Vertex coloring: limits of approximation

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Outline

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Learning objectives

At the end of this chapter you will be able to:

- prove that no polynomial algorithm has a performance ratio below $\frac{4}{3}$ for the VCP, unless $P = NP$;
- state the result of Zuckerman and explain why the VCP has no constant guarantee;
- color bipartite graphs and interval graphs optimally, and bound the colors of the greedy coloring by $\Delta(G) + 1$;
- prove that planar graphs can be colored with 5 colors in polynomial time, and derive an additive guarantee for them.

Prerequisites

- From this module, the chapter «Principles of approximation algorithms»: performance ratio, α-approximation algorithm, additive performance guarantee
- From the module *Combinatorial Optimization Problems*, the chapter «Coloring problems»: the definition of the VCP, the chromatic number $\chi(G)$, the clique bound $\omega(G) \leq \chi(G)$ and the Petersen graph
- From the module *Network Optimization*, the chapter «Graphs»: the breadth-first search
- From the module *Combinatorial Optimization Problems*, the chapter «Combinatorial optimization: definitions»: polynomial time, the classes P and NP, NP-completeness

Why limits of approximation?

Introduction: Why limits of approximation?

- An approximation algorithm comes with a guarantee: its value is never too far from the optimal one
- For some problems such a guarantee does not exist, unless $P = NP$
- The vertex coloring problem is one of them: in this chapter we see why, and what can still be done

The problem of this chapter

- The problem of this chapter is the vertex coloring problem (VCP): its definition and its assumptions are on its page
- For the vertex coloring problem we prove the opposite of a guarantee: unless $P = NP$, no polynomial algorithm can have a good guarantee
- The negative results tell us when we must give up asking

Summary

Guarantees and limits:

- The positive results bound the gap of an algorithm on every instance
- The negative results prove that, unless $P = NP$, no polynomial algorithm has a good guarantee

The vertex coloring problem:

- It is the problem of this chapter, and it is on the negative side

Notation

Introduction: Notation

- We fix the notation on graphs used in the whole chapter: neighbours, degrees, maximum degree
- We recall how the quality of a coloring algorithm is measured
- We note that a trivial ratio exists, and ask how much better a polynomial algorithm can do

Graphs and degrees

- As on the problem page, $G = (V, E)$ is a simple graph with n vertices and m edges
- $N(v)$ is the set of the neighbours of v, and $\deg(v) = |N(v)|$ its degree
- The **maximum degree** is

$$\Delta(G) = \max_{v \in V} \deg(v)$$

- The optimal value of the VCP is the chromatic number, $z(I) = \chi(G)$

Recall: the performance ratio

- As in the chapter «Principles of approximation algorithms», $\mathscr{A}(I)$ is the number of colors of the coloring returned by the algorithm $\mathscr{A}$
- The VCP is a minimization problem, hence

$$\mathscr{A}(I) \geq \chi(G) \qquad \text{and} \qquad \frac{\mathscr{A}(I)}{\chi(G)} \geq 1$$

for every instance I

The trivial ratio

- Giving each vertex its own color is always feasible, so

$$\chi(G) \leq n,$$

  and the ratio n is trivial

- This chapter shows that a polynomial algorithm cannot do much better on all graphs

- And then the classes of graphs on which it can

Summary

Notation:

- $G = (V, E)$ with n vertices and m edges, degree $\deg(v)$, maximum degree $\Delta(G)$
- $z(I) = \chi(G)$, and $\mathscr{A}(I)$ is the number of colors used by the algorithm $\mathscr{A}$

The performance ratio:

- $\frac{\mathscr{A}(I)}{\chi(G)} \geq 1$ for every instance
- The ratio n is trivial: one color per vertex

