3 + x_4 \leq 1$	$x_3 + x_4 + x_5 \leq 1$
$x_3 + x_5 \leq 1$	$x_3 + x_4 + x_5 \leq 1$
$x_4 + x_5 \leq 1$	$x_3 + x_4 + x_5 \leq 1$
$x_5 + x_6 \leq 1$	$x_5 + x_6 \leq 1$

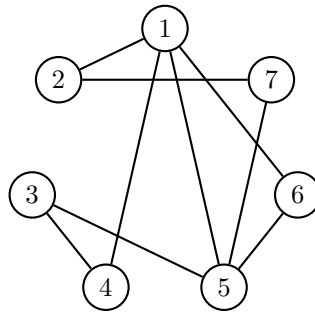
5. Both models give the optimal value 3, with the clique $\{2, 4, 6\}$ of G : it is a stable set of $\overline{G}$, and it is the unique maximum clique (enumeration).



Exercise 12.3 — LP bounds: edge versus clique formulation (MCP)

We consider the graph $G = (V, E)$ with $V = \{1, \dots, 7\}$ and the 9 edges

$$E = \{\{1, 2\}, \{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 7\}, \\ \{3, 4\}, \{3, 5\}, \{5, 6\}, \{5, 7\}\}.$$



Questions

1. Write the LP_{MCP} model (edge formulation with $0 \leq x_v \leq 1$). Show that $\bar{x}_v = \frac{1}{2}$ for every vertex is feasible, and compute $z(\text{LP}_{\text{MCP}})$ with Gurobi.
2. List the maximal cliques of $\bar{G}$ and write the $\text{LP}_{\text{MCP}}^{\text{CL}}$ model.
3. Is the point $\bar{x}_v = \frac{1}{2}$ feasible for $\text{LP}_{\text{MCP}}^{\text{CL}}$? Compute $z(\text{LP}_{\text{MCP}}^{\text{CL}})$ with Gurobi, and prove the value by summing some constraints.
4. Find $\omega(G)$ and compare the two LP bounds.

Solution

1. The 12 non-edges give the model

$$\begin{array}{llllllll}
\max & x_1 & + x_2 & + x_3 & + x_4 & + x_5 & + x_6 & + x_7 \\
\text{subject to} & x_1 & & + x_3 & & & & \leq 1 \\
& & x_1 & & & & + x_7 & \leq 1 \\
& & & x_2 & + x_3 & & & \leq 1 \\
& & & & x_2 & + x_4 & & \leq 1 \\
& & & & x_2 & & + x_5 & \leq 1 \\
& & & & & x_2 & + x_6 & \leq 1 \\
& & & & & & x_3 & + x_6 \leq 1 \\
& & & & & & & x_3 + x_7 \leq 1 \\
& & & & & & & & x_4 + x_5 \leq 1 \\
& & & & & & & & x_4 + x_6 \leq 1 \\
& & & & & & & & x_4 + x_7 \leq 1 \\
& & & & & & & & & x_6 + x_7 \leq 1 \\
& & & & & & & & & & x_1 \leq 1 \\
& & & & & & & & & & & x_2 \leq 1 \\
& & & & & & & & & & & & x_3 \leq 1 \\
& & & & & & & & & & & & & x_4 \leq 1 \\
& & & & & & & & & & & & & & x_5 \leq 1 \\
& & & & & & & & & & & & & & & x_6 \leq 1 \\
& & & & & & & & & & & & & & & & x_7 \leq 1 \\
& & & & & & & & & & & & & & & & & x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad x_6, \quad x_7 \geq 0
\end{array}$$

With $\bar{x}_v = \frac{1}{2}$ every constraint reads $\frac{1}{2} + \frac{1}{2} = 1 \leq 1$, and the value is $\frac{7}{2}$. Gurobi returns $z(\text{LP}_{\text{MCP}}) = \frac{7}{2}$: the point is optimal.

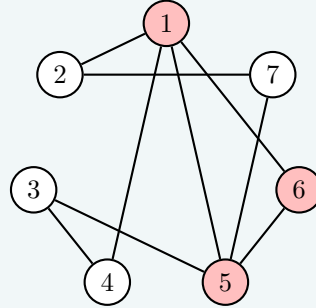
2. The 6 maximal cliques of $\overline{G}$ (maximal stable sets of G) are $\{1, 3, 7\}$, $\{2, 3, 6\}$, $\{2, 4, 5\}$, $\{2, 4, 6\}$, $\{3, 6, 7\}$, $\{4, 6, 7\}$, and the model is

$$\begin{array}{llllllll}
\max & x_1 & + x_2 & + x_3 & + x_4 & + x_5 & + x_6 & + x_7 \\
\text{subject to} & x_1 & & + x_3 & & & & + x_7 \leq 1 \\
& & x_2 & + x_3 & & & + x_6 & \leq 1 \\
& & & & x_2 & + x_4 & + x_5 & \leq 1 \\
& & & & & x_2 & + x_4 & + x_6 \leq 1 \\
& & & & & & & x_3 + x_6 + x_7 \leq 1 \\
& & & & & & & & x_4 + x_6 + x_7 \leq 1 \\
& & & & & & & & & x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad x_6, \quad x_7 \geq 0
\end{array}$$

3. No: $\bar{x}_1 + \bar{x}_3 + \bar{x}_7 = \frac{3}{2} > 1$. Gurobi returns $z(\text{LP}_{\text{MCP}}^{\text{CL}}) = 3$. The three constraints

of $\{1, 3, 7\}$, $\{2, 3, 6\}$, $\{2, 4, 5\}$ cover every vertex at least once; summing them, every variable appears with coefficient at least 1, so $\sum_{v=1}^7 x_v \leq 3$ for every feasible point.

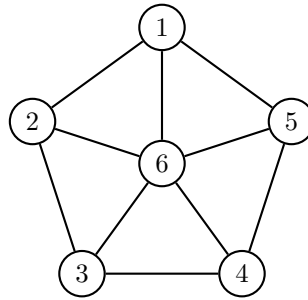
4. The triangle $\{1, 5, 6\}$ is the unique maximum clique (enumeration), so $\omega(G) = 3$.



The edge formulation gives $\frac{7}{2}$ (and $\lfloor \frac{7}{2} \rfloor = 3$ after rounding down, since $\omega(G)$ is an integer); the clique formulation gives 3 directly, with no gap.

Exercise 12.4 — Clique number, LP bound and chromatic number (MCP)

The wheel W_5 has the vertices $V = \{1, \dots, 6\}$: the vertices $1, \dots, 5$ form a cycle, and vertex 6 is adjacent to all of them.



Questions

1. List the maximal stable sets of W_5 , that is, the maximal cliques of $\overline{W_5}$.
2. Write the $\text{LP}_{\text{MCP}}^{\text{CL}}$ model and its dual $D(\text{LP}_{\text{MCP}}^{\text{CL}})$, and say which coloring formulation the dual relaxes.
3. Find a primal solution and a dual solution of value $\frac{7}{2}$. What can you conclude?
4. Deduce $\omega(W_5)$ and $\chi(W_5)$ from the integrality of the two numbers, and write the chain of the chapter.

Solution

1. A stable set contains at most 2 vertices of the cycle, never two consecutive ones, and vertex 6 is adjacent to all the others. The maximal stable sets are $\{1, 3\}$, $\{1, 4\}$, $\{2, 4\}$, $\{2, 5\}$, $\{3, 5\}$, $\{6\}$.

2. With one constraint per maximal stable set:

$$\begin{aligned}
 \max \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\
 \text{subject to} \quad & x_1 + x_3 \leq 1 \\
 & x_1 + x_4 \leq 1 \\
 & x_2 + x_4 \leq 1 \\
 & x_2 + x_5 \leq 1 \\
 & x_3 + x_5 \leq 1 \\
 & x_6 \leq 1 \\
 & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
 \end{aligned}$$

With a non-negative dual variable π_Q per constraint (index Q written as its vertices), the dual $D(\text{LP}_{\text{MCP}}^{\text{CL}})$ is

$$\begin{aligned}
 \min \quad & \pi_{13} + \pi_{14} + \pi_{24} + \pi_{25} + \pi_{35} + \pi_6 \\
 \text{subject to} \quad & \pi_{13} + \pi_{14} \geq 1 \\
 & \pi_{24} + \pi_{25} \geq 1 \\
 & \pi_{13} + \pi_{35} \geq 1 \\
 & \pi_{14} + \pi_{24} \geq 1 \\
 & \pi_{25} + \pi_{35} \geq 1 \\
 & \pi_6 \geq 1 \\
 & \pi_{13}, \pi_{14}, \pi_{24}, \pi_{25}, \pi_{35}, \pi_6 \geq 0
 \end{aligned}$$

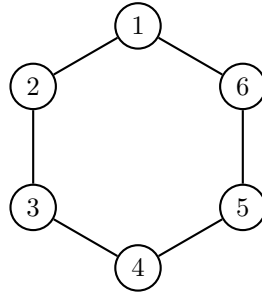
It is the LP relaxation of the stable set formulation $\text{ILP}_{\text{VCP}}^{\text{ST}}$ of the VCP, with the maximal stable sets: every vertex must be covered by a chosen stable set.

3. The primal point $\bar{x}_1 = \dots = \bar{x}_5 = \frac{1}{2}$, $\bar{x}_6 = 1$ satisfies all the constraints and has value $\frac{5}{2} + 1 = \frac{7}{2}$. The dual point $\bar{\pi}_Q = \frac{1}{2}$ for the five pairs and $\bar{\pi}_6 = 1$ covers every vertex of the cycle with $\frac{1}{2} + \frac{1}{2} = 1$ and vertex 6 with 1, and has value $\frac{7}{2}$. Equal values: by weak duality both are optimal, and $z(\text{LP}_{\text{MCP}}^{\text{CL}}) = \frac{7}{2}$ (Gurobi agrees).
4. $\omega(W_5) \leq \frac{7}{2}$ and $\omega(W_5)$ is an integer, so $\omega(W_5) \leq 3$; the triangle $\{1, 2, 6\}$ gives $\omega(W_5) = 3$. $\chi(W_5) \geq \frac{7}{2}$, so $\chi(W_5) \geq 4$; the coloring $\{1, 3\}, \{2, 4\}, \{5\}, \{6\}$ gives $\chi(W_5) = 4$. The chain reads

$$\omega(W_5) = 3 < z(\text{LP}_{\text{MCP}}^{\text{CL}}) = \frac{7}{2} < \chi(W_5) = 4.$$

Exercise 12.5 — Feasibility in the two formulations (MCP)

We consider the cycle C_6 with the vertices $V = \{1, \dots, 6\}$ and the edges $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{1, 6\}$,



and the five points $\mathbf{x} = (x_1, \dots, x_6)$:

$$\bar{\mathbf{x}}^a = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \quad \bar{\mathbf{x}}^b = (1, 1, 0, 0, 0, 0), \quad \bar{\mathbf{x}}^c = (1, 0, 1, 0, 0, 0),$$

$$\bar{\mathbf{x}}^d = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0\right), \quad \bar{\mathbf{x}}^e = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Questions

1. Write the constraints of LP_{MCP} and of $\text{LP}_{\text{MCP}}^{\text{CL}}$ for C_6 .
2. For each point, decide whether it is feasible for LP_{MCP} , for $\text{LP}_{\text{MCP}}^{\text{CL}}$ and for ILP_{MCP} .
3. What do the values of the feasible points say about the two LP bounds?
4. Compute the two LP bounds with Gurobi and compare them with $\omega(C_6)$.

