



SAPIENZA
UNIVERSITÀ DI ROMA

Algorithms

Volume 3 — Asymptotic notation and growth

How the functions of a natural variable that count the steps of an algorithm are compared: the asymptotic notations Theta, big O, Omega, little o and omega, their properties and the limit criterion; then the families of increasing functions (logarithms, powers, exponentials, factorial), the hierarchy of infinities and the comparison of orders of growth written with the symbols. Limits and functions of a natural variable are taken from the course of Mathematical Analysis 1.

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Asymptotic notation

Introduction

The number of steps executed by an algorithm is a function of a natural number: the size of its input. Two algorithms that solve the same problem are compared by looking at how fast these functions grow when the input grows, neglecting constant factors and the values on a few small inputs. In this chapter we give the precise language used to express these comparisons: the **asymptotic notations** Θ , $\mathcal{O}$, Ω , o and ω .

Prerequisites. The functions of a natural variable are the **sequences** of the course *Mathematical Analysis 1* (<https://fabiofurini.github.io/mathematical-analysis-1/>, Italian version <https://fabiofurini.github.io/analisi-matematica-1/>). Their limits, and the theorems used to compute them, are not explained again in this chapter: we use the definition of limit and the rules to compute limits of the part *Limits of sequences* of that course.

The plan of the chapter is the following.

- In Section 9.1 we define the notations Θ , $\mathcal{O}$ and Ω , we interpret them graphically and we prove that Θ is the same as $\mathcal{O}$ and Ω together. We also study the sum and the maximum of two functions.
- In Section 9.2 we define the notations o and ω , which express bounds that are **not** asymptotically tight.
- In Section 9.3 we link the five notations to the limit of the ratio $\frac{f(n)}{g(n)}$. When the limit exists, its value decides which of the five notations hold (Proposition 9.18, the **limit criterion**); when it does not exist, we use the definitions.
- In Section 9.4 we prove the relational properties of the notations (transitivity, reflexivity, symmetry and transpose symmetry), which make the asymptotic comparison of two functions similar to the comparison of two real numbers, with one important difference.
- In Section 9.5 we see how the notation is used in the analysis of algorithms of the chapters «Input size» and «Running time».

Every constant (c, c_1, c_2, n_1) used in the examples of the chapter has been checked numerically with a script, for all the values of n from the threshold n_1 up to at least $2 \cdot 10^5$.

Throughout the chapter we use the following terminology.

Definition 9.1: asymptotically non-negative and asymptotically positive functions

A function f of a natural variable is **asymptotically non-negative** if there exists $\tilde{n} \in \mathbb{N}$ such that $f(n) \geq 0$ for all $n \geq \tilde{n}$. It is **asymptotically positive** if there exists $\tilde{n} \in \mathbb{N}$ such that $f(n) > 0$ for all $n \geq \tilde{n}$.

9.1 Θ , $\mathcal{O}$ and Ω notations

9.1.1 Definitions

Definition 9.2: Θ -notation

Given two functions f and g of a natural variable, we write $f(n) = \Theta(g(n))$ if there exist $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n), \quad \forall n \geq n_1. \quad (9.1)$$

- If $f(n) = \Theta(g(n))$, then, for sufficiently large values of n , the function f is bounded from above and from below by the function g scaled by two positive constants.
- In this case we say that the function g is an **asymptotic tight bound** for the function f .
- We write $f(n) = \Theta(1)$ to mean that the values $f(n)$ are, for sufficiently large n , bounded from above and from below by two positive constants (here g is the constant function $g(n) = 1$).

Definition 9.3: $\mathcal{O}$ -notation

Given two functions f and g of a natural variable, we write $f(n) = \mathcal{O}(g(n))$ if there exist $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq f(n) \leq c g(n), \quad \forall n \geq n_1. \quad (9.2)$$

- If $f(n) = \mathcal{O}(g(n))$, then, for sufficiently large values of n , the function f is bounded from above by the function g scaled by a positive constant.
- In this case we say that the function g is an **asymptotic upper bound** for the function f .

Definition 9.4: Ω -notation

Given two functions f and g of a natural variable, we write $f(n) = \Omega(g(n))$ if there exist $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c g(n) \leq f(n), \quad \forall n \geq n_1. \quad (9.3)$$

- If $f(n) = \Omega(g(n))$, then, for sufficiently large values of n , the function f is bounded from below by the function g scaled by a positive constant.
- In this case we say that the function g is an **asymptotic lower bound** for the function f .

If $f(n) = \Theta(g(n))$, $f(n) = \mathcal{O}(g(n))$ or $f(n) = \Omega(g(n))$ then, since the constants c_1 and c are positive, we have $g(n) \geq 0$ and $f(n) \geq 0$ for all $n \geq n_1$ (for $\mathcal{O}$, the inequality $0 \leq f(n) \leq c g(n)$ gives $g(n) \geq 0$ as well). Therefore, the two functions f and g are **asymptotically non-negative**.

- To prove one of these statements we **determine** the constants (c , or c_1 and c_2) and the threshold n_1 . In general, the threshold n_1 is not the smallest possible one and the constants are not unique: any larger threshold, any smaller c_1 and any larger c_2 (or c in the definition of $\mathcal{O}$) also work.
- To prove that a statement is false, we proceed by contradiction: we suppose that the constants exist and we derive an inequality that cannot hold for arbitrarily large n .

9.1.2 Examples of asymptotic tight bounds**Example 9.1: asymptotic tight bound**

- To prove that

$$-3n + \frac{1}{2}n^2 = \Theta(n^2)$$

we determine $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^2 \leq -3n + \frac{1}{2}n^2 \leq c_2 n^2, \quad \forall n \geq n_1.$$

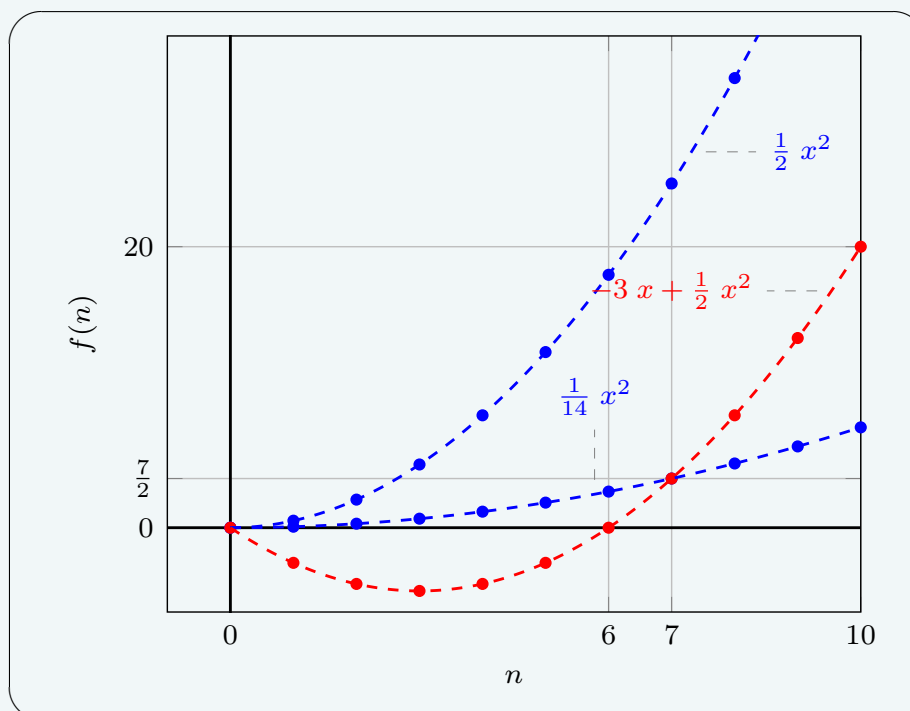
With $n \geq 1$, dividing by n^2 and since $c_1 > 0$, this is equivalent to

$$0 < c_1 \leq \frac{1}{2} - \frac{3}{n} \leq c_2, \quad \forall n \geq n_1.$$

The middle term increases with n and it is always smaller than $\frac{1}{2}$. Thus, by choosing for example $c_1 = \frac{1}{14}$, $c_2 = \frac{1}{2}$ and $n_1 = 7$ (we selected $\frac{1}{14}$ since $\frac{1}{14} = \frac{1}{2} - \frac{3}{7}$), we have

$$0 \leq \frac{1}{14} n^2 \leq -3n + \frac{1}{2} n^2 \leq \frac{1}{2} n^2, \quad \forall n \geq 7.$$

For $n = 6$ the lower bound fails, since $-18 + 18 = 0 < \frac{36}{14}$.



- The previous proof also proves that

$$-3n + \frac{1}{2}n^2 = \mathcal{O}(n^2) \quad \text{and} \quad -3n + \frac{1}{2}n^2 = \Omega(n^2).$$

Example 9.2: asymptotic tight bound

- To prove that

$$n^2 (2 - \sin n) = \Theta(n^2)$$

we determine $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

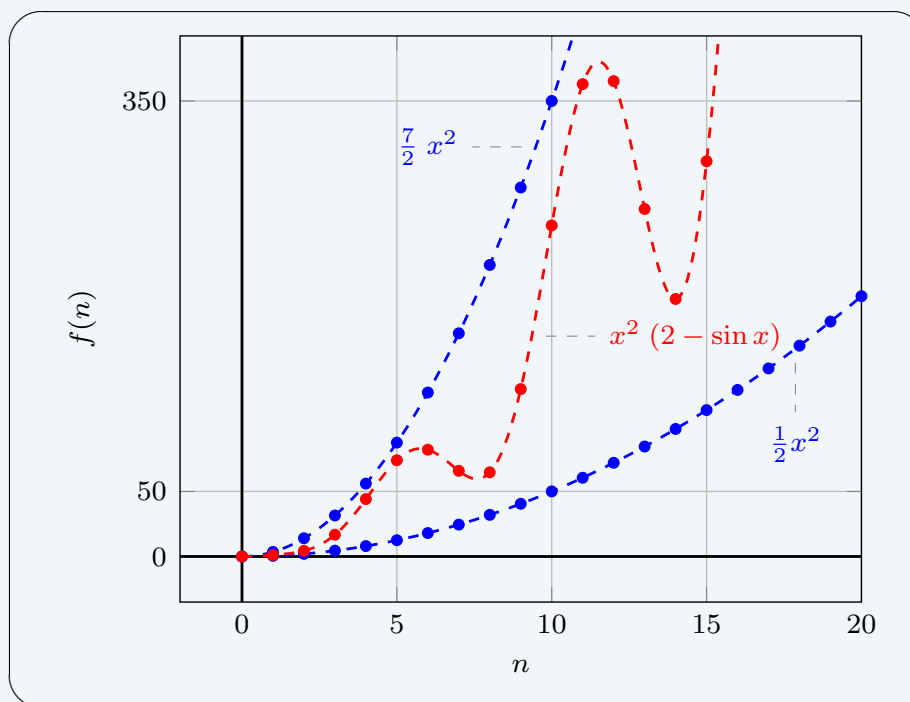
$$0 \leq c_1 n^2 \leq n^2 (2 - \sin n) \leq c_2 n^2, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^2 and since $c_1 > 0$, this is equivalent to

$$0 < c_1 \leq 2 - \sin n \leq c_2, \quad \forall n \geq n_1.$$

Since $-1 \leq \sin n \leq 1$, we have $1 \leq 2 - \sin n \leq 3$. Thus, by choosing for example $c_1 = \frac{1}{2}$, $c_2 = \frac{7}{2}$ and $n_1 = 1$, we have

$$0 \leq \frac{1}{2} n^2 \leq n^2 (2 - \sin n) \leq \frac{7}{2} n^2, \quad \forall n \geq 1.$$

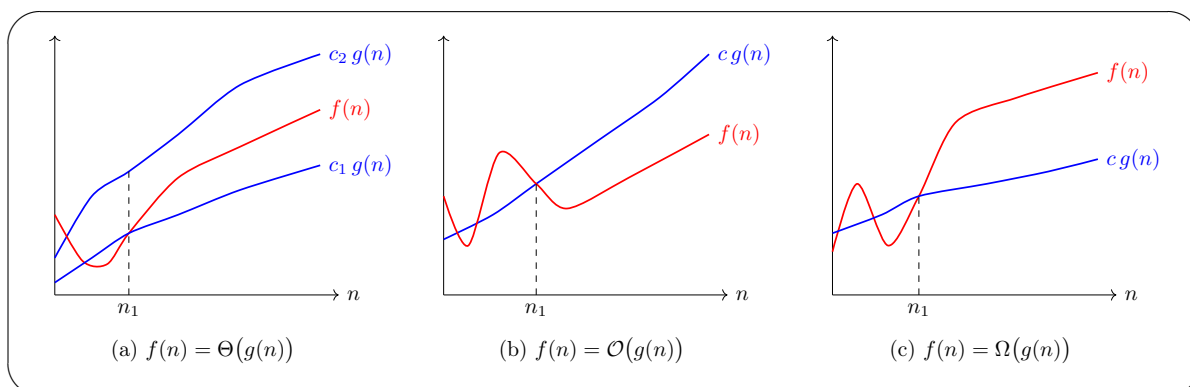


- The previous proof also proves that

$$n^2 (2 - \sin n) = \mathcal{O}(n^2) \quad \text{and} \quad n^2 (2 - \sin n) = \Omega(n^2).$$

9.1.3 Graphical interpretation

The next figure gives a graphical interpretation of the three notations.



- In (a), from n_1 on, the values of f lie between $c_1 g(n)$ and $c_2 g(n)$; in (b) they lie below $c g(n)$; in (c) they lie above $c g(n)$. Before n_1 nothing is required.
- In the figure, n_1 is the smallest possible threshold, but any larger value would also work. The figure also illustrates one possible choice of the constants c_1, c_2 and c , but other valid choices exist.

9.1.4 Θ versus $\mathcal{O}$ and Ω **Observation 9.5: Θ is the same as $\mathcal{O}$ and Ω**

Given two functions f and g of a natural variable, we have

$$f(n) = \Theta(g(n)) \iff f(n) = \mathcal{O}(g(n)) \text{ and } f(n) = \Omega(g(n)). \quad (9.4)$$

In particular,

$$f(n) = \Theta(g(n)) \implies f(n) = \mathcal{O}(g(n)) \text{ and } f(n) = \Theta(g(n)) \implies f(n) = \Omega(g(n)). \quad (9.5)$$

Proof. If $f(n) = \Theta(g(n))$, the two inequalities $0 \leq f(n) \leq c_2 g(n)$ and $0 \leq c_1 g(n) \leq f(n)$ contained in (9.1) give $f(n) = \mathcal{O}(g(n))$ (with $c = c_2$) and $f(n) = \Omega(g(n))$ (with $c = c_1$), with the same threshold n_1 .

