



SAPIENZA
UNIVERSITÀ DI ROMA

Combinatorial Optimization

Exercises 1 — Combinatorial optimization problems

Worked exercises on the chapters of Volume 1, combinatorial optimization problems: each exercise is followed by its complete solution.

Fabio Furini

fabiofurini.github.io

Department of Computer, Control and Management Engineering — Sapienza University of Rome

Online version and downloadable material:
fabiofurini.github.io/combinatorial-optimization/
github.com/fabiofurini/combinatorial-optimization

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Combinatorial optimization problems

Exercise 1.1 — Ground set, feasible solutions and objective

We consider the ground set $\mathcal{U} = \{1, \dots, 6\}$ with the weights

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
c_j	5	8	6	3	7	4

A subset $S \subseteq \mathcal{U}$ is feasible if it contains no two consecutive elements (j and $j+1$), and the objective $f(S) = \sum_{j \in S} c_j$ is to be maximized.

Questions

1. How many candidate solutions are there? Is $\{1, 3, 4\}$ feasible? And $\{2, 4, 6\}$?
2. Count the feasible subsets, grouped by number of elements.
3. Find the optimal solution and $z(I)$.
4. Apply the greedy rule: take the element of largest weight, discard its neighbours, and repeat on the elements left. Give the primal bound and its relative gap with respect to $z(I)$.

Solution

1. The candidates are the $2^6 = 64$ subsets of $\mathcal{U}$. The subset $\{1, 3, 4\}$ is not feasible, since it contains 3 and 4; the subset $\{2, 4, 6\}$ is feasible.
2. A feasible subset of 6 elements has at most 3 elements. By enumeration:

elements	feasible subsets
0	1
1	6
2	10
3	4
total	21

The subsets with 2 elements are the $\binom{6}{2} = 15$ pairs minus the 5 consecutive pairs: 10. Only 21 of the 64 candidates are feasible.

3. The feasible subsets with 3 elements are $\{1, 3, 5\}$ ($f = 18$), $\{1, 3, 6\}$ ($f = 15$), $\{1, 4, 6\}$ ($f = 12$), $\{2, 4, 6\}$ ($f = 15$). A subset with at most 2 elements has value at most $7 + 8 = 15$. Hence the optimal solution is $\tilde{S} = \{1, 3, 5\}$, with $z(I) = 18$; it is the only optimal solution.

4. The greedy rule proceeds as follows:

- we take element 2 (weight 8) and discard its neighbours 1, 3;
- we take element 5 (weight 7) and discard its neighbours 4, 6;

It returns $\bar{S} = \{2, 5\}$, of value 15: a primal bound $LB = 15$. The relative gap with respect to $z(I)$ is $\frac{18-15}{15} = \frac{1}{5} \approx 20.0\%$: taking the heaviest element first forbids two of its neighbours, whose weights together are larger.

Only 21 of the 64 subsets are feasible; the optimum is $\tilde{S} = \{1, 3, 5\}$ with $z(I) = 18$, while the greedy rule gives 15.

Exercise 1.2 — Counting the solutions

We count the combinatorial objects of small finite sets.

- The subsets of a set with 6 elements.
- The orderings and the cyclic orderings of 6 elements.
- The partitions of the set $\{1, 2, 3, 4, 5\}$.
- The assignments of 4 elements to 2 labels.

Questions

- How many subsets does set (a) have, and how many of them have exactly 2 elements?
- How many orderings of the 6 elements of (b) are there, how many of them describe the same cyclic ordering, and how many distinct cyclic orderings are there?

3. Compute the Bell numbers $B(1), \dots, B(5)$ with the recursion of the chapter: how many partitions does set (c) have?
4. Count, by enumeration, the partitions of set (c), grouped by number of blocks.
5. How many assignments does (d) have? How many distinct solutions remain if the two labels are interchangeable?

Solution

1. The subsets are $2^6 = 64$; those with 2 elements are $\binom{6}{2} = 15$.
2. There are $6! = 720$ orderings. A cyclic ordering is described by 6 orderings, one for every starting element, and by 6 more read in the opposite direction: $2 \cdot 6 = 12$ orderings each. The distinct cyclic orderings are $\frac{720}{12} = 60 = \frac{(6-1)!}{2}$.
3. With $B(0) = 1$ and $B(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} B(k)$, each term is a binomial coefficient times a Bell number already computed:

$$B(1) = 1 \cdot 1 = 1$$

$$B(2) = 1 \cdot 1 + 1 \cdot 1 = 2$$

$$B(3) = 1 \cdot 1 + 2 \cdot 1 + 1 \cdot 2 = 5$$

$$B(4) = 1 \cdot 1 + 3 \cdot 1 + 3 \cdot 2 + 1 \cdot 5 = 15$$

$$B(5) = 1 \cdot 1 + 4 \cdot 1 + 6 \cdot 2 + 4 \cdot 5 + 1 \cdot 15 = 52.$$

Set (c) has $B(5) = 52$ partitions.

4. By enumeration:

blocks	partitions
1	1
2	15
3	25
4	10
5	1
total	52

For example, the 10 partitions into 4 blocks put two elements together, and there are $\binom{5}{2} = 10$ pairs.

5. Every element goes to one of the 2 labels: $2^4 = 16$ assignments. If the labels are interchangeable, an assignment and the one with the two labels exchanged are the same solution. The assignments with all the elements on one label give 1 solution (counted twice among the 16), the other 14 give $\frac{14}{2} = 7$ solutions with two non-empty blocks: 8 distinct solutions, the partitions of the 4 elements into at most 2 blocks.

The candidates are 2^n subsets, $n!$ orderings, $(n-1)!/2$ cyclic orderings, $B(n)$ partitions and m^n assignments; with interchangeable labels, the assignments become partitions into at most m blocks.

Exercise 1.3 — Brute force and computing time

A computer evaluates 10^9 candidate solutions per second, and we allow one day of computation (86 400 seconds).

Questions

1. How many candidates can be evaluated in one day?
2. What is the largest number n of elements for which all the subsets, all the cyclic orderings and all the partitions can be enumerated in one day?
3. Answer the same question for a computer 1000 times faster. How many elements does it gain in each case?

Solution

1. $86\,400 \cdot 10^9 \approx 8.64 \cdot 10^{13}$ candidates.
2. We look for the largest n whose number of candidates does not exceed the budget:
 - subsets: $2^{46} \approx 7.04 \cdot 10^{13} \leq 8.64 \cdot 10^{13} < 2^{47} \approx 1.41 \cdot 10^{14}$, hence $n = 46$;
 - cyclic orderings: $16!/2 \approx 1.05 \cdot 10^{13} \leq 8.64 \cdot 10^{13} < 17!/2 \approx 1.78 \cdot 10^{14}$, hence $n = 17$;
 - partitions: $B(20) \approx 5.17 \cdot 10^{13} \leq 8.64 \cdot 10^{13} < B(21) \approx 4.75 \cdot 10^{14}$, hence $n = 20$;
3. The budget becomes $\approx 8.64 \cdot 10^{16}$:
 - subsets: $2^{56} \approx 7.21 \cdot 10^{16} \leq 8.64 \cdot 10^{16} < 2^{57} \approx 1.44 \cdot 10^{17}$, hence $n = 56$;
 - cyclic orderings: $19!/2 \approx 6.08 \cdot 10^{16} \leq 8.64 \cdot 10^{16} < 20!/2 \approx 1.22 \cdot 10^{18}$, hence $n = 20$;
 - partitions: $B(23) \approx 4.42 \cdot 10^{16} \leq 8.64 \cdot 10^{16} < B(24) \approx 4.46 \cdot 10^{17}$, hence $n = 23$;

The faster computer gains 10 elements for the subsets, 3 for the partitions and 3 for the cyclic orderings: a factor 1000 in speed moves the limit only by a few elements.

In one day brute force handles about 46 elements for the subsets, 20 for the partitions and 17 for the cyclic orderings; a computer 1000 times faster adds only a few elements.

Exercise 1.4 — Primal bound, dual bound and gap

We consider the ground set $\mathcal{U} = \{1, \dots, 5\}$, the weights $\mathbf{c} = (4, 6, 3, 5, 2)$, the objective $f(S) = \sum_{j \in S} c_j$ to be **minimized**, and the feasible solutions

$$\mathcal{F} = \{\{1, 2\}, \{1, 4\}, \{3, 4\}, \{2, 3, 5\}, \{1, 3, 5\}, \{2, 4, 5\}, \{1, 2, 3, 4, 5\}\}.$$

Questions

1. How many candidate and how many feasible solutions are there?
2. The solution $\{1, 2\}$ is feasible: which bound does it give?
3. Every feasible subset has at least 2 elements: which dual bound follows? Compute the absolute and relative gaps.
4. Improve the dual bound by looking separately at the feasible subsets with 2 elements and at those with at least 3 elements. Find an optimal solution and explain why it is optimal.

Solution

1. The candidates are the $2^5 = 32$ subsets of $\mathcal{U}$; 7 of them are feasible.
2. $f(\{1, 2\}) = 4 + 6 = 10$ is the value of a feasible solution of a minimization problem, hence a primal bound: $UB = 10$.
3. A subset with at least 2 elements weighs at least the two smallest weights: $f(S) \geq 2 + 3 = 5$ for every $S \in \mathcal{F}$, so $LB = 5$. The absolute gap is $10 - 5 = 5$ and the relative gap, divided by the lower bound, is $\frac{5}{5} \approx 100.0\%$.
4. The feasible subsets with 2 elements are $\{1, 2\}$ ($f = 10$), $\{1, 4\}$ ($f = 9$), $\{3, 4\}$ ($f = 8$): their minimum is 8. A subset with at least 3 elements weighs at least $2 + 3 + 4 = 9$. Hence $LB = 8$. The feasible solution $\{3, 4\}$ has value $3 + 5 = 8 = LB$: the gap is zero, so it is optimal and $z(I) = 8$.

$\{1, 2\}$ gives $UB = 10$, the cardinality argument gives $LB = 5$, and the argument by cases gives $LB = 8$, attained by $\{3, 4\}$: $z(I) = 8$.

Exercise 1.5 — Recognizing the combinatorial object

For each situation, name the combinatorial object of a solution (subset, ordering, cyclic ordering, partition, assignment) and the number of candidate solutions.

Questions

1. A research fund must choose which of 12 proposals to finance with a fixed budget.
2. A technician must visit 8 sites (her office included) and come back to the office.

3. 8 boxes must be loaded into identical vans of the same capacity (count the ways of grouping the boxes).
4. 8 jobs must be assigned to 3 machines (count the assignments).
5. A factory must decide in which of the next 12 weeks to set up its line.

Solution

1. A subset of the proposals: $2^{12} = 4096$ subsets.
2. A cyclic ordering of the 8 sites: $(8 - 1)!/2 = 5040/2 = 2520$ cyclic orderings.
3. A partition of the boxes: $B(8) = 4140$ partitions.
4. An assignment of the jobs to the machines: $3^8 = 6561$ assignments.
5. A subset of the weeks: $2^{12} = 4096$ choices.

Exercise 1.6 — Brute force, bounds and gap

Consider the ground set $\mathcal{U} = \{1, 2, 3, 4\}$, the weights $\mathbf{c} = (8, 5, 4, 3)$, the objective $f(S) = \sum_{j \in S} c_j$ to be maximized, and the feasible solutions

$$\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}\}.$$

Questions

1. How many candidate and how many feasible solutions are there?
2. The solution $\{1, 4\}$ is feasible: which bound does it give?
3. Find a dual bound by distinguishing the feasible subsets that contain element 1 from those that do not, and compute the absolute and relative gaps.
4. Find an optimal solution and explain why it is optimal.

Solution

1. The candidates are the $2^4 = 16$ subsets of $\mathcal{U}$; 10 of them are feasible.
2. $f(\{1, 4\}) = 8 + 3 = 11$ is the value of a feasible solution of a maximization problem, hence a primal bound: $\text{LB} = 11$.
3. The feasible subsets that contain 1 are $\{1\}$ and $\{1, 4\}$, of value at most $8 + 3 = 11$. A feasible subset without 1 has value at most $5 + 4 + 3 = 12$. Hence $f(S) \leq 12$ for every $S \in \mathcal{F}$, and $\text{UB} = 12$. The absolute gap is $12 - 11 = 1$ and the relative gap is $\frac{1}{11} \approx 9.1\%$.
4. The feasible solution $\{2, 3, 4\}$ has value $5 + 4 + 3 = 12 = \text{UB}$. The gap is zero, so it is optimal and $z(I) = 12$: the element of largest weight is not in the optimal solution.

Exercise 1.7 — How large can n be?

A computer evaluates 10^9 candidate solutions per second, and we allow one hour of computation.

Questions

1. What is the largest number n of elements for which all the subsets can be enumerated?
2. What is the largest number n of elements for which all the cyclic orderings can be enumerated?
3. Answer the same questions for a computer 1000 times faster.

Solution

In one hour the computer evaluates $3600 \cdot 10^9 = 3.6 \cdot 10^{12}$ candidates.

1. $2^{41} \approx 2.20 \cdot 10^{12} \leq 3.6 \cdot 10^{12} < 2^{42} \approx 4.40 \cdot 10^{12}$, hence $n = 41$ elements.
2. For 16 elements $15!/2 \approx 6.54 \cdot 10^{11} \leq 3.6 \cdot 10^{12}$, while for 17 elements $16!/2 \approx 1.05 \cdot 10^{13} > 3.6 \cdot 10^{12}$; hence $n = 16$ elements.
3. The budget becomes $3.6 \cdot 10^{15}$. For the subsets, $2^{51} \approx 2.25 \cdot 10^{15} \leq 3.6 \cdot 10^{15} < 2^{52} \approx 4.50 \cdot 10^{15}$, hence $n = 51$ elements: 10 more. For the cyclic orderings, $18!/2 \approx 3.20 \cdot 10^{15} \leq 3.6 \cdot 10^{15} < 19!/2 \approx 6.08 \cdot 10^{16}$, hence $n = 19$ elements: only 3 more.

Exercise 1.8 — Rounding the dual bounds

In two instances all the feasible solutions have integer value.

Questions

1. In a maximization instance, a heuristic finds a solution of value 11 and a relaxation gives the dual bound 12.6. Give LB, UB and the relative gap, before and after rounding the dual bound.
2. A solution of value 12 is then found: what can you conclude?
3. In a minimization instance, a heuristic finds a solution of value 15 and an argument gives the dual bound 13.2. Give LB, UB and the relative gap after rounding.

Solution

1. LB = 11 (primal) and UB = 12.6 (dual): the relative gap is $\frac{1.6}{11} \approx 14.5\%$. Since the values are integer, $\lfloor 12.6 \rfloor = 12$ is also a dual bound, and the gap becomes $\frac{1}{11} \approx 9.1\%$.
2. Now LB = 12 = UB: the gap is zero and the solution of value 12 is optimal.
3. In the minimization form the primal bound is UB = 15 and the dual bound is LB = $\lceil 13.2 \rceil = 14$. The relative gap, divided by the lower bound, is $\frac{15-14}{14} = \frac{1}{14} \approx 7.1\%$.