Solution

1. With the 9 non-edge constraints, the LP_{MCP} model is

$$\begin{aligned}
 & \max \quad x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\
 & \text{subject to} \quad x_1 + x_3 \leq 1 \\
 & \quad x_1 + x_4 \leq 1 \\
 & \quad x_1 + x_5 \leq 1 \\
 & \quad x_2 + x_4 \leq 1 \\
 & \quad x_2 + x_5 \leq 1 \\
 & \quad x_2 + x_6 \leq 1 \\
 & \quad x_3 + x_5 \leq 1 \\
 & \quad x_3 + x_6 \leq 1 \\
 & \quad x_4 + x_6 \leq 1 \\
 & \quad x_1 \leq 1 \\
 & \quad x_2 \leq 1 \\
 & \quad x_3 \leq 1 \\
 & \quad x_4 \leq 1 \\
 & \quad x_5 \leq 1 \\
 & \quad x_6 \leq 1 \\
 & \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
 \end{aligned}$$

With one constraint per maximal stable set of C_6 , the $\text{LP}_{\text{MCP}}^{\text{CL}}$ model is

$$\begin{aligned}
 & \max \quad x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\
 & \text{subject to} \quad x_1 + x_3 + x_5 \leq 1 \\
 & \quad x_2 + x_4 + x_6 \leq 1 \\
 & \quad x_1 + x_4 \leq 1 \\
 & \quad x_2 + x_5 \leq 1 \\
 & \quad x_3 + x_6 \leq 1 \\
 & \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
 \end{aligned}$$

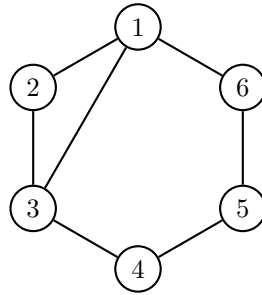
2.
 - $\bar{x}^a$: feasible for LP_{MCP} ; the clique inequality of $\{1, 3, 5\}$ gives $\frac{3}{2} > 1$, so infeasible for $\text{LP}_{\text{MCP}}^{\text{CL}}$ and ILP_{MCP} .
 - $\bar{x}^b$: feasible for all three models: it is the clique $\{1, 2\}$.
 - $\bar{x}^c$: $x_1 + x_3 = 2 > 1$ violates a non-edge constraint: infeasible for all three models (every constraint of LP_{MCP} is implied by $\text{LP}_{\text{MCP}}^{\text{CL}}$).
 - $\bar{x}^d$: feasible for LP_{MCP} and $\text{LP}_{\text{MCP}}^{\text{CL}}$, not for ILP_{MCP} (fractional).
 - $\bar{x}^e$: feasible for LP_{MCP} and $\text{LP}_{\text{MCP}}^{\text{CL}}$, not for ILP_{MCP} (fractional).
3. $\bar{x}^a$ has value 3, so $z(\text{LP}_{\text{MCP}}) \geq 3$. In $\text{LP}_{\text{MCP}}^{\text{CL}}$ the best feasible point among the five is $\bar{x}^e$, of value 2; $\bar{x}^d$ has value $\frac{3}{2}$ and $\bar{x}^b$ has value 2.

4. Gurobi returns $z(\text{LP}_{\text{MCP}}) = 3$ and $z(\text{LP}_{\text{MCP}}^{\text{CL}}) = 2$. The second value is proved by summing the constraints of $\{1, 3, 5\}$ and $\{2, 4, 6\}$, which give $\sum_{v=1}^6 x_v \leq 2$. C_6 has no triangle, so $\omega(C_6) = 2$: the clique formulation closes the gap, the edge formulation leaves 3 against 2.

Exercise 12.6 — ILP models of the Maximum Stable Set Problem (MSSP)

We consider the graph $G = (V, E)$ with $V = \{1, \dots, 6\}$ and the 7 edges

$$E = \{\{1, 2\}, \{1, 3\}, \{1, 6\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}\}.$$



We look for a maximum stable set of G .

Questions

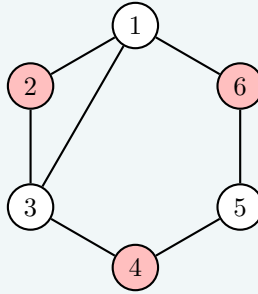
1. How many variables and constraints does the edge formulation ILP_{MSSP} of G have? Write it.
2. Solve the model with Gurobi: give $z(\text{ILP}_{\text{MSSP}})$ and an optimal solution. Is it unique?
3. Write the maximal cliques of G and the clique formulation $\text{ILP}_{\text{MSSP}}^{\text{CL}}$.
4. Compare the optimal values of the LP relaxations of the two formulations (Gurobi).

Solution

1. There are $n = 6$ binary variables and one constraint $x_u + x_v \leq 1$ per edge of G : 7 linear constraints. The ILP_{MSSP} model is:

$$\begin{aligned}
 \max \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\
 \text{subject to} \quad & x_1 + x_2 \leq 1 \\
 & x_1 + x_3 \leq 1 \\
 & x_1 + x_6 \leq 1 \\
 & x_2 + x_3 \leq 1 \\
 & x_3 + x_4 \leq 1 \\
 & x_4 + x_5 \leq 1 \\
 & x_5 + x_6 \leq 1 \\
 & x_1, x_2, x_3, x_4, x_5, x_6 \in \{0, 1\}
 \end{aligned}$$

2. Gurobi returns $z(\text{ILP}_{\text{MSSP}}) = 3$ with the stable set $\{2, 4, 6\}$. The optimal solution is unique.



3. The maximal cliques of G are the triangle $\{1, 2, 3\}$ and the edges $\{1, 6\}$, $\{3, 4\}$, $\{4, 5\}$, $\{5, 6\}$: 5 cliques instead of 7 edges, since the three edges of the triangle are replaced by $x_1 + x_2 + x_3 \leq 1$. The $\text{ILP}_{\text{MSSP}}^{\text{CL}}$ model is:

$$\begin{aligned}
 \max \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\
 \text{subject to} \quad & x_1 + x_2 + x_3 \leq 1 \\
 & x_1 + x_6 \leq 1 \\
 & x_3 + x_4 \leq 1 \\
 & x_4 + x_5 \leq 1 \\
 & x_5 + x_6 \leq 1 \\
 & x_1, x_2, x_3, x_4, x_5, x_6 \in \{0, 1\}
 \end{aligned}$$

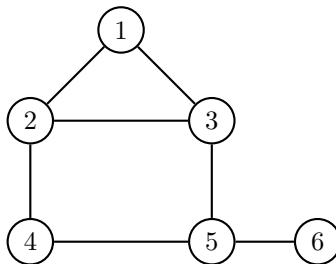
Its optimal value is again 3.

4. The LP relaxation of the edge formulation, with $0 \leq x_v \leq 1$, has optimal value 3; the LP relaxation of the clique formulation has optimal value 3. On this graph both bounds are equal to $\alpha(G) = 3$: the point $x_v = \frac{1}{2}$, of value 3, is feasible for the edge formulation but violates $x_1 + x_2 + x_3 \leq 1$, and the bound does not change because the integer optimum also has value 3.

Exercise 12.7 — ILP model of the vertex-weighted maximum clique problem

We consider the graph $G = (V, E)$ with $V = \{1, \dots, 6\}$ and the 7 edges

$$E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 5\}, \{4, 5\}, \{5, 6\}\}.$$



and the vertex weights

	$v = 1$	$v = 2$	$v = 3$	$v = 4$	$v = 5$	$v = 6$
w_v	2	3	4	6	5	7

Questions

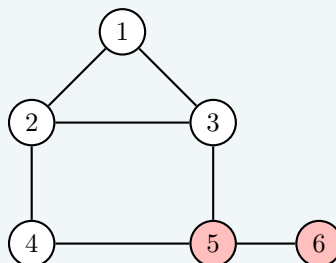
1. Write the $\text{ILP}_{\text{MCP}}^{\text{VW}}$ model of the vertex-weighted maximum clique problem on this instance.
2. Solve the model with Gurobi: give the optimal value and the optimal clique. Is it unique?
3. Compare the optimal clique with the maximum clique of G , and explain the difference.

Solution

1. G has $\binom{6}{2} - 7 = 8$ non-edges, one constraint each:

$$\begin{aligned}
 & \max \quad 2x_1 + 3x_2 + 4x_3 + 6x_4 + 5x_5 + 7x_6 \\
 & \text{subject to} \quad x_1 + x_4 \leq 1 \\
 & \quad \quad \quad x_1 + x_5 \leq 1 \\
 & \quad \quad \quad x_1 + x_6 \leq 1 \\
 & \quad \quad \quad x_2 + x_5 \leq 1 \\
 & \quad \quad \quad x_2 + x_6 \leq 1 \\
 & \quad \quad \quad x_3 + x_4 \leq 1 \\
 & \quad \quad \quad x_3 + x_6 \leq 1 \\
 & \quad \quad \quad x_4 + x_6 \leq 1 \\
 & \quad \quad \quad x_1, x_2, x_3, x_4, x_5, x_6 \in \{0, 1\}
 \end{aligned}$$

2. Gurobi returns the optimal value 12 with $\tilde{x}_5 = \tilde{x}_6 = 1$: the clique $\{5, 6\}$, of weight $5 + 7 = 12$. The enumeration of the cliques gives the second best value 11 (clique $\{4, 5\}$): the optimal solution is unique.



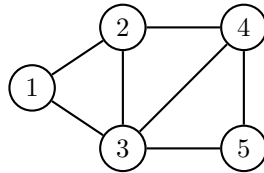
3. The maximum clique of G is the triangle $\{1, 2, 3\}$, with 3 vertices but weight only $2 + 3 + 4 = 9$. The two models have the same constraints, hence the same feasible

solutions (the cliques of G); only the objective function changes, and the light vertices of the triangle lose against the heavy edge $\{5, 6\}$.

Exercise 12.8 — ILP model of the edge-weighted maximum clique problem

We consider the graph $G = (V, E)$ with $V = \{1, \dots, 5\}$ and the 7 edges

$$E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}.$$



and the edge weights

edge	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{2, 4\}$	$\{3, 4\}$	$\{3, 5\}$	$\{4, 5\}$
r_{uv}	2	3	1	4	2	5	6

Questions

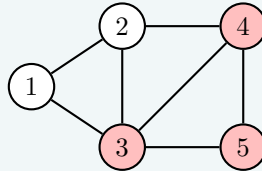
1. Count the variables and the constraints of the $\text{ILP}_{\text{MCP}}^{\text{EW}}$ model of the edge-weighted maximum clique problem on this instance, and write it.
2. Solve the model with Gurobi: give the optimal value, the optimal clique and the variables y_{uv} equal to 1.
3. Verify the result by listing the maximal cliques of G with their total edge weight.
4. Explain why the constraints $y_{uv} \geq x_u + x_v - 1$ are not needed in the model.

Solution

1. There are $n = 5$ variables x_v and $m = 7$ variables y_{uv} , so 12 binary variables. There are $2m = 14$ constraints $y_{uv} \leq x_u$, $y_{uv} \leq x_v$ and one constraint per non-edge, 3 of them: 17 linear constraints. The model is:

$$\begin{aligned}
 & \max && 2y_{12} + 3y_{13} + y_{23} + 4y_{24} + 2y_{34} + 5y_{35} + 6y_{45} \\
 & \text{subject to} && y_{12} - x_1 \leq 0 \\
 & && y_{12} - x_2 \leq 0 \\
 & && y_{13} - x_1 \leq 0 \\
 & && y_{13} - x_3 \leq 0 \\
 & && y_{23} - x_2 \leq 0 \\
 & && y_{23} - x_3 \leq 0 \\
 & && y_{24} - x_2 \leq 0 \\
 & && y_{24} - x_4 \leq 0 \\
 & && y_{34} - x_3 \leq 0 \\
 & && y_{34} - x_4 \leq 0 \\
 & && y_{35} - x_3 \leq 0 \\
 & && y_{35} - x_5 \leq 0 \\
 & && y_{45} - x_4 \leq 0 \\
 & && y_{45} - x_5 \leq 0 \\
 & && x_1 + x_4 \leq 1 \\
 & && x_1 + x_5 \leq 1 \\
 & && x_2 + x_5 \leq 1 \\
 & && x_1, x_2, x_3, x_4, x_5 \in \{0, 1\} \\
 & && y_{12}, y_{13}, y_{23}, y_{24}, y_{34}, y_{35}, y_{45} \in \{0, 1\}
 \end{aligned}$$

2. Gurobi returns the optimal value 13 with $\tilde{x}_3 = \tilde{x}_4 = \tilde{x}_5 = 1$ and $\tilde{y}_{34} = \tilde{y}_{35} = \tilde{y}_{45} = 1$, all the other variables equal to 0: the clique $\{3, 4, 5\}$, of weight $2 + 5 + 6 = 13$.



3. The maximal cliques of G and their weights are:

- $\{1, 2, 3\}$: $2 + 3 + 1 = 6$;
- $\{2, 3, 4\}$: $1 + 4 + 2 = 7$;
- $\{3, 4, 5\}$: $2 + 5 + 6 = 13$;

Every clique is contained in one of them and the weights are positive, so the optimal value is 13, attained only by $\{3, 4, 5\}$.

4. If both endpoints of $\{u, v\}$ are selected, the constraints allow $y_{uv} = 1$, and since $r_{uv} > 0$ the objective function prefers $y_{uv} = 1$ to $y_{uv} = 0$: every optimal solution has $y_{uv} = \min\{x_u, x_v\}$. The constraint $y_{uv} \geq x_u + x_v - 1$ would be needed only with negative weights.