Conversely, let $0 \leq f(n) \leq c' g(n)$ for all $n \geq n'_1$, and $0 \leq c'' g(n) \leq f(n)$ for all $n \geq n''_1$. By choosing $c_1 = c''$, $c_2 = c'$ and $n_1 = \max\{n'_1, n''_1\}$, we obtain

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n), \quad \forall n \geq n_1,$$

that is, $f(n) = \Theta(g(n))$. □

- It follows that “ $f(n) = \Theta(g(n))$ ” is a sufficient condition for “ $f(n) = \mathcal{O}(g(n))$ ”, and “ $f(n) = \mathcal{O}(g(n))$ ” is a necessary condition for “ $f(n) = \Theta(g(n))$ ”. By contraposition we also have

$$f(n) \neq \mathcal{O}(g(n)) \implies f(n) \neq \Theta(g(n)).$$

The implication does not work in the other direction (see the next two examples), that is,

$$f(n) = \mathcal{O}(g(n)) \not\Rightarrow f(n) = \Theta(g(n)).$$

- In the same way, “ $f(n) = \Theta(g(n))$ ” is a sufficient condition for “ $f(n) = \Omega(g(n))$ ”, and “ $f(n) = \Omega(g(n))$ ” is a necessary condition for “ $f(n) = \Theta(g(n))$ ”. By contraposition we also have

$$f(n) \neq \Omega(g(n)) \implies f(n) \neq \Theta(g(n)),$$

and the implication does not work in the other direction (see the third example below):

$$f(n) = \Omega(g(n)) \not\Rightarrow f(n) = \Theta(g(n)).$$

Example 9.3: asymptotic upper bound that is not tight

- To prove that

$$3n^2 = \mathcal{O}(n^3)$$

we determine $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq 3n^2 \leq cn^3, \quad \forall n \geq n_1.$$

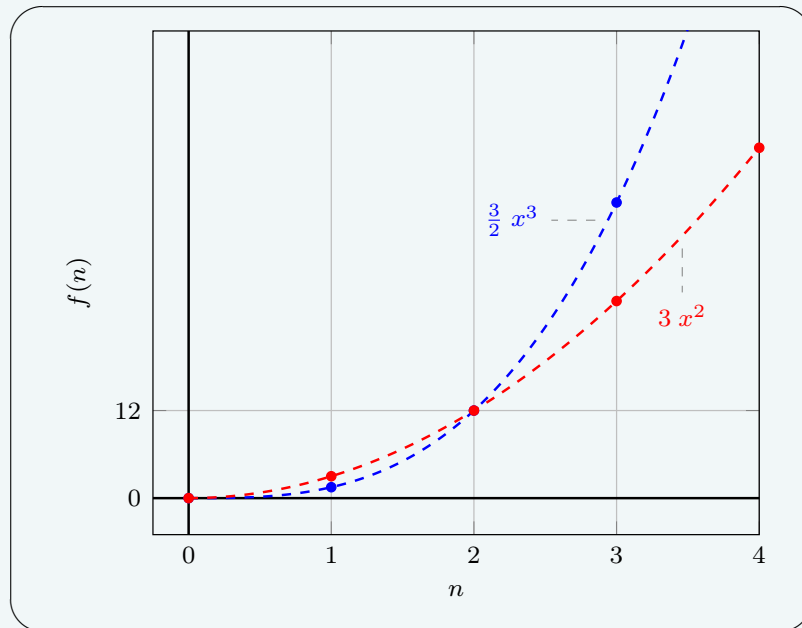
With $n \geq 1$, dividing by n^3 , this is equivalent to

$$0 \leq \frac{3}{n} \leq c, \quad \forall n \geq n_1.$$

Thus, by choosing for example $c = \frac{3}{2}$ and $n_1 = 2$, we have

$$0 \leq 3n^2 \leq \frac{3}{2}n^3, \quad \forall n \geq 2.$$

(For $n = 1$ the inequality fails, since $3 > \frac{3}{2}$.)



- To prove that

$$3n^2 \neq \Theta(n^3)$$

we proceed by contradiction and we suppose that there exist $c_1 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^3 \leq 3n^2, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^3 and since $c_1 > 0$, this is equivalent to

$$0 < c_1 \leq \frac{3}{n}, \quad \forall n \geq n_1,$$

which cannot hold for arbitrarily large n , since c_1 is a positive constant (it fails for every $n > \frac{3}{c_1}$).

Example 9.4: asymptotic upper bound that is not tight

- To prove that

$$n^2 \sin^2 n = \mathcal{O}(n^2)$$

we determine $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

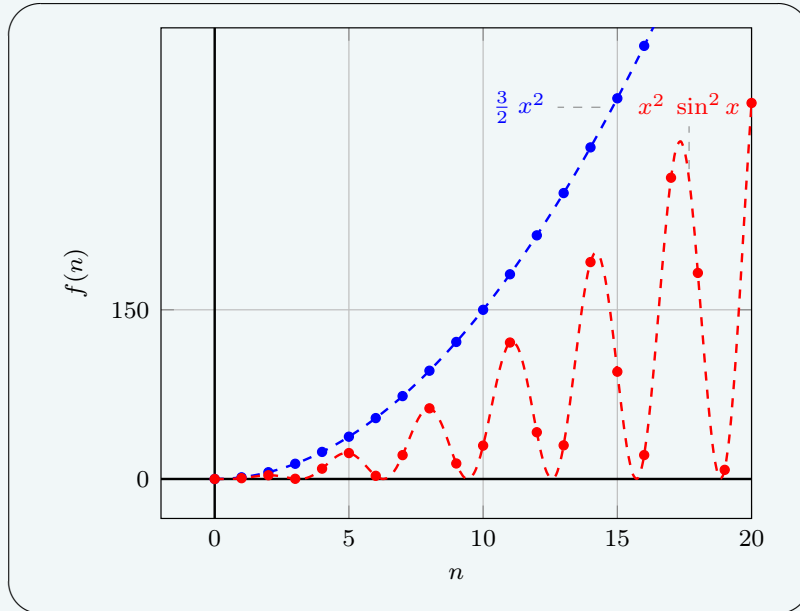
$$0 \leq n^2 \sin^2 n \leq c n^2, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^2 , this is equivalent to

$$0 \leq \sin^2 n \leq c, \quad \forall n \geq n_1.$$

Since $0 \leq \sin^2 n \leq 1$, by choosing for example $c = \frac{3}{2}$ and $n_1 = 1$ we have

$$0 \leq n^2 \sin^2 n \leq \frac{3}{2} n^2, \quad \forall n \geq 1.$$



- To prove that

$$n^2 \sin^2 n \neq \Theta(n^2)$$

we proceed by contradiction and we suppose that there exist $c_1 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^2 \leq n^2 \sin^2 n, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^2 and since $c_1 > 0$, this is equivalent to

$$0 < c_1 \leq \sin^2 n, \quad \forall n \geq n_1,$$

which cannot hold for arbitrarily large n , since c_1 is a positive constant.^a

^aSince π is irrational, the values $n \bmod \pi$, $n \in \mathbb{N}$, are dense in $[0, \pi]$. Therefore, for every $\varepsilon \in \mathbb{R}_{>0}$ there exist arbitrarily large integers n such that $|\sin n| < \varepsilon$. By choosing $\varepsilon = \frac{\sqrt{c_1}}{2}$, we obtain $\sin^2 n < \frac{c_1}{4} < c_1$, which contradicts $c_1 \leq \sin^2 n$ for all sufficiently large n .

Example 9.5: asymptotic lower bound that is not tight

- To prove that

$$2n^3 = \Omega(n^2)$$

we determine $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

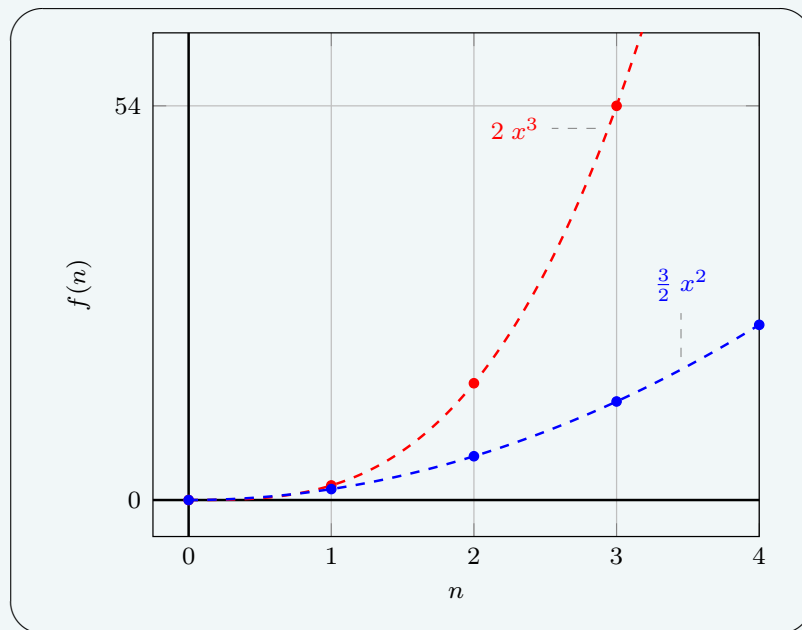
$$0 \leq c n^2 \leq 2n^3, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^2 and since $c > 0$, this is equivalent to

$$0 < c \leq 2n, \quad \forall n \geq n_1.$$

Thus, by choosing for example $c = \frac{3}{2}$ and $n_1 = 1$, we have

$$0 \leq \frac{3}{2} n^2 \leq 2n^3, \quad \forall n \geq 1.$$



- To prove that

$$2n^3 \neq \Theta(n^2)$$

we proceed by contradiction and we suppose that there exist $c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq 2n^3 \leq c_2 n^2, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^2 , this is equivalent to

$$0 \leq 2n \leq c_2, \quad \forall n \geq n_1,$$

which cannot hold for arbitrarily large n , since c_2 is a constant (it fails for every $n > \frac{c_2}{2}$).

9.1.5 Sum and maximum of two functions

Observation 9.6: adding the bound to the function

Given two functions f and g of a natural variable, we have

$$f(n) = \Theta(g(n)) \implies f(n) + g(n) = \Theta(g(n)). \quad (9.6)$$

Proof. If $f(n) = \Theta(g(n))$, there exist $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n), \quad \forall n \geq n_1.$$

Moreover, since $c_1 > 0$, we have $g(n) \geq 0$ for all $n \geq n_1$. Then, by adding $g(n)$, we get

$$0 \leq \underbrace{c_1 g(n) + g(n)}_{=\underbrace{(c_1 + 1)}_{>0} g(n)} \leq f(n) + g(n) \leq \underbrace{c_2 g(n) + g(n)}_{=\underbrace{(c_2 + 1)}_{>0} g(n)}, \quad \forall n \geq n_1.$$

Accordingly, $f(n) + g(n) = \Theta(g(n))$, with the constants $c_1 + 1$ and $c_2 + 1$ and the same threshold n_1 . $\square$

Example 9.6: asymptotic tight bound of a sum

- In Example 9.1 we proved that

$$-3n + \frac{1}{2}n^2 = \Theta(n^2)$$

with $c_1 = \frac{1}{14}$, $c_2 = \frac{1}{2}$ and $n_1 = 7$. Then, according to (9.6), we have

$$-3n + \frac{1}{2}n^2 + n^2 = \Theta(n^2),$$

with the constants $c_1 + 1 = \frac{15}{14}$, $c_2 + 1 = \frac{3}{2}$ and $n_1 = 7$.

- In Example 9.2 we proved that

$$n^2 (2 - \sin n) = \Theta(n^2)$$

with $c_1 = \frac{1}{2}$, $c_2 = \frac{7}{2}$ and $n_1 = 1$. Then, according to (9.6), we have

$$n^2 (2 - \sin n) + n^2 = \Theta(n^2),$$

with the constants $\frac{3}{2}$, $\frac{9}{2}$ and $n_1 = 1$.

- By the implication (9.6), the statement “ $f(n) = \Theta(g(n))$ ” is a sufficient condition for “ $f(n) + g(n) = \Theta(g(n))$ ”, and “ $f(n) + g(n) = \Theta(g(n))$ ” is a necessary condition for “ $f(n) = \Theta(g(n))$ ”. By contraposition of (9.6) we also have

$$f(n) + g(n) \neq \Theta(g(n)) \implies f(n) \neq \Theta(g(n)).$$

The implication (9.6) does not work in the other direction (see the next counterexample), that is,

$$f(n) + g(n) = \Theta(g(n)) \not\Rightarrow f(n) = \Theta(g(n)).$$

Example 9.7: the converse of Observation 9.6 is false

Let $f(n) = 10n$ and $g(n) = \frac{1}{10}n^2$.

- To prove that

$$f(n) + g(n) = 10n + \frac{1}{10}n^2 = \Theta\left(\frac{1}{10}n^2\right) = \Theta(g(n))$$

we determine $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

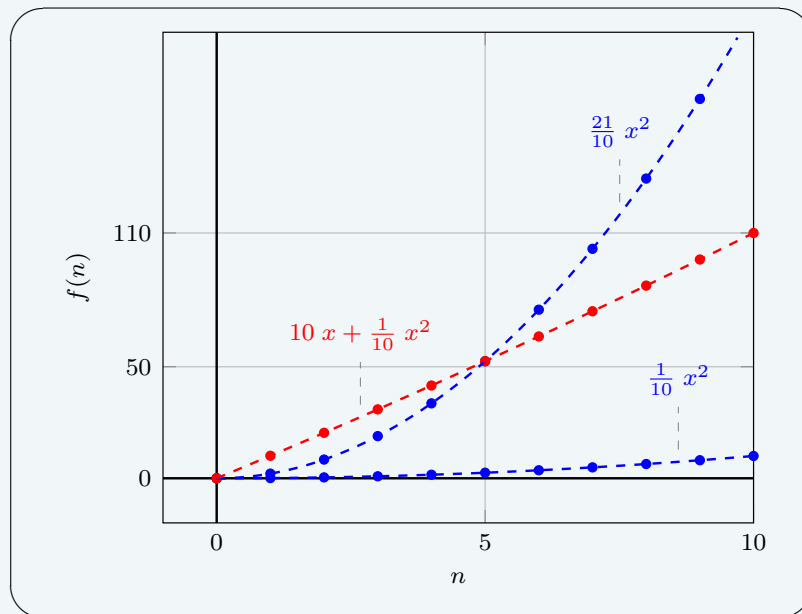
$$0 \leq c_1 \frac{1}{10}n^2 \leq 10n + \frac{1}{10}n^2 \leq c_2 \frac{1}{10}n^2, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by $\frac{1}{10}n^2$ and since $c_1 > 0$, this is equivalent to

$$0 < c_1 \leq \frac{100}{n} + 1 \leq c_2, \quad \forall n \geq n_1.$$

The middle term decreases with n and it is always larger than 1; for $n = 5$ it is equal to 21. Thus, by choosing for example $c_1 = 1$, $c_2 = 21$ and $n_1 = 5$, we have

$$0 \leq \frac{1}{10}n^2 \leq 10n + \frac{1}{10}n^2 \leq \frac{21}{10}n^2, \quad \forall n \geq 5.$$



- To prove that

$$f(n) = 10n \neq \Theta\left(\frac{1}{10}n^2\right) = \Theta(g(n))$$

we proceed by contradiction and we suppose that there exist $c_1 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 \frac{1}{10}n^2 \leq 10n, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by $\frac{1}{10} n^2$ and since $c_1 > 0$, this is equivalent to

$$0 < c_1 \leq \frac{100}{n}, \quad \forall n \geq n_1,$$

which cannot hold for arbitrarily large n , since c_1 is a positive constant.

When we add two functions, the asymptotic behaviour of the sum is determined by the larger of the two. The next two observations make this precise.

