



SAPIENZA
UNIVERSITÀ DI ROMA

Algorithms

Exercises 3 — Asymptotic notation and growth

Worked exercises on chapters 9–10: the asymptotic notations Theta, big O, Omega, little o and omega, and the comparison of orders of growth. Every exercise has a complete solution.

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Asymptotic notation

In the exercises of this chapter the asymptotic notations are used with the definitions and the limit criteria of the chapter «Asymptotic notation», summarized below. The limits of functions of a natural variable (sequences) are computed with the results of the course *Mathematical Analysis 1* (<https://fabiofurini.github.io/mathematical-analysis-1/>).

Given two functions f and g of a natural variable,

- we write $f(n) = o(g(n))$ if for all $c \in \mathbb{R}_{>0}$ there exists $n_1 \in \mathbb{N}$ such that

$$0 \leq f(n) < c g(n), \quad \forall n \geq n_1;$$

- we write $f(n) = \mathcal{O}(g(n))$ if there exist $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq f(n) \leq c g(n), \quad \forall n \geq n_1;$$

- we write $f(n) = \Theta(g(n))$ if there exist $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n), \quad \forall n \geq n_1;$$

- we write $f(n) = \Omega(g(n))$ if there exist $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c g(n) \leq f(n), \quad \forall n \geq n_1;$$

- we write $f(n) = \omega(g(n))$ if for all $c \in \mathbb{R}_{>0}$ there exists $n_1 \in \mathbb{N}$ such that

$$0 \leq c g(n) < f(n), \quad \forall n \geq n_1.$$

Limit criterion: given two asymptotically positive functions f and g of a natural variable such that the limit of $\frac{f(n)}{g(n)}$ exists, we have

$$\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = 0^+ \iff f(n) = o(g(n)),$$

$$\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = \ell \in \mathbb{R}_{>0} \text{ or } 0^+ \iff f(n) = \mathcal{O}(g(n)),$$

$$\begin{aligned}\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = \ell \in \mathbb{R}_{>0} &\iff f(n) = \Theta(g(n)), \\ \lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = \ell \in \mathbb{R}_{>0} \text{ or } +\infty &\iff f(n) = \Omega(g(n)), \\ \lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)} = +\infty &\iff f(n) = \omega(g(n)).\end{aligned}$$

When the ratio $\frac{f(n)}{g(n)}$ has no limit (for example when it contains $\sin n$ or $\cos n$), these criteria cannot be applied and we use the definitions.

Exercise 9.1 — Asymptotic notation (E1)

Consider the following asymptotically non-negative function of a natural variable:

$$f : \mathbb{N} \rightarrow \mathbb{R}, \quad f : n \mapsto 5n^4 - 19n^3 + 12n^2$$

Questions

1. Prove that:

- (a) $5n^4 - 19n^3 + 12n^2 = \mathcal{O}(n^4)$;
- (b) $5n^4 - 19n^3 + 12n^2 = \mathcal{O}(n^6)$;
- (c) $5n^4 - 19n^3 + 12n^2 = \Theta(n^4)$;
- (d) $5n^4 - 19n^3 + 12n^2 \neq \Theta(n^6)$.

Solution

1. (a) To prove that

$$5n^4 - 19n^3 + 12n^2 = \mathcal{O}(n^4)$$

we determine $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq 5n^4 - 19n^3 + 12n^2 \leq cn^4, \quad \forall n \geq n_1.$$

First, we observe that

$$5n^4 - 19n^3 + 12n^2 = n^2(5n^2 - 19n + 12) = n^2(5n - 4)(n - 3),$$

hence

$$0 \leq 5n^4 - 19n^3 + 12n^2, \quad \forall n \geq 3$$

(for $n = 2$ the value is $f(2) = -24 < 0$). With $n \geq 1$, dividing by n^4 , the inequality $f(n) \leq cn^4$ is equivalent to

$$5 - \frac{19}{n} + \frac{12}{n^2} \leq c, \quad \forall n \geq n_1.$$

Since

$$-\frac{19}{n} + \frac{12}{n^2} = \frac{12 - 19n}{n^2} \leq 0, \quad \forall n \geq 1,$$

we have

$$5 - \frac{19}{n} + \frac{12}{n^2} \leq 5, \quad \forall n \geq 1.$$

By choosing for example $c = 5$ and $n_1 = 3$, we have

$$0 \leq 5n^4 - 19n^3 + 12n^2 \leq 5n^4, \quad \forall n \geq 3.$$

Since the functions are asymptotically positive, an alternative way is to compute the limit

$$\lim_{n \rightarrow +\infty} \frac{5n^4 - 19n^3 + 12n^2}{n^4} = \lim_{n \rightarrow +\infty} \left(5 - \frac{19}{n} + \frac{12}{n^2} \right) = 5 \in \mathbb{R}_{>0}.$$

(b) To prove that

$$5n^4 - 19n^3 + 12n^2 = \mathcal{O}(n^6)$$

we determine $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq 5n^4 - 19n^3 + 12n^2 \leq cn^6, \quad \forall n \geq n_1.$$

From part (a), $0 \leq 5n^4 - 19n^3 + 12n^2 \leq 5n^4$ for all $n \geq 3$. Since $n^4 \leq n^6$ for $n \geq 1$, we also have $5n^4 - 19n^3 + 12n^2 \leq 5n^6$. By choosing for example $c = 5$ and $n_1 = 3$, we have

$$0 \leq 5n^4 - 19n^3 + 12n^2 \leq 5n^6, \quad \forall n \geq 3.$$

Since the functions are asymptotically positive, an alternative way is to compute the limit

$$\lim_{n \rightarrow +\infty} \frac{5n^4 - 19n^3 + 12n^2}{n^6} = \lim_{n \rightarrow +\infty} \left(\frac{5}{n^2} - \frac{19}{n^3} + \frac{12}{n^4} \right) = 0^+.$$

(c) To prove that

$$5n^4 - 19n^3 + 12n^2 = \Theta(n^4)$$

we determine $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^4 \leq 5n^4 - 19n^3 + 12n^2 \leq c_2 n^4, \quad \forall n \geq n_1.$$

For the upper bound, by part (a), $5n^4 - 19n^3 + 12n^2 \leq 5n^4$ for all $n \geq 1$. For the lower bound, we observe that

$$5n^4 - 19n^3 + 12n^2 - n^4 = 4n^4 - 19n^3 + 12n^2 = n^2(4n - 3)(n - 4),$$

which is non-negative for $n \geq 4$ (for $n = 3$ it is equal to $-81 < 0$). Hence $n^4 \leq 5n^4 - 19n^3 + 12n^2$ for all $n \geq 4$. By choosing for example $c_1 = 1$, $c_2 = 5$ and $n_1 = 4$, we have

$$0 \leq n^4 \leq 5n^4 - 19n^3 + 12n^2 \leq 5n^4, \quad \forall n \geq 4.$$

The limit computed in part (a) gives the same result: it is equal to $5 \in \mathbb{R}_{>0}$.