Formulations of routing problems

Formulations used in these exercises. The models of Chapter 13 «Formulations of routing problems», on the arcs $A = \{(i, j) : i, j \in \mathcal{V}, i \neq j\}$. The model $\text{ILP}_{\text{TSP}}^{\text{MTZ}}$:

$$\min \sum_{(i,j) \in A} d_{ij} x_{ij} \quad (\text{T0})$$

$$\text{subject to } \sum_{j \in \mathcal{V}, j \neq i} x_{ij} = 1, \quad \forall i \in \mathcal{V}, \quad (\text{T1})$$

$$\sum_{i \in \mathcal{V}, i \neq j} x_{ij} = 1, \quad \forall j \in \mathcal{V}, \quad (\text{T2})$$

$$u_i - u_j + (n - 1) x_{ij} \leq n - 2, \quad \forall i, j \in \mathcal{V} \setminus \{1\}, i \neq j, \quad (\text{MTZ})$$

$$x_{ij} \in \{0, 1\}, \quad \forall (i, j) \in A, \quad (\text{T3})$$

$$1 \leq u_i \leq n - 1, \quad \forall i \in \mathcal{V} \setminus \{1\}. \quad (\text{T4})$$

The model $\text{ILP}_{\text{TSP}}^{\text{DFJ}}$:

$$\min \sum_{(i,j) \in A} d_{ij} x_{ij} \quad (\text{T0})$$

$$\text{subject to } \sum_{j \in \mathcal{V}, j \neq i} x_{ij} = 1, \quad \forall i \in \mathcal{V}, \quad (\text{T1})$$

$$\sum_{i \in \mathcal{V}, i \neq j} x_{ij} = 1, \quad \forall j \in \mathcal{V}, \quad (\text{T2})$$

$$\sum_{i \in S} \sum_{j \in S, j \neq i} x_{ij} \leq |S| - 1, \quad \forall S \subseteq \mathcal{V} \setminus \{1\}, |S| \geq 2, \quad (\text{SEC})$$

$$x_{ij} \in \{0, 1\}, \quad \forall (i, j) \in A. \quad (\text{T3})$$

The assignment relaxation LP-0_{TSP} keeps only (T0), (T1), (T2) and $x_{ij} \geq 0$. The relaxations $\text{LP}_{\text{TSP}}^{\text{MTZ}}$ and $\text{LP}_{\text{TSP}}^{\text{DFJ}}$ relax (T3) into $x_{ij} \geq 0$ (the bounds $x_{ij} \leq 1$ are implied by (T1)). The model ILP_{ATSP} of the asymmetric TSP is the model $\text{ILP}_{\text{TSP}}^{\text{DFJ}}$ with distances d_{ij} that may differ from d_{ji} .

Exercise 13.1 — The MTZ model of the TSP (1)

We consider a TSP instance with $\mathcal{V} = \{1, \dots, 4\}$ ($n = 4$ cities) and the symmetric distance matrix

d_{ij}	1	2	3	4
1	—	3	5	4
2	3	—	4	6
3	5	4	—	2
4	4	6	2	—

Questions

1. Count the variables and the constraints (besides the domains) of the $\text{ILP}_{\text{TSP}}^{\text{MTZ}}$ model for $n = 4$.
2. Write the $\text{ILP}_{\text{TSP}}^{\text{MTZ}}$ model for this instance.
3. List the distinct tours with their lengths, and give the optimal tour σ .
4. Write the values of the variables x_{ij} and u_i that represent σ , and verify the constraints (MTZ) one by one.
5. Solve the LP relaxation $\text{LP}_{\text{TSP}}^{\text{MTZ}}$ with Gurobi and compare its value with the optimal value.

Solution

1. There are $n(n-1) = 12$ binary variables x_{ij} and $n-1 = 3$ continuous variables u_i . There are $2n = 8$ degree constraints and $(n-1)(n-2) = 6$ constraints (MTZ), so 14 linear constraints.
2. With $A = \{(i, j) : i, j \in \mathcal{V}, i \neq j\}$ the model is:

$$\begin{aligned}
 \min \quad & 3x_{12} + 5x_{13} + 4x_{14} + 3x_{21} + 4x_{23} + 6x_{24} + 5x_{31} \\
 & + 4x_{32} + 2x_{34} + 4x_{41} + 6x_{42} + 2x_{43} \\
 \text{subject to} \quad & x_{12} + x_{13} + x_{14} = 1 \\
 & x_{21} + x_{23} + x_{24} = 1 \\
 & x_{31} + x_{32} + x_{34} = 1 \\
 & x_{41} + x_{42} + x_{43} = 1 \\
 & x_{21} + x_{31} + x_{41} = 1 \\
 & x_{12} + x_{32} + x_{42} = 1 \\
 & x_{13} + x_{23} + x_{43} = 1 \\
 & x_{14} + x_{24} + x_{34} = 1 \\
 & u_2 - u_3 + 3x_{23} \leq 2 \\
 & u_2 - u_4 + 3x_{24} \leq 2 \\
 & u_3 - u_2 + 3x_{32} \leq 2 \\
 & u_3 - u_4 + 3x_{34} \leq 2 \\
 & u_4 - u_2 + 3x_{42} \leq 2 \\
 & u_4 - u_3 + 3x_{43} \leq 2 \\
 & x_{12}, x_{13}, x_{14}, \dots, x_{43} \in \{0, 1\} \\
 & u_2, u_3, u_4 \geq 1 \\
 & u_2, u_3, u_4 \leq 3
 \end{aligned}$$

3. Starting from city 1, and counting each tour once (one direction), the distinct tours are $\frac{(n-1)!}{2} = 3$:

tour	length
(1, 2, 3, 4)	$3 + 4 + 2 + 4 = 13$
(1, 2, 4, 3)	$3 + 6 + 2 + 5 = 16$
(1, 3, 2, 4)	$5 + 4 + 6 + 4 = 19$

The optimal tour is $\sigma = (1, 2, 3, 4)$, of length $L(\sigma) = 13$.

4. The tour uses the arcs

$$x_{12} = 1, \quad x_{23} = 1, \quad x_{34} = 1, \quad x_{41} = 1$$

and all the other x_{ij} are 0. The positions after city 1 are $u_2 = 1, u_3 = 2, u_4 = 3$. The constraints (MTZ) give:

- $(i, j) = (2, 3)$: $1 - 2 + 3 \cdot 1 = 2 \leq 2$
- $(i, j) = (2, 4)$: $1 - 3 + 3 \cdot 0 = -2 \leq 2$
- $(i, j) = (3, 2)$: $2 - 1 + 3 \cdot 0 = 1 \leq 2$

- $(i, j) = (3, 4)$: $2 - 3 + 3 \cdot 1 = 2 \leq 2$
- $(i, j) = (4, 2)$: $3 - 1 + 3 \cdot 0 = 2 \leq 2$
- $(i, j) = (4, 3)$: $3 - 2 + 3 \cdot 0 = 1 \leq 2$

All of them hold; the selected arcs (i, j) with $i, j \neq 1$ hold with equality.

5. Gurobi gives $z(\text{LP}_{\text{TSP}}^{\text{MTZ}}) = 12$, against the optimal value 13: the relaxation leaves a gap of 1.

Exercise 13.2 — The DFJ model of the TSP (1)

We consider a TSP instance with $\mathcal{V} = \{1, \dots, 4\}$ ($n = 4$ cities) and the symmetric distance matrix

d_{ij}	1	2	3	4
1	—	3	5	4
2	3	—	4	6
3	5	4	—	2
4	4	6	2	—

Questions

1. Count the subtour elimination constraints of the $\text{ILP}_{\text{TSP}}^{\text{DFJ}}$ model for $n = 4$, and write them.
2. The arcs $(1, 2)$, $(2, 1)$, $(3, 4)$, $(4, 3)$ satisfy the degree constraints. Find the subtour elimination constraint they violate.
3. Do the same for the arcs $(1, 3)$, $(3, 1)$, $(2, 4)$, $(4, 2)$, which form two subtours both of length 2.
4. Solve with Gurobi the assignment relaxation LP-0_{TSP} (degree constraints only), the relaxation $\text{LP}_{\text{TSP}}^{\text{DFJ}}$ and the model $\text{ILP}_{\text{TSP}}^{\text{DFJ}}$, and compare the values.

Solution

1. The sets are the subsets $S \subseteq \{2, 3, 4\}$ with $|S| \geq 2$: there are $2^{n-1} - n = 4$ of them. The constraints are

$$S = \{2, 3\} : x_{23} + x_{32} \leq 1$$

$$S = \{2, 4\} : x_{24} + x_{42} \leq 1$$

$$S = \{3, 4\} : x_{34} + x_{43} \leq 1$$

$$S = \{2, 3, 4\} : x_{23} + x_{24} + x_{32} + x_{34} + x_{42} + x_{43} \leq 2$$

2. The arcs form the subtours $1 \rightarrow 2 \rightarrow 1$ and $3 \rightarrow 4 \rightarrow 3$, of total length 10. The subtour on $\{3, 4\}$ does not contain city 1: for $S = \{3, 4\}$ the constraint reads $x_{34} + x_{43} = 2 > 1$. The other three constraints hold, for instance $S = \{2, 3, 4\}$ gives $0 + 0 + 1 + 0 + 1 + 0 = 2 \leq 2$.

3. The arcs form $1 \rightarrow 3 \rightarrow 1$ and $2 \rightarrow 4 \rightarrow 2$. The subtour without city 1 is on $\{2, 4\}$: $x_{24} + x_{42} = 2 > 1$. One set S is enough for every solution with subtours, because one of the subtours never contains city 1.
4. Gurobi gives $z(\text{LP-0}_{\text{TSP}}) = 10$ for the assignment relaxation, $z(\text{LP}_{\text{TSP}}^{\text{DFJ}}) = 13$ and $z(\text{ILP}_{\text{TSP}}^{\text{DFJ}}) = 13$, attained by the tour $(1, 2, 3, 4)$. The subtour elimination constraints raise the bound from 10 to 13, which is the optimal value.

Exercise 13.3 — Feasible solutions of the TSP models (1)

We consider a TSP instance with $\mathcal{V} = \{1, \dots, 5\}$ ($n = 5$ cities) and the symmetric distance matrix

d_{ij}	1	2	3	4	5
1	—	2	3	3	7
2	2	—	4	12	6
3	3	4	—	6	11
4	3	12	6	—	5
5	7	6	11	5	—

Consider the following candidate solutions (the arcs not listed have $x_{ij} = 0$):

- (a) the arcs of the tour $(1, 3, 2, 5, 4)$, with $u_2 = 2, u_3 = 1, u_4 = 4, u_5 = 3$;
- (b) the arcs of the same tour, with $u_2 = 1, u_3 = 2, u_4 = 3, u_5 = 4$;
- (c) the arcs $(1, 2), (2, 3), (3, 1), (4, 5), (5, 4)$, with $u_2 = 1, u_3 = 2, u_4 = 3, u_5 = 4$;
- (d) $x_{ij} = \frac{1}{4}$ for every arc (i, j) , with $u_2 = u_3 = u_4 = u_5 = 1$.

Questions

1. Check the degree constraints for every candidate.
2. For every candidate, decide whether it satisfies the constraints (MTZ) with the given positions, and whether its arc variables satisfy the subtour elimination constraints. Name a violated constraint when there is one.
3. Which candidates represent a feasible solution of the integer models? For candidate (c), show that no choice of the positions u_i satisfies the constraints (MTZ).
4. Give the objective value of candidate (d), and compare it with the optimal value and with $z(\text{LP}_{\text{TSP}}^{\text{MTZ}})$ (use Gurobi).