The VCP is not approximable

Introduction: The VCP is not approximable

- We prove that no polynomial algorithm can have a performance ratio below $\frac{4}{3}$, unless $P = NP$
- We then state much stronger results: not even a ratio of the form $n^{1-\varepsilon}$ is possible
- The consequence is that the VCP has no constant guarantee: this is why we will look for special classes of graphs

The scheme of the negative results

- The proof follows the scheme of all the negative results of approximation: an algorithm with a too good guarantee would decide in polynomial time an NP-complete problem
- The NP-complete problem is **3-colorability**: given a graph G, decide whether
$$\chi(G) \leq 3$$
- It remains NP-complete on planar graphs of maximum degree 4 (Garey, Johnson and Stockmeyer, 1976)

Theorem 1

Let $\mathscr{A}$ be a polynomial algorithm for the VCP. If $P \neq NP$, there is a graph G such that

$$\frac{\mathscr{A}(I)}{\chi(G)} \geq \frac{4}{3}.$$

Proof (1/2)

Suppose that $\frac{\mathscr{A}(I)}{\chi(G)} < \frac{4}{3}$ for every graph G, and run $\mathscr{A}$ on a graph G.

1. If $\chi(G) \leq 3$, then

$$\mathscr{A}(I) < \frac{4}{3}\chi(G) \leq 4.$$

   The number of colors is an integer, hence $\mathscr{A}(I) \leq 3$

2. If $\chi(G) \geq 4$, then

$$\mathscr{A}(I) \geq \chi(G) \geq 4,$$

   because $\mathscr{A}$ returns a feasible coloring

Proof (2/2)

- Hence
$$\chi(G) \leq 3 \quad \text{if and only if} \quad \mathscr{A}(I) \leq 3$$
- Running $\mathscr{A}$ and counting its colors takes polynomial time
- So 3-colorability would be in P, and $P = NP$

□

Remarks on the proof

- The proof uses only the integrality of $\chi(G)$: a ratio below $\frac{4}{3}$ cannot confuse 3 colors with 4
- With 2 colors the same argument says nothing, because deciding whether $\chi(G) \leq 2$ is easy (we will see it with the bipartite graphs)
- The planar graphs of maximum degree 4 are a special case of the VCP: the theorem holds on them too

The theorem excludes only the ratios below $\frac{4}{3}$.

Much more is known,
and we state it without proof.

Theorem 2: Zuckerman, 2007

For every $\varepsilon > 0$, if $P \neq NP$, no polynomial algorithm $\mathscr{A}$ for the VCP satisfies, for every graph G with n vertices,

$$\frac{\mathscr{A}(I)}{\chi(G)} \leq n^{1-\varepsilon}.$$

What the theorem says

- The ratio n is trivial, so the chromatic number is essentially as hard to approximate as possible
- The proof uses probabilistically checkable proofs and extractors
- Earlier, Garey and Johnson (1976) had raised the bound $\frac{4}{3}$ to 2
- Feige and Kilian (1998) had proved the bound $n^{1-\varepsilon}$ under a stronger assumption than $P \neq NP$

Corollary 1

If $P \neq NP$, the VCP has no α-approximation algorithm, for any constant $\alpha \geq 1$.

Proof

- Let $\mathscr{A}$ be a polynomial α-approximation algorithm, and take $\varepsilon = \frac{1}{2}$
- The graphs with $n > \alpha^2$ vertices satisfy

$$\frac{\mathscr{A}(I)}{\chi(G)} \leq \alpha < n^{1/2}$$

- The graphs with $n \leq \alpha^2$ vertices are solved exactly by enumeration, in a time bounded by a constant that depends only on α, with ratio $1 \leq n^{1/2}$
- The resulting algorithm is polynomial and has ratio at most $n^{1/2}$ on every graph, against the theorem of Zuckerman



The VCP is not approximable

- The VCP is not approximable: unless $P = NP$, no polynomial algorithm has a constant performance guarantee
- Not even a guarantee of the form $n^{1-\varepsilon}$
- Positive results exist only on special classes of graphs