(d) To prove that

$$5n^4 - 19n^3 + 12n^2 \neq \Theta(n^6)$$

we proceed by contradiction and we suppose that there exist $c_1 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^6 \leq 5n^4 - 19n^3 + 12n^2, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^6 and since $c_1 > 0$, this is equivalent to

$$0 < c_1 \leq \frac{5}{n^2} - \frac{19}{n^3} + \frac{12}{n^4}, \quad \forall n \geq n_1.$$

Passing to the limit as $n \rightarrow +\infty$, we obtain $0 < c_1 \leq 0$, which is impossible. Therefore

$$5n^4 - 19n^3 + 12n^2 \neq \Theta(n^6).$$

Alternatively, the limit computed in part (b) is equal to 0^+ , so that $f(n) = o(n^6)$ by the limit criterion. Then, with the constant $c = c_1$ in the definition of o , we would have $f(n) < c_1 n^6$ for all sufficiently large n , against the lower bound $c_1 n^6 \leq f(n)$.

Exercise 9.2 — Asymptotic notation (E2)

Consider the following asymptotically non-negative function of a natural variable:

$$f : \mathbb{N} \rightarrow \mathbb{R}, \quad f : n \mapsto 3n^2 + n^3(2 - \sin n)$$

Questions

1. Prove that:

- (a) $3n^2 + n^3(2 - \sin n) = \Theta(n^3)$;
- (b) $3n^2 + n^3(2 - \sin n) \neq o(n^3)$;
- (c) $3n^2 + n^3(2 - \sin n) \neq \omega(n^3)$.

Solution

The ratio

$$\frac{3n^2 + n^3(2 - \sin n)}{n^3} = \frac{3}{n} + 2 - \sin n, \quad n \geq 1,$$

does not have a limit as $n \rightarrow +\infty$, since it oscillates. Therefore the limit criteria cannot be applied, and we use the definitions. Since $-1 \leq \sin n \leq 1$, we have $1 \leq 2 - \sin n \leq 3$, and therefore $1 + \frac{3}{n} \leq \frac{3}{n} + 2 - \sin n \leq 3 + \frac{3}{n}$. Since $0 < \frac{3}{n} \leq 3$ for $n \geq 1$, we obtain

$$1 \leq \frac{3}{n} + 2 - \sin n \leq 6, \quad \forall n \geq 1.$$

1. (a) To prove that

$$3n^2 + n^3(2 - \sin n) = \Theta(n^3)$$

we determine $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^3 \leq 3n^2 + n^3(2 - \sin n) \leq c_2 n^3, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^3 and since $c_1 > 0$, this is equivalent to

$$0 < c_1 \leq \frac{3}{n} + 2 - \sin n \leq c_2, \quad \forall n \geq n_1.$$

By the bounds above, by choosing for example $c_1 = \frac{1}{2}$, $c_2 = 6$ and $n_1 = 1$, we have

$$0 \leq \frac{1}{2} n^3 \leq 3n^2 + n^3(2 - \sin n) \leq 6n^3, \quad \forall n \geq 1.$$

- (b) To prove that

$$3n^2 + n^3(2 - \sin n) \neq o(n^3)$$

we recall that $o(n^3)$ requires that for every $c \in \mathbb{R}_{>0}$ there exists $n_1 \in \mathbb{N}$ such that

$$0 \leq 3n^2 + n^3(2 - \sin n) < c n^3, \quad \forall n \geq n_1,$$

that is, dividing by $n^3 > 0$, $\frac{3}{n} + 2 - \sin n < c$ for all $n \geq n_1$. Since $\frac{3}{n} + 2 - \sin n \geq 1$ for every $n \geq 1$, for $c = 1$ there is no n_1 with this property. Therefore

$$3n^2 + n^3(2 - \sin n) \neq o(n^3).$$

Equivalently, the ratio is bounded from below by 1 and cannot converge to 0.

- (c) To prove that

$$3n^2 + n^3(2 - \sin n) \neq \omega(n^3)$$

we recall that $\omega(n^3)$ requires that for every $c \in \mathbb{R}_{>0}$ there exists $n_1 \in \mathbb{N}$ such that

$$0 \leq c n^3 < 3n^2 + n^3(2 - \sin n), \quad \forall n \geq n_1,$$

that is, dividing by $n^3 > 0$, $c < \frac{3}{n} + 2 - \sin n$ for all $n \geq n_1$. Since $\frac{3}{n} + 2 - \sin n \leq 6$ for every $n \geq 1$, for $c = 6$ there is no n_1 with this property. Therefore

$$3n^2 + n^3(2 - \sin n) \neq \omega(n^3).$$

Equivalently, the ratio is bounded from above by 6 and cannot diverge to $+\infty$.

Exercise 9.3 — Asymptotic notation (E3)

Consider the following asymptotically non-negative function of a natural variable:

$$f : \mathbb{N} \rightarrow \mathbb{R}, \quad f : n \mapsto 3n^3 - 7n^2 + 2n$$

Questions

1. Prove that:

- (a) $3n^3 - 7n^2 + 2n = \mathcal{O}(n^3)$;
- (b) $3n^3 - 7n^2 + 2n = \mathcal{O}(n^5)$;
- (c) $3n^3 - 7n^2 + 2n = \Theta(n^3)$;
- (d) $3n^3 - 7n^2 + 2n \neq \Theta(n^5)$.