Observation 9.7: sum and maximum (1)

Given two asymptotically non-negative functions f and g of a natural variable, we have

$$f(n) + g(n) = \Theta(\max\{f(n), g(n)\}).$$

Proof. Since f and g are asymptotically non-negative, there exists $\tilde{n} \in \mathbb{N}$ such that $f(n) \geq 0$ and $g(n) \geq 0$ for all $n \geq \tilde{n}$. Then the sum is at least the larger of the two terms and at most twice the larger of the two terms:

$$0 \leq \max\{f(n), g(n)\} \leq f(n) + g(n) \leq 2 \max\{f(n), g(n)\}, \quad \forall n \geq \tilde{n}.$$

This shows that the constants $c_1 = 1$, $c_2 = 2$ and $n_1 = \tilde{n}$ satisfy

$$0 \leq c_1 \max\{f(n), g(n)\} \leq f(n) + g(n) \leq c_2 \max\{f(n), g(n)\}, \quad \forall n \geq n_1.$$

Therefore, $f(n) + g(n) = \Theta(\max\{f(n), g(n)\})$. $\square$

Observation 9.8: sum and maximum (2)

Given two asymptotically non-negative functions f and g of a natural variable, we have

$$\max\{f(n), g(n)\} = \Theta(f(n) + g(n)).$$

Proof. Let $\tilde{n} \in \mathbb{N}$ be such that $f(n) \geq 0$ and $g(n) \geq 0$ for all $n \geq \tilde{n}$. The larger of two non-negative numbers is at least their average and at most their sum:

$$0 \leq \frac{1}{2} (f(n) + g(n)) \leq \max\{f(n), g(n)\} \leq f(n) + g(n), \quad \forall n \geq \tilde{n}.$$

This shows that the constants $c_1 = \frac{1}{2}$, $c_2 = 1$ and $n_1 = \tilde{n}$ satisfy

$$0 \leq c_1 (f(n) + g(n)) \leq \max\{f(n), g(n)\} \leq c_2 (f(n) + g(n)), \quad \forall n \geq n_1.$$

Therefore, $\max\{f(n), g(n)\} = \Theta(f(n) + g(n))$. $\square$

Example 9.8: sum and maximum

By Observations 9.7 and 9.8 we have

$$\frac{1}{2}n^2 + 2^n = \Theta\left(\max\left\{\frac{1}{2}n^2, 2^n\right\}\right) \quad \text{and} \quad \max\left\{\frac{1}{2}n^2, 2^n\right\} = \Theta\left(\frac{1}{2}n^2 + 2^n\right).$$

In this case $\max\left\{\frac{1}{2}n^2, 2^n\right\} = 2^n$ for every $n \in \mathbb{N}$, so that $\frac{1}{2}n^2 + 2^n = \Theta(2^n)$, with $c_1 = 1$, $c_2 = 2$ and $n_1 = 0$.

9.2 o and ω notations

The asymptotic bounds expressed by the $\mathcal{O}$ - and Ω -notations may or may not be asymptotically tight. We use the o -notation to denote an asymptotic upper bound that is **not asymptotically tight**, and the ω -notation to denote an asymptotic lower bound that is **not asymptotically tight**.

Example 9.9: asymptotically tight and non-tight upper bounds

- We have $3n^3 = \mathcal{O}(n^3)$ and $3n^3 = \Theta(n^3)$: n^3 is an asymptotic upper bound for $3n^3$, and it is also an asymptotically tight bound.
- We have $3n^3 = \mathcal{O}(n^4)$ but $3n^3 \neq \Theta(n^4)$: n^4 is an asymptotic upper bound for $3n^3$, but it is not an asymptotically tight bound.

Definition 9.9: o -notation

Given two functions f and g of a natural variable, we write $f(n) = o(g(n))$ if for all $c \in \mathbb{R}_{>0}$ there exists $n_1 \in \mathbb{N}$ such that

$$0 \leq f(n) < c g(n), \quad \forall n \geq n_1. \quad (9.7)$$

- If $f(n) = o(g(n))$, then, for sufficiently large values of n , the function f is bounded from above by the function g scaled by **any** positive constant, that is, the function f becomes insignificant relative to the function g .
- In this case we say that the function g is an **asymptotic upper bound that is not tight** for the function f .
- The difference with the $\mathcal{O}$ -notation is in the quantifier on the constant: for $\mathcal{O}$ the inequality must hold for **one** constant c , for o it must hold for **every** constant c . The threshold n_1 depends on c : the smaller c , the larger, in general, n_1 .

Definition 9.10: ω -notation

Given two functions f and g of a natural variable, we write $f(n) = \omega(g(n))$ if for all

$c \in \mathbb{R}_{>0}$ there exists $n_1 \in \mathbb{N}$ such that

$$0 \leq c g(n) < f(n), \quad \forall n \geq n_1. \quad (9.8)$$

- If $f(n) = \omega(g(n))$, then, for sufficiently large values of n , the function f is bounded from below by the function g scaled by **any** positive constant, that is, the function f becomes arbitrarily large relative to the function g .
- In this case we say that the function g is an **asymptotic lower bound that is not tight** for the function f .

If $f(n) = o(g(n))$ or $f(n) = \omega(g(n))$ then, since $c > 0$, we have $g(n) \geq 0$ and $f(n) \geq 0$ for all $n \geq n_1$. Therefore, the two functions f and g are **asymptotically non-negative**.

Example 9.10: asymptotic upper bound that is not tight

- To prove that

$$3n = o(n^2)$$

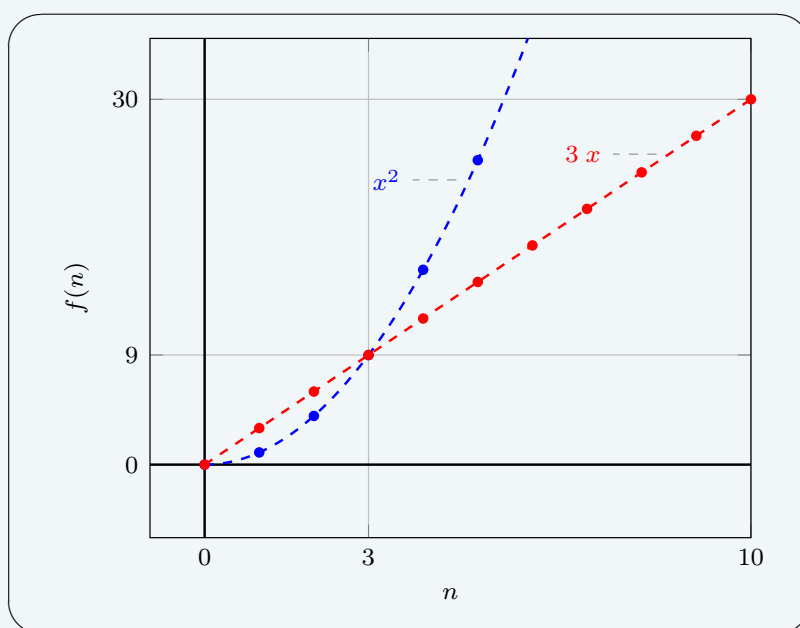
we need to verify that for every $c \in \mathbb{R}_{>0}$ there exists $n_1 \in \mathbb{N}$ such that

$$0 \leq 3n < c n^2, \quad \forall n \geq n_1.$$

For $n \geq 1$, dividing by n^2 , the condition is equivalent to

$$0 \leq \frac{3}{n} < c, \quad \forall n \geq n_1.$$

This holds whenever $n > \frac{3}{c}$, so we can choose any $n_1 > \frac{3}{c}$, for example $n_1 = \lfloor \frac{3}{c} \rfloor + 1$. For instance, $c = 1$ gives $n_1 = 4$, $c = \frac{1}{2}$ gives $n_1 = 7$, and $c = \frac{1}{10}$ gives $n_1 = 31$.



- The previous proof also proves that

$$3n = \mathcal{O}(n^2)$$

(it is enough to take one value of c , for example $c = 1$ with $n_1 = 4$).

- To prove that

$$3n \neq \Theta(n^2)$$

we proceed by contradiction and we suppose that there exist $c_1 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^2 \leq 3n, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^2 and since $c_1 > 0$, this is equivalent to

$$0 < c_1 \leq \frac{3}{n}, \quad \forall n \geq n_1,$$

which cannot hold for arbitrarily large n , since c_1 is a positive constant.

Example 9.11: asymptotic lower bound that is not tight

- To prove that

$$\frac{1}{5}n^2 = \omega(n)$$

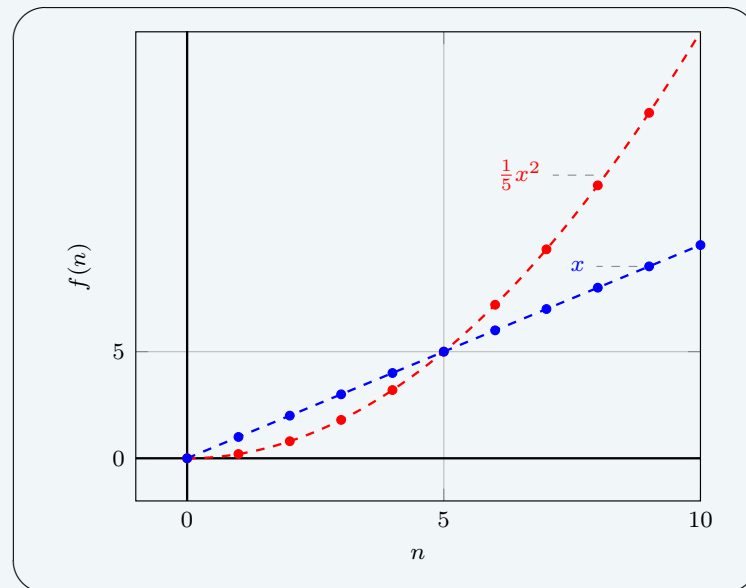
we need to verify that for every $c \in \mathbb{R}_{>0}$ there exists $n_1 \in \mathbb{N}$ such that

$$0 \leq cn < \frac{1}{5}n^2, \quad \forall n \geq n_1.$$

For $n \geq 1$, dividing by n , the condition is equivalent to

$$0 < c < \frac{1}{5}n, \quad \forall n \geq n_1.$$

This holds whenever $n > 5c$, so we can choose any $n_1 > 5c$, for example $n_1 = \lfloor 5c \rfloor + 1$. For instance, $c = 1$ gives $n_1 = 6$ and $c = 10$ gives $n_1 = 51$.



- The previous proof also proves that

$$\frac{1}{5}n^2 = \Omega(n).$$

- To prove that

$$\frac{1}{5}n^2 \neq \Theta(n)$$

we proceed by contradiction and we suppose that there exist $c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq \frac{1}{5}n^2 \leq c_2 n, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n , this is equivalent to

$$0 \leq \frac{n}{5} \leq c_2, \quad \forall n \geq n_1,$$

which cannot hold for arbitrarily large n , since c_2 is a constant.

9.3 Limits and asymptotic notations

When the ratio $\frac{f(n)}{g(n)}$ has a limit, its value tells us immediately which notations hold. In this section the functions f and g are **asymptotically positive**, so that the ratio is defined for all sufficiently large n . The limits are computed with the rules of Mathematical Analysis 1.

9.3.1 Limits and the notations o and ω

Observation 9.11: limit equal to zero

Given two asymptotically positive functions f and g of a natural variable, we have

$$\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = 0^+ \iff f(n) = o(g(n)). \quad (9.9)$$

Proof. Let $\tilde{n} \in \mathbb{N}$ be such that $f(n) > 0$ and $g(n) > 0$ for all $n \geq \tilde{n}$.

- If $\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = 0^+$, then for all $\varepsilon \in \mathbb{R}_{>0}$ there exists $n(\varepsilon) \in \mathbb{N}$, which we can take with $n(\varepsilon) \geq \tilde{n}$, such that

$$0 \leq \frac{f(n)}{g(n)} < \varepsilon, \quad \forall n \geq n(\varepsilon) \implies 0 \leq f(n) < \varepsilon g(n), \quad \forall n \geq n(\varepsilon),$$

where we multiplied by $g(n) > 0$. By choosing $c = \varepsilon$ and $n_1 = n(\varepsilon)$, this is the definition (9.7), so $f(n) = o(g(n))$.

- If $f(n) = o(g(n))$, then for all $c \in \mathbb{R}_{>0}$ there exists $n_1 \in \mathbb{N}$, which we can take with $n_1 \geq \tilde{n}$, such that

$$0 \leq f(n) < c g(n), \quad \forall n \geq n_1 \implies 0 \leq \frac{f(n)}{g(n)} < c, \quad \forall n \geq n_1,$$

where we divided by $g(n) > 0$. Since c is arbitrary, this is the definition of $\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = 0^+$. □

- If the functions f and g are asymptotically positive, “ $\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = 0^+$ ” is a necessary and sufficient condition for “ $f(n) = o(g(n))$ ”.

Observation 9.12: o implies $\mathcal{O}$

Given two functions f and g of a natural variable, we have

$$f(n) = o(g(n)) \implies f(n) = \mathcal{O}(g(n)). \quad (9.10)$$

Proof. The definition (9.7) holds for every $c \in \mathbb{R}_{>0}$; in particular it holds for $c = 1$: there exists $n_1 \in \mathbb{N}$ such that $0 \leq f(n) < g(n) \leq 1 \cdot g(n)$ for all $n \geq n_1$. This is the definition (9.2) with $c = 1$. $\square$

- It follows that “ $f(n) = o(g(n))$ ” is a sufficient condition for “ $f(n) = \mathcal{O}(g(n))$ ”, and “ $f(n) = \mathcal{O}(g(n))$ ” is a necessary condition for “ $f(n) = o(g(n))$ ”. By contraposition of (9.10) we also have

$$f(n) \neq \mathcal{O}(g(n)) \implies f(n) \neq o(g(n)).$$

The implication does not work in the other direction (see the next counterexample), that is,

$$f(n) = \mathcal{O}(g(n)) \not\Rightarrow f(n) = o(g(n)).$$

Example 9.12: an upper bound that is not an o

- To prove that

$$3n^3 = \mathcal{O}(n^3)$$

we determine $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq 3n^3 \leq cn^3, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^3 , this is equivalent to

$$0 \leq 3 \leq c, \quad \forall n \geq n_1.$$

Thus, by choosing for example $c = \frac{7}{2}$ and $n_1 = 1$, we have

$$0 \leq 3n^3 \leq \frac{7}{2}n^3, \quad \forall n \geq 1.$$

- To prove that

$$3n^3 \neq o(n^3)$$

we compute the limit

$$\lim_{n \rightarrow +\infty} \frac{3n^3}{n^3} = 3 \neq 0^+$$

and we apply (9.9).

Observation 9.13: limit equal to $+\infty$

Given two asymptotically positive functions f and g of a natural variable, we have

$$\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = +\infty \iff f(n) = \omega(g(n)). \quad (9.11)$$

Proof. Let $\tilde{n} \in \mathbb{N}$ be such that $f(n) > 0$ and $g(n) > 0$ for all $n \geq \tilde{n}$.

- If $\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = +\infty$, then for all $\lambda \in \mathbb{R}_{>0}$ there exists $n(\lambda) \in \mathbb{N}$, which we can take with $n(\lambda) \geq \tilde{n}$, such that

$$\frac{f(n)}{g(n)} > \lambda, \quad \forall n \geq n(\lambda) \quad \implies \quad 0 \leq \lambda g(n) < f(n), \quad \forall n \geq n(\lambda).$$

By choosing $c = \lambda$ and $n_1 = n(\lambda)$, this is the definition (9.8), so $f(n) = \omega(g(n))$.

- If $f(n) = \omega(g(n))$, then for all $c \in \mathbb{R}_{>0}$ there exists $n_1 \in \mathbb{N}$, which we can take with $n_1 \geq \tilde{n}$, such that

$$0 \leq c g(n) < f(n), \quad \forall n \geq n_1 \quad \implies \quad \frac{f(n)}{g(n)} > c, \quad \forall n \geq n_1.$$

Since c is arbitrary, this is the definition of $\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = +\infty$.