Summary

No ratio below $\frac{4}{3}$:

- Otherwise 3-colorability would be decided in polynomial time
- The proof only uses that the number of colors is an integer

Much stronger results:

- Zuckerman (2007): not even the ratio $n^{1-\varepsilon}$, unless $P = NP$
- Hence no α-approximation algorithm for any constant $\alpha \geq 1$

Special cases

Introduction: Special cases

- On the four classes of this section the VCP is solved exactly, or with a guarantee, in polynomial time
- The classes are recognized from the graph, or given with it (the intervals of an interval graph)
- We see, in order: bipartite graphs, graphs of bounded maximum degree, planar graphs and interval graphs
- Where the general problem gives up, the structure of the graph lets us color well

Definition 1: bipartite graph

A graph $G = (V, E)$ *is* ***bipartite*** *if* $\chi(G) \leq 2$*, i.e., if* V *can be split into two stable sets.*

Proposition 1

Deciding whether $\chi(G) \leq 2$, and finding a coloring with 2 colors when it exists, takes $O(n+m)$ time.

- The idea of the proof: a breadth-first search, and the parity of the levels

Proof (1/2)

- Run a breadth-first search from a vertex of each connected component
- Let $\ell(v)$ be the level of v, its distance from the root of its component
- Color v with color 1 if $\ell(v)$ is even and with color 2 otherwise

1. If no edge joins two vertices of levels with the same parity, the coloring is feasible

Proof (2/2)

2. Otherwise, let $\{u, v\}$ be such an edge
   - The paths of the search tree from the root to u and to v, with the edge $\{u, v\}$, form a closed walk of odd length

$$\ell(u) + \ell(v) + 1$$

   - A closed walk of odd length contains a cycle of odd length, which needs 3 colors: $\chi(G) \geq 3$

The search and the check of the edges take $O(n + m)$ time. □

Odd cycles as certificates

- The proof shows the classical characterization: a graph is bipartite if and only if it has no cycle of odd length (Kőnig, 1936)
- On a graph with at least one edge, the search gives $\chi(G) = 2$, or the certificate $\chi(G) \geq 3$ given by an odd cycle

The greedy coloring

- The **greedy coloring** GC examines the vertices in a given order
- It gives each vertex the smallest color, among $1, 2, 3, \ldots$, not used by its neighbours already colored
- It runs in $O(n + m)$ time

Proposition 2

For every graph G and every order of the vertices,

$$GC(I) \leq \Delta(G) + 1.$$

Proof

- When the vertex v is colored, its neighbours already colored use at most

$$\deg(v) \leq \Delta(G)$$

colors

- One of the colors $1, 2, \ldots, \Delta(G) + 1$ is therefore free

- The smallest free color is at most $\Delta(G) + 1$

□

Is $\Delta(G)+1$ a performance guarantee?

- The bound compares $GC(I)$ with $\Delta(G)+1$, not with $\chi(G)$
- On all graphs it is not a performance guarantee, because $\Delta(G)$ can be $n-1$ while $\chi(G)=2$ (a star)
- On the graphs of maximum degree at most a *fixed* constant Δ, it is one
- Every graph with an edge has $\chi(G)\geq 2$, hence

$$\frac{GC(I)}{\chi(G)} \leq \frac{\Delta+1}{2}.$$

When the bound is attained

- The bound $\Delta(G) + 1$ is attained by complete graphs and by cycles of odd length, whatever the order
- **Remark (Brooks, 1941).** A connected graph that is neither complete nor a cycle of odd length satisfies

$$\chi(G) \leq \Delta(G),$$

and such a coloring can be found in linear time

Planar graphs

- A graph is **planar** if it can be drawn in the plane without crossing edges
- Whether a graph is planar, and a drawing when it is, can be found in linear time (Hopcroft and Tarjan, 1974)
- We first show that a planar graph always has a vertex of small degree, then we use it to color with 5 colors

Lemma 1

Every planar graph with $n \geq 3$ vertices has $m \leq 3n-6$ edges, and a vertex of degree at most 5.