Solution

1. (a) To prove that

$$3n^3 - 7n^2 + 2n = \mathcal{O}(n^3)$$

we determine $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq 3n^3 - 7n^2 + 2n \leq cn^3, \quad \forall n \geq n_1.$$

First, we observe that

$$3n^3 - 7n^2 + 2n = n(3n^2 - 7n + 2) = n(3n - 1)(n - 2),$$

hence

$$0 \leq 3n^3 - 7n^2 + 2n, \quad \forall n \geq 2$$

(for $n = 1$ the value is $f(1) = -2 < 0$). With $n \geq 1$, dividing by n^3 , the inequality $f(n) \leq cn^3$ is equivalent to

$$3 - \frac{7}{n} + \frac{2}{n^2} \leq c, \quad \forall n \geq n_1.$$

Since

$$-\frac{7}{n} + \frac{2}{n^2} = \frac{2 - 7n}{n^2} \leq 0, \quad \forall n \geq 1,$$

we have

$$3 - \frac{7}{n} + \frac{2}{n^2} \leq 3, \quad \forall n \geq 1.$$

By choosing for example $c = 3$ and $n_1 = 2$, we have

$$0 \leq 3n^3 - 7n^2 + 2n \leq 3n^3, \quad \forall n \geq 2.$$

Since the functions are asymptotically positive, an alternative way is to compute the limit

$$\lim_{n \rightarrow +\infty} \frac{3n^3 - 7n^2 + 2n}{n^3} = \lim_{n \rightarrow +\infty} \left(3 - \frac{7}{n} + \frac{2}{n^2} \right) = 3 \in \mathbb{R}_{>0}.$$

(b) To prove that

$$3n^3 - 7n^2 + 2n = \mathcal{O}(n^5)$$

we determine $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq 3n^3 - 7n^2 + 2n \leq cn^5, \quad \forall n \geq n_1.$$

From part (a), $0 \leq 3n^3 - 7n^2 + 2n \leq 3n^3$ for all $n \geq 2$. Since $n^3 \leq n^5$ for $n \geq 1$, we also have $3n^3 - 7n^2 + 2n \leq 3n^5$. By choosing for example $c = 3$ and $n_1 = 2$, we have

$$0 \leq 3n^3 - 7n^2 + 2n \leq 3n^5, \quad \forall n \geq 2.$$

Since the functions are asymptotically positive, an alternative way is to compute the limit

$$\lim_{n \rightarrow +\infty} \frac{3n^3 - 7n^2 + 2n}{n^5} = \lim_{n \rightarrow +\infty} \left(\frac{3}{n^2} - \frac{7}{n^3} + \frac{2}{n^4} \right) = 0^+.$$

(c) To prove that

$$3n^3 - 7n^2 + 2n = \Theta(n^3)$$

we determine $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^3 \leq 3n^3 - 7n^2 + 2n \leq c_2 n^3, \quad \forall n \geq n_1.$$

For the upper bound, by part (a), $3n^3 - 7n^2 + 2n \leq 3n^3$ for all $n \geq 1$. For the lower bound, we observe that

$$3n^3 - 7n^2 + 2n - n^3 = 2n^3 - 7n^2 + 2n = n(2n^2 - 7n + 2),$$

where the roots of $2n^2 - 7n + 2$ are $\frac{7 \pm \sqrt{33}}{4}$, that is, about 0.31 and 3.19; hence $2n^2 - 7n + 2 > 0$ for every integer $n \geq 4$ (for $n = 4$ it is equal to 6, for $n = 3$ it is equal to -1). Hence $n^3 \leq 3n^3 - 7n^2 + 2n$ for all $n \geq 4$. By choosing for example $c_1 = 1$, $c_2 = 3$ and $n_1 = 4$, we have

$$0 \leq n^3 \leq 3n^3 - 7n^2 + 2n \leq 3n^3, \quad \forall n \geq 4.$$

The limit computed in part (a) gives the same result: it is equal to $3 \in \mathbb{R}_{>0}$.

(d) To prove that

$$3n^3 - 7n^2 + 2n \neq \Theta(n^5)$$

we proceed by contradiction and we suppose that there exist $c_1 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^5 \leq 3n^3 - 7n^2 + 2n, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^5 and since $c_1 > 0$, this is equivalent to

$$0 < c_1 \leq \frac{3}{n^2} - \frac{7}{n^3} + \frac{2}{n^4}, \quad \forall n \geq n_1.$$

Passing to the limit as $n \rightarrow +\infty$, we obtain $0 < c_1 \leq 0$, which is impossible. Therefore

$$3n^3 - 7n^2 + 2n \neq \Theta(n^5).$$

Alternatively, the limit computed in part (b) is equal to 0^+ , so that $f(n) = o(n^5)$ by the limit criterion. Then, with the constant $c = c_1$ in the definition of o , we would have $f(n) < c_1 n^5$ for all sufficiently large n , against the lower bound $c_1 n^5 \leq f(n)$.

Exercise 9.4 — Asymptotic notation (E4)

Consider the following asymptotically non-negative function of a natural variable:

$$f : \mathbb{N} \rightarrow \mathbb{R}, \quad f : n \mapsto n^2 (3 + \cos n)$$

Questions

1. Prove that:

- (a) $n^2 (3 + \cos n) = \Theta(n^2)$;
- (b) $n^2 (3 + \cos n) \neq o(n^2)$;
- (c) $n^2 (3 + \cos n) \neq \omega(n^2)$.

Solution

The ratio

$$\frac{n^2 (3 + \cos n)}{n^2} = 3 + \cos n, \quad n \geq 1,$$

does not have a limit as $n \rightarrow +\infty$, since it oscillates. Therefore the limit criteria cannot be applied, and we use the definitions. Since $-1 \leq \cos n \leq 1$, we have

$$2 \leq 3 + \cos n \leq 4, \quad \forall n \geq 1.$$

1. (a) To prove that

$$n^2 (3 + \cos n) = \Theta(n^2)$$

we determine $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^2 \leq n^2 (3 + \cos n) \leq c_2 n^2, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^2 and since $c_1 > 0$, this is equivalent to

$$0 < c_1 \leq 3 + \cos n \leq c_2, \quad \forall n \geq n_1.$$

By the bounds above, by choosing for example $c_1 = 2$, $c_2 = 4$ and $n_1 = 1$, we have

$$0 \leq 2n^2 \leq n^2 (3 + \cos n) \leq 4n^2, \quad \forall n \geq 1.$$

(b) To prove that

$$n^2 (3 + \cos n) \neq o(n^2)$$

we recall that $o(n^2)$ requires that for every $c \in \mathbb{R}_{>0}$ there exists $n_1 \in \mathbb{N}$ such that

$$0 \leq n^2 (3 + \cos n) < c n^2, \quad \forall n \geq n_1,$$

that is, dividing by $n^2 > 0$, $3 + \cos n < c$ for all $n \geq n_1$. Since $3 + \cos n \geq 2$ for every $n \geq 1$, for $c = 2$ there is no n_1 with this property. Therefore

$$n^2 (3 + \cos n) \neq o(n^2).$$

Equivalently, the ratio is bounded from below by 2 and cannot converge to 0.