Knapsack problems

Exercise 2.1 — Feasibility, value and bounds (KP)

We consider the knapsack instance with capacity $c = 12$ and $n = 5$ items, $\mathcal{I} = \{1, \dots, 5\}$, with the following profits and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
p_j	6	12	5	13	8
w_j	3	4	5	6	8

Questions

1. Check the two assumptions on the data of the KP.
2. For each of the subsets $S_1 = \{1, 2, 5\}$, $S_2 = \{2, 4\}$, $S_3 = \{1, 2, 3\}$, compute the total weight, decide whether it is feasible and, if so, give its profit.
3. Compute the cardinality bound $\bar{k}$ and the dual bound it gives.
4. Compute the dual bound of the continuous relaxation (the items can be taken fractionally, in order of non-increasing efficiency) and round it.
5. Find an optimal solution by listing the feasible subsets, and compute the gap between the best dual bound and $z(I)$.

Solution

1. The total weight is $3 + 4 + 5 + 6 + 8 = 26 > 12$, so not all the items fit; the heaviest item weighs $8 \leq 12$, so every item fits alone. Both assumptions hold.
2.
 - S_1 : weight $3 + 4 + 8 = 15 > 12$, not feasible;
 - S_2 : weight $4 + 6 = 10 \leq 12$, feasible, profit $12 + 13 = 25$;
 - S_3 : weight $3 + 4 + 5 = 12 \leq 12$, feasible, profit $6 + 12 + 5 = 23$;
3. The weights in non-decreasing order are 3, 4, 5, 6, 8: the 3 lightest items weigh $3 + 4 +$

$5 = 12 \leq 12$, the 4 lightest weigh $18 > 12$. Hence $\bar{k} = 3$: a feasible subset has at most 3 items, and the 3 largest profits give

$$\text{UB} = 13 + 12 + 8 = 33.$$

4. The efficiencies are $\frac{6}{3} = 2$, $\frac{12}{4} = 3$, $\frac{5}{5} = 1$, $\frac{13}{6} = 2$, $\frac{8}{8} = 1$, so the order is 2, 4, 1, 3, 5. Items 2, 4 weigh 10, and the fraction $\frac{2}{3}$ of item 1 fills the residual capacity 2:

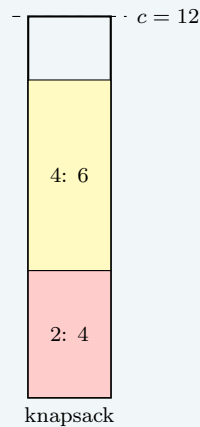
$$\text{UB} = 12 + 13 + \frac{2}{3} \cdot 6 = 29.$$

The profits are integer, so $\text{UB} = \lfloor 29 \rfloor = 29$; it is stronger than the cardinality bound.

5. There are 15 feasible subsets:

S	weight	profit	S	weight	profit
$\emptyset$	0	0	$\{1, 4\}$	9	19
$\{1\}$	3	6	$\{1, 5\}$	11	14
$\{2\}$	4	12	$\{2, 3\}$	9	17
$\{3\}$	5	5	$\{2, 4\}$	10	25
$\{4\}$	6	13	$\{2, 5\}$	12	20
$\{5\}$	8	8	$\{3, 4\}$	11	18
$\{1, 2\}$	7	18	$\{1, 2, 3\}$	12	23
$\{1, 3\}$	8	11			

The optimal solution is $\tilde{S} = \{2, 4\}$, the only one with profit $z(I) = 25$:



The gap between the continuous bound and the optimum is $29 - 25 = 4$: the bound alone does not certify the optimum, the enumeration does.

The optimal solution is $\tilde{S} = \{2, 4\}$ with $z(I) = 25$; the dual bounds are 33 (cardinality) and 29 (continuous relaxation).

Exercise 2.2 — Subset sum and its solution as a KP (SSP)

We consider the subset sum instance with $n = 6$ items and the weights

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
w_j	5	14	3	6	15	4

with the target value $c = 15$.

Questions

1. Check the two assumptions of the SSP.
2. Is the subset $\{1, 4, 6\}$ feasible? And the subset $\{3, 4, 6\}$?
3. Find all the feasible subsets for the target $c = 15$, with a short justification.
4. Now the target is $c = 16$. Solve the instance as a KP with $p_j = w_j$ and give the answer to the SSP.

Solution

1. The total weight is $47 \geq 15$ and the heaviest item weighs $15 \leq 15$: both assumptions hold.
2. $w_1 + w_4 + w_6 = 5 + 6 + 4 = 15 = 15$: feasible. $w_3 + w_4 + w_6 = 3 + 6 + 4 = 13 \neq 15$: not feasible.
3. We look at the subsets by their heaviest item, in the order 5, 2, 4, 1, 6, 3 of non-increasing weight.
 - Item 5 (weight 15) reaches the target alone: $\{5\}$.
 - Item 2 (weight 14) leaves 1, and no item weighs 1.
 - Without items 5 and 2 the remaining weights 6, 5, 4, 3 sum up to 18: a subset of them weighs 15 exactly when the items left out weigh 3, and the only such choice is to leave out item 3. This gives $\{1, 4, 6\}$: $5 + 6 + 4 = 15$.

The feasible subsets are $\{5\}$ and $\{1, 4, 6\}$: the answer is yes.

4. With $p_j = w_j$ the KP maximizes the total weight of a subset with weight at most 16. No subset weighs 16:
 - with item 5 or item 2 we need 1 or 2 more, and no item weighs so;
 - without them, the items left out should weigh $18 - 16 = 2$, less than the lightest weight 3.

The best subsets weigh $z = 15$, for example $\{5\}$. Since $z = 15 < 16$, the answer to the SSP with target 16 is no.

Target 15: yes, with $\{5\}$ and $\{1, 4, 6\}$. Target 16: no, the KP with $p_j = w_j$ reaches at most 15.

Exercise 2.3 — Copies of the items (BKP and UKP)

We consider $n = 3$ items with the following profits, weights and available copies, and a knapsack of capacity $c = 10$:

	$j = 1$	$j = 2$	$j = 3$
p_j	4	13	13
w_j	3	5	2
b_j	1	2	2

Questions

1. Check the assumptions of the BKP and write the extended set of items $\hat{\mathcal{J}}$.
2. A solution of the BKP is the vector (x_1, x_2, x_3) of the numbers of copies, with $0 \leq x_j \leq b_j$. How many such vectors are there? Is $(1, 1, 1)$ feasible? And $(0, 2, 1)$?
3. Find an optimal solution of the BKP by enumeration, with a short justification.
4. Now the items are available in an unlimited number of copies (UKP). Compute the maximum feasible numbers of copies $\hat{b}_j$, the number of candidate vectors, and an optimal solution.

Solution

1. $\sum_j b_j w_j = 1 \cdot 3 + 2 \cdot 5 + 2 \cdot 2 = 17 > 10$; every item fits ($w_j \leq 10$) and so do all its copies: $b_j w_j = 3, 10, 4 \leq 10$. The extended set has $\sum_j b_j = 5$ copies:

$$\hat{\mathcal{J}} = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2)\}.$$

2. There are $\prod_j (b_j + 1) = 2 \cdot 3 \cdot 3 = 18$ vectors, of which 12 are feasible. The vector $(1, 1, 1)$ weighs $10 \leq 10$, feasible, profit 30; the vector $(0, 2, 1)$ weighs 12, not feasible.
3. Item 3 has the largest efficiency, $\frac{13}{2}$, and item 2 the second, $\frac{13}{5}$. Taking all the copies of items 3 and 2 would weigh $14 > 10$. With $x_3 = 2$ (weight 4) the residual capacity 6 holds one copy of item 2, and the last 1 units hold nothing. The enumeration of the 12 feasible vectors confirms that $\tilde{x} = (0, 1, 2)$, of weight 9, is the only optimal solution, with $z(I) = 39$.
4. $\hat{b}_j = \lfloor c/w_j \rfloor$ gives $\hat{b} = (3, 2, 5)$, and there are $\prod_j (\hat{b}_j + 1) = 4 \cdot 3 \cdot 6 = 72$ candidate vectors, 20 of them feasible. Item 3 is the most efficient and its weight 2 divides $c = 10$: the solution $\tilde{x} = (0, 0, 5)$ fills the knapsack with the most efficient item, so no solution has a larger profit than $\frac{13}{2} \cdot 10 = 65$. It is the only optimal solution, $z(I) = 65$.

BKP: $\tilde{x} = (0, 1, 2)$, $z(I) = 39$. UKP: $\tilde{x} = (0, 0, 5)$, $z(I) = 65$. More copies can only increase the optimal value.

Exercise 2.4 — Two knapsacks (MKP)

We consider the multiple knapsack instance with $m = 2$ knapsacks of capacities $c_1 = 10$ and $c_2 = 6$, and $n = 5$ items with the following profits and weights:

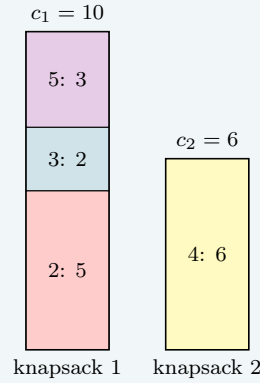
	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
p_j	4	8	9	10	13
w_j	2	5	2	6	3

Questions

1. Can all the items be inserted? How many candidate solutions are there, if each item goes into one of the knapsacks or into none?
2. Is the solution $S_1 = \{1, 2, 3\}$, $S_2 = \{4, 5\}$ feasible? What is its profit?
3. Every feasible solution of the MKP is also a feasible solution of a KP with capacity $c_1 + c_2 = 16$. Use this to compute a dual bound.
4. Find an optimal solution of the MKP.

Solution

1. The total weight is $18 > c_1 + c_2 = 16$: not all the items can be inserted. Every item has $m + 1 = 3$ choices, so there are $3^5 = 243$ candidate solutions; 100 of them are feasible.
2. S_1 weighs $2 + 5 + 2 = 9 \leq 10$ and S_2 weighs $6 + 3 = 9 > 6$: the solution is not feasible.
3. If S_1, S_2 is feasible for the MKP, the union $S_1 \cup S_2$ weighs at most $c_1 + c_2 = 16$, so its profit is at most the optimal value of the KP with capacity 16. The KP has the optimal solution $\{2, 3, 4, 5\}$, of weight 16 and profit 40: $UB = 40$.
4. The items $\{2, 3, 4, 5\}$ of the KP solution can be split between the two knapsacks: $S_1 = \{2, 3, 5\}$ weighs $10 \leq 10$ and $S_2 = \{4\}$ weighs $6 \leq 6$. Its profit 40 attains the dual bound, so it is optimal: $z(I) = 40$. The enumeration of the 100 feasible solutions shows that it is the only optimal one.



The optimal solution puts $\{2, 3, 5\}$ in knapsack 1 and $\{4\}$ in knapsack 2, with $z(I) = 40$, equal to the dual bound of the KP with capacity $c_1 + c_2$.

Exercise 2.5 — One item per class (MCKP)

We consider the multiple-choice knapsack instance with capacity $c = 16$ and $n = 7$ items, partitioned into the classes $\mathcal{J}_1 = \{1, 2\}$, $\mathcal{J}_2 = \{3, 4, 5\}$, $\mathcal{J}_3 = \{6, 7\}$, with the following profits and weights:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$
p_j	5	8	14	13	11	12	6
w_j	6	9	4	6	7	8	2

Questions

1. Check the feasibility condition $\sum_h \min_{j \in \mathcal{J}_h} w_j \leq c$ of the assumptions.
2. How many candidate solutions are there? Is the choice of the most profitable item of every class feasible?
3. Find the optimal solution by enumeration.

Solution

1. The lightest items of the classes weigh 6, 4, 2, and $6 + 4 + 2 = 12 \leq 16$: the instance has a feasible solution.
2. A candidate solution takes one item per class: $2 \cdot 3 \cdot 2 = 12$ candidates. The most profitable items are 2, 3, 6, of total weight $9 + 4 + 8 = 21 > 16$: this choice is not feasible.
3. The candidates are:

S	weight	profit	feasible	S	weight	profit	feasible
$\{1, 3, 6\}$	18	31	no	$\{2, 3, 6\}$	21	34	no
$\{1, 3, 7\}$	12	25	yes	$\{2, 3, 7\}$	15	28	yes
$\{1, 4, 6\}$	20	30	no	$\{2, 4, 6\}$	23	33	no
$\{1, 4, 7\}$	14	24	yes	$\{2, 4, 7\}$	17	27	no
$\{1, 5, 6\}$	21	28	no	$\{2, 5, 6\}$	24	31	no
$\{1, 5, 7\}$	15	22	yes	$\{2, 5, 7\}$	18	25	no

Only 4 candidates are feasible, and the best one is $\tilde{S} = \{2, 3, 7\}$, of weight 15, with $z(I) = 28$; it is the only optimal solution.

The optimal solution takes items 2, 3, 7, one per class, with $z(I) = 28$: the most profitable item of every class is not a feasible choice.

Exercise 2.6 — A greedy choice that is not optimal (KP)

A fund has a budget of $c = 14$ and $n = 5$ candidate projects, the items $\mathcal{I} = \{1, \dots, 5\}$ of a knapsack instance, with the following profits and costs (weights):

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
p_j	14	9	15	13	11
w_j	7	2	9	6	8

Questions

1. Write the instance as a combinatorial optimization problem: ground set, feasible solutions and objective function. How many candidate solutions are there?
2. List the feasible subsets, with weight and profit, and find the optimal solution by brute force.
3. Apply the greedy algorithm by efficiency: sort the items by non-increasing efficiency p_j/w_j (ties by the smallest index) and insert each item that fits. Is the greedy solution optimal?
4. Compute the dual bound of the continuous relaxation, in which the items can be taken fractionally, and round it. What are the relative gaps of the greedy solution with respect to this bound and with respect to $z(I)$?

Solution

1. The ground set is $\mathcal{U} = \mathcal{I} = \{1, 2, 3, 4, 5\}$. A subset $S \subseteq \mathcal{U}$ is feasible if $\sum_{j \in S} w_j \leq 14$, and the objective is $f(S) = \sum_{j \in S} p_j$, to be maximized. The candidate solutions

are the $2^5 = 32$ subsets.

2. The three lightest items weigh $2 + 6 + 7 = 15 > 14$, so a feasible subset has at most 2 items. The 12 feasible subsets are:

S	weight	profit	S	weight	profit
$\emptyset$	0	0	$\{1, 2\}$	9	23
$\{1\}$	7	14	$\{1, 4\}$	13	27
$\{2\}$	2	9	$\{2, 3\}$	11	24
$\{3\}$	9	15	$\{2, 4\}$	8	22
$\{4\}$	6	13	$\{2, 5\}$	10	20
$\{5\}$	8	11	$\{4, 5\}$	14	24

The optimal solution is $\tilde{S} = \{1, 4\}$, of weight 13, with $z(I) = 27$; it is the only subset with this profit.