Proof (1/2)

- Adding edges to a drawing without creating crossings does not decrease m
- So we may assume that no such edge can be added: the drawing is connected and every face is a triangle
- Euler's formula for a connected planar drawing with f faces gives

$$n - m + f = 2$$

- Every face has 3 edges and every edge borders 2 faces, so

$$3f = 2m$$

Proof (2/2)

- Substituting $f = \frac{2m}{3}$ in Euler's formula:

$$2 = n - m + f \;=\; n - m + \frac{2m}{3} \qquad \Longrightarrow \qquad m \leq 3n - 6$$

- The degrees sum to

$$2m \leq 6n - 12 < 6n,$$

so some vertex has degree at most 5

□

Theorem 3: Heawood, 1890

Every planar graph can be colored with 5 *colors, and such a coloring can be found in polynomial time.*

- The idea of the proof: remove a vertex of degree at most 5, color the rest, and free a color for it by exchanging two colors

Proof (1/3)

- By induction on n; graphs with $n \leq 5$ are trivial
- Let v be a vertex of degree at most 5 (by the lemma above)
- Color $G - v$, which is planar, with 5 colors

1. If the neighbours of v use at most 4 colors, v takes the fifth one
2. Otherwise v has 5 neighbours $v_1, \ldots, v_5$, in this order around v in the drawing, with colors $1, \ldots, 5$. Let H_{13} be the graph induced by the vertices of colors 1 and 3

Proof (2/3)

3. If v_1 and v_3 are in different components of H_{13}, exchange the colors 1 and 3 in the component of v_1: the coloring stays feasible, and color 1 becomes free for v

4. Otherwise a path of colors 1 and 3 joins v_1 to v_3
   - With v, it closes a cycle that separates v_2 from v_4 in the drawing
   - A path from v_2 to v_4 must cross that cycle, so v_2 and v_4 are in different components of H_{24}
   - Exchanging 2 and 4 in the component of v_2 frees color 2 for v

Proof (3/3)

- Each step uses a search in time

$$O(n+m) = O(n)$$

- There are n steps: $O(n^2)$ time in total

□

The four color theorem

- **Remark (the four color theorem).** Every planar graph can be colored with 4 colors (Appel and Haken, 1977)
- The proof uses a computer
- Robertson, Sanders, Seymour and Thomas (1997) simplified it and gave an algorithm that finds such a coloring in $O(n^2)$ time

The algorithm *PL* for planar graphs

The algorithm *PL* for planar graphs combines the previous results:

- if G has no edge, it uses 1 color;
- if G is bipartite, it uses 2 colors (by the breadth-first search of the bipartite graphs);
- otherwise it uses the 5 colors of the theorem of Heawood.

Corollary 2

PL is a polynomial 2-additive approximation algorithm for the VCP on planar graphs:

$$PL(I) - \chi(G) \leq 2.$$

With the four color algorithm in place of the five color one, the additive error is at most 1.

Proof

- In the first two cases PL is exact
- In the third case G has a cycle of odd length, so $\chi(G) \geq 3$, while $PL(I) \leq 5$: the additive error is at most

$$5 - 3 = 2$$

- With 4 colors it is at most

$$4 - 3 = 1$$

□

The best possible on planar graphs

- The additive error 1 cannot be improved to 0: deciding whether a planar graph has $\chi(G) \leq 3$ is NP-complete
- For the same reason, the ratio $\frac{4}{3}$ of the four color version is the best possible on planar graphs (by the theorem of $\frac{4}{3}$)

Definition 2: interval graph

*A graph $G = (V, E)$ is an **interval graph** if every vertex v is associated with a closed interval $[a_v, b_v]$ of the real line, and $\{u, v\} \in E$ if and only if the two intervals intersect.*

- Interval graphs describe requests over time that conflict when they overlap: lectures that need a room, reservations that need a resource. A color is a room.