(c) To prove that

$$n^2 (3 + \cos n) \neq \omega(n^2)$$

we recall that $\omega(n^2)$ requires that for every $c \in \mathbb{R}_{>0}$ there exists $n_1 \in \mathbb{N}$ such that

$$0 \leq c n^2 < n^2 (3 + \cos n), \quad \forall n \geq n_1,$$

that is, dividing by $n^2 > 0$, $c < 3 + \cos n$ for all $n \geq n_1$. Since $3 + \cos n \leq 4$ for every $n \geq 1$, for $c = 4$ there is no n_1 with this property. Therefore

$$n^2 (3 + \cos n) \neq \omega(n^2).$$

Equivalently, the ratio is bounded from above by 4 and cannot diverge to $+\infty$.

Exercise 9.5 — Asymptotic notation (E5)

Consider the following asymptotically non-negative function of a natural variable:

$$f : \mathbb{N} \rightarrow \mathbb{R}, \quad f : n \mapsto 2 n^5 - 8 n^4 + 3 n^3$$

Questions

1. Prove that:

(a) $2 n^5 - 8 n^4 + 3 n^3 = \mathcal{O}(n^5);$

(b) $2 n^5 - 8 n^4 + 3 n^3 = \Theta(n^5);$

(c) $2 n^5 - 8 n^4 + 3 n^3 \neq \Theta(n^3).$

Solution

1. (a) To prove that

$$2 n^5 - 8 n^4 + 3 n^3 = \mathcal{O}(n^5)$$

we determine $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq 2n^5 - 8n^4 + 3n^3 \leq cn^5, \quad \forall n \geq n_1.$$

First, we factorize:

$$2n^5 - 8n^4 + 3n^3 = n^3(2n^2 - 8n + 3),$$

where the roots of $2n^2 - 8n + 3$ are $2 \pm \frac{\sqrt{10}}{2}$, that is, about 0.42 and 3.58. Hence $f(n) \geq 0$ for every integer $n \geq 4$ (for $n = 4$ the factor $2n^2 - 8n + 3$ is equal to 3; for $n = 3$ we have $f(3) = -81 < 0$). With $n \geq 1$, dividing by n^5 , the inequality $f(n) \leq cn^5$ is equivalent to

$$2 - \frac{8}{n} + \frac{3}{n^2} \leq c, \quad \forall n \geq n_1.$$

Since

$$-\frac{8}{n} + \frac{3}{n^2} = \frac{3 - 8n}{n^2} \leq 0, \quad \forall n \geq 1,$$

we have

$$2 - \frac{8}{n} + \frac{3}{n^2} \leq 2, \quad \forall n \geq 1.$$

By choosing for example $c = 2$ and $n_1 = 4$, we have

$$0 \leq 2n^5 - 8n^4 + 3n^3 \leq 2n^5, \quad \forall n \geq 4.$$

Since the functions are asymptotically positive, an alternative way is to compute the limit

$$\lim_{n \rightarrow +\infty} \frac{2n^5 - 8n^4 + 3n^3}{n^5} = \lim_{n \rightarrow +\infty} \left(2 - \frac{8}{n} + \frac{3}{n^2} \right) = 2 \in \mathbb{R}_{>0}.$$

(b) To prove that

$$2n^5 - 8n^4 + 3n^3 = \Theta(n^5)$$

we determine $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^5 \leq 2n^5 - 8n^4 + 3n^3 \leq c_2 n^5, \quad \forall n \geq n_1.$$

For the upper bound, by part (a), $2n^5 - 8n^4 + 3n^3 \leq 2n^5$ for all $n \geq 1$. For the lower bound, we observe that

$$2n^5 - 8n^4 + 3n^3 - n^5 = n^5 - 8n^4 + 3n^3 = n^3(n^2 - 8n + 3),$$

where the roots of $n^2 - 8n + 3$ are $4 \pm \sqrt{13}$, that is, about 0.39 and 7.61. Hence $n^5 \leq 2n^5 - 8n^4 + 3n^3$ for every integer $n \geq 8$ (for $n = 8$ the factor $n^2 - 8n + 3$ is equal to 3, for $n = 7$ it is equal to -4). By choosing for example $c_1 = 1$, $c_2 = 2$ and $n_1 = 8$, we have

$$0 \leq n^5 \leq 2n^5 - 8n^4 + 3n^3 \leq 2n^5, \quad \forall n \geq 8.$$

The limit computed in part (a) gives the same result: it is equal to $2 \in \mathbb{R}_{>0}$.

(c) To prove that

$$2n^5 - 8n^4 + 3n^3 \neq \Theta(n^3)$$

we proceed by contradiction and we suppose that there exist $c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq 2n^5 - 8n^4 + 3n^3 \leq c_2 n^3, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^3 , this is equivalent to

$$0 \leq 2n^2 - 8n + 3 \leq c_2, \quad \forall n \geq n_1.$$

Since $2n^2 - 8n + 3 \rightarrow +\infty$ as $n \rightarrow +\infty$, the middle expression cannot remain bounded by the constant c_2 : this is a contradiction.

Alternatively, we compute the limit

$$\lim_{n \rightarrow +\infty} \frac{2n^5 - 8n^4 + 3n^3}{n^3} = \lim_{n \rightarrow +\infty} (2n^2 - 8n + 3) = +\infty,$$

so that $f(n) = \omega(n^3)$ by the limit criterion. Then, with the constant $c = c_2$ in the definition of ω , we would have $f(n) > c_2 n^3$ for all sufficiently large n , against the upper bound $f(n) \leq c_2 n^3$. Hence $f(n) \neq \Theta(n^3)$.