3. The efficiencies are

$$\frac{p_1}{w_1} = \frac{14}{7} = 2, \quad \frac{p_2}{w_2} = \frac{9}{2}, \quad \frac{p_3}{w_3} = \frac{15}{9} = \frac{5}{3}, \quad \frac{p_4}{w_4} = \frac{13}{6}, \quad \frac{p_5}{w_5} = \frac{11}{8},$$

so the order is 2, 4, 1, 3, 5. The greedy algorithm examines the items in this order:

- item 2 fits (weight 2): residual capacity 12;
- item 4 fits (weight 6): residual capacity 6;
- item 1 does not fit (weight $7 > 6$);
- item 3 does not fit (weight $9 > 6$);
- item 5 does not fit (weight $8 > 6$);

It returns $\bar{S} = \{2, 4\}$, of profit 22: a primal bound $LB = 22$. It is not optimal, since $z(I) = 27$. The two efficient items fill 8 units of the budget, and the residual capacity 6 is too small for any other item; the optimal solution gives up item 2, the most efficient one, and uses 13 of the 14 units of the budget with items 1 and 4.

4. In the continuous relaxation the items are taken in the same order: items 2, 4 fill 8 units, and the fraction $\frac{6}{7}$ of item 1 fills the residual capacity 6:

$$UB = 9 + 13 + \frac{6}{7} \cdot 14 = 34.$$

The bound is an integer, no rounding is needed. With respect to the bound, the relative gap of the greedy solution is $\frac{34-22}{22} = \frac{6}{11} \approx 54.5\%$; with respect to the optimum it is $\frac{27-22}{22} = \frac{5}{22} \approx 22.7\%$. Without the enumeration, the bound alone cannot certify the optimum.

The greedy choice by efficiency gives profit 22, while the optimum, found by brute force over the 12 feasible subsets, is $z(I) = 27$ with $\tilde{S} = \{1, 4\}$: $22 \leq z(I) = 27 \leq 34$.

Packing problems

Exercise 3.1 — Checking a packing (BPP)

We consider the bin packing instance with capacity $c = 12$ and $n = 8$ items with weights

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$	$j = 8$
w_j	7	5	4	6	3	8	2	4

and the three proposed solutions

$$\mathcal{P}_a = \{\{1, 2\}, \{3, 6\}, \{4, 5, 7\}, \{8\}\},$$

$$\mathcal{P}_b = \{\{1, 4\}, \{2, 3, 5\}, \{6, 8\}, \{7\}\},$$

$$\mathcal{P}_c = \{\{1, 2\}, \{3, 6\}, \{4, 5\}, \{8\}\}.$$

Questions

1. Check the assumptions of the BPP.
2. For each proposed solution, compute the weight of every bin and decide whether it is feasible.
3. Compute the weight bound. Is one of the proposed solutions optimal?
4. Draw the feasible solution.

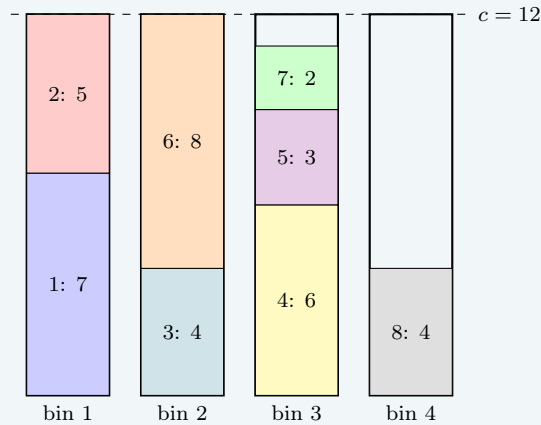
Solution

1. The total weight is $39 > 12$ and the heaviest item weighs $8 \leq 12$: both assumptions hold.
2.
 - $\mathcal{P}_a$: the bins weigh $7 + 5 = 12$, $4 + 8 = 12$, $6 + 3 + 2 = 11$, $4 = 4$. It is a partition and every bin weighs at most 12: feasible, with 4 bins.
 - $\mathcal{P}_b$: the bins weigh $7 + 6 = 13$, $5 + 4 + 3 = 12$, $8 + 4 = 12$, $2 = 2$. It is a partition, but the bin $\{1, 4\}$ weighs $13 > 12$. Not feasible.
 - $\mathcal{P}_c$: the bins weigh $7 + 5 = 12$, $4 + 8 = 12$, $6 + 3 = 9$, $4 = 4$. It is not a partition

of $\mathcal{I}$: item 7 is in no bin. Not feasible.

3. $LB = \lceil 39/12 \rceil = 4$. The feasible solution $\mathcal{P}_a$ uses 4 bins, so it attains the bound: it is optimal and $z(I) = 4$.

4. The four bins of $\mathcal{P}_a$, with the dashed line at the capacity:



Only $\mathcal{P}_a$ is feasible; it uses $4 = \lceil 39/12 \rceil$ bins and is therefore optimal.

Exercise 3.2 — When the weight bound is not tight (BPP)

We consider the bin packing instance with capacity $c = 10$ and $n = 6$ items with weights

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
w_j	7	6	6	5	3	3

Questions

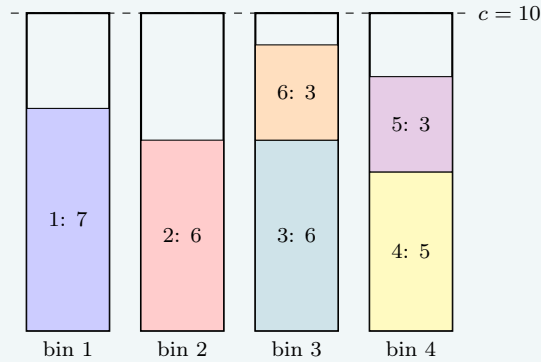
1. Compute the weight bound.
2. Show that no two of the items 1, 2, 3, 4 fit together in a bin. What lower bound on $z(I)$ follows?
3. Find an optimal solution. Is it unique?
4. What is the gap between the weight bound and $z(I)$?

Solution

1. $\sum_j w_j = 7 + 6 + 6 + 5 + 3 + 3 = 30$, hence $\lceil 30/10 \rceil = 3$.
2. The pairs give $w_1 + w_2 = 13$, $w_1 + w_3 = 13$, $w_1 + w_4 = 12$, $w_2 + w_3 = 12$, $w_2 + w_4 = 11$, $w_3 + w_4 = 11$, all larger than 10. Hence items 1, 2, 3, 4 are in four different bins in

every feasible solution, and $z(I) \geq 4$.

3. The solution $\{\{1\}, \{2\}, \{3, 6\}, \{4, 5\}\}$ has bins of weight 7, 6, 9, 8, all at most 10: it uses 4 bins and attains the bound of Question 2, so it is optimal, $z(I) = 4$.



It is not unique: the enumeration gives 12 optimal partitions, since items 5 and 6 can go with any of the four large items that has room for them.

4. The weight bound gives 3 and $z(I) = 4$: an absolute gap of 1. The weight bound does not see that the large items cannot share a bin.

$z(I) = 4$, while the weight bound gives only 3: the four items of weight greater than $c/2$ pairwise conflict and need four bins.

Exercise 3.3 — Counting the packings (BPP)

We consider the bin packing instance with capacity $c = 12$ and $n = 5$ items with weights

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
w_j	7	6	5	4	2

Questions

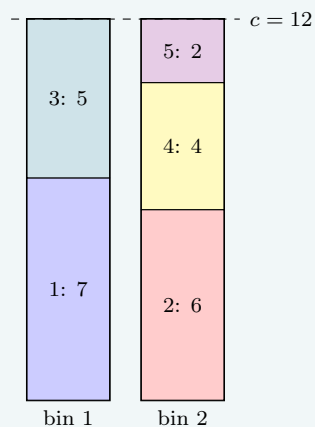
1. How many candidate solutions does the instance have?
2. Which pairs of items fit together in a bin? Which triples?
3. Count the feasible solutions, grouped by number of bins.
4. Compute the weight bound and list all the optimal solutions.

Solution

1. The candidates are the partitions of the 5 items: $B(5) = 52$.
2. The pairs that fit are $\{1, 3\}$ (12), $\{1, 4\}$ (11), $\{1, 5\}$ (9), $\{2, 3\}$ (11), $\{2, 4\}$ (10), $\{2, 5\}$ (8), $\{3, 4\}$ (9), $\{3, 5\}$ (7), $\{4, 5\}$ (6); the triples that fit are $\{2, 4, 5\}$ (12), $\{3, 4, 5\}$ (11). No four items fit together, since the four lightest weigh $17 > 12$.
3. A feasible partition uses only singletons and the pairs and triples above. The enumeration gives:

bins	feasible partitions
2	1
3	14
4	9
5	1
total	25

4. $LB = \lceil 24/12 \rceil = 2$, and there are partitions with 2 bins: $z(I) = 2$. The optimal solution is unique, $\{\{1, 3\}, \{2, 4, 5\}\}$: both bins are filled exactly.



Of the 52 partitions, 25 are feasible and only one is optimal, with $z(I) = 2 = \lceil 24/12 \rceil$.

Exercise 3.4 — Copies of the items (CSP)

We consider the cutting stock instance with capacity $c = 10$ and $n = 3$ items, with the following weights and required copies:

	$j = 1$	$j = 2$	$j = 3$
w_j	5	4	3
d_j	2	3	2

Questions

1. Write the extended set of items $\hat{\mathcal{J}}$ and check the assumptions.
2. Is the family $\{\{(1, 1), (2, 1)\}, \{(1, 2), (2, 2)\}, \{(3, 1), (3, 2)\}\}$ a feasible solution? How many bins does it use?
3. Compute the weight bound, with the weights of all the copies.
4. Find an optimal solution.

Solution

1. The extended set has $\sum_j d_j = 7$ copies:

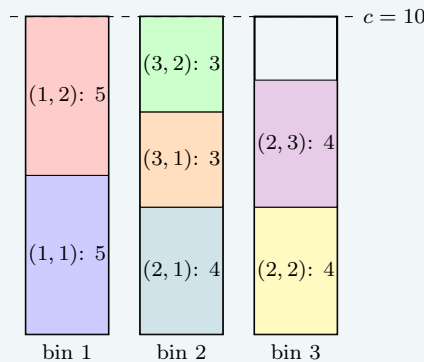
$$\hat{\mathcal{J}} = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2)\}.$$

The total weight is $\sum_j d_j w_j = 2 \cdot 5 + 3 \cdot 4 + 2 \cdot 3 = 28 > 10$ and every item fits in a bin, $w_j \leq 10$: both assumptions hold.

2. The bins weigh 9, 9, 6, all at most 10, but the copy (2,3) is in no bin: the family is not a partition of $\hat{\mathcal{J}}$ and it is not feasible.
3. $\text{LB} = \left\lceil \sum_j d_j w_j / c \right\rceil = \lceil 28/10 \rceil = 3$.
4. The two copies of item 1 fill a bin (10); one copy of item 2 and the two copies of item 3 fill another one ($4 + 3 + 3 = 10$); the last two copies of item 2 weigh 8. The solution

$$\mathcal{P} = \{\{(1, 1), (1, 2)\}, \{(2, 1), (3, 1), (3, 2)\}, \{(2, 2), (2, 3)\}\}$$

uses 3 bins and attains the bound: it is optimal, $z(I) = 3$.



With the weights of the 7 copies the weight bound is 3, and the packing found by pairing the copies attains it: $z(I) = 3$.

Exercise 3.5 — Filling every bin exactly (BPP)

We consider the bin packing instance with capacity $c = 10$ and $n = 8$ items with weights

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$	$j = 8$
w_j	5	5	4	4	4	3	3	2

Questions

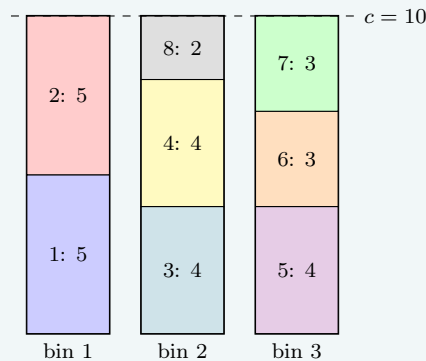
1. Compute the weight bound.
2. Show that a solution with 3 bins must fill every bin exactly.
3. List the ways to obtain a total weight of exactly 10 with the weights of the instance, and use them to find a solution with 3 bins.
4. If item 8 weighed 3 instead of 2, how many bins would be needed?

Solution

1. $\sum_j w_j = 30$, hence $\text{LB} = \lceil 30/10 \rceil = 3$.
2. 3 bins hold at most $3 \cdot 10 = 30$, and the items weigh exactly 30: if a bin weighed less than 10, the others could not hold the rest.
3. As multisets of weights, a total of exactly 10 is obtained as

$$\{5, 5\}, \quad \{5, 3, 2\}, \quad \{4, 4, 2\}, \quad \{4, 3, 3\}.$$

Each item of weight 5 must be in a full bin, and the only multisets with a 5 are $\{5, 5\}$, $\{5, 3, 2\}$. There is only one item of weight 2, so the two items of weight 5 cannot both be in a bin $\{5, 3, 2\}$: they share the bin $\{5, 5\}$. Then the remaining weights 4, 4, 4, 3, 3, 2 must form two full bins: $\{4, 4, 2\}$ and $\{4, 3, 3\}$. The solution is $\{\{1, 2\}, \{3, 4, 8\}, \{5, 6, 7\}\}$:



It attains the bound, so $z(I) = 3$.

4. The total weight would be 31, and $\lceil 31/10 \rceil = 4$: at least 4 bins. A solution with 4 bins is obtained from the one above by moving item 8 to a bin of its own, so $z(I) = 4$.

When $\sum_j w_j = \text{LB} \cdot c$, a solution with LB bins fills every bin exactly; here the bins $\{5, 5\}$, $\{4, 4, 2\}$, $\{4, 3, 3\}$ give $z(I) = 3$.

Exercise 3.6 — A greedy choice that is not optimal (BPP)

We consider the bin packing instance with capacity $c = 10$ and $n = 6$ items, with weights

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
w_j	5	4	5	6	8	2

Questions

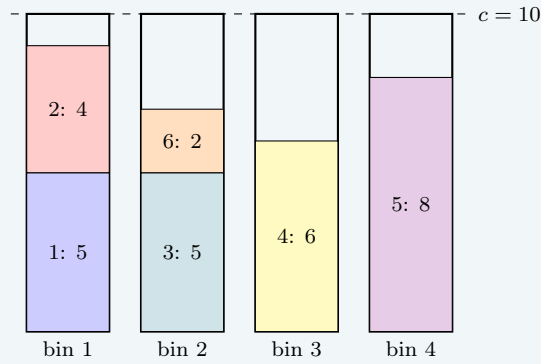
1. Apply First Fit: take the items in the order $1, 2, \dots, n$ and put each one into the first open bin where it fits, opening a new bin when it fits nowhere. How many bins does it use?
2. Compute the weight bound $\lceil \sum_j w_j / c \rceil$.
3. Find a solution that attains the bound. What can you conclude on the solution of First Fit?
4. Compute the absolute and the relative gap between the First Fit solution and the dual bound.