Proposition 3

On an interval graph, the greedy coloring with the vertices in order of non-decreasing left endpoint a_v uses $\omega(G)$ colors. It is therefore optimal: $GC(I) = \omega(G) = \chi(G)$.

Proof (1/2)

- Let v be a vertex that receives color k
- The colors $1, \ldots, k-1$ are used by neighbours u of v colored before v, so with

$$a_u \leq a_v$$

1. Since $[a_u, b_u]$ intersects $[a_v, b_v]$ and $a_u \leq a_v$, we have

$$b_u \geq a_v :$$

the point a_v lies in the interval of each of these neighbours, and in the interval of v

Proof (2/2)

2. The k intervals that contain a_v pairwise intersect, so their vertices form a clique and

$$k \leq \omega(G)$$

Hence, by the clique bound,

$$GC(I) \leq \omega(G) \leq \chi(G) \leq GC(I).$$

□

The order matters

- Sorting the intervals takes $O(n \log n)$ time, and the coloring $O(n + m)$
- The order matters: with another order the greedy coloring may use more than $\omega(G)$ colors on an interval graph (see the example on six intervals in the next section)

Summary (1/2)

Bipartite graphs:

- A breadth-first search decides $\chi(G) \leq 2$ in $O(n+m)$ time
- A conflict gives an odd cycle, the certificate of $\chi(G) \geq 3$

Graphs of bounded maximum degree:

- The greedy coloring uses at most $\Delta(G)+1$ colors, in any order
- With a fixed Δ, the ratio is at most $\frac{\Delta+1}{2}$

Summary (2/2)

Planar graphs:

- A vertex of degree at most 5 always exists, and 5 colors suffice (Heawood)
- *PL* has additive error at most 2, or 1 with the four color theorem

Interval graphs:

- The greedy coloring by left endpoint uses $\omega(G)$ colors, hence it is optimal
- The order matters

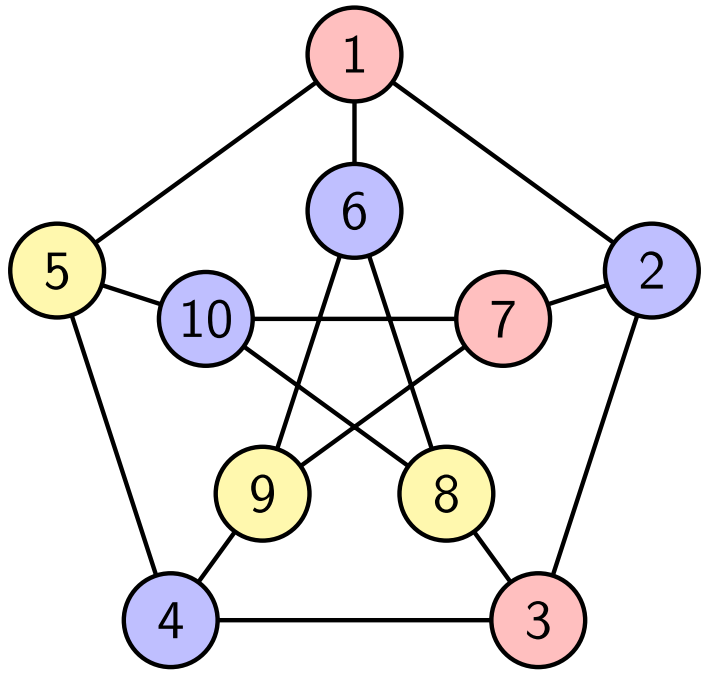
Examples

Introduction: Examples

- We apply the special cases to two small graphs
- First the Petersen graph, the reference instance of the VCP: is it bipartite, and how good is the greedy coloring?
- Then an interval graph with six intervals, to see that the order of the greedy coloring matters

Example: the special cases on the Petersen graph (1/3)

The reference instance of the VCP is the Petersen graph, with $n = 10$, $m = 15$ and $\Delta(G) = 3$.