Solution

1. In (a), (b) and (c) every city has one outgoing and one incoming arc. In (d) every city has 4 outgoing and 4 incoming arcs of value $\frac{1}{4}$: both sums are 1. All the degree constraints hold.
2. (a) The arcs form a tour, and the positions are those of the tour: every selected arc

- (i, j) with $i, j \neq 1$ gives $u_j = u_i + 1$, and every constraint holds. The subtour elimination constraints hold too.
- (b) The arcs are the same tour, so the subtour elimination constraints hold. The positions are wrong: the arc $(3, 2)$ is selected and $u_3 - u_2 + 4x_{32} = 2 - 1 + 4 = 5 > 3$.
- (c) With the given positions the arc $(5, 4)$ gives $u_5 - u_4 + 4x_{54} = 4 - 3 + 4 = 5 > 3$. The subtour on $S = \{4, 5\}$ violates $x_{45} + x_{54} \leq 1$, since $x_{45} + x_{54} = 2$.
- (d) Every arc has $x_{ij} = \frac{1}{4}$, so with $u_i = 1$ every constraint (MTZ) reads $4 \cdot \frac{1}{4} = 1 \leq 3$. A set S contains $|S|(|S| - 1)$ arcs, of total value $\frac{|S|(|S| - 1)}{4} \leq |S| - 1$ since $|S| \leq 4$: all the subtour elimination constraints hold.
3. Only (a) and (b) are integer and form a tour. Candidate (a) is feasible for both integer models; the arcs of (b) are feasible for $\text{ILP}_{\text{TSP}}^{\text{DFJ}}$, and become feasible for $\text{ILP}_{\text{TSP}}^{\text{MTZ}}$ with the positions of (a). Candidate (d) is feasible only for the two relaxations.
- For (c), the selected arcs $(4, 5)$ and $(5, 4)$ force $u_5 \geq u_4 + 1$ and $u_4 \geq u_5 + 1$. Summing gives $0 \geq 2$, which is impossible: no positions work, because the subtour $4 \rightarrow 5 \rightarrow 4$ does not contain city 1.
4. The value of (d) is one quarter of the sum of d_{ij} over all the arcs, that is $\frac{2 \cdot 59}{4} = \frac{59}{2}$. It is larger than the optimal value $L((1, 3, 2, 5, 4)) = 21$, so it does not improve the bound $z(\text{LP}_{\text{TSP}}^{\text{MTZ}}) = 20$ given by Gurobi; it shows only that the relaxations contain fractional points far from every tour.

Exercise 13.4 — Comparing the TSP relaxations (1)

We consider a TSP instance with $\mathcal{V} = \{1, \dots, 5\}$ ($n = 5$ cities) and the symmetric distance matrix

d_{ij}	1	2	3	4	5
1	—	5	12	2	9
2	5	—	2	8	12
3	12	2	—	9	9
4	2	8	9	—	5
5	9	12	9	5	—

Questions

- Count the variables and the constraints (besides the domains) of the three relaxations: LP-0_{TSP} (degree constraints only), $\text{LP}_{\text{TSP}}^{\text{MTZ}}$ and $\text{LP}_{\text{TSP}}^{\text{DFJ}}$.
- Solve the three relaxations with Gurobi and report their optimal solutions.
- Find the optimal tour by enumerating the 12 distinct tours, and compute the gap $\frac{z - \text{LB}}{\text{LB}}$ of each bound.
- Show that the optimal solution of $\text{LP}_{\text{TSP}}^{\text{MTZ}}$ found by Gurobi lies in the weak region of Proposition 13.3 of the chapter, and that it violates subtour elimination constraints.

Solution

1. All three use the 20 arc variables; the MTZ relaxation adds the 4 positions. The constraints are:

relaxation	variables	constraints
LP-0 _{TSP} (degree constraints only)	20	10
LP _{TSP} ^{MTZ}	20 + 4	10 + 12
LP _{TSP} ^{DFJ}	20	10 + 11

2. • LP-0_{TSP}: value 20, with $x_{15} = x_{23} = x_{32} = x_{41} = x_{54} = 1$: the subtours $1 \rightarrow 5 \rightarrow 4 \rightarrow 1$, $2 \rightarrow 3 \rightarrow 2$.
- LP_{TSP}^{MTZ}: value $\frac{43}{2}$, with $x_{14} = x_{23} = x_{32} = x_{45} = \frac{3}{4}$, $x_{51} = \frac{1}{2}$, $x_{12} = x_{21} = x_{35} = x_{41} = x_{53} = x_{54} = \frac{1}{4}$ and $u_2 = u_3 = u_4 = u_5 = 1$.
- LP_{TSP}^{DFJ}: value 23, with $x_{12} = x_{23} = x_{35} = x_{41} = x_{54} = 1$, which is the tour (1, 2, 3, 5, 4).
3. The optimal tour is $\sigma = (1, 2, 3, 5, 4)$, of length $L(\sigma) = 23$. The gaps are:

bound	value	gap from 23
LP-0 _{TSP} (degree constraints only)	20	15.0%
LP _{TSP} ^{MTZ}	$\frac{43}{2}$	7.0%
LP _{TSP} ^{DFJ}	23	0.0%

The DFJ relaxation closes the gap on this instance.

4. Every arc (i, j) with $i, j \neq 1$ has $x_{ij} \leq \frac{3}{4} = \frac{n-2}{n-1}$, and all the positions are 1: this is the weak region of Proposition 13.3, where every constraint (MTZ) reads $4x_{ij} \leq 3$. For $S = \{2, 3\}$ the arcs inside S have total value $\frac{3}{2} > 1$: the subtour elimination constraint of S is violated, and the DFJ relaxation cuts this point off.

Exercise 13.5 — Adding subtour elimination constraints (1)

We consider a TSP instance with $\mathcal{V} = \{1, \dots, 5\}$ ($n = 5$ cities) and the symmetric distance matrix

d_{ij}	1	2	3	4	5
1	—	4	10	2	8
2	4	—	11	2	5
3	10	11	—	4	2
4	2	2	4	—	2
5	8	5	2	2	—

We start from the degree constraints only, and we add the subtour elimination constraints one at a time, only when they are violated.

Questions

1. Solve the assignment relaxation $LP-0_{TSP}$ (degree constraints only) with Gurobi, and describe its optimal solution.
2. Add the subtour elimination constraint of the subtour that does not contain city 1, solve again and repeat, until the solution is a tour. Give the value of each relaxation.
3. Compute the optimal value by enumeration and the gap $\frac{z-LB}{LB}$ of each relaxation solved, together with the MTZ relaxation.

Solution

1. Gurobi gives value 12 with $x_{14} = 1, x_{21} = 1, x_{35} = 1, x_{42} = 1, x_{53} = 1$. The arcs form the subtours $1 \rightarrow 4 \rightarrow 2 \rightarrow 1$ and $3 \rightarrow 5 \rightarrow 3$. It satisfies the degree constraints but it is not a tour: the iterations below start from this solution.

2. Iteration 1

Gurobi gives value 12 with $x_{14} = 1, x_{21} = 1, x_{35} = 1, x_{42} = 1, x_{53} = 1$. The arcs form the subtours $1 \rightarrow 4 \rightarrow 2 \rightarrow 1$ and $3 \rightarrow 5 \rightarrow 3$. The subtour on $S = \{3, 5\}$ violates its constraint, which we add:

$$x_{35} + x_{53} \leq 1.$$

Iteration 2

Gurobi gives value 16 with $x_{12} = 1, x_{21} = 1, x_{34} = 1, x_{45} = 1, x_{53} = 1$. The arcs form the subtours $1 \rightarrow 2 \rightarrow 1$ and $3 \rightarrow 4 \rightarrow 5 \rightarrow 3$. The subtour on $S = \{3, 4, 5\}$ violates its constraint, which we add:

$$x_{34} + x_{35} + x_{43} + x_{45} + x_{53} + x_{54} \leq 2.$$

Iteration 3

Gurobi gives value 17 with $x_{12} = 1, x_{25} = 1, x_{34} = 1, x_{41} = 1, x_{53} = 1$. The arcs form the tour $(1, 2, 5, 3, 4)$. No subtour elimination constraint is violated, and the procedure stops.

3. Enumeration gives the optimal tour $(1, 2, 5, 3, 4)$, of length 17, which is the tour found at the last iteration. Gurobi gives $z(LP_{TSP}^{MTZ}) = 14$. The gaps are:

relaxation	value	gap from 17
$LP-0_{TSP}$ (degree constraints only)	12	41.7%
LP_{TSP}^{MTZ}	14	21.4%
with 1 subtour elimination constraint	16	6.2%
with 2 subtour elimination constraints	17	0.0%

Two subtour elimination constraints, out of the 11 of the DFJ model, are enough to close the gap.

Exercise 13.6 — ILP model of the asymmetric TSP (ATSP)

We consider an ATSP instance with $\mathcal{V} = \{1, \dots, 4\}$ ($n = 4$ cities) and the asymmetric distance matrix

d_{ij}	1	2	3	4
1	—	8	7	5
2	8	—	3	6
3	9	8	—	8
4	2	9	9	—

where d_{ij} is the distance from city i to city j .

Questions

1. Count the variables and the constraints of the ILP_{ATSP} model of the chapter (the DFJ model on the arcs, with $d_{ij} \neq d_{ji}$ allowed), and write it for this instance.
2. Solve with Gurobi the model with only the degree constraints. Describe the solution and find the subtour elimination constraint that it violates.
3. Solve the ILP_{ATSP} model with Gurobi: give the optimal tour and its length.
4. List the lengths of all the tours, each in its direction, and verify the result. Compare the optimal tour with the same tour travelled backwards.

Solution

1. There are $n(n-1) = 12$ binary variables x_{ij} , $2n = 8$ degree constraints and $2^{n-1} - n = 4$ subtour elimination constraints, one for every $S \subseteq \{2, 3, 4\}$ with $|S| \geq 2$. The model is:

$$\begin{aligned}
 \min \quad & 8x_{12} + 7x_{13} + 5x_{14} + 8x_{21} + 3x_{23} + 6x_{24} + 9x_{31} \\
 & + 8x_{32} + 8x_{34} + 2x_{41} + 9x_{42} + 9x_{43} \\
 \text{subject to} \quad & x_{12} + x_{13} + x_{14} = 1 \\
 & x_{21} + x_{23} + x_{24} = 1 \\
 & x_{31} + x_{32} + x_{34} = 1 \\
 & x_{41} + x_{42} + x_{43} = 1 \\
 & x_{21} + x_{31} + x_{41} = 1 \\
 & x_{12} + x_{32} + x_{42} = 1 \\
 & x_{13} + x_{23} + x_{43} = 1 \\
 & x_{14} + x_{24} + x_{34} = 1 \\
 & x_{23} + x_{32} \leq 1 \\
 & x_{24} + x_{42} \leq 1 \\
 & x_{34} + x_{43} \leq 1 \\
 & x_{23} + x_{24} + x_{32} + x_{34} + x_{42} + x_{43} \leq 2 \\
 & x_{12}, x_{13}, x_{14}, \dots, x_{43} \in \{0, 1\}
 \end{aligned}$$

2. Gurobi returns the value 18 with $x_{14} = x_{23} = x_{32} = x_{41} = 1$: two subtours, $1 \rightarrow 4 \rightarrow 1$ and $2 \rightarrow 3 \rightarrow 2$. The subtour on $S = \{2, 3\}$ uses $2 > |S| - 1 = 1$ arcs inside S : it violates the constraint $x_{23} + x_{32} \leq 1$.
3. Gurobi returns $z(\text{ILP}_{\text{ATSP}}) = 21$ with the tour $\tilde{\sigma} = (1, 2, 3, 4)$, that is $\tilde{x}_{12} = \tilde{x}_{23} = \tilde{x}_{34} = \tilde{x}_{41} = 1$, of length $8 + 3 + 8 + 2 = 21$. The MTZ model gives the same value.
4. The $(n - 1)! = 6$ tours from city 1 are:

tour	length
(1, 2, 3, 4)	$8 + 3 + 8 + 2 = 21$
(1, 2, 4, 3)	$8 + 6 + 9 + 9 = 32$
(1, 3, 2, 4)	$7 + 8 + 6 + 2 = 23$
(1, 3, 4, 2)	$7 + 8 + 9 + 8 = 32$
(1, 4, 2, 3)	$5 + 9 + 3 + 9 = 26$
(1, 4, 3, 2)	$5 + 9 + 8 + 8 = 30$

The minimum 21 is attained only by (1, 2, 3, 4): the optimal tour is unique. Travelled backwards it is the tour (1, 4, 3, 2), of length 30: in the ATSP the two directions are different solutions.