Solution

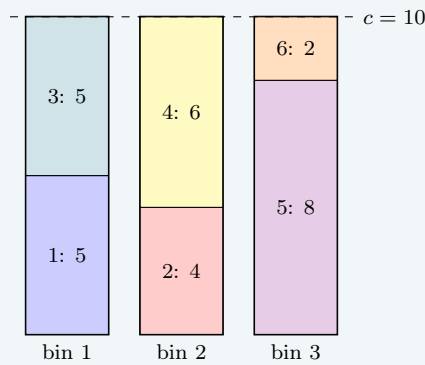
1. The loads of the open bins after each item are:

item j	w_j	placed in	loads
1	5	new bin 1	(5)
2	4	bin 1	(9)
3	5	new bin 2	(9, 5)
4	6	new bin 3	(9, 5, 6)
5	8	new bin 4	(9, 5, 6, 8)
6	2	bin 2	(9, 7, 6, 8)

First Fit returns $\{\{1, 2\}, \{3, 6\}, \{4\}, \{5\}\}$, with 4 bins: a primal bound $\text{UB} = 4$.



2. $\sum_j w_j = 5 + 4 + 5 + 6 + 8 + 2 = 30$, hence $\text{LB} = \lceil 30/10 \rceil = 3$.
3. The partition $\{\{1, 3\}, \{2, 4\}, \{5, 6\}\}$ has bins of weight 10, 10, 10, all at most 10: it is feasible and uses 3 bins.



It attains the dual bound, so it is optimal and $z(I) = 3$; the solution of First Fit is not optimal. First Fit puts item 2 (weight 4) with item 1 (load 9) and wastes the residual capacity 1; the optimal packing puts item 1 with item 3 and item 2 with item 4, two bins filled exactly.

4. The absolute gap is $4 - 3 = 1$ and the relative gap is $\frac{1}{3} \approx 33.3\%$.

First Fit, a greedy choice, uses 4 bins, while the weight bound gives 3 and the partition $\{\{1, 3\}, \{2, 4\}, \{5, 6\}\}$ attains it: $z(I) = 3$.

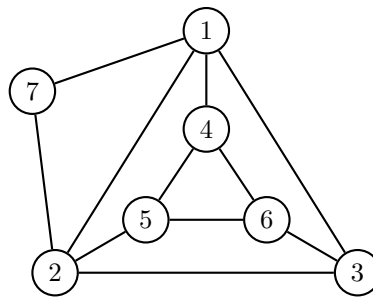
Coloring problems

Exercise 4.1 — Checking a coloring (VCP)

We consider the graph $G = (V, E)$ with $V = \{1, \dots, 7\}$ and the edges

$$E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 7\}, \{2, 3\}, \{2, 5\}, \{2, 7\}, \{3, 6\}, \{4, 5\}, \{4, 6\}, \{5, 6\}\},$$

drawn below:



and the three proposed solutions

$$\mathcal{Q}_a = \{\{1, 6\}, \{2, 4\}, \{3, 5, 7\}\},$$

$$\mathcal{Q}_b = \{\{1, 5, 6\}, \{2, 4\}, \{3, 7\}\},$$

$$\mathcal{Q}_c = \{\{1, 6\}, \{2, 4\}, \{3, 5\}\}.$$

Questions

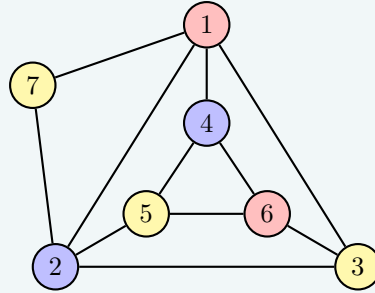
1. For each proposed solution, decide whether it is a feasible solution of the VCP.
2. Find a clique with 3 vertices. What dual bound does it give?
3. Is the feasible solution of Question 1 optimal? Draw it.

Solution

1.
 - $\mathcal{Q}_a$ is a partition of V and no edge joins two vertices of the same class: feasible, with 3 colors.
 - $\mathcal{Q}_b$ is a partition of V , but the class $\{1, 5, 6\}$ contains the edge $\{5, 6\}$: it is not a

stable set. Not feasible.

- $\mathcal{Q}_c$ is not a partition of V : vertex 7 has no color. Not feasible.
2. The vertices $\{1, 2, 3\}$ are pairwise adjacent, so they form a clique; G has 3 cliques with 3 vertices, $\{1, 2, 3\}$, $\{1, 2, 7\}$, $\{4, 5, 6\}$. By the clique bound, $\chi(G) \geq \omega(G) \geq 3$.
 3. $\mathcal{Q}_a$ uses 3 colors and attains the clique bound: it is optimal and $\chi(G) = 3$.



Only $\mathcal{Q}_a$ is feasible; it uses 3 colors, as many as the vertices of a clique: $\chi(G) = 3$.

Exercise 4.2 — Neighbourhoods, cliques and stable sets (VCP)

We consider the graph $G = (V, E)$ with $V = \{1, \dots, 6\}$ and the edges

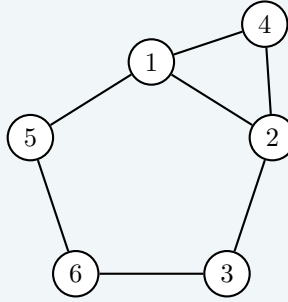
$$E = \{\{1, 2\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{3, 6\}, \{5, 6\}\}.$$

Questions

1. Draw the graph and compute the neighbourhood $N(v)$ and the degree of every vertex.
2. Check the two assumptions of the VCP.
3. Write the edges of the complement graph $\overline{G}$.
4. Find $\omega(G)$ and $\alpha(G)$, with a maximum clique and a maximum stable set, and compute the clique bound and the stable set bound.
5. Find an optimal coloring.

Solution

1. The graph, and the neighbourhoods and degrees of its vertices:



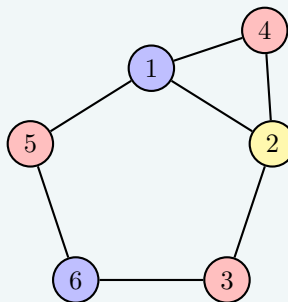
v	1	2	3	4	5	6
$N(v)$	$\{2, 4, 5\}$	$\{1, 3, 4\}$	$\{2, 6\}$	$\{1, 2\}$	$\{1, 6\}$	$\{3, 5\}$
$\deg(v)$	3	3	2	2	2	2

The degrees sum up to $14 = 2|E|$, since every edge is counted at both its endpoints.

- Every vertex has degree at least 1, so there are no isolated vertices. No neighbourhood is contained in another one: for every pair $u \neq v$ the table shows a vertex of $N(u)$ that is not in $N(v)$. For example, $N(4) = \{1, 2\}$ is contained in no other neighbourhood, since no other vertex is adjacent to both 1 and 2. Both assumptions hold.
- $\overline{G}$ has the 8 pairs of non-adjacent vertices of G :

$$\overline{E} = \{\{1, 3\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{4, 6\}\}.$$

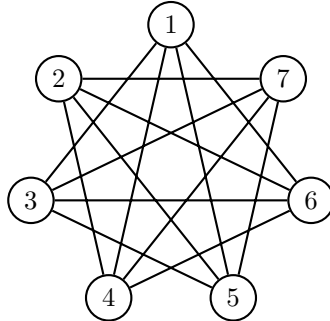
- The vertices $\{1, 2, 4\}$ form a triangle, so $\omega(G) \geq 3$. A clique with 4 vertices would need 4 vertices of degree at least 3, and only vertices 1 and 2 have degree 3: hence $\omega(G) = 3$. The set $\{3, 4, 5\}$ is stable; a stable set with 4 vertices would be a clique with 4 vertices of $\overline{G}$, and the enumeration shows there is none: $\alpha(G) = 3$. The clique bound gives 3, the stable set bound $\lceil 6/3 \rceil = 2$.
- The coloring $\{\{3, 4, 5\}, \{1, 6\}, \{2\}\}$ has stable classes and uses 3 colors, as many as the clique bound: it is optimal and $\chi(G) = 3$.



$\omega(G) = 3$ with the clique $\{1, 2, 4\}$, $\alpha(G) = 3$ with the stable set $\{3, 4, 5\}$; the clique bound is tight and $\chi(G) = 3$, while the stable set bound gives only 2.

Exercise 4.3 — The stable set bound on a complement graph (VCP)

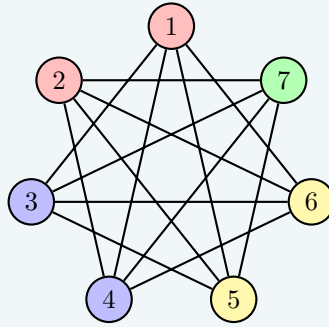
The graph $G = (V, E)$ has the vertices $V = \{1, \dots, 7\}$, and two distinct vertices are adjacent if and only if they are *not* consecutive on the cycle $1, 2, \dots, 7, 1$. In other words, G is the complement of the cycle C_7 with 7 vertices:

**Questions**

1. How many edges does G have? What is the degree of every vertex?
2. Compute $\alpha(G)$, using the complement graph.
3. Compute $\omega(G)$ and the clique bound.
4. Compute the stable set bound and find an optimal coloring.

Solution

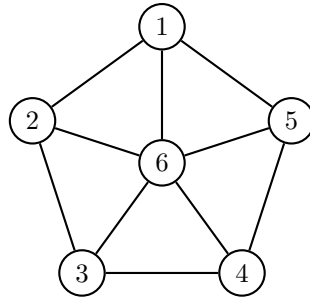
1. C_7 has 7 edges out of the $\binom{7}{2} = 21$ pairs, so G has $21 - 7 = 14$ edges. Every vertex is adjacent to all the vertices except itself and its two neighbours on the cycle: degree $6 - 2 = 4$.
2. A stable set of G is a clique of $\bar{G} = C_7$. The largest cliques of a cycle with 7 vertices are its edges, since it has no triangle: $\alpha(G) = 2$, for example the stable set $\{1, 2\}$.
3. A clique of G is a stable set of C_7 : a set of vertices with no two consecutive on the cycle. $\{1, 3, 5\}$ is such a set; with 4 vertices out of 7, two of them would be consecutive. Hence $\omega(G) = 3$ and the clique bound gives $\chi(G) \geq 3$.
4. $\lceil n/\alpha(G) \rceil = \lceil 7/2 \rceil = 4$: every color class has at most 2 vertices, and 7 vertices need at least 4 classes. The coloring $\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7\}\}$ uses pairs of consecutive vertices, which are not adjacent in G : it is feasible, uses 4 colors and attains the bound, so $\chi(G) = 4$.



On the complement of C_7 the clique bound gives 3, the stable set bound gives $\lceil 7/2 \rceil = 4$ and a coloring with 4 colors closes the gap: $\chi(G) = 4$.

Exercise 4.4 — When both bounds are not tight (VCP)

The wheel W_5 is the graph with the vertices $V = \{1, \dots, 6\}$: the vertices $1, \dots, 5$ form a cycle and vertex 6, the hub, is adjacent to all of them.



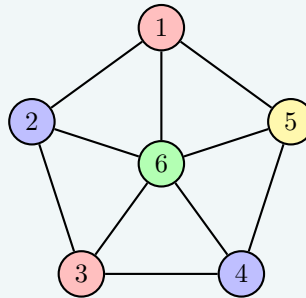
Questions

1. Compute $\omega(W_5)$, $\alpha(W_5)$ and the two dual bounds of the chapter.
2. Show that the cycle $1, 2, 3, 4, 5$ cannot be colored with 2 colors.
3. Show that $\chi(W_5) = 4$ and draw an optimal coloring.

Solution

1. Every triangle uses the hub and an edge of the cycle, for example $\{1, 2, 6\}$; 4 pairwise adjacent vertices would need 3 pairwise adjacent vertices on the cycle, which has no triangle. Hence $\omega(W_5) = 3$. A stable set contains at most 2 vertices of the cycle with 5 vertices, and the hub is adjacent to all of them: $\alpha(W_5) = 2$, for example $\{1, 3\}$. The clique bound gives 3 and the stable set bound $\lceil 6/2 \rceil = 3$.
2. With 2 colors the colors must alternate along the cycle: 1 gets color A, 2 color B, 3 color A, 4 color B, 5 color A. Then 5 and 1, which are adjacent, have the same color. A cycle with an odd number of vertices needs 3 colors.

3. The cycle needs at least 3 colors, and the hub is adjacent to all the vertices of the cycle, so it needs a color of its own: $\chi(W_5) \geq 4$. The coloring $\{\{1, 3\}, \{2, 4\}, \{5\}, \{6\}\}$ is feasible and uses 4 colors, hence $\chi(W_5) = 4$.



Both bounds of the chapter give 3: the gap is closed by the argument on the odd cycle.

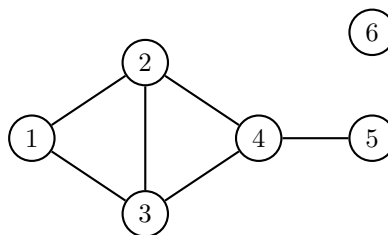
$\omega(W_5) = 3$ and $\lceil 6/\alpha(W_5) \rceil = 3$, but $\chi(W_5) = 4$: the odd cycle needs 3 colors and the hub a fourth one. Neither bound is tight.

Exercise 4.5 — The assumptions and the reduction of the graph (VCP)

We consider the graph $G = (V, E)$ with $V = \{1, \dots, 6\}$ and the edges

$$E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\},$$

drawn below:

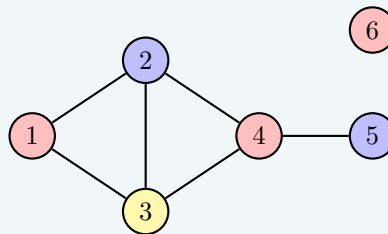


Questions

1. Compute the neighbourhoods of the vertices. Which assumptions of the VCP are violated, and by which vertices?
2. Remove the vertices allowed by the assumptions, one at a time, and compute the chromatic number of the graph that remains.
3. Extend a coloring of the reduced graph to an optimal coloring of G .
4. How many candidate solutions does G have? How many of them are feasible, by number of colors?

Solution

- The neighbourhoods are $N(1) = \{2, 3\}$, $N(2) = \{1, 3, 4\}$, $N(3) = \{1, 2, 4\}$, $N(4) = \{2, 3, 5\}$, $N(5) = \{4\}$, $N(6) = \emptyset$.
 - Vertex 6 is isolated: the first assumption is violated.
 - $N(1) = \{2, 3\} \subseteq N(4) = \{2, 3, 5\}$ and $N(5) = \{4\} \subseteq N(2) = \{1, 3, 4\}$: the second assumption is violated by the pairs $(1, 4)$ and $(5, 2)$.
- We remove the isolated vertex 6, the vertex 1 (dominated by 4) and the vertex 5 (dominated by 2). The vertices 2, 3, 4 remain, with the edges $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$: a triangle, whose chromatic number is 3. Removing a vertex does not change χ , so $\chi(G) = 3$.
- We color the triangle with $\{2\}, \{3\}, \{4\}$ and put each removed vertex back: 1 takes the color of 4, 5 the color of 2, and 6 any color, for example that of 4. The coloring is $\{\{1, 4, 6\}, \{2, 5\}, \{3\}\}$:



- The candidates are the partitions of the 6 vertices: $B(6) = 203$. The feasible ones, by enumeration, are:

colors	feasible partitions
3	6
4	18
5	9
6	1
total	34

There are 6 optimal colorings with 3 colors.

The isolated vertex and the two dominated vertices can be removed: the triangle $\{2, 3, 4\}$ remains, and $\chi(G) = 3$.