Exercise 9.6 — Asymptotic notation (E6)

Consider the following asymptotically non-negative function of a natural variable:

$$f : \mathbb{N} \rightarrow \mathbb{R}, \quad f : n \mapsto n^3 (4 - \sin n) + 5n^2$$

Questions

1. Prove that:

(a) $n^3 (4 - \sin n) + 5n^2 = \Theta(n^3)$;

(b) $n^3 (4 - \sin n) + 5n^2 \neq o(n^3)$;

(c) $n^3 (4 - \sin n) + 5n^2 \neq \omega(n^3)$.

Solution

The ratio

$$\frac{n^3 (4 - \sin n) + 5n^2}{n^3} = 4 - \sin n + \frac{5}{n}, \quad n \geq 1,$$

does not have a limit as $n \rightarrow +\infty$, since it oscillates. Therefore the limit criteria cannot be applied, and we use the definitions. Since $-1 \leq \sin n \leq 1$, we have $3 \leq 4 - \sin n \leq 5$. Since $0 < \frac{5}{n} \leq 5$ for $n \geq 1$, we obtain

$$3 \leq 4 - \sin n + \frac{5}{n} \leq 10, \quad \forall n \geq 1.$$

1. (a) To prove that

$$n^3 (4 - \sin n) + 5 n^2 = \Theta(n^3)$$

we determine $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^3 \leq n^3 (4 - \sin n) + 5 n^2 \leq c_2 n^3, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^3 and since $c_1 > 0$, this is equivalent to

$$0 < c_1 \leq 4 - \sin n + \frac{5}{n} \leq c_2, \quad \forall n \geq n_1.$$

By the bounds above, by choosing for example $c_1 = 3$, $c_2 = 10$ and $n_1 = 1$, we have

$$0 \leq 3 n^3 \leq n^3 (4 - \sin n) + 5 n^2 \leq 10 n^3, \quad \forall n \geq 1.$$

- (b) To prove that

$$n^3 (4 - \sin n) + 5 n^2 \neq o(n^3)$$

we recall that $o(n^3)$ requires that for every $c \in \mathbb{R}_{>0}$ there exists $n_1 \in \mathbb{N}$ such that

$$0 \leq n^3 (4 - \sin n) + 5 n^2 < c n^3, \quad \forall n \geq n_1,$$

that is, dividing by $n^3 > 0$, $4 - \sin n + \frac{5}{n} < c$ for all $n \geq n_1$. Since $4 - \sin n + \frac{5}{n} \geq 3$ for every $n \geq 1$, for $c = 3$ there is no n_1 with this property. Therefore

$$n^3 (4 - \sin n) + 5 n^2 \neq o(n^3).$$

Equivalently, the ratio is bounded from below by 3 and cannot converge to 0.

- (c) To prove that

$$n^3 (4 - \sin n) + 5 n^2 \neq \omega(n^3)$$

we recall that $\omega(n^3)$ requires that for every $c \in \mathbb{R}_{>0}$ there exists $n_1 \in \mathbb{N}$ such that

$$0 \leq c n^3 < n^3 (4 - \sin n) + 5 n^2, \quad \forall n \geq n_1,$$

that is, dividing by $n^3 > 0$, $c < 4 - \sin n + \frac{5}{n}$ for all $n \geq n_1$. Since $4 - \sin n + \frac{5}{n} \leq 10$ for every $n \geq 1$, for $c = 10$ there is no n_1 with this property. Therefore

$$n^3 (4 - \sin n) + 5 n^2 \neq \omega(n^3).$$

Equivalently, the ratio is bounded from above by 10 and cannot diverge to $+\infty$.

Exercise 9.7 — Asymptotic notation (E7)

Consider the following asymptotically non-negative function of a natural variable:

$$f : \mathbb{N} \rightarrow \mathbb{R}, \quad f : n \mapsto 4 n^4 - n^3 - 6 n^2$$

Questions

1. Prove that:

- (a) $4n^4 - n^3 - 6n^2 = \mathcal{O}(n^4)$;
- (b) $4n^4 - n^3 - 6n^2 = \Theta(n^4)$;
- (c) $4n^4 - n^3 - 6n^2 \neq \Theta(n^2)$.

Solution

1. (a) To prove that

$$4n^4 - n^3 - 6n^2 = \mathcal{O}(n^4)$$

we determine $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq 4n^4 - n^3 - 6n^2 \leq cn^4, \quad \forall n \geq n_1.$$

First, we factorize:

$$4n^4 - n^3 - 6n^2 = n^2(4n^2 - n - 6),$$

where the roots of $4n^2 - n - 6$ are $\frac{1 \pm \sqrt{97}}{8}$, that is, about -1.11 and 1.36 . Hence $f(n) \geq 0$ for every integer $n \geq 2$ (for $n = 2$ the factor $4n^2 - n - 6$ is equal to 8 ; for $n = 1$ we have $f(1) = -3 < 0$). With $n \geq 1$, dividing by n^4 , the inequality $f(n) \leq cn^4$ is equivalent to

$$4 - \frac{1}{n} - \frac{6}{n^2} \leq c, \quad \forall n \geq n_1.$$

Since

$$-\frac{1}{n} - \frac{6}{n^2} = -\frac{n+6}{n^2} \leq 0, \quad \forall n \geq 1,$$

we have

$$4 - \frac{1}{n} - \frac{6}{n^2} \leq 4, \quad \forall n \geq 1.$$

By choosing for example $c = 4$ and $n_1 = 2$, we have

$$0 \leq 4n^4 - n^3 - 6n^2 \leq 4n^4, \quad \forall n \geq 2.$$

Since the functions are asymptotically positive, an alternative way is to compute the limit

$$\lim_{n \rightarrow +\infty} \frac{4n^4 - n^3 - 6n^2}{n^4} = \lim_{n \rightarrow +\infty} \left(4 - \frac{1}{n} - \frac{6}{n^2} \right) = 4 \in \mathbb{R}_{>0}.$$

(b) To prove that

$$4n^4 - n^3 - 6n^2 = \Theta(n^4)$$

we determine $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^4 \leq 4n^4 - n^3 - 6n^2 \leq c_2 n^4, \quad \forall n \geq n_1.$$

For the upper bound, by part (a), $4n^4 - n^3 - 6n^2 \leq 4n^4$ for all $n \geq 1$. For the lower bound, we observe that

$$4n^4 - n^3 - 6n^2 - 3n^4 = n^4 - n^3 - 6n^2 = n^2(n-3)(n+2),$$

which is non-negative for every integer $n \geq 3$ (it is equal to 0 for $n = 3$ and positive for $n \geq 4$). Hence $3n^4 \leq 4n^4 - n^3 - 6n^2$ for every $n \geq 3$; we choose $n_1 = 4$, but $n_1 = 3$ would also work. By choosing for example $c_1 = 3$, $c_2 = 4$ and $n_1 = 4$, we have

$$0 \leq 3n^4 \leq 4n^4 - n^3 - 6n^2 \leq 4n^4, \quad \forall n \geq 4.$$

The limit computed in part (a) gives the same result: it is equal to $4 \in \mathbb{R}_{>0}$.