□

- If the functions f and g are asymptotically positive, “ $\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = +\infty$ ” is a necessary and sufficient condition for “ $f(n) = \omega(g(n))$ ”.

Observation 9.14: ω implies Ω

Given two functions f and g of a natural variable, we have

$$f(n) = \omega(g(n)) \implies f(n) = \Omega(g(n)). \quad (9.12)$$

Proof. The definition (9.8) holds for every $c \in \mathbb{R}_{>0}$; in particular it holds for $c = 1$: there exists $n_1 \in \mathbb{N}$ such that $0 \leq 1 \cdot g(n) < f(n)$ for all $n \geq n_1$. This is the definition (9.3) with $c = 1$. □

- It follows that “ $f(n) = \omega(g(n))$ ” is a sufficient condition for “ $f(n) = \Omega(g(n))$ ”, and “ $f(n) = \Omega(g(n))$ ” is a necessary condition for “ $f(n) = \omega(g(n))$ ”. By contraposition of (9.12) we also have

$$f(n) \neq \Omega(g(n)) \implies f(n) \neq \omega(g(n)).$$

The implication does not work in the other direction (see the next counterexample), that is,

$$f(n) = \Omega(g(n)) \not\Rightarrow f(n) = \omega(g(n)).$$

Example 9.13: a lower bound that is not an ω

- To prove that

$$3n^3 = \Omega(n^3)$$

we determine $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c n^3 \leq 3n^3, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^3 and since $c > 0$, this is equivalent to

$$0 < c \leq 3, \quad \forall n \geq n_1.$$

Thus, by choosing for example $c = \frac{5}{2}$ and $n_1 = 1$, we have

$$0 \leq \frac{5}{2} n^3 \leq 3n^3, \quad \forall n \geq 1.$$

- To prove that

$$3n^3 \neq \omega(n^3)$$

we compute the limit

$$\lim_{n \rightarrow +\infty} \frac{3n^3}{n^3} = 3 \neq +\infty$$

and we apply (9.11).

9.3.2 Limits and the notations Θ , $\mathcal{O}$ and Ω **Observation 9.15: finite positive limit**

Given two asymptotically positive functions f and g of a natural variable, we have

$$\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = \ell \in \mathbb{R}_{>0} \implies f(n) = \Theta(g(n)). \quad (9.13)$$

Proof. If $\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = \ell \in \mathbb{R}_{>0}$, then for all $\varepsilon \in \mathbb{R}_{>0}$ there exists $n(\varepsilon) \in \mathbb{N}$ such that $g(n) > 0$ and

$$\ell - \varepsilon < \frac{f(n)}{g(n)} < \ell + \varepsilon, \quad \forall n \geq n(\varepsilon).$$

Since $\ell > 0$, we can choose ε small enough to have $\ell - \varepsilon > 0$, for example $\varepsilon = \frac{\ell}{2}$. By setting $c_1 = \ell - \varepsilon = \frac{\ell}{2} > 0$ and $c_2 = \ell + \varepsilon = \frac{3\ell}{2} > 0$, and multiplying by $g(n) > 0$, we have

$$0 < c_1 g(n) < f(n) < c_2 g(n), \quad \forall n \geq n(\varepsilon),$$

and, as a consequence, $f(n) = \Theta(g(n))$ with $n_1 = n(\varepsilon)$. □

- It follows that “ $\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = \ell \in \mathbb{R}_{>0}$ ” is a sufficient condition for “ $f(n) = \Theta(g(n))$ ”, and “ $f(n) = \Theta(g(n))$ ” is a necessary condition for “ $\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = \ell \in \mathbb{R}_{>0}$ ”. By

contraposition of (9.13) we also have

$$f(n) \neq \Theta(g(n)) \implies \text{the limit of } \frac{f(n)}{g(n)} \text{ is not a number } \ell \in \mathbb{R}_{>0}.$$

The implication does not work in the other direction. A counterexample is given by Example 9.2:

$$n^2 (2 - \sin n) = \Theta(n^2), \quad \text{but} \quad \lim_{n \rightarrow +\infty} \frac{n^2 (2 - \sin n)}{n^2} = \lim_{n \rightarrow +\infty} (2 - \sin n) \text{ does not exist.}$$

Accordingly, we have

$$f(n) = \Theta(g(n)) \not\Rightarrow \lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = \ell \in \mathbb{R}_{>0}.$$

It is possible that $f(n) = \Theta(g(n))$ even though $\frac{f(n)}{g(n)}$ does not converge to a limit. In this case the limit criterion cannot be applied, and we use the definition.

Example 9.14: asymptotic tight bound without a limit

To prove that

$$(2 + (-1)^n) n^2 = \Theta(n^2)$$

we determine $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

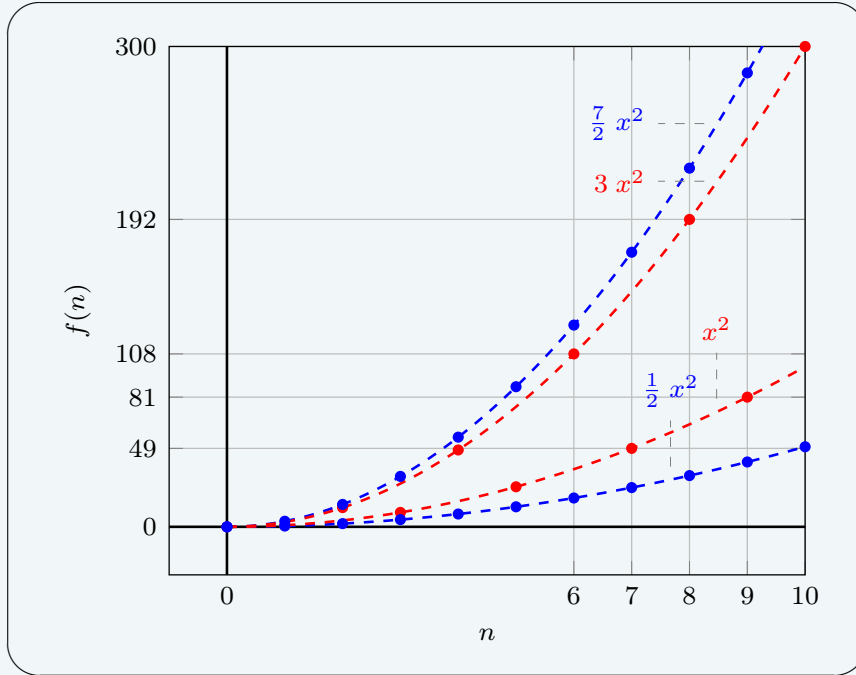
$$0 \leq c_1 n^2 \leq (2 + (-1)^n) n^2 \leq c_2 n^2, \quad \forall n \geq n_1.$$

Since

$$2 + (-1)^n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 3 & \text{if } n \text{ is even,} \end{cases} \quad \text{we have} \quad (2 + (-1)^n) n^2 = \begin{cases} n^2 & \text{if } n \text{ is odd,} \\ 3 n^2 & \text{if } n \text{ is even.} \end{cases}$$

Thus, by choosing for example $c_1 = \frac{1}{2}$, $c_2 = \frac{7}{2}$ and $n_1 = 1$, we have

$$0 \leq \frac{1}{2} n^2 \leq (2 + (-1)^n) n^2 \leq \frac{7}{2} n^2, \quad \forall n \geq 1.$$



On the other hand,

$$\lim_{n \rightarrow +\infty} \frac{(2 + (-1)^n) n^2}{n^2} = \lim_{n \rightarrow +\infty} (2 + (-1)^n) \text{ does not exist,}$$

since the ratio is 1 for all odd n and 3 for all even n .

Observation 9.16: limit criterion for $\mathcal{O}$

Given two asymptotically positive functions f and g of a natural variable, we have

$$\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = \ell \in \mathbb{R}_{>0} \text{ or } 0^+ \implies f(n) = \mathcal{O}(g(n)). \quad (9.14)$$

Proof. By Observations 9.15, 9.5, 9.11 and 9.12, we have

$$\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = \ell \in \mathbb{R}_{>0} \implies f(n) = \Theta(g(n)) \implies f(n) = \mathcal{O}(g(n)),$$

$$\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = 0^+ \iff f(n) = o(g(n)) \implies f(n) = \mathcal{O}(g(n)).$$

In both cases $f(n) = \mathcal{O}(g(n))$. □

- It follows that “ $\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = \ell \in \mathbb{R}_{>0} \text{ or } 0^+$ ” is a sufficient condition for “ $f(n) = \mathcal{O}(g(n))$ ”. By contraposition of (9.14) we also have

$$f(n) \neq \mathcal{O}(g(n)) \implies \text{the limit of } \frac{f(n)}{g(n)} \text{ is neither a number } \ell \in \mathbb{R}_{>0} \text{ nor } 0^+.$$

The implication does not work in the other direction. A counterexample is given by Example 9.4:

$$n^2 \sin^2 n = \mathcal{O}(n^2), \quad \text{but} \quad \lim_{n \rightarrow +\infty} \frac{n^2 \sin^2 n}{n^2} = \lim_{n \rightarrow +\infty} \sin^2 n \text{ does not exist.}$$

Accordingly, we have

$$f(n) = \mathcal{O}(g(n)) \not\Rightarrow \lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = \ell \in \mathbb{R}_{>0} \text{ or } 0^+.$$

It is possible that $f(n) = \mathcal{O}(g(n))$ even though $\frac{f(n)}{g(n)}$ does not converge to a limit.

Observation 9.17: limit criterion for Ω

Given two asymptotically positive functions f and g of a natural variable, we have

$$\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = \ell \in \mathbb{R}_{>0} \text{ or } +\infty \implies f(n) = \Omega(g(n)). \quad (9.15)$$

Proof. By Observations 9.15, 9.5, 9.13 and 9.14, we have

$$\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = \ell \in \mathbb{R}_{>0} \implies f(n) = \Theta(g(n)) \implies f(n) = \Omega(g(n)),$$

$$\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = +\infty \iff f(n) = \omega(g(n)) \implies f(n) = \Omega(g(n)).$$

In both cases $f(n) = \Omega(g(n))$. □

- It follows that “ $\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = \ell \in \mathbb{R}_{>0} \text{ or } +\infty$ ” is a sufficient condition for “ $f(n) = \Omega(g(n))$ ”. By contraposition of (9.15) we also have

$$f(n) \neq \Omega(g(n)) \implies \text{the limit of } \frac{f(n)}{g(n)} \text{ is neither a number } \ell \in \mathbb{R}_{>0} \text{ nor } +\infty.$$

The implication does not work in the other direction. A counterexample is given by Example 9.14: since $(2 + (-1)^n) n^2 = \Theta(n^2)$, by Observation 9.5 we have

$$(2 + (-1)^n) n^2 = \Omega(n^2), \quad \text{but} \quad \lim_{n \rightarrow +\infty} \frac{(2 + (-1)^n) n^2}{n^2} = \lim_{n \rightarrow +\infty} (2 + (-1)^n) \text{ does not exist.}$$

Accordingly, we have

$$f(n) = \Omega(g(n)) \not\Rightarrow \lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = \ell \in \mathbb{R}_{>0} \text{ or } +\infty.$$

It is possible that $f(n) = \Omega(g(n))$ even though $\frac{f(n)}{g(n)}$ does not converge to a limit.

Example 9.15: asymptotic bounds from limits

- We have

$$\lim_{n \rightarrow +\infty} \frac{6n - n^3 + 2n^4}{n^4} = \lim_{n \rightarrow +\infty} \left(\underbrace{\frac{6}{n^3}}_{\rightarrow 0} - \underbrace{\frac{1}{n}}_{\rightarrow 0} + 2 \right) = 2 \in \mathbb{R}_{>0}.$$

Then, according to (9.14), we have $6n - n^3 + 2n^4 = \mathcal{O}(n^4)$ and, according to (9.15), we have $6n - n^3 + 2n^4 = \Omega(n^4)$. Finally, according to (9.13), we also have $6n - n^3 + 2n^4 = \Theta(n^4)$.

- We have

$$\lim_{n \rightarrow +\infty} \frac{n + 3n^3}{n^4} = \lim_{n \rightarrow +\infty} \left(\underbrace{\frac{1}{n^3}}_{\rightarrow 0} + \underbrace{\frac{3}{n}}_{\rightarrow 0} \right) = 0^+.$$

Then, according to (9.9), we have $n + 3n^3 = o(n^4)$. Moreover, according to (9.10) or to (9.14), we have $n + 3n^3 = \mathcal{O}(n^4)$.

- We have

$$\lim_{n \rightarrow +\infty} \frac{n + 3n^4}{n^3} = \lim_{n \rightarrow +\infty} \left(\underbrace{\frac{1}{n^2}}_{\rightarrow 0} + 3n \right) = +\infty.$$

Then, according to (9.11), we have $n + 3n^4 = \omega(n^3)$. Moreover, according to (9.12) or to (9.15), we have $n + 3n^4 = \Omega(n^3)$.

9.3.3 The limit criterion

When the limit of the ratio exists, the previous observations can be reversed: the value of the limit tells us exactly which of the five notations hold and which ones do not.

Proposition 9.18: limit criterion

Let f and g be two asymptotically positive functions of a natural variable such that the limit

$$L = \lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)}$$

exists (equal to 0^+ , to a number $\ell \in \mathbb{R}_{>0}$, or to $+\infty$). Then:

$$f(n) = o(g(n)) \iff L = 0^+, \quad (9.16)$$

$$f(n) = \mathcal{O}(g(n)) \iff L = 0^+ \text{ or } L \in \mathbb{R}_{>0}, \quad (9.17)$$

$$f(n) = \Theta(g(n)) \iff L \in \mathbb{R}_{>0}, \quad (9.18)$$

$$f(n) = \Omega(g(n)) \iff L \in \mathbb{R}_{>0} \text{ or } L = +\infty, \quad (9.19)$$

$$f(n) = \omega(g(n)) \iff L = +\infty. \quad (9.20)$$

Proof. The implications from right to left are (9.9), (9.14), (9.13), (9.15) and (9.11), and (9.16) and (9.20) are (9.9) and (9.11). For the other implications we first observe that $f(n) = \mathcal{O}(g(n))$ and $f(n) = \omega(g(n))$ cannot hold together. Indeed, if $0 \leq f(n) \leq c g(n)$ for all $n \geq n'_1$, the definition (9.8) with the same constant c gives n''_1 such that $c g(n) < f(n)$ for all $n \geq n''_1$, and for $n \geq \max\{n'_1, n''_1\}$ we would have $c g(n) < f(n) \leq c g(n)$, which is impossible. In the same way, $f(n) = \Omega(g(n))$ and $f(n) = o(g(n))$ cannot hold together, since they would give $c g(n) \leq f(n) < c g(n)$. Since the limit exists, exactly one of the three cases $L = 0^+$, $L \in \mathbb{R}_{>0}$ and $L = +\infty$ holds.

- If $f(n) = \mathcal{O}(g(n))$, then $L \neq +\infty$, since otherwise $f(n) = \omega(g(n))$ by (9.11). Hence $L = 0^+$ or $L \in \mathbb{R}_{>0}$.
- If $f(n) = \Omega(g(n))$, then $L \neq 0^+$, since otherwise $f(n) = o(g(n))$ by (9.9). Hence $L \in \mathbb{R}_{>0}$ or $L = +\infty$.
- If $f(n) = \Theta(g(n))$, then $f(n) = \mathcal{O}(g(n))$ and $f(n) = \Omega(g(n))$ by Observation 9.5, and by the two previous points $L \in \mathbb{R}_{>0}$.