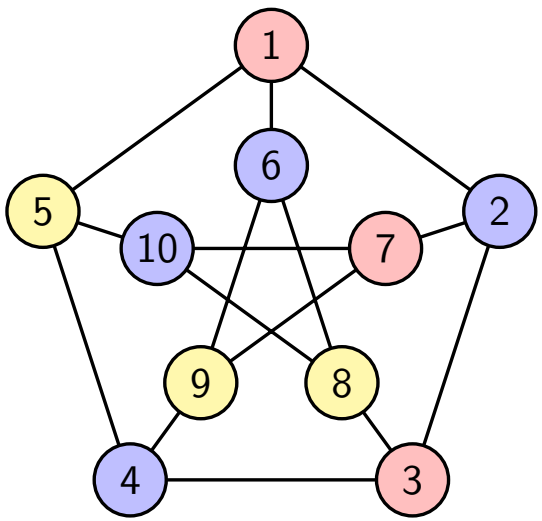
Example: the special cases on the Petersen graph (2/3)

Is it bipartite?

- The search from vertex 1 gives the levels 0 to vertex 1, 1 to the vertices $2, 5, 6$, and 2 to the other six
- The edge $\{3, 4\}$ joins two vertices of level 2
- With the paths $1, 2, 3$ and $1, 5, 4$ it closes the odd cycle $1, 2, 3, 4, 5$, so

$$\chi(G) \geq 3$$

Example: the special cases on the Petersen graph (3/3)



Greedy coloring.

- In the order $1, 2, \ldots, 10$, GC gives the colors

$$1, 2, 1, 2, 3, 2, 1, 3, 3, 2$$

(red, blue, yellow in the figure)

- $GC(I) = 3 \leq \Delta(G) + 1 = 4$
- Together,

$$3 \leq \chi(G) \leq GC(I) = 3 :$$

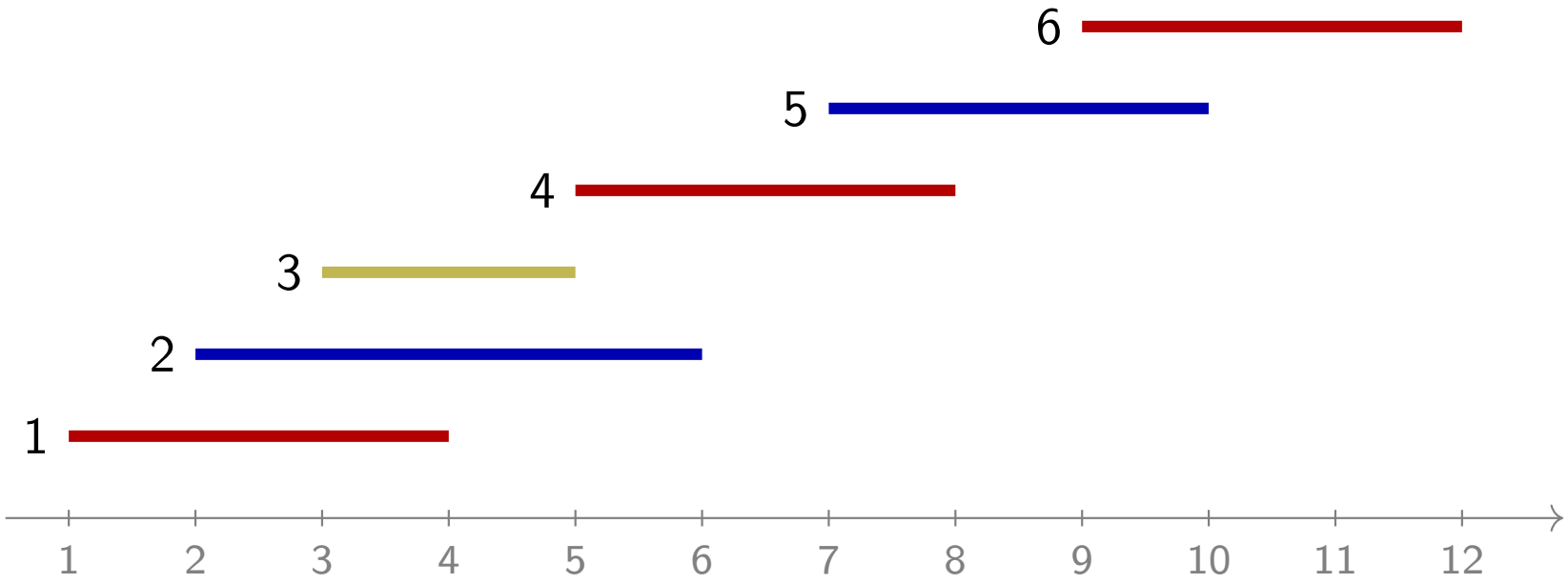
the greedy coloring is optimal on this instance