(c) To prove that

$$4n^4 - n^3 - 6n^2 \neq \Theta(n^2)$$

we proceed by contradiction and we suppose that there exist $c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq 4n^4 - n^3 - 6n^2 \leq c_2 n^2, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^2 , this is equivalent to

$$0 \leq 4n^2 - n - 6 \leq c_2, \quad \forall n \geq n_1.$$

Since $4n^2 - n - 6 \rightarrow +\infty$ as $n \rightarrow +\infty$, the middle expression cannot remain bounded by the constant c_2 : this is a contradiction.

Alternatively, we compute the limit

$$\lim_{n \rightarrow +\infty} \frac{4n^4 - n^3 - 6n^2}{n^2} = \lim_{n \rightarrow +\infty} (4n^2 - n - 6) = +\infty,$$

so that $f(n) = \omega(n^2)$ by the limit criterion. Then, with the constant $c = c_2$ in the definition of ω , we would have $f(n) > c_2 n^2$ for all sufficiently large n , against the upper bound $f(n) \leq c_2 n^2$. Hence $f(n) \neq \Theta(n^2)$.

Exercise 9.8 — Asymptotic notation (E8)

Consider the following asymptotically non-negative function of a natural variable:

$$f : \mathbb{N} \rightarrow \mathbb{R}, \quad f : n \mapsto n^4(1 + \sin^2 n)$$

Questions

1. Prove that:

$$(a) \quad n^4(1 + \sin^2 n) = \Theta(n^4);$$

- (b) $n^4 (1 + \sin^2 n) \neq o(n^4)$;
(c) $n^4 (1 + \sin^2 n) \neq \omega(n^4)$.

Solution

The ratio

$$\frac{n^4 (1 + \sin^2 n)}{n^4} = 1 + \sin^2 n, \quad n \geq 1,$$

does not have a limit as $n \rightarrow +\infty$, since it oscillates. Therefore the limit criteria cannot be applied, and we use the definitions. Since $0 \leq \sin^2 n \leq 1$, we have

$$1 \leq 1 + \sin^2 n \leq 2, \quad \forall n \geq 1.$$

1. (a) To prove that

$$n^4 (1 + \sin^2 n) = \Theta(n^4)$$

we determine $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^4 \leq n^4 (1 + \sin^2 n) \leq c_2 n^4, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^4 and since $c_1 > 0$, this is equivalent to

$$0 < c_1 \leq 1 + \sin^2 n \leq c_2, \quad \forall n \geq n_1.$$

By the bounds above, by choosing for example $c_1 = 1$, $c_2 = 2$ and $n_1 = 1$, we have

$$0 \leq 1 n^4 \leq n^4 (1 + \sin^2 n) \leq 2 n^4, \quad \forall n \geq 1.$$

- (b) To prove that

$$n^4 (1 + \sin^2 n) \neq o(n^4)$$

we recall that $o(n^4)$ requires that for every $c \in \mathbb{R}_{>0}$ there exists $n_1 \in \mathbb{N}$ such that

$$0 \leq n^4 (1 + \sin^2 n) < c n^4, \quad \forall n \geq n_1,$$

that is, dividing by $n^4 > 0$, $1 + \sin^2 n < c$ for all $n \geq n_1$. Since $1 + \sin^2 n \geq 1$ for every $n \geq 1$, for $c = 1$ there is no n_1 with this property. Therefore

$$n^4 (1 + \sin^2 n) \neq o(n^4).$$

Equivalently, the ratio is bounded from below by 1 and cannot converge to 0.

- (c) To prove that

$$n^4 (1 + \sin^2 n) \neq \omega(n^4)$$

we recall that $\omega(n^4)$ requires that for every $c \in \mathbb{R}_{>0}$ there exists $n_1 \in \mathbb{N}$ such that

$$0 \leq c n^4 < n^4 (1 + \sin^2 n), \quad \forall n \geq n_1,$$

that is, dividing by $n^4 > 0$, $c < 1 + \sin^2 n$ for all $n \geq n_1$. Since $1 + \sin^2 n \leq 2$ for every $n \geq 1$, for $c = 2$ there is no n_1 with this property. Therefore

$$n^4 (1 + \sin^2 n) \neq \omega(n^4).$$

Equivalently, the ratio is bounded from above by 2 and cannot diverge to $+\infty$.

Exercise 9.9 — Asymptotic notation (E9)

Consider the following asymptotically non-negative function of a natural variable:

$$f : \mathbb{N} \rightarrow \mathbb{R}, \quad f : n \mapsto 6n^3 - 11n^2 + 4n$$

Questions

1. Prove that:

- (a) $6n^3 - 11n^2 + 4n = \mathcal{O}(n^3)$;
- (b) $6n^3 - 11n^2 + 4n = \mathcal{O}(n^4)$;
- (c) $6n^3 - 11n^2 + 4n = \Theta(n^3)$;
- (d) $6n^3 - 11n^2 + 4n \neq \Theta(n^4)$.