□

For two asymptotically positive functions f and g whose ratio has a limit L , the limit criterion reads:

- if $L = 0^+$, then $f(n) = o(g(n))$ and $f(n) = \mathcal{O}(g(n))$, and the other three notations do not hold;
- if $L \in \mathbb{R}_{>0}$, then $f(n) = \Theta(g(n))$, $f(n) = \mathcal{O}(g(n))$ and $f(n) = \Omega(g(n))$, while $f(n) \neq o(g(n))$ and $f(n) \neq \omega(g(n))$;
- if $L = +\infty$, then $f(n) = \omega(g(n))$ and $f(n) = \Omega(g(n))$, and the other three notations do not hold.

If the ratio has no limit, the criterion cannot be applied, and we use the definitions (Examples 9.2 and 9.14).

Example 9.16: the limit criterion

- In the course *Mathematical Analysis 1* (chapter *Hierarchies of infinities and the ratio test*) it is proved that

$$\lim_{n \rightarrow +\infty} \frac{\ln n}{n} = 0^+.$$

By Proposition 9.18, we have $\ln n = o(n)$ and $\ln n = \mathcal{O}(n)$, while $\ln n \neq \Theta(n)$, $\ln n \neq \Omega(n)$ and $\ln n \neq \omega(n)$.

- In the same chapter of that course it is proved that $\lim_{n \rightarrow +\infty} \frac{n^2}{2^n} = 0^+$, hence

$$\lim_{n \rightarrow +\infty} \frac{2^n + n^2}{2^n} = \lim_{n \rightarrow +\infty} \left(1 + \underbrace{\frac{n^2}{2^n}}_{\rightarrow 0^+} \right) = 1 \in \mathbb{R}_{>0}.$$

By Proposition 9.18, we have $2^n + n^2 = \Theta(2^n)$, $2^n + n^2 = \mathcal{O}(2^n)$ and $2^n + n^2 = \Omega(2^n)$, while $2^n + n^2 \neq o(2^n)$ and $2^n + n^2 \neq \omega(2^n)$.

9.4 Relational properties of asymptotic notations

Proposition 9.19: relational properties

Given three functions f , g and h of a natural variable, the following properties hold.

1. Transitivity:

$$\begin{aligned} f(n) = o(g(n)) \text{ and } g(n) = o(h(n)) &\implies f(n) = o(h(n)), \\ f(n) = \mathcal{O}(g(n)) \text{ and } g(n) = \mathcal{O}(h(n)) &\implies f(n) = \mathcal{O}(h(n)), \\ f(n) = \Theta(g(n)) \text{ and } g(n) = \Theta(h(n)) &\implies f(n) = \Theta(h(n)), \\ f(n) = \Omega(g(n)) \text{ and } g(n) = \Omega(h(n)) &\implies f(n) = \Omega(h(n)), \\ f(n) = \omega(g(n)) \text{ and } g(n) = \omega(h(n)) &\implies f(n) = \omega(h(n)). \end{aligned}$$

2. Reflexivity (for an asymptotically non-negative function f):

$$f(n) = \Theta(f(n)), \quad f(n) = \mathcal{O}(f(n)), \quad f(n) = \Omega(f(n)).$$

3. Symmetry:

$$f(n) = \Theta(g(n)) \iff g(n) = \Theta(f(n)).$$

4. Transpose symmetry:

$$\begin{aligned} f(n) = \mathcal{O}(g(n)) &\iff g(n) = \Omega(f(n)), \\ f(n) = o(g(n)) &\iff g(n) = \omega(f(n)). \end{aligned}$$

Proof. Transitivity. Let $0 \leq f(n) \leq c' g(n)$ for $n \geq n'_1$ and $0 \leq g(n) \leq c'' h(n)$ for $n \geq n''_1$. Then $0 \leq f(n) \leq c' c'' h(n)$ for all $n \geq \max\{n'_1, n''_1\}$, and $f(n) = \mathcal{O}(h(n))$ with $c = c' c'' > 0$. The proofs for Ω and Θ are the same, with the products of the constants. For o , let $c \in \mathbb{R}_{>0}$: the definition of $f(n) = o(g(n))$ with the constant 1 and the definition of $g(n) = o(h(n))$ with the constant c give two thresholds n'_1 and n''_1 such that

$$0 \leq f(n) < g(n) < c h(n), \quad \forall n \geq \max\{n'_1, n''_1\}.$$

Since c is arbitrary, $f(n) = o(h(n))$. The proof for ω is the same, with $0 \leq c h(n) < g(n) < f(n)$.

Reflexivity. Let $\tilde{n}$ be such that $f(n) \geq 0$ for all $n \geq \tilde{n}$. Then $0 \leq 1 \cdot f(n) \leq f(n) \leq 1 \cdot f(n)$ for all $n \geq \tilde{n}$: the definitions of Θ , $\mathcal{O}$ and Ω hold with all the constants equal to 1 and $n_1 = \tilde{n}$.

Symmetry. If $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_1$, dividing the left inequality by c_1 and the right inequality by c_2 we obtain

$$0 \leq \frac{1}{c_2} f(n) \leq g(n) \leq \frac{1}{c_1} f(n), \quad \forall n \geq n_1,$$

that is, $g(n) = \Theta(f(n))$. Exchanging the roles of f and g we obtain the other direction.

Transpose symmetry. We have $0 \leq f(n) \leq c g(n)$ if and only if $0 \leq \frac{1}{c} f(n) \leq g(n)$: the first inequality defines $f(n) = \mathcal{O}(g(n))$ with the constant c , the second one defines $g(n) = \Omega(f(n))$ with the constant $\frac{1}{c}$, and the threshold is the same. For o and ω , let $f(n) = o(g(n))$ and let $c \in \mathbb{R}_{>0}$. The definition of $f(n) = o(g(n))$ with the constant $\frac{1}{c}$ gives n_1 such that $0 \leq f(n) < \frac{1}{c} g(n)$, that is, $0 \leq c f(n) < g(n)$, for all $n \geq n_1$. Since c is arbitrary, $g(n) = \omega(f(n))$. The other direction is proved in the same way. $\square$

Because these relational properties hold for asymptotic notations, we can draw an analogy between the **asymptotic comparison** of two functions and the comparison of two real numbers a and b :

$$f(n) = o(g(n)) \quad \text{is like} \quad a < b,$$

$$f(n) = \mathcal{O}(g(n)) \quad \text{is like} \quad a \leq b,$$

$$f(n) = \Theta(g(n)) \quad \text{is like} \quad a = b,$$

$$f(n) = \Omega(g(n)) \quad \text{is like} \quad a \geq b,$$

$$f(n) = \omega(g(n)) \quad \text{is like} \quad a > b.$$

- For any two real numbers a and b , exactly one of the following three cases holds:

$$a < b, \quad a = b, \quad a > b \quad (\text{trichotomy property}).$$

- The analogous property **does not hold** for the asymptotic comparison of two functions: there exist two functions f and g such that none of the five notations holds, as the next example shows.

Example 9.17: two functions that cannot be compared

We consider the asymptotically positive function

$$f : \mathbb{N} \rightarrow \mathbb{R}, \quad f : n \mapsto n^{1+\sin n}$$

and the function $g(n) = n$. For every $n \geq 1$ we have $\frac{f(n)}{g(n)} = n^{\sin n}$. Since π is irrational, the values $n \bmod 2\pi$, $n \in \mathbb{N}$, are dense in $[0, 2\pi]$. Hence there exist arbitrarily large integers n

with $\sin n > \frac{1}{2}$, for which

$$n^{\sin n} \geq n^{1/2} = \sqrt{n},$$

and arbitrarily large integers m with $\sin m < -\frac{1}{2}$, for which

$$m^{\sin m} \leq m^{-1/2} = \frac{1}{\sqrt{m}}.$$

- To prove that

$$n^{1+\sin n} \neq \mathcal{O}(n),$$

we proceed by contradiction and we suppose that there exist $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that $0 \leq n^{1+\sin n} \leq cn$, that is, $n^{\sin n} \leq c$, for all $n \geq n_1$. This is false for the arbitrarily large integers n with $\sin n > \frac{1}{2}$ and $\sqrt{n} > c$.

- To prove that

$$n^{1+\sin n} \neq \Omega(n),$$

we proceed by contradiction and we suppose that there exist $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that $0 \leq cn \leq n^{1+\sin n}$, that is, $c \leq n^{\sin n}$, for all $n \geq n_1$. This is false for the arbitrarily large integers m with $\sin m < -\frac{1}{2}$ and $\frac{1}{\sqrt{m}} < c$.

- By Observation 9.5, it follows that

$$n^{1+\sin n} \neq \Theta(n).$$

- By the contrapositions of (9.10) and (9.12), it also follows that

$$n^{1+\sin n} \neq o(n) \quad \text{and} \quad n^{1+\sin n} \neq \omega(n).$$

The same conclusion follows from the limit

$$\lim_{n \rightarrow +\infty} \frac{n^{1+\sin n}}{n} = \lim_{n \rightarrow +\infty} n^{\sin n} = \lim_{n \rightarrow +\infty} e^{\sin n \ln n},$$

which does not exist (along the integers n with $\sin n > \frac{1}{2}$ the ratio tends to $+\infty$, along the integers m with $\sin m < -\frac{1}{2}$ it tends to 0^+): the limit is neither 0^+ nor $+\infty$, and we apply (9.9) and (9.11).

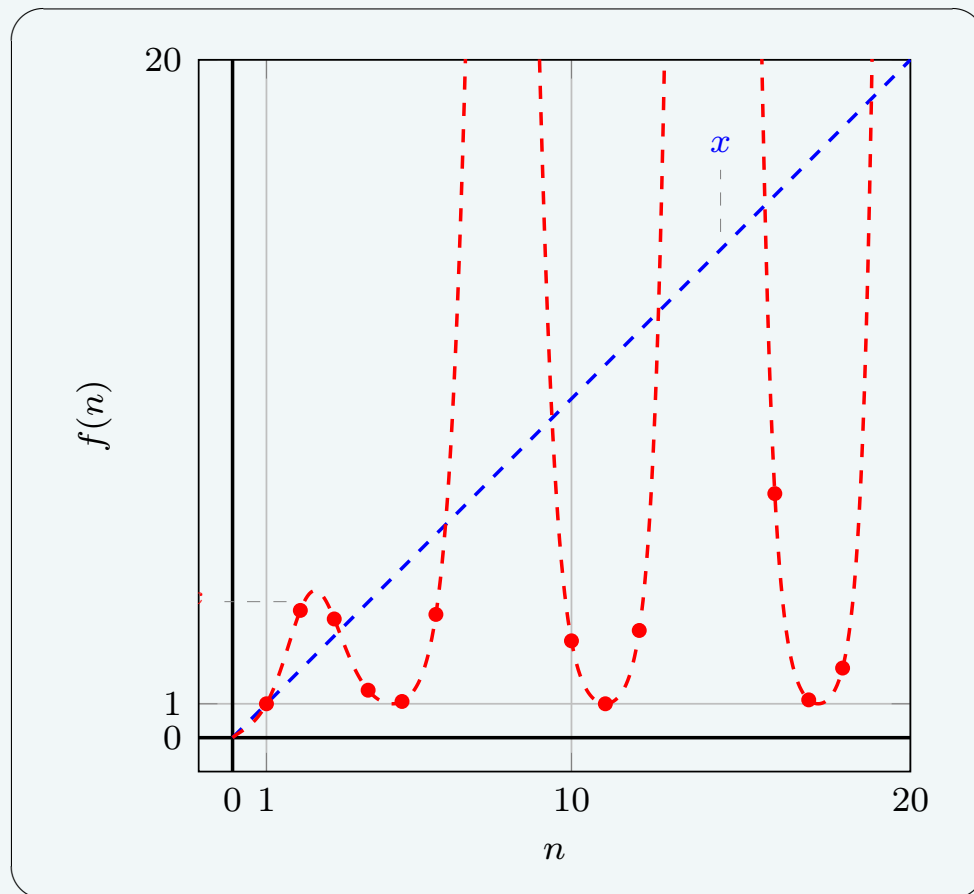
Therefore none of the five notations holds between $n^{1+\sin n}$ and n . Note that this is a stronger failure than the failure of trichotomy: not even the analogues of $a \leq b$ or $a \geq b$ hold.

Example 9.18: the graph of $n^{1+\sin n}$

This is the graph of the function

$$f : \mathbb{N} \rightarrow \mathbb{R}, \quad f : n \mapsto n^{1+\sin n},$$

together with the graph of the function $g : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$, $g : x \mapsto x$. The values $f(n)$ are alternately above and below the line g , farther and farther from it.



9.5 Asymptotic notation in the analysis of algorithms

In the chapters «Input size» and «Running time» we measure the size $|I|$ of the input of an algorithm and we count the number of steps $S(n)$ that the algorithm executes in the worst case, as a function of a natural parameter n . The exact expression of $S(n)$ depends on the conventions used to count the steps, and the time of one step depends on the computer. The asymptotic notation keeps only what does not depend on these details: the **order of growth**.

- **Constant factors do not matter.** If $f(n) = \Theta(g(n))$ and $\lambda \in \mathbb{R}_{>0}$, then $\lambda f(n) = \Theta(g(n))$, with the constants λc_1 and λc_2 and the same threshold. For example, an algorithm executing $4n$ steps and an algorithm executing n steps both have a number of steps $\Theta(n)$.
- **Lower-order terms do not matter.** This is made precise by the next observation, which follows from Observation 9.7.

Observation 9.20: dropping the lower-order terms

Given two asymptotically non-negative functions f and g of a natural variable, we have

$$g(n) = \mathcal{O}(f(n)) \implies f(n) + g(n) = \Theta(f(n)).$$

Proof. Let $0 \leq g(n) \leq c f(n)$ for all $n \geq n'_1$, and let $f(n) \geq 0$ for all $n \geq n''_1$. Then, for all $n \geq n_1 = \max\{n'_1, n''_1\}$,

$$0 \leq 1 \cdot f(n) \leq f(n) + g(n) \leq f(n) + c f(n) = (1 + c) f(n),$$

that is, $f(n) + g(n) = \Theta(f(n))$ with the constants $c_1 = 1$ and $c_2 = 1 + c$. $\square$

Example 9.19: the number of steps of two algorithms

In the chapter «Running time» we count the steps of two algorithms on an array of n entries.

- **FindMaximum** executes $S(n) = 4n$ steps. By choosing $c_1 = c_2 = 4$ and $n_1 = 1$ we have $0 \leq 4n \leq S(n) \leq 4n$ for all $n \geq 1$, so $S(n) = \Theta(n)$: the number of steps is **linear** in n .
- **InsertionSort** executes, in the worst case,

$$S(n) = \frac{3n^2 + 9n - 8}{2}$$

steps. To prove that $S(n) = \Theta(n^2)$ we determine $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^2 \leq \frac{3n^2 + 9n - 8}{2} \leq c_2 n^2, \quad \forall n \geq n_1.$$

For the lower bound, $9n - 8 \geq 0$ for all $n \geq 1$, hence $\frac{3n^2 + 9n - 8}{2} \geq \frac{3}{2} n^2$. For the upper bound, $9n - 8 \leq 9n \leq 9n^2$ for all $n \geq 1$, hence $\frac{3n^2 + 9n - 8}{2} \leq \frac{12n^2}{2} = 6n^2$. By choosing

$c_1 = \frac{3}{2}$, $c_2 = 6$ and $n_1 = 1$, we have

$$0 \leq \frac{3}{2}n^2 \leq \frac{3n^2 + 9n - 8}{2} \leq 6n^2, \quad \forall n \geq 1.$$

The same result follows from the limit $\lim_{n \rightarrow +\infty} \frac{3n^2 + 9n - 8}{2n^2} = \frac{3}{2} \in \mathbb{R}_{>0}$ and (9.13). The number of steps is **quadratic** in n .