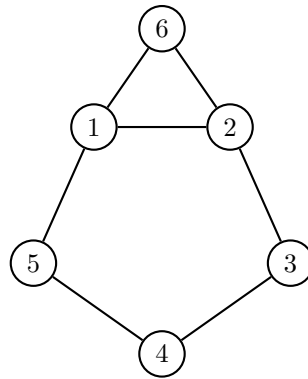
Clique problems

Exercise 5.1 — Cliques, stable sets and the complement graph (MCP)

We consider the graph $G = (V, E)$ with $V = \{1, \dots, 6\}$ and the edges

$$E = \{\{1, 2\}, \{1, 5\}, \{1, 6\}, \{2, 3\}, \{2, 6\}, \{3, 4\}, \{4, 5\}\}.$$

drawn below:



and the four subsets of vertices

$$A = \{1, 2, 6\}, \quad B = \{1, 3\}, \quad C = \{2, 4, 5\}, \quad D = \{3, 5, 6\}.$$

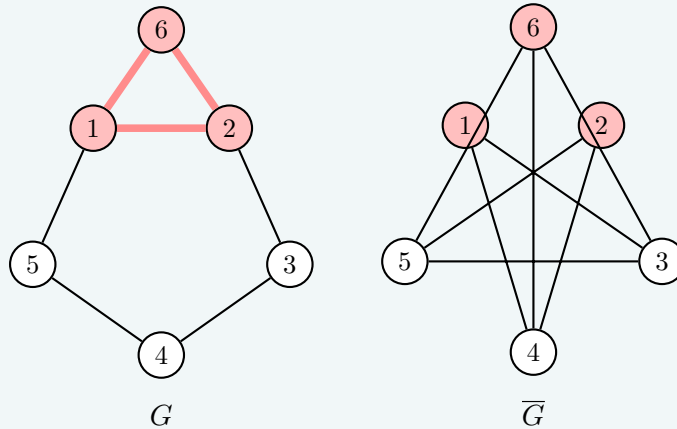
Questions

1. For each subset, decide whether it is a clique of G , a stable set of G , or neither.
2. List all the maximal cliques of G (cliques to which no vertex can be added).
3. Write the edges of the complement graph $\overline{G}$ and draw it next to G .
4. Find $\omega(G)$ and $\alpha(G)$, and check that $\omega(G) = \alpha(\overline{G})$ and $\alpha(G) = \omega(\overline{G})$.

Solution

1.
 - A : all the pairs are edges, so A is a clique (a triangle).
 - B : no pair is an edge, so B is a stable set.
 - C : the pair $\{4, 5\}$ is an edge and the pair $\{2, 4\}$ is not, so C is neither a clique nor a stable set.
 - D : no pair is an edge, so D is a stable set.
2. The maximal cliques are $\{1, 2, 6\}$ and the edges $\{1, 5\}$, $\{2, 3\}$, $\{3, 4\}$, $\{4, 5\}$: every other edge lies in the triangle $\{1, 2, 6\}$, and no vertex is adjacent to both endpoints of the four edges listed.
3. $\overline{G}$ has $\binom{6}{2} - 7 = 8$ edges:

$$\overline{E} = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{3, 6\}, \{4, 6\}, \{5, 6\}\}.$$



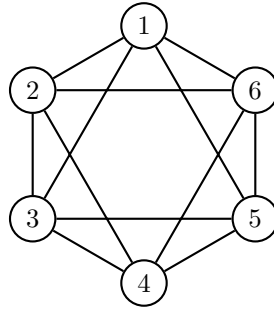
The red vertices are a clique of G and a stable set of $\overline{G}$.

4. $\omega(G) = 3$: the only triangle is $\{1, 2, 6\}$, and a clique with 4 vertices would contain 4 triangles. $\alpha(G) = 3$ with the unique maximum stable set $D = \{3, 5, 6\}$: a stable set takes at most 2 vertices of the cycle $1, 2, 3, 4, 5$, and with vertex 6 it cannot use 1 or 2. In $\overline{G}$ the set $\{1, 2, 6\}$ is a maximum stable set and D is a maximum clique (a triangle of $\overline{G}$): $\alpha(\overline{G}) = 3 = \omega(G)$ and $\omega(\overline{G}) = 3 = \alpha(G)$.

A is the unique maximum clique, $\omega(G) = 3$; D is the unique maximum stable set, $\alpha(G) = 3$. In $\overline{G}$ the two roles exchange.

Exercise 5.2 — Counting the candidate solutions (MCP)

We consider the graph $G = (V, E)$ with $V = \{1, \dots, 6\}$ in which every vertex is adjacent to all the others except the opposite one on the circle: the non-adjacent pairs are $\{1, 4\}$, $\{2, 5\}$ and $\{3, 6\}$.

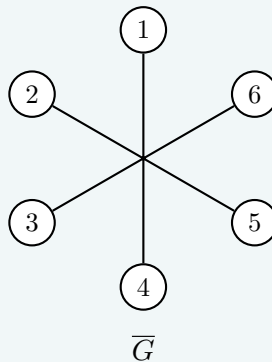


Questions

1. How many edges does G have? Draw the complement graph $\overline{G}$.
2. How many candidate solutions does the MCP have on G ?
3. Count the cliques of G with 0, 1, 2, 3 vertices, reasoning on the three pairs $\{1, 4\}$, $\{2, 5\}$ and $\{3, 6\}$. Is there a clique with 4 vertices?
4. Find $\omega(G)$ and the number of maximum cliques.

Solution

1. G has $\binom{6}{2} - 3 = 12$ edges. $\overline{G}$ has only the 3 edges $\{1, 4\}$, $\{2, 5\}$, $\{3, 6\}$: three disjoint edges.



2. The candidate solutions are the subsets of V : $2^6 = 64$.
3. A subset is a clique if and only if it contains at most one vertex of each pair. For each pair there are 3 choices (no vertex, the first one, the second one), so there are $3^3 = 27$ cliques:

number of vertices	0	1	2	3	total
cliques	1	6	12	8	27

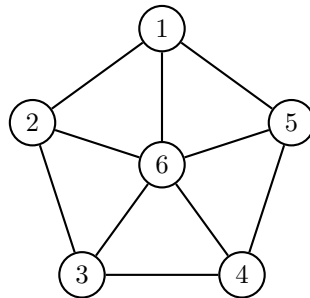
With 2 vertices: $\binom{3}{2} = 3$ pairs of pairs, times $2 \cdot 2$ choices, gives 12; with 3 vertices: $2^3 = 8$. A clique with 4 vertices would take two vertices of the same pair: there is none.

4. $\omega(G) = 3$, and there are 8 maximum cliques, for example $\{1, 2, 3\}$ and $\{4, 5, 6\}$. Only 27 of the 64 candidate solutions are feasible.

27 cliques out of 64 subsets; $\omega(G) = 3$ with 8 maximum cliques, one vertex from each non-adjacent pair.

Exercise 5.3 — A clique by hand and the simple bounds (MCP)

The wheel W_5 is the graph with the vertices $V = \{1, \dots, 6\}$: the vertices $1, \dots, 5$ form a cycle and vertex 6, the hub, is adjacent to all of them.



Questions

1. Build a clique by hand: start from a vertex of maximum degree and add, each time, the vertex of smallest index adjacent to all the vertices already chosen. Which lower bound do you get?
2. Compute the degree bound and the edge bound.
3. Compute the degree count bound $k^{\max}$.
4. Find a coloring of W_5 with 4 colors. Which upper bound does it give?
5. Find $\omega(W_5)$. Which bound is tight?

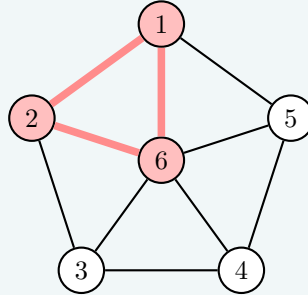
Solution

1. Vertex 6 has degree 5, the maximum. The vertices adjacent to 6 are $1, \dots, 5$: we add 1. The vertices adjacent to both 6 and 1 are 2 and 5: we add 2. No vertex is adjacent to 6, 1 and 2, so we stop with $K = \{1, 2, 6\}$ and $\text{LB} = 3$.
2. The degrees are:

v	1	2	3	4	5	6
$\deg(v)$	3	3	3	3	3	5

The degree bound gives $\Delta + 1 = 6$. The graph has $m = 10$ edges; since $5 \cdot 4 / 2 = 10 \leq 10 < 15 = 6 \cdot 5 / 2$, the edge bound gives 5.

3. For $k = 4$ we need 4 vertices of degree at least 3: all the 6 vertices qualify. For $k = 5$ we need 5 vertices of degree at least 4: only vertex 6 qualifies. Hence $k^{\max} = 4$.
4. The classes $\{\{1, 3\}, \{2, 4\}, \{5\}, \{6\}\}$ are stable sets: 1, 3 and 2, 4 are not consecutive on the cycle. The coloring uses 4 colors, so $\omega(W_5) \leq 4$.



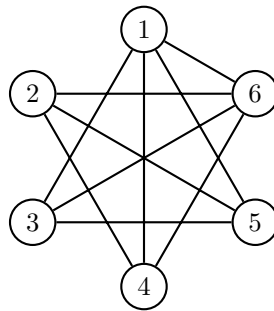
5. A clique with 4 vertices would contain at least 3 vertices of the cycle, pairwise adjacent, and the cycle with 5 vertices has no triangle. Hence $\omega(W_5) = 3$, attained by the clique of Question 1 (drawn in red above). None of the upper bounds is tight: the best ones give 4, and the gap $(4 - 3)/3$ is closed only by the argument on the cycle.

LB = 3 from the clique $\{1, 2, 6\}$; upper bounds 6 (degree), 5 (edges), 4 (degree count) and 4 (coloring); $\omega(W_5) = 3$.

Exercise 5.4 — Working on the complement graph (MCP)

We consider the graph $G = (V, E)$ with $V = \{1, \dots, 6\}$ and the 10 edges

$$E = \{\{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 6\}\}.$$



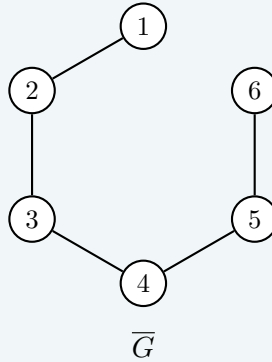
Questions

1. Write the edges of $\overline{G}$ and draw it. What kind of graph is it?
2. Compute $\alpha(\overline{G})$ and list all the maximum stable sets of $\overline{G}$.
3. Deduce $\omega(G)$ and the maximum cliques of G .

4. Compute $\alpha(G)$.

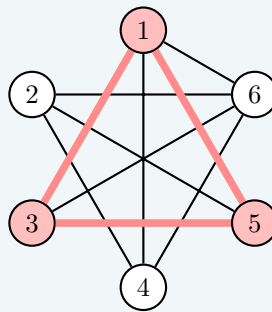
Solution

1. $\overline{G}$ has $\binom{6}{2} - 10 = 5$ edges, $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}$: it is the path 1, 2, 3, 4, 5, 6.



On a dense graph it is easier to work on the complement.

2. A stable set of the path contains no two consecutive vertices. With 4 vertices out of 6, two of them would be consecutive, so $\alpha(\overline{G}) \leq 3$. The stable sets with 3 vertices are $\{1, 3, 5\}, \{1, 3, 6\}, \{1, 4, 6\}, \{2, 4, 6\}$: $\alpha(\overline{G}) = 3$, with 4 maximum stable sets.
3. By the proposition of the chapter, the cliques of G are the stable sets of $\overline{G}$: $\omega(G) = \alpha(\overline{G}) = 3$, and the maximum cliques of G are the four sets above. The optimal solution is not unique.

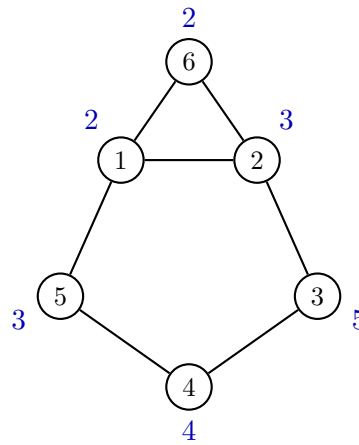


4. $\alpha(G) = \omega(\overline{G})$: the path has no triangle and has edges, so $\omega(\overline{G}) = 2$ and $\alpha(G) = 2$. For example, $\{1, 2\}$ is a maximum stable set of G .

$\overline{G}$ is a path: $\omega(G) = \alpha(\overline{G}) = 3$ with 4 maximum cliques, and $\alpha(G) = \omega(\overline{G}) = 2$.

Exercise 5.5 — Vertex-weighted maximum clique (MCP)

We consider the graph $G = (V, E)$ of the figure, with the vertex weights in blue:



$$w = (2, 3, 5, 4, 3, 2).$$

Questions

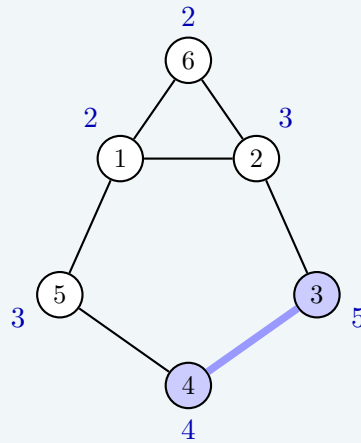
1. List the maximal cliques of G with their total weight $w(K)$.
2. Why is it enough to look at the maximal cliques? Find the vertex-weighted maximum clique.
3. Compare it with the optimal solution of the MCP on the same graph.

Solution

1. The maximal cliques are the triangle $\{1, 2, 6\}$ and four edges of the cycle:

maximal clique K	$w(K)$
$\{1, 2, 6\}$	$2 + 3 + 2 = 7$
$\{1, 5\}$	$2 + 3 = 5$
$\{2, 3\}$	$3 + 5 = 8$
$\{3, 4\}$	$5 + 4 = 9$
$\{4, 5\}$	$4 + 3 = 7$

2. All the weights are positive, and every clique is contained in a maximal clique: adding vertices to a clique increases its weight. The heaviest maximal clique is $\{3, 4\}$, with weight 9: it is the unique optimal solution (Gurobi on the model with weights confirms it).



3. The MCP chooses the triangle $\{1, 2, 6\}$, with 3 vertices but weight 7. The vertex-weighted problem prefers the two heavy vertices 3 and 4: the number of vertices does not matter, only their weight.

The vertex-weighted maximum clique is $\{3, 4\}$, of weight 9; the maximum clique $\{1, 2, 6\}$ weighs 7.

Routing problems

Exercise 6.1 — Tours and their length (TSP)

We consider the TSP instance with 6 cities, $\mathcal{V} = \{1, \dots, 6\}$, and the symmetric distances

d_{ij}	1	2	3	4	5	6
1	—	4	7	6	3	5
2	4	—	3	5	6	8
3	7	3	—	4	7	6
4	6	5	4	—	3	4
5	3	6	7	3	—	5
6	5	8	6	4	5	—

and the permutations

$$\sigma^1 = (1, 2, 3, 4, 5, 6), \quad \sigma^2 = (4, 5, 6, 1, 2, 3), \quad \sigma^3 = (1, 6, 5, 4, 3, 2), \quad \sigma^4 = (1, 5, 4, 3, 2, 6).$$

Questions

1. Compute the length of each permutation, read as a tour.
2. Which permutations describe the same tour? Explain.
3. How many permutations of the 6 cities are there, and how many distinct tours?
4. The two cycles $(1, 2, 3)$ and $(4, 5, 6)$ visit every city exactly once. Do they form a tour? Compare their total length with the length of σ^4 .