Solution

1. (a) To prove that

$$6n^3 - 11n^2 + 4n = \mathcal{O}(n^3)$$

we determine $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq 6n^3 - 11n^2 + 4n \leq cn^3, \quad \forall n \geq n_1.$$

First, we observe that

$$6n^3 - 11n^2 + 4n = n(6n^2 - 11n + 4) = n(2n - 1)(3n - 4),$$

hence

$$0 \leq 6n^3 - 11n^2 + 4n, \quad \forall n \geq 2$$

(for $n = 1$ the value is $f(1) = -1 < 0$). With $n \geq 1$, dividing by n^3 , the inequality $f(n) \leq cn^3$ is equivalent to

$$6 - \frac{11}{n} + \frac{4}{n^2} \leq c, \quad \forall n \geq n_1.$$

Since

$$-\frac{11}{n} + \frac{4}{n^2} = \frac{4 - 11n}{n^2} \leq 0, \quad \forall n \geq 1,$$

we have

$$6 - \frac{11}{n} + \frac{4}{n^2} \leq 6, \quad \forall n \geq 1.$$

By choosing for example $c = 6$ and $n_1 = 2$, we have

$$0 \leq 6n^3 - 11n^2 + 4n \leq 6n^3, \quad \forall n \geq 2.$$

Since the functions are asymptotically positive, an alternative way is to compute the limit

$$\lim_{n \rightarrow +\infty} \frac{6n^3 - 11n^2 + 4n}{n^3} = \lim_{n \rightarrow +\infty} \left(6 - \frac{11}{n} + \frac{4}{n^2} \right) = 6 \in \mathbb{R}_{>0}.$$

(b) To prove that

$$6n^3 - 11n^2 + 4n = \mathcal{O}(n^4)$$

we determine $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq 6n^3 - 11n^2 + 4n \leq cn^4, \quad \forall n \geq n_1.$$

From part (a), $0 \leq 6n^3 - 11n^2 + 4n \leq 6n^3$ for all $n \geq 2$. Since $n^3 \leq n^4$ for $n \geq 1$, we also have $6n^3 - 11n^2 + 4n \leq 6n^4$. By choosing for example $c = 6$ and $n_1 = 2$, we have

$$0 \leq 6n^3 - 11n^2 + 4n \leq 6n^4, \quad \forall n \geq 2.$$

Since the functions are asymptotically positive, an alternative way is to compute the limit

$$\lim_{n \rightarrow +\infty} \frac{6n^3 - 11n^2 + 4n}{n^4} = \lim_{n \rightarrow +\infty} \left(\frac{6}{n} - \frac{11}{n^2} + \frac{4}{n^3} \right) = 0^+.$$

(c) To prove that

$$6n^3 - 11n^2 + 4n = \Theta(n^3)$$

we determine $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^3 \leq 6n^3 - 11n^2 + 4n \leq c_2 n^3, \quad \forall n \geq n_1.$$

For the upper bound, by part (a), $6n^3 - 11n^2 + 4n \leq 6n^3$ for all $n \geq 1$. For the lower bound, we observe that

$$6n^3 - 11n^2 + 4n - 5n^3 = n^3 - 11n^2 + 4n = n(n^2 - 11n + 4),$$

where the roots of $n^2 - 11n + 4$ are $\frac{11 \pm \sqrt{105}}{2}$, that is, about 0.38 and 10.62; hence $n^2 - 11n + 4 > 0$ for every integer $n \geq 11$ (for $n = 11$ it is equal to 4, for $n = 10$ it is equal to -6). Hence $5n^3 \leq 6n^3 - 11n^2 + 4n$ for all $n \geq 11$. By choosing for example $c_1 = 5$, $c_2 = 6$ and $n_1 = 11$, we have

$$0 \leq 5n^3 \leq 6n^3 - 11n^2 + 4n \leq 6n^3, \quad \forall n \geq 11.$$

The limit computed in part (a) gives the same result: it is equal to $6 \in \mathbb{R}_{>0}$.

(d) To prove that

$$6n^3 - 11n^2 + 4n \neq \Theta(n^4)$$

we proceed by contradiction and we suppose that there exist $c_1 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^4 \leq 6n^3 - 11n^2 + 4n, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^4 and since $c_1 > 0$, this is equivalent to

$$0 < c_1 \leq \frac{6}{n} - \frac{11}{n^2} + \frac{4}{n^3}, \quad \forall n \geq n_1.$$

Passing to the limit as $n \rightarrow +\infty$, we obtain $0 < c_1 \leq 0$, which is impossible. Therefore

$$6n^3 - 11n^2 + 4n \neq \Theta(n^4).$$

Alternatively, the limit computed in part (b) is equal to 0^+ , so that $f(n) = o(n^4)$ by the limit criterion. Then, with the constant $c = c_1$ in the definition of o , we would have $f(n) < c_1 n^4$ for all sufficiently large n , against the lower bound $c_1 n^4 \leq f(n)$.

Exercise 9.10 — Asymptotic notation (E10)

Consider the following asymptotically non-negative function of a natural variable:

$$f : \mathbb{N} \rightarrow \mathbb{R}, \quad f : n \mapsto n^2 (2 + \cos^2 n)$$

Questions

1. Prove that:

- (a) $n^2 (2 + \cos^2 n) = \Theta(n^2)$;
- (b) $n^2 (2 + \cos^2 n) \neq o(n^2)$;
- (c) $n^2 (2 + \cos^2 n) \neq \omega(n^2)$.

Solution

The ratio

$$\frac{n^2 (2 + \cos^2 n)}{n^2} = 2 + \cos^2 n, \quad n \geq 1,$$

does not have a limit as $n \rightarrow +\infty$, since it oscillates. Therefore the limit criteria cannot be applied, and we use the definitions. Since $0 \leq \cos^2 n \leq 1$, we have

$$2 \leq 2 + \cos^2 n \leq 3, \quad \forall n \geq 1.$$

1. (a) To prove that

$$n^2 (2 + \cos^2 n) = \Theta(n^2)$$

we determine $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^2 \leq n^2 (2 + \cos^2 n) \leq c_2 n^2, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^2 and since $c_1 > 0$, this is equivalent to

$$0 < c_1 \leq 2 + \cos^2 n \leq c_2, \quad \forall n \geq n_1.$$

By the bounds above, by choosing for example $c_1 = 2$, $c_2 = 3$ and $n_1 = 1$, we have

$$0 \leq 2n^2 \leq n^2 (2 + \cos^2 n) \leq 3n^2, \quad \forall n \geq 1.$$

(b) To prove that

$$n^2 (2 + \cos^2 n) \neq o(n^2)$$

we recall that $o(n^2)$ requires that for every $c \in \mathbb{R}_{>0}$ there exists $n_1 \in \mathbb{N}$ such that

$$0 \leq n^2 (2 + \cos^2 n) < c n^2, \quad \forall n \geq n_1,$$

that is, dividing by $n^2 > 0$, $2 + \cos^2 n < c$ for all $n \geq n_1$. Since $2 + \cos^2 n \geq 2$ for every $n \geq 1$, for $c = 2$ there is no n_1 with this property. Therefore

$$n^2 (2 + \cos^2 n) \neq o(n^2).$$

Equivalently, the ratio is bounded from below by 2 and cannot converge to 0.