- Since $n = o(n^2)$, for large arrays **InsertionSort** executes, in the worst case, many more steps than **FindMaximum**: by Observation 9.11, the ratio $\frac{4n}{(3n^2 + 9n - 8)/2}$ tends to 0^+ .

- The three notations $\mathcal{O}$, Ω and Θ say different things about the number of steps $S(n)$ in the worst case. If $S(n) = \mathcal{O}(g(n))$, then, for all $n \geq n_1$, the algorithm executes at most $cg(n)$ steps on **every** input with parameter n . If $S(n) = \Omega(g(n))$, then, for all $n \geq n_1$, there is **at least one** input with parameter n on which the algorithm executes at least $cg(n)$ steps. The statement $S(n) = \Theta(g(n))$ gives both, and it is the most informative one.
- In the chapter «Running time» the order of growth of $S(n)$ is then expressed as a function of the input size $|I|$, using the order of growth of $|I|$ obtained in the chapter «Input size».

Summary

- $f(n) = \Theta(g(n))$, $\mathcal{O}(g(n))$, $\Omega(g(n))$: there exist positive constants (c_1 and c_2 , or c) and a threshold n_1 such that, for all $n \geq n_1$, $f(n)$ lies between $c_1g(n)$ and $c_2g(n)$, below $cg(n)$, above $cg(n)$. To prove these statements we determine the constants.
- $f(n) = o(g(n))$, $\omega(g(n))$: the inequality $f(n) < cg(n)$, or $cg(n) < f(n)$, holds for **every** $c > 0$ from a threshold n_1 that depends on c .
- Θ is the same as $\mathcal{O}$ and Ω together; o implies $\mathcal{O}$ and ω implies Ω . None of these implications can be reversed.
- Limit criterion: for asymptotically positive functions whose ratio $\frac{f(n)}{g(n)}$ has a limit, the limit decides: 0^+ gives o and $\mathcal{O}$, a number $\ell > 0$ gives Θ , $\mathcal{O}$ and Ω , $+\infty$ gives ω and Ω , and in each case the other notations do not hold. When the limit does not exist, as for $2 - \sin n$ or $2 + (-1)^n$, we use the definitions.
- The notations are transitive; Θ , $\mathcal{O}$, Ω are reflexive; Θ is symmetric; $\mathcal{O}$ and Ω , and o and ω , are transpose symmetric. Unlike real numbers, two functions may not be comparable at all, as $n^{1+\sin n}$ and n .
- In the analysis of algorithms, the notation drops constant factors and lower-order terms and keeps the order of growth of the number of steps.

Asymptotic growth

Introduction

The number of steps executed by an algorithm is a function of a natural number: the size of its input. Two algorithms that solve the same problem are compared by looking at how fast these functions grow when the input grows, and not at their values on a few small inputs. In this chapter we study the functions of a natural variable that appear in the analysis of algorithms and we compare their **orders of growth** with the asymptotic notations o , $\mathcal{O}$, Θ , Ω and ω introduced in chapter 9 «Asymptotic notation».

The plan of the chapter is the following.

- In Section 10.1 we introduce the five families of increasing functions of a natural variable used in the course: logarithmic, power, exponential, factorial and self-power functions.
- In Section 10.2 we recall the four limits that put these families in order (the **hierarchy of infinities**).
- In Section 10.3 we express the comparison of two orders of growth with the asymptotic notation, through the limit criterion of chapter 9 «Asymptotic notation». We then derive the hierarchy of the orders of growth, a classification of the orders of growth, and the properties of logarithmic, power, polynomial and exponential functions that are used in the analysis of algorithms.

Prerequisites. The functions of a natural variable are the **sequences** of the course *Mathematical Analysis 1* (<https://fabiofurini.github.io/mathematical-analysis-1/>, Italian version <https://fabiofurini.github.io/analisi-matematica-1/>). Their limits, and the theorems used to compute them, are not explained again in this chapter: we use them, and we refer to the part *Limits of sequences* of that course, in particular to the chapters *Computing limits of sequences* (comparison theorem), *Euler's number* and *Hierarchies of infinities and the ratio test*.

10.1 Families of increasing functions of a natural variable

A function of a natural variable is defined on a set of natural numbers of the form $\mathbb{N}_{\geq n_0} = \{n_0, n_0 + 1, n_0 + 2, \dots\}$, and in the analysis of algorithms only its behaviour for large values of n matters. In chapter 9 «Asymptotic notation» we called a function **asymptotically positive** if $f(n) > 0$ for all n from some value onwards: in the course *Mathematical Analysis 1* such properties are said to hold *eventually*. The following definition describes another property of the same kind.

Definition 10.1: asymptotically increasing function

A function of a natural variable $f : \mathbb{N}_{\geq n_0} \rightarrow \mathbb{R}$ is **asymptotically increasing** if there exists $n_1 \in \mathbb{N}_{\geq n_0}$ such that:

$$f(n) < f(n+1), \quad \forall n \in \mathbb{N}_{\geq n_1} \quad (10.1)$$

- The principal families of asymptotically increasing functions of a natural variable are the following ones.

1. The **logarithmic** functions of a natural variable:

$$f : \mathbb{N}_{\geq 1} \rightarrow \mathbb{R}, \quad f : n \mapsto \log_a n \quad \forall a \in \mathbb{R}_{>1} \quad (10.2)$$

2. The **power** functions of a natural variable:

$$f : \mathbb{N} \rightarrow \mathbb{R}, \quad f : n \mapsto n^\alpha \quad \forall \alpha \in \mathbb{R}_{>0} \quad (10.3)$$

3. The **exponential** functions of a natural variable:

$$f : \mathbb{N} \rightarrow \mathbb{R}, \quad f : n \mapsto b^n \quad \forall b \in \mathbb{R}_{>1} \quad (10.4)$$

4. The **factorial** function of a natural variable:

$$f : \mathbb{N} \rightarrow \mathbb{R}, \quad f : n \mapsto n! \quad (10.5)$$

5. The **self-power** function of a natural variable:

$$f : \mathbb{N} \rightarrow \mathbb{R}, \quad f : n \mapsto n^n \quad (10.6)$$

where, by convention, we set $0^0 = 1$.

- All these functions are asymptotically positive, and all of them diverge to $+\infty$ as $n \rightarrow +\infty$. The factorial and the self-power functions are not increasing on their whole domain, since $0! = 1! = 1$ and $0^0 = 1^1 = 1$, but they are strictly increasing from $n = 1$ onwards: this is why we speak of *asymptotically* increasing functions.
- In these notes $\ln n = \log_e n$ denotes the natural logarithm, where e is Euler's number.

Example 10.1: increasing functions of a natural variable

The table reports the values of some increasing functions of a natural variable, computed by increasing n by one order of magnitude at a time, from 1 to 100 million. The values of $\ln n$ and the leading factors of 2^n and $n!$ are rounded to one decimal digit; the values of n^2 and n^n are exact.

n	$\ln n$	n^2	2^n	$n!$	n^n
$1 = 10^0$	0	1	2	1	1
$10 = 10^1$	≈ 2.3	10^2	$\approx 1.0 \cdot 10^3$	$\approx 3.6 \cdot 10^6$	10^{10}
$100 = 10^2$	≈ 4.6	10^4	$\approx 1.3 \cdot 10^{30}$	$\approx 9.3 \cdot 10^{157}$	10^{200}
$1,000 = 10^3$	≈ 6.9	10^6	$\approx 1.1 \cdot 10^{301}$	$\approx 4.0 \cdot 10^{2,567}$	$10^{3,000}$
$10,000 = 10^4$	≈ 9.2	10^8	$\approx 2.0 \cdot 10^{3,010}$	$\approx 2.8 \cdot 10^{35,659}$	$10^{40,000}$
$100,000 = 10^5$	≈ 11.5	10^{10}	$\approx 1.0 \cdot 10^{30,103}$	$\approx 2.8 \cdot 10^{456,573}$	$10^{500,000}$
10^6	≈ 13.8	10^{12}	$\approx 9.9 \cdot 10^{301,029}$	$\approx 8.3 \cdot 10^{5,565,708}$	$10^{6,000,000}$
10^7	≈ 16.1	10^{14}	$\approx 9.0 \cdot 10^{3,010,299}$	$\approx 1.2 \cdot 10^{65,657,059}$	$10^{70,000,000}$
10^8	≈ 18.4	10^{16}	$\approx 3.7 \cdot 10^{30,102,999}$	$\approx 1.6 \cdot 10^{756,570,556}$	$10^{800,000,000}$

To appreciate these numbers, recall that the number of atoms in the observable universe is estimated at about 10^{82} , and that a personal computer of 2024 executes about $10 \cdot 10^9 = 10^{10}$ operations per second (10 GFLOPS). An algorithm that executes 2^n operations on an input with $n = 100$ needs

$$\frac{2^{100}}{10^{10}} \text{ seconds} \approx 1.3 \cdot 10^{20} \text{ seconds} \approx 4.0 \cdot 10^{12} \text{ years},$$

while an algorithm that executes n^2 operations needs $10^4/10^{10} = 10^{-6}$ seconds.

- Figure 10.1 shows the first values of the same functions. Since the values grow very fast, the vertical axis is in logarithmic scale: each horizontal line of the grid corresponds to a multiplication by 100.

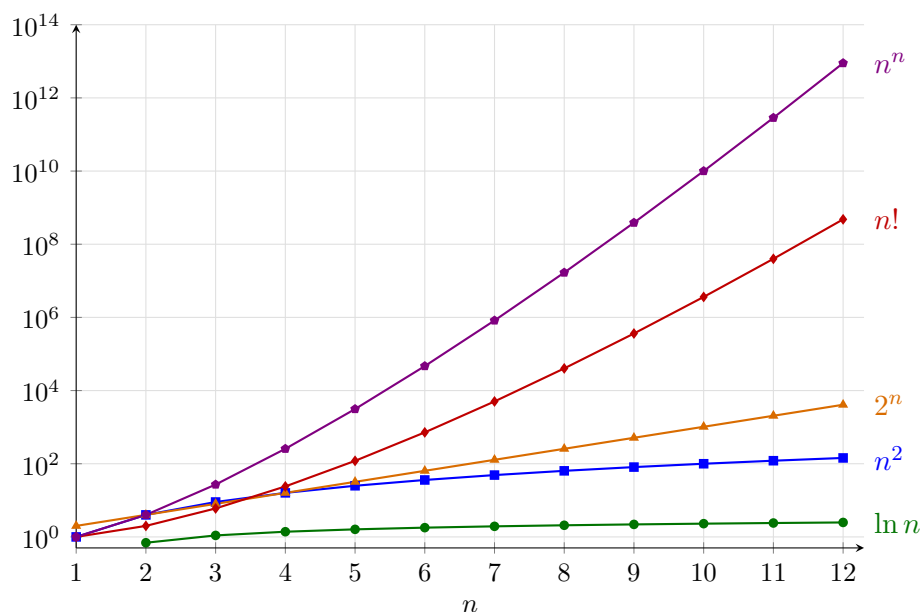


Figure 10.1: The values of $\ln n$, n^2 , 2^n , $n!$ and n^n for $n \in \{1, 2, \dots, 12\}$ (vertical axis in logarithmic scale; the points are joined by segments only for readability). For small values of n the order of the functions can be different from the final one: $2^3 = 8 < 9 = 3^2$, and $2^n > n!$ for $n \in \{1, 2, 3\}$.

10.2 The hierarchy of infinities

The five families of Section 10.1 all diverge to $+\infty$. To decide which one diverges faster, we compute the limit of the ratio of two of them. The limits of this section are the *hierarchy of infinities* for sequences of the course *Mathematical Analysis 1* (chapter *Hierarchies of infinities and the ratio test*), where they are proved; here we recall them, together with the idea of their proofs, and we extend the first one to the powers of the logarithms.

- Given $a \in \mathbb{R}_{>1}$ and $\beta \in \mathbb{R}_{>0}$, the asymptotically increasing function of a natural variable

$$f : \mathbb{N}_{\geq 1} \rightarrow \mathbb{R}, \quad f : n \mapsto \log_a^\beta n = (\log_a n)^\beta \quad (10.7)$$

is called a **polylogarithmic** function of a natural variable.

Proposition 10.2: logarithms and powers

For every $\alpha \in \mathbb{R}_{>0}$, $\beta \in \mathbb{R}_{>0}$ and $a \in \mathbb{R}_{>1}$, we have:

$$\lim_{n \rightarrow +\infty} \frac{\log_a^\beta n}{n^\alpha} = 0^+ \quad (10.8)$$

Proof. For $\beta = 1$ this is the hierarchy of infinities (part I) of the course *Mathematical*

Analysis 1:

$$\lim_{n \rightarrow +\infty} \frac{\log_a n}{n^\alpha} = 0^+, \quad \forall \alpha \in \mathbb{R}_{>0}, \quad \forall a \in \mathbb{R}_{>1}.$$

It is proved with the inequality $\log_a x < x \log_a 2$, valid for every $x \in \mathbb{R}_{>0}$ (a consequence of Bernoulli's inequality), applied to $x = n^{\alpha/2}$, and with the comparison theorem.

For every $\beta \in \mathbb{R}_{>0}$, since $\frac{\alpha}{\beta} \in \mathbb{R}_{>0}$, we then have:

$$\lim_{n \rightarrow +\infty} \frac{\log_a^\beta n}{n^\alpha} = \lim_{n \rightarrow +\infty} \underbrace{\left(\frac{\log_a n}{n^{\alpha/\beta}} \right)^\beta}_{\rightarrow 0^+} = 0^+$$

□

Proposition 10.3: powers and exponentials

For every $\alpha \in \mathbb{R}_{>0}$ and $b \in \mathbb{R}_{>1}$, we have:

$$\lim_{n \rightarrow +\infty} \frac{n^\alpha}{b^n} = 0^+ \quad (10.9)$$

- This is the hierarchy of infinities (part II). It follows from Proposition 10.2 by replacing n with 2^n , which gives $\frac{n}{(2^\mu)^n} \rightarrow 0^+$ for every $\mu \in \mathbb{R}_{>0}$, hence $\frac{n}{b^n} \rightarrow 0^+$ for every $b \in \mathbb{R}_{>1}$ (choose μ such that $2^\mu = b$); the general case follows from the identity

$$\frac{n^\alpha}{b^n} = \left(\frac{n}{(b^{1/\alpha})^n} \right)^\alpha, \quad b^{1/\alpha} \in \mathbb{R}_{>1}.$$

Proposition 10.4: exponentials and factorial

For every $b \in \mathbb{R}_{>1}$, we have:

$$\lim_{n \rightarrow +\infty} \frac{b^n}{n!} = 0^+ \quad (10.10)$$

- This is the hierarchy of infinities (part III). It follows from the ratio test, since

$$\frac{b^{n+1}}{(n+1)!} \frac{n!}{b^n} = \frac{b}{n+1} \rightarrow 0^+ < 1.$$

Proposition 10.5: factorial and self-power

$$\lim_{n \rightarrow +\infty} \frac{n!}{n^n} = 0^+ \quad (10.11)$$

- This is the hierarchy of infinities (part IV). It follows from the ratio test, since

$$\frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \left(\frac{n}{n+1} \right)^n = \frac{1}{\left(1 + \frac{1}{n}\right)^n} \rightarrow \frac{1}{e} < 1.$$

Polylogarithmic functions (with base greater than one) diverge more slowly than power functions (with positive exponent), which diverge more slowly than exponential functions (with base greater than one), which diverge more slowly than the factorial function, which diverges more slowly than the self-power function.