Formulations of scheduling problems

Formulations used in these exercises. The model $\text{ILP}_{P\|C_{\max}}$ of the chapter [Formulations of scheduling problems](#), with the set of machines $\mathcal{H} = \{1, \dots, m\}$:

$$\min C_{\max} \quad (\text{P0})$$

$$\text{subject to } \sum_{h \in \mathcal{H}} x_{jh} = 1, \quad \forall j \in \mathcal{J}, \quad (\text{P1})$$

$$- \sum_{j \in \mathcal{J}} t_j x_{jh} + C_{\max} \geq 0, \quad \forall h \in \mathcal{H}, \quad (\text{P2})$$

$$x_{jh} \in \{0, 1\}, \quad \forall j \in \mathcal{J}, \forall h \in \mathcal{H}, \quad (\text{P3})$$

$$C_{\max} \geq 0. \quad (\text{P4})$$

The relaxation $\text{LP-0}_{P\|C_{\max}}$ relaxes (P3) into $x_{jh} \geq 0$; the relaxation $\text{LP}_{P\|C_{\max}}$ also has $x_{jh} \leq 1$. For every instance

$$z(\text{LP-0}_{P\|C_{\max}}) = z(\text{LP}_{P\|C_{\max}}) = \frac{\sum_{j \in \mathcal{J}} t_j}{m}, \quad (\text{P5})$$

attained by $\tilde{x}_{jh} = \frac{1}{m}$; an optimal dual solution is $\tilde{\pi}_h = \frac{1}{m}$ and $\tilde{\mu}_j = \frac{t_j}{m}$. The inequalities

$$C_{\max} \geq t_j \quad (j \in \mathcal{J}), \quad C_{\max} \geq \left\lceil \frac{\sum_j t_j}{m} \right\rceil \quad (\text{P6})$$

are valid for the model $\text{ILP}_{P\|C_{\max}}$.

Exercise 14.1 — ILP model of $P\|C_{\max}$ (1)

We consider $m = 2$ identical machines and a set $\mathcal{J} = \{1, \dots, 5\}$ of 5 jobs ($n = 5$) with the following processing times:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
t_j	7	5	4	3	3

Questions

1. Count the variables and the constraints (besides the domains) of the $\text{ILP}_{P\|C_{\max}}$ model for this instance.
2. Write the $\text{ILP}_{P\|C_{\max}}$ model for this instance.
3. Compute the lower bound $\max \left\{ \max_j t_j, \left\lceil \frac{\sum_j t_j}{m} \right\rceil \right\}$ on the makespan.
4. Solve the model with Gurobi, give an optimal schedule, and say how many optimal solutions the model has.

Solution

1. There are $n m = 5 \cdot 2 = 10$ binary variables x_{jh} and the continuous variable $C_{\max}$, so 11 variables. There are $n = 5$ assignment constraints and $m = 2$ load constraints, so 7 linear constraints.
2. With $\mathcal{H} = \{1, 2\}$ the model is:

$$\begin{array}{llll}
 \min & C_{\max} & & \\
 \text{subject to} & x_{11} + x_{12} & = & 1 \\
 & x_{21} + x_{22} & = & 1 \\
 & x_{31} + x_{32} & = & 1 \\
 & x_{41} + x_{42} & = & 1 \\
 & x_{51} + x_{52} & = & 1 \\
 & -7x_{11} - 5x_{21} - 4x_{31} - 3x_{41} - 3x_{51} + C_{\max} & \geq & 0 \\
 & -7x_{12} - 5x_{22} - 4x_{32} - 3x_{42} - 3x_{52} + C_{\max} & \geq & 0 \\
 & x_{11}, x_{12}, x_{21}, \dots, x_{52} & \in & \{0, 1\} \\
 & C_{\max} & \geq & 0
 \end{array}$$

3. The total processing time is $7 + 5 + 4 + 3 + 3 = 22$, so

$$\max \{7, \lceil 22/2 \rceil\} = \max \{7, 11\} = 11.$$

4. Gurobi gives $z(\text{ILP}_{P\|C_{\max}}) = 11$, equal to the lower bound. An optimal schedule is $J_1 = \{1, 3\}$ and $J_2 = \{2, 4, 5\}$, with loads 11 and 11: $\tilde{x}_{11} = 1, \tilde{x}_{31} = 1, \tilde{x}_{22} = 1, \tilde{x}_{42} = 1, \tilde{x}_{52} = 1, \tilde{C}_{\max} = 11$.

By enumeration of the $2^5 = 32$ assignments, the partition of the jobs is unique: job 1 can share a machine only with job 3. Exchanging the two machines gives the second optimal solution of the model, so there are 2 optimal solutions.

Exercise 14.2 — LP relaxation of the $P\|C_{\max}$ model (1)

We consider $m = 3$ identical machines and a set $\mathcal{J} = \{1, \dots, 6\}$ of 6 jobs ($n = 6$) with the following processing times:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
t_j	8	6	5	4	4	2

Questions

1. Solve $\text{LP-}P_{\|C_{\max}}$ with Gurobi and compare its value with formula (P5).
2. Write the primal and the dual solutions of the chapter for this instance, and verify feasibility and equal objective values.
3. Solve the $\text{ILP}_{P_{\|C_{\max}}}$ model with Gurobi, and compute the ratio $\frac{z(\text{LP}_{P_{\|C_{\max}}})}{z(\text{ILP}_{P_{\|C_{\max}}})}$. Compare it with $\frac{1}{m}$.

Solution

We use the value (P5) of the relaxations, the primal and dual solutions given with it and the valid inequalities (P6) of the box at the beginning of the chapter.

1. Gurobi gives $z(\text{LP-}P_{\|C_{\max}}) = \frac{29}{3}$, and the formula gives $\frac{\sum_j t_j}{m} = \frac{29}{3}$, the same value. Adding the bounds $x_{jh} \leq 1$ does not change it.
2. Primal: $\tilde{x}_{jh} = \frac{1}{3}$ for every job and machine, and $\tilde{C}_{\max} = \frac{29}{3}$. Each job has $3 \cdot \frac{1}{3} = 1$, and each machine has load $\frac{1}{3} \sum_j t_j = \frac{29}{3} = \tilde{C}_{\max}$: all the constraints hold with equality.

Dual: $\tilde{\pi}_h = \frac{1}{3}$ for every machine and $\tilde{\mu}_j = \frac{t_j}{3}$, that is

$$\tilde{\mu}_1 = \frac{8}{3}, \quad \tilde{\mu}_2 = 2, \quad \tilde{\mu}_3 = \frac{5}{3}, \quad \tilde{\mu}_4 = \frac{4}{3}, \quad \tilde{\mu}_5 = \frac{4}{3}, \quad \tilde{\mu}_6 = \frac{2}{3}.$$

Every constraint $\mu_j - t_j \pi_h \leq 0$ holds with equality, and $\sum_h \tilde{\pi}_h = 3 \cdot \frac{1}{3} = 1$. The dual value is $\sum_j \tilde{\mu}_j = \frac{29}{3}$, equal to the primal value: both solutions are optimal.

3. Gurobi gives $z(\text{ILP}_{P_{\|C_{\max}}}) = 10$ (for instance $J_1 = \{1, 6\}$, $J_2 = \{2, 4\}$, $J_3 = \{3, 5\}$, with loads 10, 10, 9). The ratio is $\frac{\frac{29}{3}}{10} = \frac{29}{30}$, much larger than $\frac{1}{3}$: the worst case of the chapter needs one very long job.

Exercise 14.3 — Feasible solutions of the $P_{\|C_{\max}}$ model (1)

We consider $m = 2$ identical machines and a set $\mathcal{J} = \{1, \dots, 5\}$ of 5 jobs ($n = 5$) with the following processing times:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
t_j	6	5	4	3	2

Consider the following candidate solutions of the $\text{ILP}_{P_{\|C_{\max}}}$ model, given by the jobs on each machine:

- (a) machine 1: $\{1, 3\}$, machine 2: $\{2, 4, 5\}$, and $C_{\max} = 10$;

- (b) machine 1: $\{1, 2\}$, machine 2: $\{3, 4, 5\}$, and $C_{\max} = 10$;
- (c) machine 1: $\{1, 2\}$, machine 2: $\{3, 4, 5\}$, and $C_{\max} = 12$;
- (d) $x_{j1} = x_{j2} = \frac{1}{2}$ for every job j , and $C_{\max} = 10$;
- (e) machine 1: $\{1, 3, 5\}$, machine 2: $\{2, 4, 5\}$, and $C_{\max} = 12$.

Questions

1. For each candidate, check the assignment constraints and the load constraints.
2. Decide which candidates are feasible for $\text{ILP}_{P\|C_{\max}}$ and which for $\text{LP}_{P\|C_{\max}}$. For the feasible ones, give the value and the smallest value of $C_{\max}$ that keeps the same assignment feasible.
3. Find the optimal value of the instance, and decide which candidates are optimal for $\text{ILP}_{P\|C_{\max}}$ or for $\text{LP}_{P\|C_{\max}}$ (use Gurobi).

Solution

1. (a) Every job is on one machine. The loads are 10 and 10, and $C_{\max} = 10$: the load constraints hold.
- (b) Every job is on one machine. The loads are 11 and 9, and $C_{\max} = 10$: the load constraint of machine 1 is violated, since $11 > 10$.
- (c) Every job is on one machine. The loads are 11 and 9, and $C_{\max} = 12$: the load constraints hold.
- (d) Every job is on one machine. The loads are 10 and 10, and $C_{\max} = 10$: the load constraints hold.
- (e) Job 5 is on both machines: its assignment constraint gives $2 \neq 1$. The loads are 12 and 10, and $C_{\max} = 12$: the load constraints hold.
2. (a) Feasible for both models, of value 10, which is also the largest load.
- (b) Infeasible for both models.
- (c) Feasible for both models, of value 12; the largest load is 11, so $C_{\max} = 11$ keeps the assignment feasible and is better.
- (d) Feasible for $\text{LP}_{P\|C_{\max}}$ only, of value 10: the fractional assignment balances the loads exactly.
- (e) Infeasible for both models.
3. The lower bound is $\lceil 20/2 \rceil = 10$ and candidate (a) attains it: $z(\text{ILP}_{P\|C_{\max}}) = 10$, and (a) is optimal. Gurobi gives $z(\text{LP}_{P\|C_{\max}}) = 10$, so (a) and (d) are both optimal for the relaxation: on this instance there is no integrality gap.

Exercise 14.4 — Valid inequalities and symmetry for $P\|C_{\max}$ (1)

We consider $m = 3$ identical machines and a set $\mathcal{J} = \{1, \dots, 6\}$ of 6 jobs ($n = 6$) with the following processing times:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
t_j	12	3	3	2	2	2

Questions

1. Compute $z(\text{LP}_{P\|C_{\max}})$ with Gurobi.
2. Add the valid inequality $C_{\max} \geq \left\lceil \frac{\sum_j t_j}{m} \right\rceil$ and solve the relaxation again. Then add the valid inequalities $C_{\max} \geq t_j$ instead, and solve again.
3. Compute the optimal value with Gurobi, and compare it with the three bounds.
4. Count, by enumeration, the optimal solutions of the model without and with the symmetry-breaking constraints $\sum_j t_j x_{jh} \geq \sum_j t_j x_{j,h+1}$ ($h = 1, 2$).

Solution

We use the value (P5) of the relaxations, the primal and dual solutions given with it and the valid inequalities (P6) of the box at the beginning of the chapter.

1. Gurobi gives $z(\text{LP}_{P\|C_{\max}}) = 24/3 = 8$.
2. The inequality $C_{\max} \geq \lceil 24/3 \rceil = 8$ does not change the bound: Gurobi gives again 8, since 24/3 is already an integer. The inequalities $C_{\max} \geq t_j$ include $C_{\max} \geq 12$, and Gurobi gives 12: the long job alone fixes the bound.
3. Gurobi gives $z(\text{ILP}_{P\|C_{\max}}) = 12$: job 1 alone on one machine, the other jobs on the other two machines. The LP bound 8 has gap $\frac{12-8}{8} = 50.0\%$; with the inequalities $C_{\max} \geq t_j$ the bound is 12, equal to the optimal value.
4. Without the symmetry-breaking constraints: job 1 must be alone (any other job would give a load larger than 12), it can be on any of the 3 machines, and the other 5 jobs can be split in any way between the other two machines, whose loads stay at most 12. This gives $3 \cdot 2^5 = 96$ optimal solutions.

With the constraints the loads must be non-increasing. The enumeration gives 18 optimal solutions: job 1 on machine 1 with the other jobs split so that machine 2 has load at least machine 3, plus the solution with all the short jobs on machine 1 and job 1 on machine 2 (loads 12, 12, 0). The constraints remove most, but not all, of the equivalent solutions.

Exercise 14.5 — Integrality gap of the $P\|C_{\max}$ model (1)

We consider $m = 2$ identical machines and a set $\mathcal{J} = \{1, \dots, 3\}$ of 3 jobs ($n = 3$) with the following processing times:

	$j = 1$	$j = 2$	$j = 3$
t_j	5	5	3

Questions

1. Compute $z(\text{LP}_{P\|C_{\max}})$ and $z(\text{ILP}_{P\|C_{\max}})$ with Gurobi, and explain the optimal value.
2. Compute the absolute gap and the ratio $\frac{z(\text{LP}_{P\|C_{\max}})}{z(\text{ILP}_{P\|C_{\max}})}$.
3. Add both valid inequalities (P6) and solve the relaxation again. How much of the gap is closed?
4. Consider the family of the chapter with $m = 3$ machines, one job of time $M = 10$ and 3 jobs of time 1. Compute the ratio with Gurobi, and its limit for $M \rightarrow +\infty$.