Solution

1. Including the return edge $d_{\sigma_n \sigma_1}$:

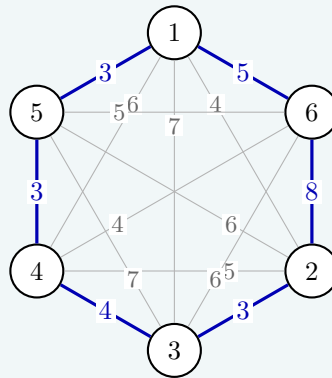
$$L(\sigma^1) = 4 + 3 + 4 + 3 + 5 + 5 = 24$$

$$L(\sigma^2) = 3 + 5 + 5 + 4 + 3 + 4 = 24$$

$$L(\sigma^3) = 5 + 5 + 3 + 4 + 3 + 4 = 24$$

$$L(\sigma^4) = 3 + 3 + 4 + 3 + 8 + 5 = 26.$$

2. σ^2 starts from city 4 and visits the cities in the same cyclic order as σ^1 ; σ^3 is σ^1 travelled backwards. The three permutations use the same 6 edges and describe the same tour, of length 24. The permutation σ^4 uses the edge $\{1, 5\}$, which is not in σ^1 : it is a different tour.
3. There are $6! = 720$ permutations; each tour is described by 6 starting cities times 2 directions, so there are $\frac{720}{12} = \frac{(6-1)!}{2} = 60$ tours.
4. No: a tour is a single cycle through all the cities, and the two cycles are disjoint. Their total length is $(4 + 3 + 7) + (3 + 5 + 4) = 26$, not smaller than $L(\sigma^4) = 26$, but it is not the length of a feasible solution.



$\sigma^1, \sigma^2, \sigma^3$ are the same tour, of length 24; σ^4 is a different tour, of length 26. Two disjoint cycles are not a tour.

Exercise 6.2 — Enumeration of the tours (TSP)

We consider the TSP instance with 5 cities and the symmetric distances

d_{ij}	1	2	3	4	5
1	—	12	3	5	3
2	12	—	9	7	10
3	3	9	—	4	5
4	5	7	4	—	4
5	3	10	5	4	—

Questions

1. Build a tour with the nearest neighbour rule from city 1 (ties broken by the smallest index), and compute its length.
2. Enumerate all the tours and find an optimal one. Is it unique?
3. Draw the optimal tour on the complete graph of the instance.

Solution

1.
 - from 1 the closest unvisited cities are 3, 5, at distance 3: we go to 3;
 - from 3 the closest unvisited city is 4, at distance 4;
 - from 4 the closest unvisited city is 5, at distance 4;
 - from 5 the closest unvisited city is 2, at distance 10;
 - we return from 2 to 1, at distance 12.

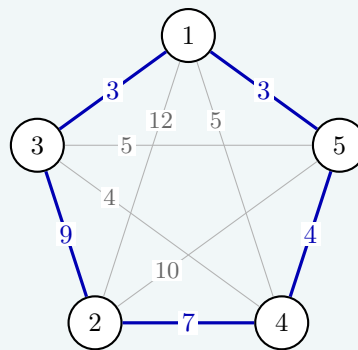
The tour is $(1, 3, 4, 5, 2)$, of length $3 + 4 + 4 + 10 + 12 = 33$.

2. There are $(5 - 1)!/2 = 12$ tours, all written from city 1:

tour	L	tour	L	tour	L
$(1, 3, 2, 4, 5)$	26	$(1, 2, 4, 3, 5)$	31	$(1, 2, 3, 4, 5)$	32
$(1, 3, 4, 2, 5)$	27	$(1, 2, 4, 5, 3)$	31	$(1, 2, 5, 4, 3)$	33
$(1, 4, 2, 3, 5)$	29	$(1, 3, 2, 5, 4)$	31	$(1, 2, 3, 5, 4)$	35
$(1, 3, 5, 2, 4)$	30	$(1, 4, 3, 2, 5)$	31	$(1, 2, 5, 3, 4)$	36

The tour $\tilde{\sigma} = (1, 3, 2, 4, 5)$ is the only optimal one, with $z(I) = 26$; the nearest neighbour tour is 7 longer.

3. The optimal tour on the complete graph of the instance:



The bold edges cost $3 + 9 + 7 + 4 + 3 = 26$. The tour avoids the long edges at city 2, which is far from all the other cities, and enters and leaves it through its two shortest edges, 7 and 9.

Nearest neighbour from city 1 gives 33; the enumeration of the 12 tours gives the unique optimal tour $(1, 3, 2, 4, 5)$, with $z(I) = 26$.

Exercise 6.3 — Nearest neighbour from every city and a lower bound (TSP)

We consider again the TSP instance with 5 cities and the symmetric distances

d_{ij}	1	2	3	4	5
1	—	12	3	5	3
2	12	—	9	7	10
3	3	9	—	4	5
4	5	7	4	—	4
5	3	10	5	4	—

whose optimal tour has length $z(I) = 26$.

Questions

1. Apply the nearest neighbour rule from every starting city (ties broken by the smallest index). Which starting city gives the best tour?
2. Compute the dual bound $\lceil \frac{1}{2} \sum_i (a_i + b_i) \rceil$ of the two shortest edges.
3. Compute the absolute and relative gaps between the best nearest neighbour tour and the dual bound. Could the bound alone certify that this tour is optimal?

Solution

1. The tours built from the five starting cities are:

start	tour	length
1	(1, 3, 4, 5, 2)	$3 + 4 + 4 + 10 + 12 = 33$
2	(2, 4, 3, 1, 5)	$7 + 4 + 3 + 3 + 10 = 27$
3	(3, 1, 5, 4, 2)	$3 + 3 + 4 + 7 + 9 = 26$
4	(4, 3, 1, 5, 2)	$4 + 3 + 3 + 10 + 7 = 27$
5	(5, 1, 3, 4, 2)	$3 + 3 + 4 + 7 + 10 = 27$

The best tour starts from city 3: written from city 1 it is (1, 3, 2, 4, 5), of length 26, which is the optimal tour. The starting city matters: from city 1 the rule leaves the long edges for the end.

2. The two smallest distances of every city are:

city i	1	2	3	4	5	sum
a_i	3	7	3	4	3	
b_i	3	9	4	4	4	
$a_i + b_i$	6	16	7	8	7	44

Hence $LB = 44/2 = 22$.

3. The absolute gap is $26 - 22 = 4$ and the relative gap is $\frac{4}{22} = \frac{2}{11} \approx 18.2\%$. The bound alone cannot certify the optimum: only the enumeration of the 12 tours shows that no tour is shorter than 26.

Starting from city 3 the nearest neighbour rule finds the optimal tour, of length 26; the two shortest edges give $LB = 22$, and the gap 4 stays open without enumeration.

Exercise 6.4 — The triangle inequality (metric TSP)

We consider the TSP instance with 4 cities and the symmetric distances

d_{ij}	1	2	3	4
1	—	4	10	3
2	4	—	5	2
3	10	5	—	6
4	3	2	6	—

Questions

1. How many triangle inequalities $d_{ij} \leq d_{ik} + d_{kj}$ must be checked? Which ones are violated?
2. What is the largest value of d_{13} that makes the instance metric, all the other distances being unchanged?
3. Compute the lengths of the tours for the original value of d_{13} and for the value of Question 2. Does the optimal tour change?

Solution

1. One inequality for every ordered triple of distinct cities: $4 \cdot 3 \cdot 2 = 24$. The violated ones are those with $d_{13} = d_{31} = 10$ on the left:

$$d_{13} = 10 > d_{12} + d_{23} = 4 + 5 = 9, \quad d_{13} = 10 > d_{14} + d_{43} = 3 + 6 = 9,$$

and the same two with d_{31} . All the other 20 inequalities hold. Going from 1 to 3 directly is longer than going through city 2 or city 4.

2. d_{13} must satisfy $d_{13} \leq d_{12} + d_{23} = 9$ and $d_{13} \leq d_{14} + d_{43} = 9$; when d_{13} is on the right-hand side, decreasing it from 10 cannot violate an inequality that already holds. The largest value is $d_{13} = 9$.
3. The three tours are:

tour	$d_{13} = 10$	$d_{13} = 9$
(1, 2, 3, 4)	$4 + 5 + 6 + 3 = 18$	$4 + 5 + 6 + 3 = 18$
(1, 2, 4, 3)	$4 + 2 + 6 + 10 = 22$	$4 + 2 + 6 + 9 = 21$
(1, 3, 2, 4)	$10 + 5 + 2 + 3 = 20$	$9 + 5 + 2 + 3 = 19$

In both cases the optimal tour is (1, 2, 3, 4), of length 18: it does not use the edge $\{1, 3\}$, so it is not affected. The two tours that use the edge $\{1, 3\}$ get shorter by 1.

The instance is not metric because of $d_{13} = 10$; with $d_{13} \leq 9$ it becomes metric. The optimal tour $(1, 2, 3, 4)$, of length 18, does not change.

Exercise 6.5 — Asymmetric distances (ATSP)

We consider the ATSP instance with 4 cities and the asymmetric distances (d_{ij} is the distance from city i to city j)

d_{ij}	1	2	3	4
1	—	3	8	5
2	6	—	2	9
3	4	7	—	3
4	2	6	5	—

Questions

1. How many distinct tours does the instance have? Why are there more than for a symmetric instance with 4 cities?
2. Compute the length of every tour, pairing each tour with the same tour travelled backwards.
3. Find the optimal tour and draw it.

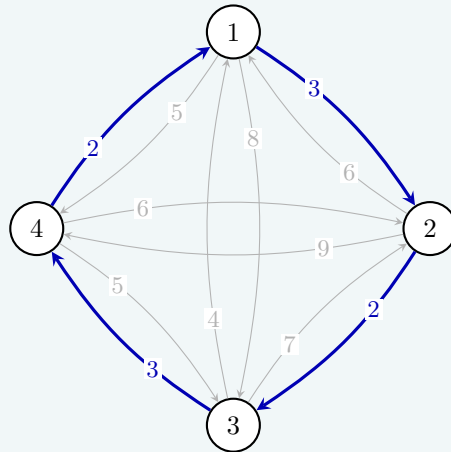
Solution

1. Every tour can start from city 1, and the other 3 cities can be ordered in $3! = 6$ ways: $(4 - 1)! = 6$ tours. In the symmetric case a tour and its reverse have the same length and are one solution, $(4 - 1)!/2 = 3$; here the direction matters.
2. Each tour, written from city 1, is paired with its reverse:

tour	length	reverse	length
$(1, 2, 3, 4)$	$3 + 2 + 3 + 2 = \mathbf{10}$	$(1, 4, 3, 2)$	23
$(1, 2, 4, 3)$	$3 + 9 + 5 + 4 = 21$	$(1, 3, 4, 2)$	23
$(1, 3, 2, 4)$	$8 + 7 + 9 + 2 = 26$	$(1, 4, 2, 3)$	17

A tour and its reverse use the same pairs of cities, but in opposite directions, and their lengths differ.

3. The optimal tour is $\tilde{\sigma} = (1, 2, 3, 4)$, with $z(I) = 10$; its reverse has length 23.



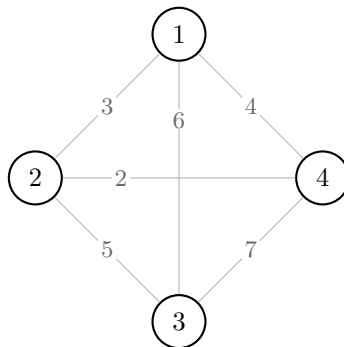
The ATSP instance has $(4-1)! = 6$ tours; the optimal one is $(1, 2, 3, 4)$, with $z(I) = 10$, while the same cycle travelled backwards is much longer.

Exercise 6.6 — Ground set, feasible solutions and objective (TSP)

We consider the TSP instance with 4 cities, $\mathcal{V} = \{1, 2, 3, 4\}$, and the symmetric distances

$$d_{12} = 3, \quad d_{13} = 6, \quad d_{14} = 4, \quad d_{23} = 5, \quad d_{24} = 2, \quad d_{34} = 7.$$

The instance is drawn as a complete graph, with the distance on every edge:



Questions

1. Write the instance as a combinatorial optimization problem in the form of the definition of [Chapter 1 «Combinatorial optimization problems»](#): take as ground set $\mathcal{U}$ the set of the edges, and describe the feasible solutions $\mathcal{F}$ and the objective function f . How many subsets does $\mathcal{U}$ have?
2. How many subsets of $\mathcal{U}$ have exactly 4 edges? How many of them are tours? Explain why the others are not feasible.
3. Compute f for every feasible solution and find the optimal solution and $z(I)$.
4. Compare the number of feasible solutions with the formula $(n-1)!/2$ of the chapter.

Solution

1. The ground set is the set of the 6 edges of the complete graph,

$$\mathcal{U} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}.$$

A subset $S \subseteq \mathcal{U}$ is feasible, $S \in \mathcal{F}$, when its edges form a single cycle through all the cities, that is a tour. The objective function is additive, with the distances as weights:

$$f(S) = \sum_{\{i,j\} \in S} d_{ij}.$$

The ground set has $2^6 = 64$ subsets.

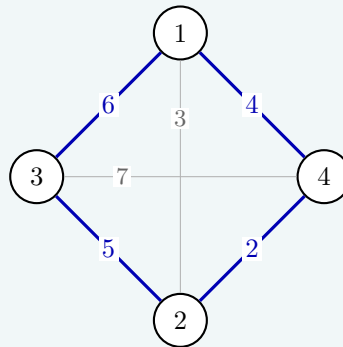
2. A tour of 4 cities uses exactly 4 edges, and there are $\binom{6}{4} = 15$ subsets with 4 edges. Each of them leaves out 2 edges.
 - If the 2 missing edges have no city in common, every city keeps 2 edges: the 4 edges form a cycle through all the cities. There are 3 such pairs, $\{\{1, 2\}, \{3, 4\}\}$, $\{\{1, 3\}, \{2, 4\}\}$ and $\{\{1, 4\}, \{2, 3\}\}$: 3 tours.
 - If the 2 missing edges meet in a city v , the city v keeps only 1 edge and the other three cities form a triangle: this is not a tour. There are $4 \cdot 3 = 12$ such subsets (4 choices of v and $\binom{3}{2} = 3$ pairs of edges at v).

Hence only 3 of the 64 subsets of $\mathcal{U}$ are feasible.

3. The three tours, written as sequences starting from city 1 and as sets of edges, are:

tour σ	edges S	$f(S) = L(\sigma)$
(1, 2, 3, 4)	$\{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}$	$3 + 4 + 5 + 7 = 19$
(1, 2, 4, 3)	$\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}$	$3 + 6 + 2 + 7 = 18$
(1, 3, 2, 4)	$\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}$	$6 + 4 + 5 + 2 = \mathbf{17}$

The optimal solution is the tour $\tilde{\sigma} = (1, 3, 2, 4)$, with $z(I) = 17$:



A short way to see it: the 6 distances sum up to 27, and a tour leaves out two edges with no city in common. The best tour leaves out the most expensive such pair, $\{1, 2\}$ and $\{3, 4\}$, of total $3 + 7 = 10$, and $27 - 10 = 17$.