10.3 Comparison of orders of growth

- The **order of growth** of an asymptotically increasing function of a natural variable describes how rapidly its values increase compared to the values of another asymptotically increasing function of a natural variable.
- We compare orders of growth with the asymptotic notations o , $\mathcal{O}$, Θ , Ω and ω of chapter 9 «Asymptotic notation». For the functions of this chapter the ratio $\frac{f(n)}{g(n)}$ always has a limit, and the notations that hold are decided by the **limit criterion** proved there, which we recall.

Proposition 10.6: limit criterion

Let f and g be two asymptotically positive functions of a natural variable such that the limit

$$L = \lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)}$$

exists ($L = 0^+$, $L \in \mathbb{R}_{>0}$ or $L = +\infty$). Then:

$$\begin{aligned} f(n) = o(g(n)) &\iff L = 0^+, \\ f(n) = \mathcal{O}(g(n)) &\iff L = 0^+ \text{ or } L \in \mathbb{R}_{>0}, \\ f(n) = \Theta(g(n)) &\iff L \in \mathbb{R}_{>0}, \\ f(n) = \Omega(g(n)) &\iff L \in \mathbb{R}_{>0} \text{ or } L = +\infty, \\ f(n) = \omega(g(n)) &\iff L = +\infty. \end{aligned}$$

- The proof is in chapter 9 «Asymptotic notation» (Proposition «limit criterion»), where the five notations are defined through positive constants and a threshold. In this chapter we compute the limit of the ratio and apply Proposition 10.6.

Given two asymptotically increasing functions of a natural variable f and g :

- if $f(n) = o(g(n))$, f has a strictly **smaller order of growth** than g , $\rightsquigarrow$ the function f **grows slower than** the function g ;
- if $f(n) = \mathcal{O}(g(n))$, f has a **smaller or equal order of growth** compared to g , $\rightsquigarrow$ the function f **grows at most as fast as** the function g ;
- if $f(n) = \Theta(g(n))$, f has the **same order of growth** as g , $\rightsquigarrow$ the function f **grows as fast as** the function g ;

- if $f(n) = \Omega(g(n))$, f has a **greater or equal order of growth** compared to g , $\rightsquigarrow$ the function f **grows at least as fast as** the function g ;
- if $f(n) = \omega(g(n))$, f has a strictly **greater order of growth** than g , $\rightsquigarrow$ the function f **grows faster than** the function g .

- The notations do not depend on the values of f and g on finitely many natural numbers: changing them changes neither the limit of the ratio nor, with a larger threshold, the inequalities of the definitions.
- By the relational properties of chapter 9 «Asymptotic notation» (symmetry and transpose symmetry), exchanging the two functions exchanges o with ω and $\mathcal{O}$ with Ω , while Θ does not change:

$$f(n) = o(g(n)) \iff g(n) = \omega(f(n)), \quad f(n) = \Theta(g(n)) \iff g(n) = \Theta(f(n)).$$

All five notations are also transitive: for example, $f(n) = o(g(n))$ and $g(n) = o(h(n))$ give $f(n) = o(h(n))$.

Example 10.2: the limit of the ratio

We compare three pairs of functions; the three ratios are drawn in Figure 10.2.

- We have $\ln n = o(\sqrt{n})$, since by Proposition 10.2 (with $a = e$, $\beta = 1$, $\alpha = \frac{1}{2}$):

$$\lim_{n \rightarrow +\infty} \frac{\ln n}{\sqrt{n}} = 0^+$$

- We have $2n^2 + 3n = \Theta(n^2)$, since:

$$\lim_{n \rightarrow +\infty} \frac{2n^2 + 3n}{n^2} = \lim_{n \rightarrow +\infty} \left(2 + \underbrace{\frac{3}{n}}_{\rightarrow 0} \right) = 2 \in \mathbb{R}_{>0}$$

- We have $n \ln n = \omega(n)$, since:

$$\lim_{n \rightarrow +\infty} \frac{n \ln n}{n} = \lim_{n \rightarrow +\infty} \ln n = +\infty$$

The first ratio is not monotone: it increases up to its largest value, about 0.74, reached for $n = 7$, and only then it decreases towards 0.

- Proposition 10.6 requires the limit of the ratio to exist. For example, for $f(n) = n$ if n is even and $f(n) = 2n$ if n is odd, the ratio $\frac{f(n)}{n}$ takes alternately the values 1 and 2 and has no limit; nevertheless $f(n) = \Theta(n)$, by the definition of chapter 9 «Asymptotic notation» with $c_1 = 1$, $c_2 = 2$ and $n_1 = 0$, since $0 \leq n \leq f(n) \leq 2n$ for every $n \in \mathbb{N}$. All the comparisons of this chapter are made between functions whose ratio has a limit.

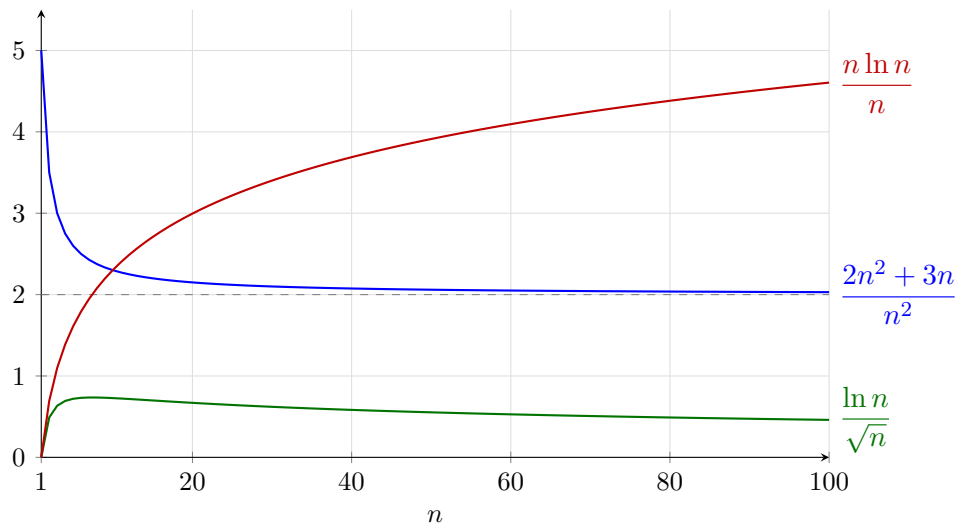


Figure 10.2: The three ratios of Example 10.2 for $n \in \{1, 2, \dots, 100\}$ (the points are joined by segments). The first one tends to 0, the second one to 2 (dashed line), the third one diverges to $+\infty$.

10.3.1 Hierarchy of the orders of growth

By Proposition 10.6, the limits of Section 10.2 translate directly into relations o between the families.

- Any polylogarithmic function of a natural variable with a base strictly greater than one is o of any power function of a natural variable with a positive exponent.

Observation 10.7: polylogarithms and powers

For every $\alpha \in \mathbb{R}_{>0}$, $\beta \in \mathbb{R}_{>0}$ and $a \in \mathbb{R}_{>1}$, we have $\log_a^\beta n = o(n^\alpha)$.

Proof. By Proposition 10.2, for every $\alpha \in \mathbb{R}_{>0}$, $\beta \in \mathbb{R}_{>0}$ and $a \in \mathbb{R}_{>1}$, we have:

$$\lim_{n \rightarrow +\infty} \frac{\log_a^\beta n}{n^\alpha} = 0^+$$

and the claim follows from Proposition 10.6. □

- Any power function of a natural variable with a positive exponent is o of any exponential function of a natural variable with a base strictly greater than one.

Observation 10.8: powers and exponentials

For every $\alpha \in \mathbb{R}_{>0}$ and $b \in \mathbb{R}_{>1}$, we have $n^\alpha = o(b^n)$.

Proof. By Proposition 10.3, for every $\alpha \in \mathbb{R}_{>0}$ and $b \in \mathbb{R}_{>1}$, we have:

$$\lim_{n \rightarrow +\infty} \frac{n^\alpha}{b^n} = 0^+$$

and the claim follows from Proposition 10.6. $\square$

Example 10.3: comparison of orders of growth

- We have $\ln^6 n + 2 + 2^{1/n} + 3^{-n} + 9n^2 = o(n^3)$, since:

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \frac{\ln^6 n + 2 + 2^{1/n} + 3^{-n} + 9n^2}{n^3} \\ &= \lim_{n \rightarrow +\infty} \left(\underbrace{\frac{\ln^6 n}{n^3}}_{\rightarrow 0^+} + \underbrace{\frac{2}{n^3}}_{\rightarrow 0^+} + \underbrace{\frac{2^{1/n}}{n^3}}_{\rightarrow 0^+} + \underbrace{\frac{3^{-n}}{n^3}}_{\rightarrow 0^+} + \underbrace{\frac{9}{n}}_{\rightarrow 0^+} \right) = 0^+ \end{aligned}$$

- We have $3n^2 + \sqrt{n} + \ln n + 5 + 4^{-n} + 10n^{1/3} = o(n^3)$, since:

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \frac{3n^2 + \sqrt{n} + \ln n + 5 + 4^{-n} + 10n^{1/3}}{n^3} \\ &= \lim_{n \rightarrow +\infty} \left(\underbrace{\frac{3}{n}}_{\rightarrow 0^+} + \underbrace{\frac{1}{n^{5/2}}}_{\rightarrow 0^+} + \underbrace{\frac{\ln n}{n^3}}_{\rightarrow 0^+} + \underbrace{\frac{5}{n^3}}_{\rightarrow 0^+} + \underbrace{\frac{4^{-n}}{n^3}}_{\rightarrow 0^+} + \underbrace{\frac{10}{n^{8/3}}}_{\rightarrow 0^+} \right) = 0^+ \end{aligned}$$

- We have $n^{1/4} + \ln^2 n + \frac{7}{\sqrt{n}} + e^{-n} + 20n^{1/2} = o(n^2)$, since:

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \frac{n^{1/4} + \ln^2 n + \frac{7}{\sqrt{n}} + e^{-n} + 20n^{1/2}}{n^2} \\ &= \lim_{n \rightarrow +\infty} \left(\underbrace{\frac{1}{n^{7/4}}}_{\rightarrow 0^+} + \underbrace{\frac{\ln^2 n}{n^2}}_{\rightarrow 0^+} + \underbrace{\frac{7}{n^{5/2}}}_{\rightarrow 0^+} + \underbrace{\frac{e^{-n}}{n^2}}_{\rightarrow 0^+} + \underbrace{\frac{20}{n^{3/2}}}_{\rightarrow 0^+} \right) = 0^+ \end{aligned}$$

In each case the terms with a logarithm go to 0^+ by Proposition 10.2, the other terms by elementary limits, and the claim follows from Proposition 10.6.

- Any exponential function of a natural variable with a base strictly greater than one is o of the factorial function of a natural variable.

Observation 10.9: exponentials and factorial

For every $b \in \mathbb{R}_{>1}$, we have $b^n = o(n!)$.

Proof. By Proposition 10.4, for every $b \in \mathbb{R}_{>1}$, we have:

$$\lim_{n \rightarrow +\infty} \frac{b^n}{n!} = 0^+$$

and the claim follows from Proposition 10.6. $\square$

- The factorial function of a natural variable is o of the self-power function of a natural variable.

Observation 10.10: factorial and self-power

We have $n! = o(n^n)$.

Proof. By Proposition 10.5, we have:

$$\lim_{n \rightarrow +\infty} \frac{n!}{n^n} = 0^+$$

and the claim follows from Proposition 10.6. $\square$

Example 10.4: comparison with the self-power function

We have $n! + 2^n + \sqrt{n} + \ln^3 n + n^{n-1} + \frac{1}{n} = o(n^n)$, since:

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \frac{n! + 2^n + \sqrt{n} + \ln^3 n + n^{n-1} + \frac{1}{n}}{n^n} \\ &= \lim_{n \rightarrow +\infty} \left(\underbrace{\frac{n!}{n^n}}_{\rightarrow 0^+} + \underbrace{\frac{2^n}{n^n}}_{\rightarrow 0^+} + \underbrace{\frac{\sqrt{n}}{n^n}}_{\rightarrow 0^+} + \underbrace{\frac{\ln^3 n}{n^n}}_{\rightarrow 0^+} + \underbrace{\frac{1}{n}}_{\rightarrow 0^+} + \underbrace{\frac{1}{n^{n+1}}}_{\rightarrow 0^+} \right) = 0^+ \end{aligned}$$

For example, $\frac{2^n}{n^n} = \frac{2^n}{n!} \frac{n!}{n^n}$ is the product of two functions that tend to 0^+ , by Propositions 10.4 and 10.5.

Hierarchy of the orders of growth. For every $a \in \mathbb{R}_{>1}$, $\beta \in \mathbb{R}_{>0}$, $\alpha \in \mathbb{R}_{>0}$ and $b \in \mathbb{R}_{>1}$, we have

$$\log_a^\beta n = o(n^\alpha), \quad n^\alpha = o(b^n), \quad b^n = o(n!), \quad n! = o(n^n).$$

By the transitivity of o (chapter 9 «Asymptotic notation»), each function of the chain is o of all the functions that follow it, for example $\log_a^\beta n = o(n!)$.

10.3.2 Classification of orders of growth

The families of the hierarchy are used as yardsticks to classify the order of growth of any other function.

Definition 10.11: bounded orders of growth

An asymptotically increasing function of a natural variable f has:

- a **polylogarithmically bounded order of growth**, if there exists $\beta \in \mathbb{R}_{>0}$ such that $f(n) = \mathcal{O}(\ln^\beta n)$; and a **logarithmically bounded order of growth**, if $f(n) = \mathcal{O}(\ln n)$;
- a **polynomially bounded order of growth**, if there exists $\alpha \in \mathbb{R}_{>0}$ such that $f(n) = \mathcal{O}(n^\alpha)$;
- an **exponentially bounded order of growth**, if there exists $b \in \mathbb{R}_{>1}$ such that $f(n) = \mathcal{O}(b^n)$;
- a **factorially bounded order of growth**, if $f(n) = \mathcal{O}(n!)$.

Definition 10.12: linear, sub-linear and super-linear order of growth

An asymptotically increasing function of a natural variable f has:

- a **sub-linear order of growth**, if $f(n) = o(n)$;
- a **linear order of growth**, if $f(n) = \Theta(n)$;
- a **super-linear order of growth**, if $f(n) = \omega(n)$.

- By the hierarchy of the orders of growth, a function with a polylogarithmically bounded order of growth has also a polynomially bounded order of growth, which in turn is also exponentially bounded and factorially bounded. For example, $f(n) = \mathcal{O}(\ln^\beta n)$ and $\ln^\beta n = o(n)$ give $f(n) = \mathcal{O}(n)$, since o implies $\mathcal{O}$ and $\mathcal{O}$ is transitive (chapter 9 «Asymptotic notation»).

Example 10.5: classification of orders of growth

We classify four functions, drawn in Figure 10.3.