Solution

We use the value (P5) of the relaxations, the primal and dual solutions given with it and the valid inequalities (P6) of the box at the beginning of the chapter.

1. Gurobi gives $z(\text{LP}_{P\|C_{\max}}) = \frac{13}{2} = \frac{13}{2}$ and $z(\text{ILP}_{P\|C_{\max}}) = 8$. The two jobs of time 5 on the same machine give load 10; otherwise the job of time 3 shares a machine with a job of time 5, with load 8.
2. The absolute gap is $8 - \frac{13}{2} = \frac{3}{2}$, and the ratio is $\frac{\frac{13}{2}}{8} = \frac{13}{16}$, which is larger than $\frac{1}{m} = \frac{1}{2}$.
3. The inequalities give $C_{\max} \geq \max\{5, \lceil \frac{13}{2} \rceil\} = 7$, and Gurobi gives 7. The gap goes from $\frac{3}{2}$ to 1: the rounding closes one third of it, and the remaining unit comes from the integer assignment.
4. Gurobi gives $z(\text{LP}_{P\|C_{\max}}) = \frac{10+3}{3} = \frac{13}{3}$ and $z(\text{ILP}_{P\|C_{\max}}) = 10$, the long job. The ratio is $\frac{\frac{13}{3}}{10} = \frac{13}{30}$. In general it is $\frac{M+m}{mM} = \frac{1}{m} + \frac{1}{M}$, which tends to $\frac{1}{3}$ as $M \rightarrow +\infty$.

Formulations of lot sizing problems

Formulations used in these exercises. The models of Chapter 15 «Formulations of lot sizing problems», on the periods $T = \{1, \dots, m\}$, with $D_{t,k} = \sum_{s=t}^k d_s$. The big-M formulation ILP_{CLSP}:

$$\min \sum_{t \in T} (sc_t y_t + vc_t x_t + hc_t s_t) \quad (\text{L0})$$

$$\text{subject to } s_{t-1} + x_t = d_t + s_t, \quad \forall t \in T, \quad (\text{L1})$$

$$x_t \leq M_t y_t, \quad \forall t \in T, \quad (\text{L2})$$

$$x_t \leq cap_t, \quad \forall t \in T, \quad (\text{L3})$$

$$s_0 = s_m = 0, \quad (\text{L4})$$

$$x_t \in \mathbb{Z}_{\geq 0}, \quad \forall t \in T, \quad (\text{L5})$$

$$s_t \in \mathbb{Z}_{\geq 0}, \quad \forall t \in T, \quad (\text{L6})$$

$$y_t \in \{0, 1\}, \quad \forall t \in T. \quad (\text{L7})$$

Big-M values in (L2): $M_t^{\text{dem}} = D_{t,m}$, $M_t^{\text{cap}} = cap_t$, and the strong value $M_t = \min\{cap_t, D_{t,m}\}$. The shortest-path formulation MIP_{CLSP}^{SP}, where zv_{tk} is the fraction of the demands of the periods $t, \dots, k$ produced in period t :

$$\min \sum_{t \in T} sc_t y_t + \sum_{t \in T} \sum_{k=t}^m cv_{tk} zv_{tk} \quad (\text{S0})$$

$$\text{subject to } \sum_{k=1}^m zv_{1k} = 1, \quad (\text{S1})$$

$$\sum_{u=1}^{t-1} zv_{u,t-1} - \sum_{k=t}^m zv_{tk} = 0, \quad \forall t \in T \setminus \{1\}, \quad (\text{S2})$$

$$\sum_{k=t}^m zv_{tk} \leq y_t, \quad \forall t \in T, \quad (\text{S3})$$

$$\sum_{k=t}^m D_{t,k} zv_{tk} \leq cap_t, \quad \forall t \in T, \quad (\text{S4})$$

$$zv_{tk} \geq 0, \quad \forall t, k \in T, k \geq t, \quad (\text{S5})$$

$$y_t \in \{0, 1\}, \quad \forall t \in T, \quad (\text{S6})$$

with the costs

$$cv_{tk} = vc_t D_{t,k} + \sum_{s=t+1}^k \sum_{u=t}^{s-1} hc_u d_s. \quad (S7)$$

The LP relaxations relax the domains into $x_t, s_t \geq 0$ and $0 \leq y_t \leq 1$: they are the models LP_{CLSP}^{dem} , LP_{CLSP}^{cap} and LP_{CLSP}^{strong} of ILP_{CLSP} with the three big-M values, and the model LP_{CLSP}^{SP} of MIP_{CLSP}^{SP} .

The model ILP_{ULS} of the uncapacitated lot sizing problem (ULS) is ILP_{CLSP} without the capacity constraints (L3) and with $M_t = D_{t,m}$ in (L2).

The model ILP_{CLSP}^P of the lot sizing problem with several products $p \in P$ and setup times (MLS), with $sd_{ptk} = \sum_{u=t}^k d_{pu}$:

$$\min \sum_{p \in P} \sum_{t \in T} (sc_{pt} y_{pt} + vc_{pt} x_{pt} + hc_{pt} s_{pt}) \quad (P0)$$

$$\text{subject to } s_{p,t-1} + x_{pt} = d_{pt} + s_{pt}, \quad \forall p \in P, \forall t \in T, \quad (P1)$$

$$\sum_{p \in P} (st_{pt} y_{pt} + vt_{pt} x_{pt}) \leq cap_t, \quad \forall t \in T, \quad (P2)$$

$$x_{pt} \leq M_{pt} y_{pt}, \quad \forall p \in P, \forall t \in T, \quad (P3)$$

$$s_{p0} = s_{pm} = 0, \quad \forall p \in P, \quad (P4)$$

$$x_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T, \quad (P5)$$

$$s_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T, \quad (P6)$$

$$y_{pt} \in \{0, 1\}, \quad \forall p \in P, \forall t \in T, \quad (P7)$$

with the strong big-M values $M_{pt} = \min\{[(cap_t - st_{pt})/vt_{pt}], sd_{ptm}\}$.

The model ILP_{CLSP}^{OT} with overtime (LS-OT) adds $\sum_{t \in T} (cw_t \mu_t + co_t \nu_t)$ to (P0), replaces cap_t with $\mu_t + \nu_t$ in (P2), uses $cap_t^N + cap_t^O$ in place of cap_t in the big-M values, and adds:

$$\mu_t \leq cap_t^N, \quad \forall t \in T, \quad (O1)$$

$$\nu_t \leq cap_t^O, \quad \forall t \in T, \quad (O2)$$

$$\mu_t \in \mathbb{Z}_{\geq 0}, \quad \forall t \in T, \quad (O3)$$

$$\nu_t \in \mathbb{Z}_{\geq 0}, \quad \forall t \in T. \quad (O4)$$

The model ILP_{CLSP}^{UD} with unsatisfied demand (LS-UD) adds $\sum_{p \in P} \sum_{t \in T} cu_{pt} \varepsilon_{pt}$ to (P0), replaces d_{pt} with $d_{pt} - \varepsilon_{pt}$ in (P1), and adds:

$$\varepsilon_{pt} \leq d_{pt}, \quad \forall p \in P, \forall t \in T, \quad (U1)$$

$$\varepsilon_{pt} \in \mathbb{Z}_{\geq 0}, \quad \forall p \in P, \forall t \in T. \quad (U2)$$

With one product the index p is dropped.

Exercise 15.1 — The big-M formulation of lot sizing (1)

We consider a capacitated lot sizing instance with $m = 4$ periods, $s_0 = s_4 = 0$ and the following data:

period t	1	2	3	4
demand d_t	1	4	1	3
capacity cap_t	4	4	4	4
setup cost sc_t	40	40	40	40
production cost vc_t	0	0	0	0
holding cost hc_t	5	5	5	5

Questions

1. Compute the big-M values M_t^{dem} , M_t^{cap} and the strong value M_t for every period.
2. Write the big-M formulation ILP_{CLSP} with the strong values M_t .
3. Count the variables and the constraints (besides the domains) of the formulation.
4. Solve the formulation with Gurobi, and give the optimal plan, its inventories, its setups and its cost.

Solution

We use the big-M values and the costs (S7) of the box at the beginning of the chapter.

1. The demands of the periods $t, \dots, m$ give $D_{t,m}$, and the strong value is the smaller of the two:

period t	1	2	3	4
$M_t^{\text{dem}} = D_{t,4}$	9	8	4	3
$M_t^{\text{cap}} = cap_t$	4	4	4	4
$M_t = \min\{cap_t, D_{t,m}\}$	4	4	4	3

2. With the strong values the formulation is:

$$\begin{array}{llll}
\min & 40 y_1 + 40 y_2 + 40 y_3 + 40 y_4 + 5 s_1 + 5 s_2 + 5 s_3 \\
& + 5 s_4 \\
\text{subject to} & x_1 - s_1 & = & 1 \\
& s_1 + x_2 - s_2 & = & 4 \\
& s_2 + x_3 - s_3 & = & 1 \\
& s_3 + x_4 - s_4 & = & 3 \\
& x_1 - 4 y_1 & \leq & 0 \\
& x_2 - 4 y_2 & \leq & 0 \\
& x_3 - 4 y_3 & \leq & 0 \\
& x_4 - 3 y_4 & \leq & 0 \\
& x_1 & \leq & 4 \\
& x_2 & \leq & 4 \\
& x_3 & \leq & 4 \\
& x_4 & \leq & 4 \\
& s_4 & = & 0 \\
& x_1, x_2, x_3, x_4 & \in & \mathbb{Z}_{\geq 0} \\
& s_1, s_2, s_3, s_4 & \in & \mathbb{Z}_{\geq 0} \\
& y_1, y_2, y_3, y_4 & \in & \{0, 1\}
\end{array}$$

3. There are $m = 4$ variables x_t , 4 variables $s_1, \dots, s_4$ and 4 variables y_t , so 12 variables ($s_0 = 0$ is a constant). There are 4 balance constraints, 4 big-M constraints, 4 capacity constraints and the constraint $s_4 = 0$, so 13 linear constraints.
4. Gurobi gives $z(\text{ILP}_{\text{CLSP}}) = 130$, with the plan $\tilde{\mathbf{x}} = (2, 4, 0, 3)$, the inventories $(s_1, \dots, s_4) = (1, 1, 0, 0)$ and the setups $\tilde{\mathbf{y}} = (1, 1, 0, 1)$. The cost is $3 \cdot 40$ for the setups plus $5 \cdot (1 + 1) = 10$ for the holding, that is 130. By enumeration of the plans, the optimal plan is unique.

Exercise 15.2 — Big-M values and LP bounds for lot sizing (1)

We consider a capacitated lot sizing instance with $m = 4$ periods, $s_0 = s_4 = 0$ and the following data:

period t	1	2	3	4
demand d_t	1	4	1	3
capacity cap_t	4	4	4	4
setup cost sc_t	40	40	40	40
production cost vc_t	0	0	0	0
holding cost hc_t	5	5	5	5

Questions

1. Compute the cost of the best plan when the setups are ignored.
2. Solve with Gurobi the LP relaxations $\text{LP}_{\text{CLSP}}^{\text{dem}}$, $\text{LP}_{\text{CLSP}}^{\text{cap}}$ and $\text{LP}_{\text{CLSP}}^{\text{strong}}$ of the big-M formulation, and give the values of y_t in their optimal solutions.
3. Explain the values of y_t using the big-M constraints.
4. Compute the optimal value with Gurobi, and the gap $\frac{z - \text{LB}}{\text{LB}}$ of every bound.

Solution

We use the big-M values and the costs (S7) of the box at the beginning of the chapter.