4. The formula gives $(4 - 1)!/2 = 3$ tours, the same number found in Question 2. The candidates of the definition, the 64 subsets of edges, are many more than the feasible solutions.

Seen as a combinatorial optimization problem on the ground set of the edges, the instance has $2^6 = 64$ candidate subsets and only 3 feasible solutions, the tours. The optimal tour is $(1, 3, 2, 4)$, with $z(I) = 17$.

Exercise 6.7 — Primal bound, dual bound and gap (TSP)

We consider the TSP instance with 5 cities and the symmetric distances

d_{ij}	1	2	3	4	5
1	—	2	2	9	9
2	2	—	4	5	9
3	2	4	—	5	4
4	9	5	5	—	8
5	9	9	4	8	—

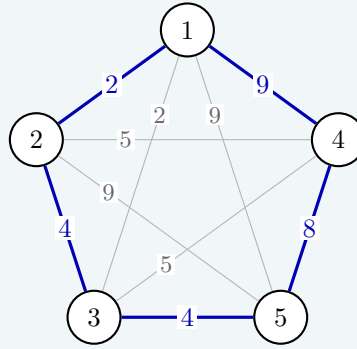
Questions

1. Build a tour with the nearest neighbour rule from city 1 (ties broken by the smallest index) and compute its length. Is it a primal or a dual bound?
2. Compute the dual bound $\lceil \frac{1}{2} \sum_i (a_i + b_i) \rceil$, where $a_i \leq b_i$ are the two smallest distances from city i .
3. Enumerate the $(5 - 1)!/2$ tours and find the optimal one.
4. Compute the absolute and relative gaps of the nearest neighbour tour, with respect to the dual bound and with respect to $z(I)$. Why is the greedy choice of the nearest city not optimal here?

Solution

1. The nearest neighbour rule proceeds as follows:
 - from 1 the closest unvisited cities are 2, 3, at distance 2: we go to 2 (smallest index);
 - from 2 the closest unvisited city is 3, at distance 4;
 - from 3 the closest unvisited city is 5, at distance 4;
 - from 5 the closest unvisited city is 4, at distance 8;
 - we return from 4 to 1, at distance $d_{41} = 9$.

The tour is $\bar{\sigma} = (1, 2, 3, 5, 4)$, of length $L(\bar{\sigma}) = 2 + 4 + 4 + 8 + 9 = 27$. It is the length of a feasible solution of a minimization problem, hence a primal bound: $UB = 27$.



2. The two smallest distances of every city are:

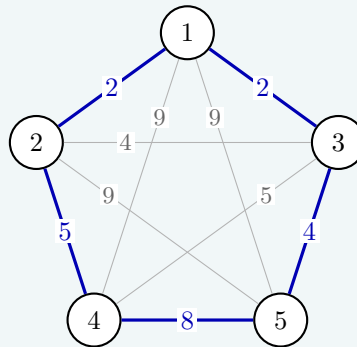
city i	1	2	3	4	5	sum
a_i	2	2	2	5	4	
b_i	2	4	4	5	8	
$a_i + b_i$	4	6	6	10	12	38

Hence $LB = \frac{38}{2} = 19$ (the sum is even, no rounding is needed).

3. The 12 tours, all starting from city 1, are:

tour	L	tour	L	tour	L
(1, 2, 4, 5, 3)	21	(1, 2, 3, 4, 5)	28	(1, 3, 4, 2, 5)	30
(1, 2, 4, 3, 5)	25	(1, 3, 2, 4, 5)	28	(1, 4, 2, 3, 5)	31
(1, 2, 5, 4, 3)	26	(1, 2, 5, 3, 4)	29	(1, 3, 2, 5, 4)	32
(1, 2, 3, 5, 4)	27	(1, 3, 5, 2, 4)	29	(1, 4, 3, 2, 5)	36

The optimal tour is $\tilde{\sigma} = (1, 2, 4, 5, 3)$, the only one of length $z(I) = 21$:



4. With respect to the dual bound the absolute gap is $27 - 19 = 8$ and the relative gap is $\frac{8}{19} \approx 42.1\%$. With respect to the optimum the absolute gap is $27 - 21 = 6$ and the relative gap is $\frac{6}{21} = \frac{2}{7} \approx 28.6\%$. The rule always takes the shortest available edge and leaves the last edges to chance: the closing edges $\{5, 4\}$ and $\{4, 1\}$ cost 8 and 9.

Nearest neighbour gives $UB = 27$, the two shortest edges give $LB = 19$, and the enumeration of the 12 tours gives $z(I) = 21$: $19 \leq z(I) = 21 \leq 27$.

Scheduling problems

Exercise 7.1 — Loads, makespan and Gantt chart ($P||C_{\max}$)

We consider the $P||C_{\max}$ instance with $m = 3$ machines and $n = 7$ jobs, with processing times

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$
t_j	4	2	5	3	6	2	3

and the three assignments

$$\text{mach}^1 = (1, 2, 3, 1, 2, 3, 1), \quad \text{mach}^2 = (3, 1, 2, 3, 1, 2, 3), \quad \text{mach}^3 = (1, 3, 1, 2, 2, 3, 3).$$

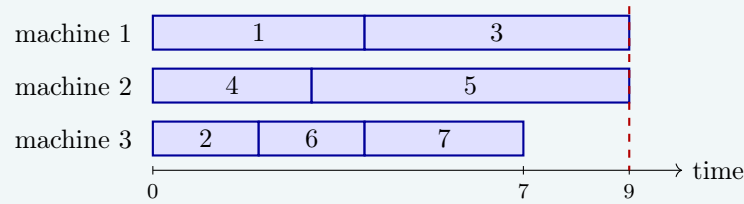
Questions

1. For each assignment, write the sets J_1, J_2, J_3 , the loads and the makespan.
2. Which assignments describe the same schedule?
3. Draw the Gantt chart of the assignment with the smallest makespan.
4. Compute the lower bound $\max\{\max_j t_j, \lceil \sum_j t_j / m \rceil\}$. Is that schedule optimal?

Solution

1.
 - mach^1 : $J_1 = \{1, 4, 7\}$, $J_2 = \{2, 5\}$, $J_3 = \{3, 6\}$, loads (10, 8, 7), makespan $C_{\max} = 10$;
 - mach^2 : $J_1 = \{2, 5\}$, $J_2 = \{3, 6\}$, $J_3 = \{1, 4, 7\}$, loads (8, 7, 10), makespan $C_{\max} = 10$;
 - mach^3 : $J_1 = \{1, 3\}$, $J_2 = \{4, 5\}$, $J_3 = \{2, 6, 7\}$, loads (9, 9, 7), makespan $C_{\max} = 9$;
2. mach^1 and mach^2 give the same three sets of jobs, with the names of the machines changed ($1 \rightarrow 3, 2 \rightarrow 1, 3 \rightarrow 2$): they are the same schedule. mach^3 gives different sets.

3. The smallest makespan, 9, is that of mach³:



4. $\sum_j t_j = 25$ and $\max_j t_j = 6$, so $\text{LB} = \max\{6, \lceil 25/3 \rceil\} = 9$. The schedule of mach³ attains it: it is optimal, $z(I) = 9$.

mach¹ and mach² are the same schedule; the schedule of mach³ has makespan 9, equal to the lower bound, and is optimal.

Exercise 7.2 — Counting the schedules ($P||C_{\max}$)

We consider a $P||C_{\max}$ instance with $n = 5$ jobs. We count the assignments and the distinct schedules for $m = 2$ and for $m = 3$ machines.

Questions

1. How many assignments mach are there with $m = 2$ and with $m = 3$?
2. Compute the Stirling number $S(5, 2)$, counting the partitions of the jobs into two non-empty sets.
3. Compute $S(5, 3)$, grouping the partitions into three non-empty sets by the sizes of the sets.
4. How many distinct schedules are there with $m = 2$ and with $m = 3$? Why are there fewer schedules than assignments?

Solution

1. Every job chooses one of the m machines: $2^5 = 32$ assignments with $m = 2$, $3^5 = 243$ with $m = 3$.
2. A partition into two sets is fixed by the set that contains job 1 together with some of the other 4 jobs, but not all of them: $2^4 - 1 = 15$ choices. Hence $S(5, 2) = 15$.
3. The sizes of three non-empty sets with 5 jobs are $(3, 1, 1)$ or $(2, 2, 1)$.
 - Sizes $(3, 1, 1)$: we choose the set of 3 jobs, $\binom{5}{3} = 10$ ways.
 - Sizes $(2, 2, 1)$: we choose the job alone, 5 ways, and split the other 4 into two pairs, $\binom{4}{2}/2 = 3$ ways: 15.

Hence $S(5, 3) = 10 + 15 = 25$.

4. With $m = 2$: $S(5, 1) + S(5, 2) = 1 + 15 = 16$ schedules. With $m = 3$: $S(5, 1) + S(5, 2) + S(5, 3) = 1 + 15 + 25 = 41$ schedules. The machines are identical: an assignment and those obtained by renaming the machines are the same schedule. With $m = 2$ every schedule with two non-empty machines is given by 2 assignments: $\frac{32-2}{2} = 15$, plus the schedule with all the jobs together.

With 5 jobs there are 32 and 243 assignments on 2 and 3 machines, but only 16 and 41 schedules.

Exercise 7.3 — A first schedule, a lower bound and the optimum ($P||C_{\max}$)

We consider the $P||C_{\max}$ instance with $m = 3$ machines and $n = 7$ jobs, with processing times

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$
t_j	3	5	2	6	4	4	3

Questions

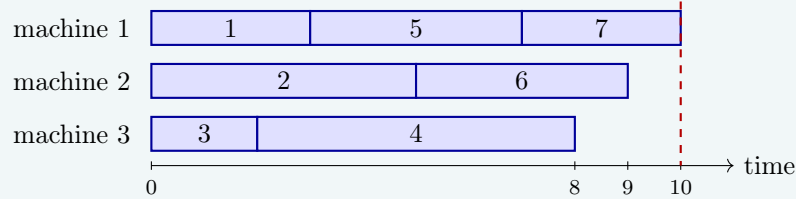
1. Build a schedule by taking the jobs in the order $1, 2, \dots, n$ and assigning each one to the machine with the smallest load (ties broken by the smallest index). Draw its Gantt chart.
2. Compute the lower bound of the chapter.
3. Find an optimal schedule. Is it unique?

Solution

1. The loads after each assignment are:

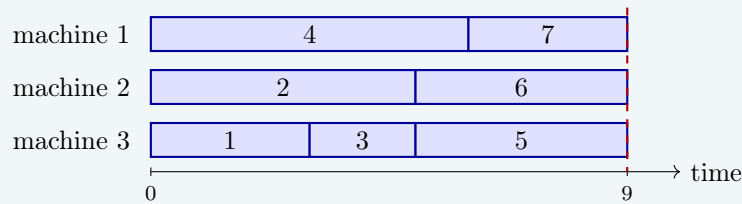
job j (t_j)	machine chosen	loads (L_1, L_2, L_3)
1 (3)	1	(3, 0, 0)
2 (5)	2	(3, 5, 0)
3 (2)	3	(3, 5, 2)
4 (6)	3	(3, 5, 8)
5 (4)	1	(7, 5, 8)
6 (4)	2	(7, 9, 8)
7 (3)	1	(10, 9, 8)

We obtain $J_1 = \{1, 5, 7\}$, $J_2 = \{2, 6\}$, $J_3 = \{3, 4\}$, with makespan 10:



2. $LB = \max\{6, \lceil 27/3 \rceil\} = 9$, so $9 \leq z(I) \leq 10$.

3. A makespan of 9 needs three loads of exactly 9, since the loads sum up to $27 = 3 \cdot 9$. Job 4 (6) needs a partner of length 3, job 2 (5) one of length 4, and the remaining jobs fill the third machine. The schedule $J_1 = \{4, 7\}$, $J_2 = \{2, 6\}$, $J_3 = \{1, 3, 5\}$ has loads (9, 9, 9):



It attains the bound, so $z(I) = 9$. It is not unique: the enumeration of the schedules gives 4 optimal ones, for instance jobs 1 and 7, both of length 3, can be exchanged.

Assigning the jobs in the input order gives makespan 10; the schedule $J_1 = \{4, 7\}$, $J_2 = \{2, 6\}$, $J_3 = \{1, 3, 5\}$ attains the lower bound, and $z(I) = 9$.

Exercise 7.4 — When the lower bound is not attained ($P||C_{\max}$)

We consider the $P||C_{\max}$ instance with $m = 2$ machines and $n = 4$ jobs, with processing times

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
t_j	8	5	4	3

Questions

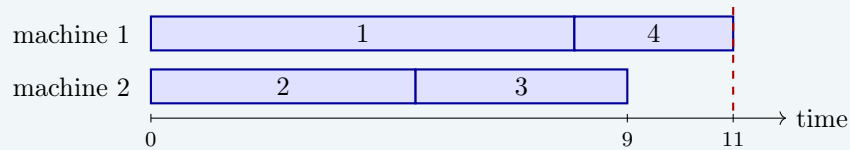
1. Compute the lower bound of the chapter.
2. Show that no schedule has makespan 10.
3. Find an optimal schedule, by enumeration of the distinct schedules.

Solution

1. $LB = \max\{8, \lceil 20/2 \rceil\} = 10$.
2. The two loads sum up to 20: a makespan of 10 needs both loads equal to 10, that is a set of jobs of total length exactly 10. The totals of the subsets of jobs are 0, 3, 4, 5, 7, 8, 9, 11, 12, 13, 15, 16, 17, 20, and 10 is not among them. Hence $z(I) \geq 11$.
3. With $m = 2$ there are $S(4, 1) + S(4, 2) = 1 + 7 = 8$ schedules:

J_1	J_2	loads	$C_{\max}$
$\{1, 4\}$	$\{2, 3\}$	(11, 9)	11
$\{1\}$	$\{2, 3, 4\}$	(8, 12)	12
$\{1, 3\}$	$\{2, 4\}$	(12, 8)	12
$\{1, 2\}$	$\{3, 4\}$	(13, 7)	13
$\{1, 3, 4\}$	$\{2\}$	(15, 5)	15
$\{1, 2, 4\}$	$\{3\}$	(16, 4)	16
$\{1, 2, 3\}$	$\{4\}$	(17, 3)	17
$\{1, 2, 3, 4\}$	$\emptyset$	(20, 0)	20

The optimal schedule is unique, $J_1 = \{1, 4\}$, $J_2 = \{2, 3\}$, with $z(I) = 11$:



The lower bound gives 10, but no set of jobs has total length 10: the optimal makespan is $z(I) = 11$, and the bound is not attained.

 Exercise 7.5 — Makespan and bin packing ($P||C_{\max}$)

We consider the $P||C_{\max}$ instance with $m = 3$ machines and $n = 5$ jobs, with processing times

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
t_j	7	6	5	5	4

We use the relation with the bin packing problem: a schedule with makespan at most T exists if and only if the jobs, as items of weight t_j , fit into m bins of capacity T .