On the Petersen graph every order would be fine.

On an interval graph, instead,
the order of the greedy coloring matters:
let us see it on six intervals.

Example: the greedy coloring on an interval graph (1/3)

Six intervals, with the edges between intersecting intervals:



$$[a_v, b_v]: \ [1,4],\ [2,6],\ [3,5],\ [5,8],\ [7,10],\ [9,12],$$

$$E = \big\{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\},\{4,5\},\{5,6\}\big\}.$$

Example: the greedy coloring on an interval graph (2/3)

- The left endpoints are already in increasing order
- GC gives the colors

  $$1, 2, 3, 1, 2, 1$$

  (red, blue, yellow in the figure): $GC(I) = 3$
- The intervals $1, 2, 3$ contain the point 3, so $\{1, 2, 3\}$ is a clique and $\omega(G) = 3$
- The coloring is optimal, as the proposition on interval graphs guarantees

Example: the greedy coloring on an interval graph (3/3)

Another order.

- With the order $1, 2, 3, 5, 4, 6$ the colors are $1, 2, 3$, then 1 to vertex 5
- Vertex 4 gets color 4: its neighbours $2, 3, 5$ use the colors $1, 2, 3$
- 4 colors, one more than $\chi(G)$

Summary

The Petersen graph:

- The breadth-first search finds the odd cycle $1, 2, 3, 4, 5$: $\chi(G) \geq 3$
- The greedy coloring in the order $1, 2, \ldots, 10$ uses 3 colors: it is optimal

The interval graph:

- By left endpoint, the greedy coloring uses $\omega(G) = 3$ colors
- With the order $1, 2, 3, 5, 4, 6$ it uses 4

Sources

Introduction: Sources

- We list the papers where the results of this chapter were proved
- First the negative results on the approximation of the chromatic number
- Then the positive results on the special classes of graphs

Sources (1/2)

- M.R. Garey, D.S. Johnson, L. Stockmeyer, Some simplified NP-complete graph problems, *Theoretical Computer Science* 1(3):237–267, 1976.
- M.R. Garey, D.S. Johnson, The complexity of near-optimal graph coloring, *Journal of the ACM* 23(1):43–49, 1976.
- U. Feige, J. Kilian, Zero knowledge and the chromatic number, *Journal of Computer and System Sciences* 57(2):187–199, 1998.
- D. Zuckerman, Linear degree extractors and the inapproximability of max clique and chromatic number, *Theory of Computing* 3:103–128, 2007.