1. Without the setups the best plan produces in every period its demand, $\mathbf{x} = (1, 4, 1, 3)$: there is no inventory and the production cost is zero, so the bound is 0.
2. Gurobi gives:

relaxation	value	$\tilde{\mathbf{y}}$
$\text{LP}_{\text{CLSP}}^{\text{dem}} (M_t = D_{t,m})$	$\frac{670}{9}$	$(\frac{1}{9}, \frac{1}{2}, \frac{1}{4}, 1)$
$\text{LP}_{\text{CLSP}}^{\text{cap}} (M_t = \text{cap}_t)$	90	$(\frac{1}{4}, 1, \frac{1}{4}, \frac{3}{4})$
$\text{LP}_{\text{CLSP}}^{\text{strong}} (M_t = \min\{\text{cap}_t, D_{t,m}\})$	100	$(\frac{1}{4}, 1, \frac{1}{4}, 1)$

3. In all three solutions the plan is the lot-for-lot plan $\mathbf{x} = (1, 4, 1, 3)$, which has no holding cost. The setup variables take the smallest value allowed by the big-M constraints, $y_t = \frac{x_t}{M_t}$: for instance, in period 1 we get $\frac{1}{9}$, $\frac{1}{4}$ and $\frac{1}{4}$. The smaller M_t , the larger y_t and the higher the bound.
4. Gurobi gives $z(\text{ILP}_{\text{CLSP}}) = 130$. The gaps are:

bound	value	gap from 130
without setups	0	undefined
$\text{LP}_{\text{CLSP}}^{\text{dem}}$	$\frac{670}{9}$	74.6%
$\text{LP}_{\text{CLSP}}^{\text{cap}}$	90	44.4%
$\text{LP}_{\text{CLSP}}^{\text{strong}}$	100	30.0%

Exercise 15.3 — Feasible solutions of the lot sizing formulation (1)

We consider a capacitated lot sizing instance with $m = 4$ periods, $s_0 = s_4 = 0$ and the following data:

period t	1	2	3	4
demand d_t	1	4	1	3
capacity cap_t	4	4	4	4
setup cost sc_t	40	40	40	40
production cost vc_t	0	0	0	0
holding cost hc_t	5	5	5	5

We use the big-M formulation with the strong values $M_t = \min\{cap_t, D_{t,m}\} = (4, 4, 4, 3)$. Consider the following candidates $(\mathbf{x}, \mathbf{y})$, with the inventories given by the balance constraints:

- (a) $\mathbf{x} = (2, 4, 0, 3)$, $\mathbf{y} = (1, 1, 0, 1)$;
- (b) $\mathbf{x} = (1, 4, 1, 3)$, $\mathbf{y} = (1, 1, 1, 1)$;
- (c) $\mathbf{x} = (5, 0, 1, 3)$, $\mathbf{y} = (1, 0, 1, 1)$;
- (d) $\mathbf{x} = (2, 4, 0, 3)$, $\mathbf{y} = (1, 1, 0, 0)$;
- (e) $\mathbf{x} = (2, 4, 0, 2)$, $\mathbf{y} = (1, 1, 0, 1)$;
- (f) $\mathbf{x} = (1, 4, 1, 3)$, $\mathbf{y} = (\frac{1}{4}, 1, \frac{1}{4}, 1)$;

Questions

1. Compute the inventories $s_1, \dots, s_4$ of every candidate and check the balance constraints with $s_0 = s_4 = 0$.
2. Check the big-M constraints and the capacity constraints, and decide which candidates are feasible for ILP_{CLSP} and which for its LP relaxation.
3. Give the cost of the feasible candidates, and compare them with $z(\text{LP}_{\text{CLSP}}^{\text{strong}})$ computed with Gurobi.

Solution

1. (a) $(s_1, \dots, s_4) = (1, 1, 0, 0)$: all the inventories are non-negative and $s_4 = 0$.
 (b) $(s_1, \dots, s_4) = (0, 0, 0, 0)$: all the inventories are non-negative and $s_4 = 0$.
 (c) $(s_1, \dots, s_4) = (4, 0, 0, 0)$: all the inventories are non-negative and $s_4 = 0$.
 (d) $(s_1, \dots, s_4) = (1, 1, 0, 0)$: all the inventories are non-negative and $s_4 = 0$.
 (e) $(s_1, \dots, s_4) = (1, 1, 0, -1)$: $s_4 < 0$, the demand is not satisfied.
 (f) $(s_1, \dots, s_4) = (0, 0, 0, 0)$: all the inventories are non-negative and $s_4 = 0$.
2. (a) All the constraints hold: feasible for both models.
 (b) All the constraints hold: feasible for both models.
 (c) $x_1 = 5 > cap_1 = 4$: infeasible.
 (d) $x_4 = 3 > M_4 y_4 = 3 \cdot 0$: infeasible.

- (e) Infeasible, because of the balance constraints.
- (f) All the constraints hold: feasible for the LP relaxation only, since y_1 and y_3 are fractional.
3. Candidate (a) costs $3 \cdot 40 + 5 \cdot (1 + 1) = 130$ and (b), the lot-for-lot plan with four setups, costs $4 \cdot 40 = 160$. Candidate (f) costs $40 \cdot (\frac{1}{4} + 1 + \frac{1}{4} + 1) = 100$. Gurobi gives $z(\text{LP}_{\text{CLSP}}^{\text{strong}}) = 100$: candidate (f) is an optimal solution of the relaxation, and (a) is the optimal plan of the instance.

Exercise 15.4 — The shortest-path formulation of lot sizing (1)

We consider a capacitated lot sizing instance with $m = 4$ periods, $s_0 = s_4 = 0$ and the following data:

period t	1	2	3	4
demand d_t	1	4	1	3
capacity cap_t	4	4	4	4
setup cost sc_t	40	40	40	40
production cost vc_t	0	0	0	0
holding cost hc_t	5	5	5	5

Questions

1. Compute the demands $D_{t,k}$ and the costs cv_{tk} of the shortest-path formulation.
2. Write the shortest-path formulation for this instance.
3. Solve its LP relaxation $\text{LP}_{\text{CLSP}}^{\text{SP}}$ with Gurobi, and compare the bound with $z(\text{LP}_{\text{CLSP}}^{\text{strong}})$ of the big-M formulation.
4. The optimal plan is $\tilde{x} = (2, 4, 0, 3)$. Write it as a combination of paths of the network, and verify its quantities and its cost.

Solution

We use the big-M values and the costs (S7) of the box at the beginning of the chapter.

1. The production cost is zero, so cv_{tk} is only holding cost:

$t \backslash k$	$D_{t,k}$				cv_{tk}			
	1	2	3	4	1	2	3	4
1	1	5	6	9	0	20	30	75
2		4	5	8		0	5	35
3			1	4			0	15
4				3				0

2. The variable $z_{v_{tk}}$ is the arc from node $t-1$ to node k , with cost cv_{tk} . The formulation is:

$$\begin{aligned}
 \min \quad & 40 y_1 + 40 y_2 + 40 y_3 + 40 y_4 + 20 z v_{12} + 30 z v_{13} + 75 z v_{14} \\
 & + 5 z v_{23} + 35 z v_{24} + 15 z v_{34} \\
 \text{subject to} \quad & z v_{11} + z v_{12} + z v_{13} + z v_{14} = 1 \\
 & z v_{11} - z v_{22} - z v_{23} - z v_{24} = 0 \\
 & z v_{12} + z v_{22} - z v_{33} - z v_{34} = 0 \\
 & z v_{13} + z v_{23} + z v_{33} - z v_{44} = 0 \\
 & z v_{11} + z v_{12} + z v_{13} + z v_{14} - y_1 \leq 0 \\
 & z v_{22} + z v_{23} + z v_{24} - y_2 \leq 0 \\
 & z v_{33} + z v_{34} - y_3 \leq 0 \\
 & z v_{44} - y_4 \leq 0 \\
 & z v_{11} + 5 z v_{12} + 6 z v_{13} + 9 z v_{14} \leq 4 \\
 & 4 z v_{22} + 5 z v_{23} + 8 z v_{24} \leq 4 \\
 & z v_{33} + 4 z v_{34} \leq 4 \\
 & 3 z v_{44} \leq 4 \\
 & z v_{11}, z v_{12}, z v_{13}, \dots, z v_{44} \geq 0 \\
 & y_1, y_2, y_3, y_4 \in \{0, 1\}
 \end{aligned}$$

3. Gurobi gives $z(\text{LP}_{\text{CLSP}}^{\text{SP}}) = 112$, with $z v_{11} = \frac{2}{5}$, $z v_{13} = \frac{3}{5}$, $z v_{24} = \frac{2}{5}$, $z v_{44} = \frac{3}{5}$. The bound is better than $z(\text{LP}_{\text{CLSP}}^{\text{strong}}) = 100$, and the optimal value is 130.

4. With the quantities fixed to $\tilde{\mathbf{x}}$, Gurobi gives $z v_{11} = \frac{4}{5}$, $z v_{13} = \frac{1}{5}$, $z v_{23} = \frac{4}{5}$, $z v_{44} = 1$, all the others 0. These arcs decompose into the paths

- $0 \rightarrow 1 \rightarrow 3 \rightarrow 4$ with weight $\frac{4}{5}$, which is the plan $(1, 5, 0, 3)$;
- $0 \rightarrow 3 \rightarrow 4$ with weight $\frac{1}{5}$, which is the plan $(6, 0, 0, 3)$;

The combination gives $x_1 = \frac{4}{5} \cdot 1 + \frac{1}{5} \cdot 6 = 2$, $x_2 = \frac{4}{5} \cdot 5 = 4$, $x_4 = \frac{4}{5} \cdot 3 + \frac{1}{5} \cdot 3 = 3$. The single paths may exceed the capacity: only the combined quantities must respect it. The cost is $3 \cdot 40 + 0 \cdot \frac{4}{5} + 30 \cdot \frac{1}{5} + 5 \cdot \frac{4}{5} + 0 \cdot 1 = 130$.

Exercise 15.5 — Comparing the lot sizing formulations (1)

We consider a capacitated lot sizing instance with $m = 5$ periods, $s_0 = s_5 = 0$ and the following data:

period t	1	2	3	4	5
demand d_t	2	1	3	3	3
capacity cap_t	4	4	4	4	4
setup cost sc_t	40	40	40	40	40
production cost vc_t	0	0	0	0	0
holding cost hc_t	10	10	10	10	10

Questions

1. Compute the three big-M values of every period.
2. Compute with Gurobi the bound without setups, the three LP bounds of the big-M formulation and the LP bound of the shortest-path formulation.
3. Compute the optimal value with Gurobi, using both formulations with integer setups, and the gap $\frac{z-LB}{LB}$ of every bound.
4. Compare the number of variables of the two formulations, and comment on the price of the stronger bound.

Solution

We use the big-M values and the costs (S7) of the box at the beginning of the chapter.

period t	1	2	3	4	5
1. $M_t^{\text{dem}} = D_{t,5}$	12	10	9	6	3
$M_t^{\text{cap}} = \text{cap}_t$	4	4	4	4	4
$M_t = \min\{\text{cap}_t, D_{t,m}\}$	4	4	4	4	3

2. Gurobi gives:

bound	value
without setups	0
$\text{LP}_{\text{CLSP}}^{\text{dem}}$	84
$\text{LP}_{\text{CLSP}}^{\text{cap}}$	120
$\text{LP}_{\text{CLSP}}^{\text{strong}}$	130
$\text{LP}_{\text{CLSP}}^{\text{SP}}$	160

Each bound improves the previous one.

3. Both formulations give $z(\text{ILP}_{\text{CLSP}}) = 170$, with the unique optimal plan $\tilde{x} = (3, 0, 3, 3, 3)$. The gaps are:

bound	value	gap from 170
without setups	0	undefined
$\text{LP}_{\text{CLSP}}^{\text{dem}}$	84	102.4%
$\text{LP}_{\text{CLSP}}^{\text{cap}}$	120	41.7%
$\text{LP}_{\text{CLSP}}^{\text{strong}}$	130	30.8%
$\text{LP}_{\text{CLSP}}^{\text{SP}}$	160	6.2%

4. The big-M formulation has 16 variables (x_t , s_t and y_t); the shortest-path formulation has $\frac{m(m+1)}{2} + m = 20$ variables (zv_{tk} and y_t). The number of variables zv_{tk} grows like m^2 : the stronger bound is paid with a larger model.

Exercise 15.6 — ILP model of the uncapacitated lot sizing problem

We consider an uncapacitated lot sizing instance (ULS) with $m = 4$ periods, $s_0 = s_4 = 0$ and the following data:

period t	1	2	3	4
demand d_t	1	4	1	3
setup cost sc_t	40	40	40	40
production cost vc_t	0	0	0	0
holding cost hc_t	5	5	5	5

Questions

1. Compute the big-M values $D_{t,m}$ and write the ILP_{ULS} model for this instance.
2. Count the variables and the constraints (besides the domains) of the model.
3. Solve the model with Gurobi, and give the optimal plan, its inventories and its cost.
4. Add the capacity $cap_t = 4$ in every period: Gurobi gives the optimal value 130. Explain why it is larger.