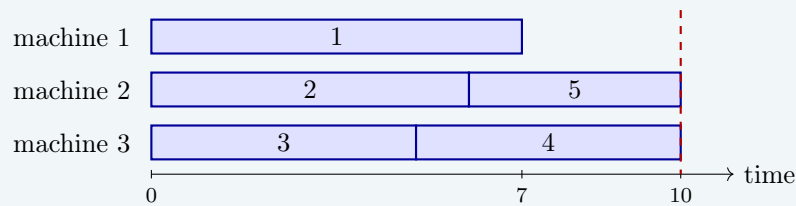
Questions

1. Compute the lower bound LB of the chapter.
2. Can the jobs be packed into 3 bins of capacity $T = LB$?

3. Find the smallest T for which the jobs fit into 3 bins, and the corresponding schedule.
4. Is the optimal schedule unique?

Solution

1. $LB = \max\{7, \lceil 27/3 \rceil\} = \max\{7, 9\} = 9$.
2. Three bins of capacity 9 hold $27 = 27$, the total weight: every bin must be filled exactly. The item of weight 7 needs items of total weight 2, and every other item weighs at least 4. Hence the jobs do not fit, and $z(I) \geq 10$.
3. With $T = 10$ the packing $\{\{1\}, \{2, 5\}, \{3, 4\}\}$ has bins of weight 7, 10, $10 \leq 10$. The schedule $J_1 = \{1\}$, $J_2 = \{2, 5\}$, $J_3 = \{3, 4\}$ has makespan 10:



Hence $z(I) = 10$.

4. Yes: the enumeration of the 41 schedules with at most 3 machines shows that it is the only schedule with makespan 10. Job 1 must be alone, since with any other job its machine would exceed 10, and the pairs $\{2, 5\}$ and $\{3, 4\}$ are the only way to split the other jobs within 10.

The jobs do not fit into 3 bins of capacity 9 and fit into 3 bins of capacity 10: $z(I) = 10$, one more than the lower bound.

Lot sizing problems

Exercise 8.1 — Feasibility and cost of a plan (CLS)

We consider the capacitated lot sizing instance with $m = 4$ periods and the following data:

period t	1	2	3	4
demand d_t	3	2	4	3
capacity cap_t	5	5	5	5
setup cost sc_t	30	40	30	40
production cost vc_t	2	2	2	2
holding cost hc_t	4	4	4	4

and the four production plans

$$\mathbf{x}^1 = (5, 0, 4, 3), \quad \mathbf{x}^2 = (3, 6, 0, 3), \quad \mathbf{x}^3 = (2, 3, 4, 3), \quad \mathbf{x}^4 = (5, 0, 5, 2).$$

Questions

1. For each plan, compute the stocks $s_t(\mathbf{x})$ and decide whether the plan is feasible.
2. Compute the cost of the feasible plans, split into setup, production and holding costs.
3. Draw the cheaper feasible plan.

Solution

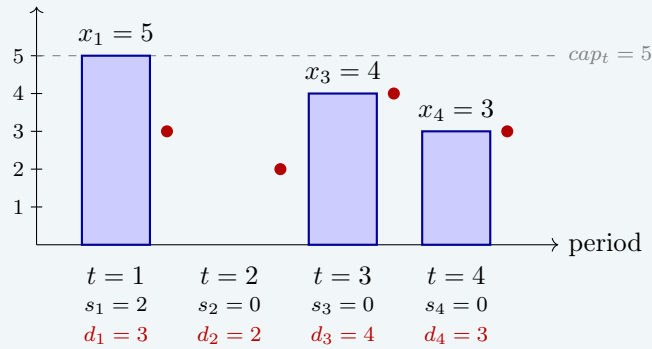
1. The stock is $s_t = s_{t-1} + x_t - d_t$, starting from $s_0 = 0$:
 - $\mathbf{x}^1$: stocks $(2, 0, 0, 0)$, feasible;
 - $\mathbf{x}^2$: stocks $(0, 4, 0, 0)$, not feasible: $x_2 = 6 > cap_2 = 5$;
 - $\mathbf{x}^3$: stocks $(-1, 0, 0, 0)$, not feasible: $s_1 = -1 < 0$, the demand of period 1 is not covered (production cannot serve an earlier period);
 - $\mathbf{x}^4$: stocks $(2, 0, 1, 0)$, feasible;
2. A setup is paid in every period with $x_t > 0$, the production cost on every unit, the

holding cost on the stock at the end of every period:

$$c(\mathbf{x}^1) = (30 + 30 + 40) + 2 \cdot 12 + 4 \cdot 2 = 100 + 24 + 8 = 132$$

$$c(\mathbf{x}^4) = (30 + 30 + 40) + 2 \cdot 12 + (4 \cdot 2 + 4 \cdot 1) = 100 + 24 + 12 = 136.$$

3. The cheaper plan is $\mathbf{x}^1$, of cost 132:



Only $\mathbf{x}^1$ and $\mathbf{x}^4$ are feasible, with costs 132 and 136; $\mathbf{x}^2$ exceeds a capacity and $\mathbf{x}^3$ produces too late.

Exercise 8.2 — Counting the setup choices (CLS)

We consider again the capacitated lot sizing instance

period t	1	2	3	4
demand d_t	3	2	4	3
capacity cap_t	5	5	5	5
setup cost sc_t	30	40	30	40
production cost vc_t	2	2	2	2
holding cost hc_t	4	4	4	4

Questions

1. How many choices of the periods with a setup are there?
2. Compute the cumulative demands $D_t = d_1 + \dots + d_t$. Which condition must the periods with a setup satisfy for a feasible plan to exist?
3. List the feasible setup choices.
4. Is the number of feasible plans equal to the number of feasible setup choices?

Solution

1. A setup choice is a subset of the 4 periods: $2^4 = 16$ choices.
2. $D = (3, 5, 9, 12)$. The production of the periods $1, \dots, t$ must cover D_t , and only the periods with a setup produce: the capacities of the periods with a setup among $1, \dots, t$ must sum up to at least D_t , for every t . In particular period 1 needs a setup, since $d_1 > 0$; with capacity 5 in every period, the periods $1, \dots, t$ need at least $\lceil D_t/5 \rceil$ setups: 1, 1, 2, 3.
3. The feasible choices, with the cumulative capacities of their setup periods, are:

setups	$t = 1$	$t = 2$	$t = 3$	$t = 4$
cumulative demand	3	5	9	12
$\{1, 2, 3\}$	5	10	15	15
$\{1, 2, 4\}$	5	10	10	15
$\{1, 3, 4\}$	5	5	10	15
$\{1, 2, 3, 4\}$	5	10	15	20

They are 4 out of 16.

4. No: a setup choice can carry several plans, since the quantities can be distributed in many ways. The enumeration of the integer plans gives 50 feasible plans for the 4 feasible setup choices.

Of the 16 setup choices, 4 are feasible: they all contain period 1 and at least 3 periods in total. They carry 50 feasible plans.

Exercise 8.3 — The best plan by enumeration of the setups (CLS)

We consider again the capacitated lot sizing instance

period t	1	2	3	4
demand d_t	3	2	4	3
capacity cap_t	5	5	5	5
setup cost sc_t	30	40	30	40
production cost vc_t	2	2	2	2
holding cost hc_t	4	4	4	4

whose feasible setup choices are those that contain period 1 and whose capacities cover the cumulative demands.

Questions

1. Once the periods with a setup are fixed, why is it best to produce every unit as late as possible?

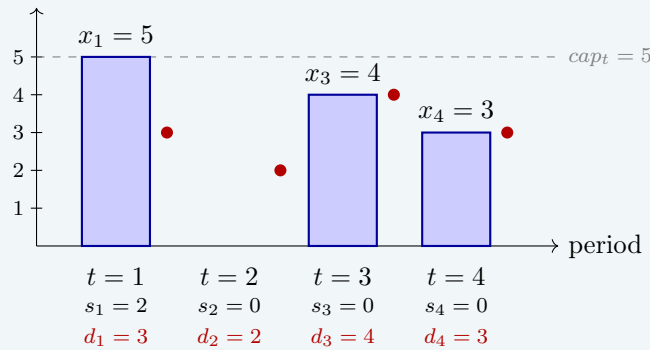
2. For every feasible setup choice, compute the best plan and its cost.
3. Which plan is optimal? Draw it.

Solution

1. The production cost is the same in every period, so it does not depend on when a unit is produced. A unit produced earlier stays longer in stock and pays more holding cost. Hence every demand is produced in the latest period with a setup and free capacity before it, starting from the last period.
2. For every feasible choice of the setups, the plan built in this way is:

setups	plan	setup	production	holding	total
$\{1, 2, 3\}$	$(3, 4, 5, 0)$	100	24	20	144
$\{1, 2, 4\}$	$(4, 5, 0, 3)$	110	24	20	154
$\{1, 3, 4\}$	$(5, 0, 4, 3)$	100	24	8	132
$\{1, 2, 3, 4\}$	$(3, 2, 4, 3)$	140	24	0	164

3. The optimal plan is $\tilde{x} = (5, 0, 4, 3)$, with setups in the periods 1, 3, 4 and $z(I) = 132$; it is the only optimal plan.



The two cheapest setups, $sc_1 = sc_3 = 30$, are not enough: with capacity 5 two setups produce at most $10 < 12$ units, and the third setup goes to period 4.

Enumerating the feasible setup choices, and producing as late as possible for each of them, gives the optimal plan $(5, 0, 4, 3)$ with $z(I) = 132$.

Exercise 8.4 — The effect of the capacity (CLS)

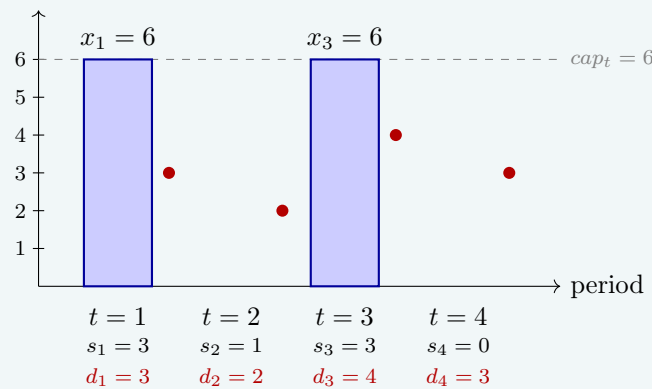
We consider the lot sizing instance of the previous exercises, with 4 periods, $\mathbf{d} = (3, 2, 4, 3)$, $\mathbf{sc} = (30, 40, 30, 40)$, $\mathbf{vc} = (2, 2, 2, 2)$, $\mathbf{hc} = (4, 4, 4, 4)$, and three values of the capacity, the same in every period: $cap_t = 5$, $cap_t = 6$, and no capacity limit.

Questions

1. For $cap_t = 5$ the optimal plan is $(5, 0, 4, 3)$. Compute its cost.
2. Find the optimal plan for $cap_t = 6$, by enumeration of the setup choices.
3. Find the optimal plan without capacities. In which periods does it produce, and with which stock?
4. Compare the three optimal costs, and explain the differences.

Solution

1. $c = (30 + 30 + 40) + 2 \cdot 12 + 4 \cdot 2 = 100 + 24 + 8 = 132$.
2. With capacity 6 two setups can produce 12 units, the whole demand: the setups in the periods 1 and 3, the cheapest ones, become feasible. Producing as late as possible gives $(6, 0, 6, 0)$, of cost $(30 + 30) + 2 \cdot 12 + (4 \cdot 3 + 4 \cdot 1 + 4 \cdot 3) = 60 + 24 + 28 = 112$. The enumeration of all the plans confirms that it is the only optimal one.



3. Without capacities, a setup in period 1 and one in period 3 give $(5, 0, 7, 0)$, of cost $(30 + 30) + 2 \cdot 12 + (4 \cdot 2 + 4 \cdot 3) = 60 + 24 + 20 = 104$. Period 1 produces for the periods 1 and 2, period 3 for the periods 3 and 4: every production starts with an empty stock.
4. The optimal costs are 132, 112 and 104: a larger capacity can only lower the optimal cost, since every plan feasible with a smaller capacity is still feasible with a larger one.
 - Without capacities, period 3 produces the 7 units of the periods 3 and 4.
 - With $cap_t = 6$ period 3 produces only 6 of them: one unit is produced in period 1 and kept in stock for two periods, $104 + 2 \cdot 4 = 112$.
 - With $cap_t = 5$ two setups produce at most $10 < 12$ units: a third setup is needed, in period 4, and the cost becomes 132.

The optimal cost is 132 with $cap_t = 5$, 112 with $cap_t = 6$ and 104 without capacities: the capacity forces a third setup and changes the structure of the optimal plan.

Exercise 8.5 — Lot-for-lot, a lower bound and the optimum (CLS)

We consider the capacitated lot sizing instance with $m = 3$ periods and the following data:

period t	1	2	3
demand d_t	4	3	5
capacity cap_t	7	6	7
setup cost sc_t	50	20	60
production cost vc_t	0	0	0
holding cost hc_t	3	3	3

Questions

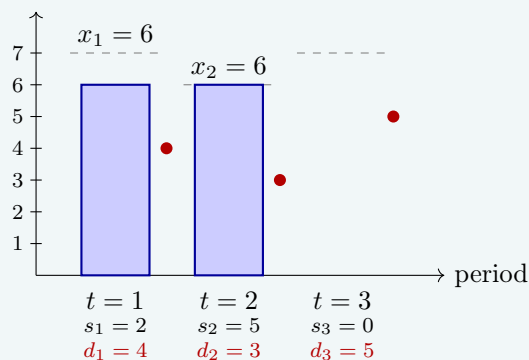
1. Compute the cost of the lot-for-lot plan.
2. Compute the lower bound of the chapter: the smallest number k of setups that the total demand requires, and the sum of the k smallest setup costs.
3. Find the optimal plan by enumeration of the setup choices. How far are the lot-for-lot plan and the lower bound from the optimal cost?

Solution

1. Lot-for-lot produces $\mathbf{x} = (4, 3, 5)$: every demand fits in the capacity of its period, there is no stock and there are 3 setups, $c = 50 + 20 + 60 = 130$.
2. The total demand is $D = 12$. The largest capacity, 7, is smaller than D , while the two largest sum up to $14 \geq 12$: $k = 2$. The two smallest setup costs are 20 and 50, so $z(I) \geq \text{LB} = 70$.
3. For every feasible setup choice, the plan that produces as late as possible is:

setups	plan	cost
$\{1, 2\}$	$(6, 6, 0)$	91
$\{1, 3\}$	$(7, 0, 5)$	119
$\{1, 2, 3\}$	$(4, 3, 5)$	130

The optimal plan is $(6, 6, 0)$, with $z(I) = 91$:



The bounds give $70 \leq z(I) \leq 130$, an absolute gap of 60: lot-for-lot is 39 above the optimum, the lower bound 21 below it. The bound pays the setups of periods 1 and 2, but these two periods cannot produce the demand of period 3 without keeping stock.

Lot-for-lot costs 130, the lower bound is 70, and the enumeration of the setup choices gives the optimal plan $(6, 6, 0)$ with $z(I) = 91$.