- $\sqrt{n} = o(n)$, since $\frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \rightarrow 0^+$: a sub-linear order of growth. It has a polynomially bounded order of growth, since $\sqrt{n} = \mathcal{O}(n^{1/2})$, but not a polylogarithmically bounded one: since $\frac{\ln^\beta n}{\sqrt{n}} \rightarrow 0^+$ (Proposition 10.2), we have $\frac{\sqrt{n}}{\ln^\beta n} \rightarrow +\infty$, that is, $\sqrt{n} = \omega(\ln^\beta n)$ and $\sqrt{n} \neq \mathcal{O}(\ln^\beta n)$ for every $\beta \in \mathbb{R}_{>0}$ (Proposition 10.6).
- $n = \Theta(n)$: a linear order of growth.
- $n \ln n = \omega(n)$, since $\frac{n \ln n}{n} = \ln n \rightarrow +\infty$: a super-linear order of growth. It has a

polynomially bounded order of growth, since $\frac{n \ln n}{n^2} = \frac{\ln n}{n} \rightarrow 0^+$, so that $n \ln n = o(n^2)$ and $n \ln n = \mathcal{O}(n^2)$.

- $n^2 = \omega(n)$, since $\frac{n^2}{n} = n \rightarrow +\infty$: a super-linear order of growth. It has a polynomially bounded order of growth, since $n^2 = \mathcal{O}(n^2)$.

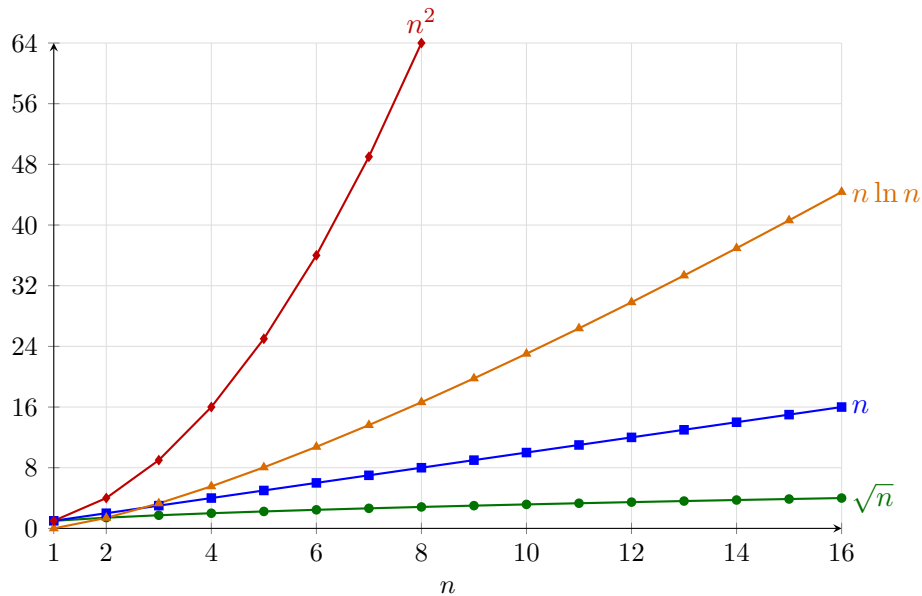


Figure 10.3: A sub-linear function ($\sqrt{n}$), the linear function n and two super-linear functions ($n \ln n$ and n^2) for $n \in \{1, 2, \dots, 16\}$ (n^2 only up to $n = 8$, where it reaches 64; the points are joined by segments).

10.3.3 Properties of logarithmic functions

In the proofs of this subsection and of the following ones we compute the limit of the ratio of the two functions and we apply Proposition 10.6.

- Any two logarithmic functions of a natural variable with bases strictly greater than one are Θ of each other; multiplying the argument by a positive constant does not change the order of growth either.

Observation 10.13: change of base

For every $a \in \mathbb{R}_{>1}$, $c \in \mathbb{R}_{>1}$ and $\lambda \in \mathbb{R}_{>0}$, we have $\log_a(\lambda n) = \Theta(\log_c n)$.

Proof. For every $a, c \in \mathbb{R}_{>1}$ and $\lambda \in \mathbb{R}_{>0}$, since $\log_a(\lambda n) = \log_a \lambda + \log_a n$ and $\log_a n =$

$\frac{\log_c n}{\log_c a}$, we have:

$$\lim_{n \rightarrow +\infty} \frac{\log_a(\lambda n)}{\log_c n} = \lim_{n \rightarrow +\infty} \left(\underbrace{\frac{\log_a \lambda}{\log_c n}}_{\rightarrow 0} + \frac{\frac{\log_c n}{\log_c a}}{\log_c n} \right) = \frac{1}{\log_c a} \in \mathbb{R}_{>0}$$

□

- For this reason, when only the order of growth matters, the base of a logarithm is often omitted. For example, $\log_2 n = \Theta(\ln n)$, and $\lim_{n \rightarrow +\infty} \frac{\log_2 n}{\ln n} = \frac{1}{\ln 2} \approx 1.44$.
- Adding a positive constant to the argument of a logarithmic function of a natural variable does not change its order of growth.

Observation 10.14: shift of the argument

For every $a \in \mathbb{R}_{>1}$ and $\lambda \in \mathbb{R}_{>0}$, we have $\log_a(n + \lambda) = \Theta(\log_a n)$.

Proof. For every $a \in \mathbb{R}_{>1}$ and $\lambda \in \mathbb{R}_{>0}$, we have:

$$\lim_{n \rightarrow +\infty} \frac{\log_a(n + \lambda)}{\log_a n} = \lim_{n \rightarrow +\infty} \frac{\log_a \left(n \left(1 + \frac{\lambda}{n} \right) \right)}{\log_a n} = \lim_{n \rightarrow +\infty} \frac{\log_a n + \overbrace{\log_a \left(1 + \frac{\lambda}{n} \right)}^{\rightarrow 0}}{\log_a n} = 1$$

□

- The **quasi-linear** function of a natural variable $n \mapsto n \log_a n$ is o of any power function of a natural variable with exponent strictly greater than one.

Observation 10.15: quasi-linear and powers

For every $\alpha \in \mathbb{R}_{>1}$ and $a \in \mathbb{R}_{>1}$, we have $n \log_a n = o(n^\alpha)$.

Proof. For every $\alpha \in \mathbb{R}_{>1}$ and $a \in \mathbb{R}_{>1}$, we have $\alpha - 1 \in \mathbb{R}_{>0}$ and, by Proposition 10.2:

$$\lim_{n \rightarrow +\infty} \frac{n \log_a n}{n^\alpha} = \lim_{n \rightarrow +\infty} \frac{\log_a n}{n^{\alpha-1}} = 0^+$$

□

- The quasi-linear function of a natural variable $n \mapsto n \log_a n$ is ω of the linear function of a natural variable: it has a super-linear order of growth.

Observation 10.16: quasi-linear and linear

For every $a \in \mathbb{R}_{>1}$, we have $n \log_a n = \omega(n)$.

Proof. For every $a \in \mathbb{R}_{>1}$, we have:

$$\lim_{n \rightarrow +\infty} \frac{n \log_a n}{n} = \lim_{n \rightarrow +\infty} \log_a n = +\infty$$

□

10.3.4 Properties of power functions

- Adding a positive constant to the argument of a power function of a natural variable does not change its order of growth.

Observation 10.17: shift of the argument

For every $\alpha \in \mathbb{R}_{>0}$ and $\lambda \in \mathbb{R}_{>0}$, we have $(n + \lambda)^\alpha = \Theta(n^\alpha)$.

Proof. For every $\alpha \in \mathbb{R}_{>0}$ and $\lambda \in \mathbb{R}_{>0}$, we have:

$$\lim_{n \rightarrow +\infty} \frac{(n + \lambda)^\alpha}{n^\alpha} = \lim_{n \rightarrow +\infty} \left(1 + \underbrace{\frac{\lambda}{n}}_{\rightarrow 0} \right)^\alpha = 1$$

□

- Multiplying the argument of a power function of a natural variable by a positive constant does not change its order of growth.

Observation 10.18: scaling of the argument

For every $\alpha \in \mathbb{R}_{>0}$ and $\lambda \in \mathbb{R}_{>0}$, we have $(\lambda n)^\alpha = \Theta(n^\alpha)$.

Proof. For every $\alpha \in \mathbb{R}_{>0}$ and $\lambda \in \mathbb{R}_{>0}$, we have:

$$\lim_{n \rightarrow +\infty} \frac{(\lambda n)^\alpha}{n^\alpha} = \lim_{n \rightarrow +\infty} \frac{\lambda^\alpha n^\alpha}{n^\alpha} = \lambda^\alpha \in \mathbb{R}_{>0}$$

□

- The order of growth of a power function of a natural variable, compared to the linear function of a natural variable, depends on whether the exponent is greater or smaller than one.

Observation 10.19: powers and the linear function

For every $\alpha \in \mathbb{R}_{>0}$, we have:

- $n^\alpha = \omega(n)$ (super-linear order of growth), if $\alpha > 1$;
- $n^\alpha = o(n)$ (sub-linear order of growth), if $\alpha < 1$.

Proof. For every $\alpha \in \mathbb{R}_{>0}$, we have:

$$\lim_{n \rightarrow +\infty} \frac{n^\alpha}{n} = \lim_{n \rightarrow +\infty} n^{\alpha-1} = \begin{cases} +\infty & \text{if } \alpha > 1 \\ 0^+ & \text{if } \alpha < 1 \end{cases}$$

□

10.3.5 Properties of polynomial functions

- Any polynomial function of a natural variable of degree k with a positive leading coefficient is $\Theta(n^k)$: it has the same order of growth as its leading term.

Observation 10.20: polynomials

For every $k \in \mathbb{N}_{\geq 1}$ and every $a_0, a_1, \dots, a_k \in \mathbb{R}$ with $a_k \in \mathbb{R}_{>0}$, we have $\sum_{i=0}^k a_i n^i = \Theta(n^k)$.

Proof. We have:

$$\lim_{n \rightarrow +\infty} \frac{\sum_{i=0}^k a_i n^i}{n^k} = a_k + \lim_{n \rightarrow +\infty} \sum_{i=0}^{k-1} \underbrace{\frac{a_i}{n^{k-i}}}_{\rightarrow 0} = a_k \in \mathbb{R}_{>0}$$

In particular, the polynomial is asymptotically positive, so Proposition 10.6 applies. □

10.3.6 Properties of exponential functions

- Any exponential function of a natural variable with a base strictly greater than one is ω of the exponential functions of a natural variable with a smaller base, and o of the exponential functions of a natural variable with a greater base.

Observation 10.21: different bases

For every $b \in \mathbb{R}_{>1}$ and $c \in \mathbb{R}_{>1}$, we have:

- $b^n = \omega(c^n)$, if $b > c$;
- $b^n = o(c^n)$, if $b < c$.

Proof. For every $b \in \mathbb{R}_{>1}$ and $c \in \mathbb{R}_{>1}$, since $b = c^{\log_c b}$, we have:

$$\lim_{n \rightarrow +\infty} \frac{b^n}{c^n} = \lim_{n \rightarrow +\infty} \frac{c^{n \log_c b}}{c^n} = \lim_{n \rightarrow +\infty} \left(c^{\log_c b - 1} \right)^n = \begin{cases} +\infty & \text{if } b > c \\ 0^+ & \text{if } b < c \end{cases}$$

since $c^{\log_c b - 1} = \frac{b}{c}$ is greater than one if $b > c$ and belongs to $(0, 1)$ if $b < c$ (limit of the geometric progression). $\square$

- Multiplying the exponent of an exponential function of a natural variable by a positive constant changes its order of growth, unless the constant is equal to one.

Observation 10.22: scaling of the exponent

For every $b \in \mathbb{R}_{>1}$ and $\lambda \in \mathbb{R}_{>0}$, we have:

- $b^{\lambda n} = \omega(b^n)$, if $\lambda > 1$;
- $b^{\lambda n} = o(b^n)$, if $0 < \lambda < 1$.

Proof. For every $b \in \mathbb{R}_{>1}$ and $\lambda \in \mathbb{R}_{>0}$, we have:

$$\lim_{n \rightarrow +\infty} \frac{b^{\lambda n}}{b^n} = \lim_{n \rightarrow +\infty} \left(\frac{b^\lambda}{b} \right)^n = \lim_{n \rightarrow +\infty} \left(b^{\lambda - 1} \right)^n = \begin{cases} +\infty & \text{if } \lambda > 1 \\ 0^+ & \text{if } 0 < \lambda < 1 \end{cases}$$

$\square$

- An alternative proof is based on the fact that

$$b^{\lambda n} = (b^\lambda)^n \quad \text{and} \quad b^\lambda \begin{cases} > b & \text{if } \lambda > 1 \\ < b & \text{if } 0 < \lambda < 1 \end{cases}$$

Accordingly, by Observation 10.21 applied to the bases b^λ and b , we have $b^{\lambda n} = \omega(b^n)$ if $\lambda > 1$, and $b^{\lambda n} = o(b^n)$ if $0 < \lambda < 1$.

Example 10.6: scaling the exponent

- We have $2^{n/2} = o(2^n)$, since

$$2^{n/2} = (2^{1/2})^n \quad \text{and} \quad 2^{1/2} = \sqrt{2} < 2.$$

- We have $2^{3n/2} = \omega(2^n)$, since

$$2^{3n/2} = (2^{3/2})^n \quad \text{and} \quad 2^{3/2} = 2\sqrt{2} > 2.$$

- Shifting the exponent of an exponential function of a natural variable by a constant does not change its order of growth.

Observation 10.23: shift of the exponent

For every $b \in \mathbb{R}_{>1}$ and $\lambda \in \mathbb{R}$, we have $b^{n+\lambda} = \Theta(b^n)$.

Proof. For every $b \in \mathbb{R}_{>1}$ and $\lambda \in \mathbb{R}$, we have:

$$\lim_{n \rightarrow +\infty} \frac{b^{n+\lambda}}{b^n} = \lim_{n \rightarrow +\infty} \frac{b^n b^\lambda}{b^n} = b^\lambda \in \mathbb{R}_{>0}$$

□

- For example, $2^{n+10} = 1024 \cdot 2^n = \Theta(2^n)$, while $2^{2n} = 4^n = \omega(2^n)$: a constant added to the exponent is a constant factor, a constant factor in the exponent changes the base.

Key points of the chapter

- The functions of a natural variable used in the analysis of algorithms are logarithmic ($\log_a n$), power (n^α), exponential (b^n), factorial ($n!$) and self-power (n^n) functions, together with their sums and products.
- The orders of growth are compared with the asymptotic notation of chapter 9 «Asymptotic notation». When the limit of $\frac{f(n)}{g(n)}$ exists it decides (limit criterion): 0^+ gives $f(n) = o(g(n))$, a positive real number gives $f(n) = \Theta(g(n))$, $+\infty$ gives $f(n) = \omega(g(n))$.
- Hierarchy (with $a, b > 1$ and $\alpha, \beta > 0$): $\log_a^\beta n = o(n^\alpha)$, $n^\alpha = o(b^n)$, $b^n = o(n!)$, $n! = o(n^n)$.

- The base of a logarithm, a constant added to the argument of a logarithm or of a power, a constant factor in the argument of a power, and a constant added to the exponent of an exponential do not change the order of growth (the two functions are Θ of each other); a polynomial of degree k with positive leading coefficient is $\Theta(n^k)$.
- A constant factor in the exponent of an exponential, and a different base of an exponential, do change the order of growth: for example $2^{n/2} = o(2^n)$ and $3^n = \omega(2^n)$.
- A function has a sub-linear, linear or super-linear order of growth if it is $o(n)$, $\Theta(n)$ or $\omega(n)$; for example $\sqrt{n}$, $5n + 3$ and $n \ln n$.