Sources (2/2)

- R.L. Brooks, On colouring the nodes of a network, *Proceedings of the Cambridge Philosophical Society* 37(2):194–197, 1941.
- P.J. Heawood, Map-colour theorem, *Quarterly Journal of Pure and Applied Mathematics* 24, 1890.
- J. Hopcroft, R. Tarjan, Efficient planarity testing, *Journal of the ACM* 21(4):549–568, 1974.
- K. Appel, W. Haken, Every planar map is four colorable, *Illinois Journal of Mathematics* 21, 1977.
- N. Robertson, D. Sanders, P. Seymour, R. Thomas, The four-colour theorem, *Journal of Combinatorial Theory, Series B* 70(1):2–44, 1997.

Summary

Negative results:

- Garey, Johnson and Stockmeyer; Garey and Johnson; Feige and Kilian; Zuckerman

Positive results:

- Brooks; Heawood; Hopcroft and Tarjan; Appel and Haken; Robertson, Sanders, Seymour and Thomas

Typical mistakes

Introduction: Typical mistakes

- In this section we list the most common mistakes on the approximation of the VCP
- They concern the meaning of the bounds, the four color theorem, the order of the intervals and the odd cycle
- Recognizing them helps to state the results correctly

Observation 1: typical mistakes (1/2)

- ***Calling the bound $\Delta(G)+1$ a performance guarantee.*** *It compares the greedy coloring with $\Delta(G)+1$, not with $\chi(G)$; it becomes a constant ratio only on graphs of bounded maximum degree.*
- ***Reading the theorem of $\frac{4}{3}$ as an upper bound.*** *It says that ratios below $\frac{4}{3}$ are impossible, not that a ratio of $\frac{4}{3}$ is achievable: by the result of Zuckerman, no constant ratio is.*
- ***Using the four color theorem as an exact algorithm.*** *It colors every planar graph with at most 4 colors, but deciding whether 3 colors suffice remains NP-complete on planar graphs.*

Observation: typical mistakes (2/2)

- ***Sorting the intervals by right endpoint, or by length.*** *The proposition on interval graphs uses the order of the left endpoints: the colors used before v all contain the point a_v.*
- ***Forgetting the odd cycle.*** *A breadth-first search that finds a conflict proves $\chi(G) \geq 3$ only through the odd cycle it closes; the cycle is the certificate.*

Summary

The meaning of the results:

- $\Delta(G) + 1$ is not a ratio with $\chi(G)$, except on graphs of bounded degree
- The theorem of $\frac{4}{3}$ is a lower bound on the ratio, not an achievable ratio
- Four colors always suffice on planar graphs, but three are hard to decide

The details of the algorithms:

- Interval graphs: sort by left endpoint
- Bipartite test: the odd cycle is the certificate

In summary

Introduction: In summary

- We collect the key ideas of the chapter in a few points
- We close with a table of the results on the classes of graphs, and with the dashboard of the reference instance

Summary (1/3)

- Unless $P = NP$, no polynomial algorithm colors every graph with fewer than $\frac{4}{3}\chi(G)$ colors: it would decide 3-colorability
- By the result of Zuckerman, not even the ratio $n^{1-\varepsilon}$ is possible: the VCP is not approximable
- Bipartite graphs and interval graphs are colored optimally in polynomial time
- On graphs of bounded degree the greedy coloring has the ratio $\frac{\Delta+1}{2}$
- On planar graphs the additive error is at most 2, or 1 with the four color theorem

Summary (2/3)

graphs	algorithm	result
all graphs	any polynomial	no constant ratio, unless $P = NP$
bipartite	breadth-first search	exact, $O(n + m)$
maximum degree $\leq \Delta$	greedy coloring	at most $\Delta + 1$ colors
planar	*PL*	additive error ≤ 2 (or ≤ 1)
interval	greedy by left endpoint	exact, $\omega(G)$ colors

Summary (3/3)

Dashboard of the reference instance.

Instance	Lower bound	Heuristic value	Optimal value
Petersen graph	3, odd cycle	GC 3	3