Solution

1. The big-M values are $D_{t,4} = (9, 8, 4, 3)$. The ILP_{ULS} model is:

$$\begin{aligned}
 \min \quad & 40 y_1 + 40 y_2 + 40 y_3 + 40 y_4 + 5 s_1 + 5 s_2 \\
 & + 5 s_3 + 5 s_4 \\
 \text{subject to} \quad & x_1 - s_1 = 1 \\
 & s_1 + x_2 - s_2 = 4 \\
 & s_2 + x_3 - s_3 = 1 \\
 & s_3 + x_4 - s_4 = 3 \\
 & x_1 - 9 y_1 \leq 0 \\
 & x_2 - 8 y_2 \leq 0 \\
 & x_3 - 4 y_3 \leq 0 \\
 & x_4 - 3 y_4 \leq 0 \\
 & s_4 = 0 \\
 & x_1, x_2, x_3, x_4 \in \mathbb{Z}_{\geq 0} \\
 & s_1, s_2, s_3, s_4 \in \mathbb{Z}_{\geq 0} \\
 & y_1, y_2, y_3, y_4 \in \{0, 1\}
 \end{aligned}$$

2. There are $3m = 12$ variables x_t, s_t, y_t ($s_0 = 0$ is a constant), 4 balance constraints, 4 big-M constraints and the constraint $s_4 = 0$: 9 linear constraints.

3. Gurobi gives $z(\text{ILP}_{\text{ULS}}) = 110$ with the plan $\tilde{x} = (6, 0, 0, 3)$, the inventories $(5, 1, 0, 0)$ and the setups in the periods 1 and 4: the cost is $2 \cdot 40 + 5 \cdot (5 + 1) = 110$. By enumeration of the plans, the optimal plan is unique.
4. The plan $(6, 0, 0, 3)$ produces $6 > 4$ units in period 1, so it is not feasible with the capacities. The capacitated problem has fewer feasible plans, hence a larger optimal value: the ULS is a relaxation of the CLS.

Exercise 15.7 — ILP model of lot sizing with several products and setup times

We consider a lot sizing problem with $|P| = 2$ products, $m = 3$ periods, the capacity $\text{cap}_t = 10$ in every period, $vc_{pt} = 0$, $s_{p0} = s_{p3} = 0$ and the following data:

period t	product 1			product 2		
	1	2	3	1	2	3
d_{pt}	2	3	1	3	1	3
vt_{pt}	1	1	1	2	2	2
st_{pt}	1	1	1	1	1	1
sc_{pt}	20	20	20	30	30	30
hc_{pt}	4	4	4	3	3	3

Questions

1. Compute the strong big-M values M_{pt} .
2. Write the $\text{ILP}_{\text{CLSP}}^P$ model with the strong big-M values for this instance.
3. Solve the model with Gurobi: give the optimal plan of each product, its cost and the time used in each period.

Solution

1. With $M_{pt} = \min\{\lfloor (\text{cap}_t - st_{pt})/vt_{pt} \rfloor, sd_{ptm}\}$: for product 1, $\lfloor 9/1 \rfloor = 9$ and $sd_{1t3} = (6, 4, 1)$, so $M_{1t} = (6, 4, 1)$; for product 2, $\lfloor 9/2 \rfloor = 4$ and $sd_{2t3} = (7, 4, 3)$, so $M_{2t} = (4, 4, 3)$.
2. The $\text{ILP}_{\text{CLSP}}^P$ model is:

$$\begin{aligned}
 \min \quad & 20y_{11} + 20y_{12} + 20y_{13} + 30y_{21} + 30y_{22} + 30y_{23} \\
 & + 4s_{11} + 4s_{12} + 4s_{13} + 3s_{21} + 3s_{22} + 3s_{23} \\
 \text{subject to} \quad & x_{11} - s_{11} = 2 \\
 & s_{11} + x_{12} - s_{12} = 3 \\
 & s_{12} + x_{13} - s_{13} = 1 \\
 & x_{21} - s_{21} = 3 \\
 & s_{21} + x_{22} - s_{22} = 1 \\
 & s_{22} + x_{23} - s_{23} = 3 \\
 & x_{11} + 2x_{21} + y_{11} + y_{21} \leq 10 \\
 & x_{12} + 2x_{22} + y_{12} + y_{22} \leq 10 \\
 & x_{13} + 2x_{23} + y_{13} + y_{23} \leq 10 \\
 & x_{11} - 6y_{11} \leq 0 \\
 & x_{12} - 4y_{12} \leq 0 \\
 & x_{13} - y_{13} \leq 0 \\
 & x_{21} - 4y_{21} \leq 0 \\
 & x_{22} - 4y_{22} \leq 0 \\
 & x_{23} - 3y_{23} \leq 0 \\
 & s_{13} = 0 \\
 & s_{23} = 0 \\
 & x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \in \mathbb{Z}_{\geq 0} \\
 & s_{11}, s_{12}, s_{13}, s_{21}, s_{22}, s_{23} \in \mathbb{Z}_{\geq 0} \\
 & y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23} \in \{0, 1\}
 \end{aligned}$$

3. Gurobi gives $z(\text{ILP}_{\text{CLSP}}^P) = 134$, and the optimal solution is unique. Product 1 follows the plan $(2, 4, 0)$ with the inventories $(0, 1, 0)$, product 2 the plan $(3, 1, 3)$ with the inventories $(0, 0, 0)$. The times used in the periods, setups included, are $(10, 8, 7)$, all at most 10: in period 1 the capacity is saturated, so product 2 needs a setup in every period.

Exercise 15.8 — ILP model of lot sizing with overtime

We consider a lot sizing problem with overtime, one product, $m = 5$ periods, $s_0 = s_5 = 0$, production time $vt_t = 1$, setup time $st_t = 0$, $vc_t = 0$ and the data

period t	1	2	3	4	5
demand d_t	2	1	3	3	3
setup cost sc_t	40	40	40	40	40
holding cost hc_t	10	10	10	10	10

In every period the normal hours are $cap_t^N = 2$, at the cost $cw_t = 1$ per unit, and the

overtime is $cap_t^O = 2$, at the cost $co_t = 5$ per unit.

Questions

1. Show that the instance has no feasible plan without overtime.
2. Compute the strong big-M values and write the ILP_{CLSP}^{OT} model for this instance.
3. Solve the model with Gurobi: give the optimal plan, the working times and the cost.

Solution

1. The total demand is 12, while the normal hours of the 5 periods give at most $5 \cdot 2 = 10$ units: some overtime is needed.
2. With $M_t = \min\{cap_t^N + cap_t^O, D_{t,5}\}$ and $D_{t,5} = (12, 10, 9, 6, 3)$ we get $M_t = (4, 4, 4, 4, 3)$. The model is:

$$\begin{aligned}
\min \quad & 40 y_1 + 40 y_2 + 40 y_3 + 40 y_4 + 40 y_5 + 10 s_1 \\
& + 10 s_2 + 10 s_3 + 10 s_4 + 10 s_5 + \mu_1 + \mu_2 \\
& + \mu_3 + \mu_4 + \mu_5 + 5 \nu_1 + 5 \nu_2 + 5 \nu_3 \\
& + 5 \nu_4 + 5 \nu_5 \\
\text{subject to} \quad & x_1 - s_1 = 2 \\
& s_1 + x_2 - s_2 = 1 \\
& s_2 + x_3 - s_3 = 3 \\
& s_3 + x_4 - s_4 = 3 \\
& s_4 + x_5 - s_5 = 3 \\
& x_1 - \mu_1 - \nu_1 \leq 0 \\
& x_2 - \mu_2 - \nu_2 \leq 0 \\
& x_3 - \mu_3 - \nu_3 \leq 0 \\
& x_4 - \mu_4 - \nu_4 \leq 0 \\
& x_5 - \mu_5 - \nu_5 \leq 0 \\
& x_1 - 4 y_1 \leq 0 \\
& x_2 - 4 y_2 \leq 0 \\
& x_3 - 4 y_3 \leq 0 \\
& x_4 - 4 y_4 \leq 0 \\
& x_5 - 3 y_5 \leq 0 \\
& \mu_1 \leq 2 \\
& \mu_2 \leq 2 \\
& \mu_3 \leq 2 \\
& \mu_4 \leq 2 \\
& \mu_5 \leq 2 \\
& \nu_1 \leq 2 \\
& \nu_2 \leq 2 \\
& \nu_3 \leq 2 \\
& \nu_4 \leq 2 \\
& \nu_5 \leq 2 \\
& s_5 = 0 \\
& x_1, x_2, x_3, x_4, x_5 \in \mathbb{Z}_{\geq 0} \\
& s_1, s_2, s_3, s_4, s_5 \in \mathbb{Z}_{\geq 0} \\
& \mu_1, \mu_2, \mu_3, \dots, \nu_5 \in \mathbb{Z}_{\geq 0} \\
& y_1, y_2, y_3, y_4, y_5 \in \{0, 1\}
\end{aligned}$$

3. Gurobi gives $z(\text{ILP}_{\text{CLSP}}^{\text{OT}}) = 198$, and the optimal solution is unique: the plan

$\tilde{x} = (3, 0, 3, 3, 3)$, the inventories $(1, 0, 0, 0, 0)$, the normal hours $(2, 0, 2, 2, 2)$ and the overtime $(1, 0, 1, 1, 1)$. The cost is $4 \cdot 40$ for the setups, $10 \cdot 1$ for the holding, $1 \cdot 8$ for the normal hours and $5 \cdot 4$ for the overtime: 198.

Exercise 15.9 — ILP model of lot sizing with unsatisfied demand

We consider a capacitated lot sizing instance with $m = 4$ periods, $s_0 = s_4 = 0$ and the following data:

period t	1	2	3	4
demand d_t	1	4	1	3
capacity cap_t	4	4	4	4
setup cost sc_t	40	40	40	40
production cost vc_t	0	0	0	0
holding cost hc_t	5	5	5	5

A unit of demand may be left unsatisfied, at the penalty $cu_t = 18$ per unit.

Questions

1. Write the $\text{ILP}_{\text{CLSP}}^{\text{UP}}$ model with the strong big-M values for this instance.
2. Solve the model with Gurobi: give the optimal plan, the unsatisfied demand and the cost.
3. Without unsatisfied demand the optimal value is 130 (Gurobi). Explain the saving.

Solution

1. The strong big-M values are $M_t = \min\{cap_t, D_{t,4}\} = (4, 4, 4, 3)$. The model is:

$$\begin{aligned}
 \min \quad & 40 y_1 + 40 y_2 + 40 y_3 + 40 y_4 + 5 s_1 + 5 s_2 \\
 & + 5 s_3 + 5 s_4 + 18 \varepsilon_1 + 18 \varepsilon_2 + 18 \varepsilon_3 + 18 \varepsilon_4 \\
 \text{subject to} \quad & x_1 - s_1 + \varepsilon_1 = 1 \\
 & s_1 + x_2 - s_2 + \varepsilon_2 = 4 \\
 & s_2 + x_3 - s_3 + \varepsilon_3 = 1 \\
 & s_3 + x_4 - s_4 + \varepsilon_4 = 3 \\
 & x_1 \leq 4 \\
 & x_2 \leq 4 \\
 & x_3 \leq 4 \\
 & x_4 \leq 4 \\
 & x_1 - 4 y_1 \leq 0 \\
 & x_2 - 4 y_2 \leq 0 \\
 & x_3 - 4 y_3 \leq 0 \\
 & x_4 - 3 y_4 \leq 0 \\
 & \varepsilon_1 \leq 1 \\
 & \varepsilon_2 \leq 4 \\
 & \varepsilon_3 \leq 1 \\
 & \varepsilon_4 \leq 3 \\
 & s_4 = 0 \\
 & x_1, x_2, x_3, x_4 \in \mathbb{Z}_{\geq 0} \\
 & s_1, s_2, s_3, s_4 \in \mathbb{Z}_{\geq 0} \\
 & \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \mathbb{Z}_{\geq 0} \\
 & y_1, y_2, y_3, y_4 \in \{0, 1\}
 \end{aligned}$$

2. Gurobi gives $z(\text{ILP}_{\text{CLSP}}^{\text{UD}}) = 113$, and the optimal solution is unique: the plan $\tilde{x} = (0, 4, 4, 0)$, the inventories $(0, 0, 3, 0)$ and the unsatisfied demand $\tilde{\varepsilon} = (1, 0, 0, 0)$. The cost is $2 \cdot 40 + 5 \cdot 3 + 18 \cdot 1 = 113$.
3. Leaving unsatisfied the demand $d_1 = 1$ removes the setup of period 1: the penalty 18 is smaller than what the first period costs in the optimal plan of the CLS, and the value drops from 130 to 113.