(c) To prove that

$$n^2 (2 + \cos^2 n) \neq \omega(n^2)$$

we recall that $\omega(n^2)$ requires that for every $c \in \mathbb{R}_{>0}$ there exists $n_1 \in \mathbb{N}$ such that

$$0 \leq c n^2 < n^2 (2 + \cos^2 n), \quad \forall n \geq n_1,$$

that is, dividing by $n^2 > 0$, $c < 2 + \cos^2 n$ for all $n \geq n_1$. Since $2 + \cos^2 n \leq 3$ for every $n \geq 1$, for $c = 3$ there is no n_1 with this property. Therefore

$$n^2 (2 + \cos^2 n) \neq \omega(n^2).$$

Equivalently, the ratio is bounded from above by 3 and cannot diverge to $+\infty$.

Exercise 9.11 — Asymptotic notation (E11)

Given $a \in \mathbb{R}_{>0}$ and $b \in \mathbb{R}$, consider the following asymptotically non-negative function of a natural variable:

$$f : \mathbb{N} \rightarrow \mathbb{R}, \quad f : n \mapsto a n + b.$$

Questions

1. Prove, using the definitions, that for all $a \in \mathbb{R}_{>0}$ and $b \in \mathbb{R}$:

- (a) $a n + b = \mathcal{O}(n)$;
- (b) $a n + b = \mathcal{O}(n^2)$;
- (c) $a n + b = \Theta(n)$;
- (d) $a n + b \neq \Theta(n^2)$.

Solution

The key remark is the following: if $n \geq \frac{2|b|}{a}$, then $|b| \leq \frac{a}{2}n$, and therefore

$$\frac{1}{2}an = an - \frac{a}{2}n \leq an - |b| \leq an + b \leq an + |b| \leq an + \frac{a}{2}n = \frac{3}{2}an.$$

To have $n \geq 1$ as well (so that we can divide by n), we use the threshold

$$n_1 = \max \left\{ 1, \left\lceil \frac{2|b|}{a} \right\rceil \right\}.$$

1. (a) To prove that $an + b = \mathcal{O}(n)$, we determine $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq an + b \leq cn, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n , this is equivalent to

$$0 \leq a + \frac{b}{n} \leq c, \quad \forall n \geq n_1.$$

By the key remark, $\frac{a}{2} \leq a + \frac{b}{n} \leq \frac{3}{2}a$ for all $n \geq \frac{2|b|}{a}$ (the upper value is reached when $b > 0$ and $n = \frac{2b}{a}$). Thus, by choosing for example $c = \frac{3}{2}a > 0$ and $n_1 = \max \left\{ 1, \left\lceil \frac{2|b|}{a} \right\rceil \right\}$, we have

$$0 \leq an + b \leq \frac{3}{2}an, \quad \forall n \geq \max \left\{ 1, \left\lceil \frac{2|b|}{a} \right\rceil \right\}.$$

- (b) To prove that $an + b = \mathcal{O}(n^2)$, we determine $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq an + b \leq cn^2, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^2 , this is equivalent to

$$0 \leq \frac{a}{n} + \frac{b}{n^2} \leq c, \quad \forall n \geq n_1.$$

For the left inequality we need $an + b \geq 0$, that is, $n \geq -\frac{b}{a}$; together with $n \geq 1$ this gives the threshold $\max \left\{ 1, -\frac{b}{a} \right\}$. For the right inequality, for all $n \geq 1$ we have

$$\frac{a}{n} + \frac{b}{n^2} \leq \frac{a}{n} + \frac{|b|}{n^2} \leq a + |b|.$$

Thus, by choosing for example $c = a + |b| > 0$ and $n_1 = \left\lceil \max \left\{ 1, -\frac{b}{a} \right\} \right\rceil$, we have

$$0 \leq an + b \leq (a + |b|)n^2, \quad \forall n \geq \left\lceil \max \left\{ 1, -\frac{b}{a} \right\} \right\rceil.$$

(c) To prove that $a n + b = \Theta(n)$, we determine $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n \leq a n + b \leq c_2 n, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n and since $c_1 > 0$, this is equivalent to

$$0 < c_1 \leq a + \frac{b}{n} \leq c_2, \quad \forall n \geq n_1.$$

By the key remark, $\frac{a}{2} \leq a + \frac{b}{n} \leq \frac{3}{2}a$ for all $n \geq \frac{2|b|}{a}$ (the lower value is reached when $b < 0$ and $n = \frac{2|b|}{a}$, the upper one when $b > 0$ and $n = \frac{2b}{a}$). Thus, by choosing for example $c_1 = \frac{1}{2}a > 0$, $c_2 = \frac{3}{2}a > 0$ and $n_1 = \max\left\{1, \left\lceil \frac{2|b|}{a} \right\rceil\right\}$, we have

$$0 \leq \frac{1}{2}a n \leq a n + b \leq \frac{3}{2}a n, \quad \forall n \geq \max\left\{1, \left\lceil \frac{2|b|}{a} \right\rceil\right\}.$$

An alternative way is to compute the limit

$$\lim_{n \rightarrow +\infty} \frac{a n + b}{n} = a + \lim_{n \rightarrow +\infty} \underbrace{\frac{b}{n}}_{\rightarrow 0} = a \in \mathbb{R}_{>0}.$$

(d) To prove that $a n + b \neq \Theta(n^2)$, we proceed by contradiction and we suppose that there exist $c_1 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^2 \leq a n + b, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^2 and since $c_1 > 0$, this is equivalent to

$$0 < c_1 \leq \frac{a}{n} + \frac{b}{n^2}, \quad \forall n \geq n_1.$$

Passing to the limit as $n \rightarrow +\infty$, we obtain $0 < c_1 \leq 0$, which is impossible: the inequality cannot hold for arbitrarily large n , since c_1 is a positive constant.

Exercise 9.12 — Asymptotic notation (E12)

Given $a \in \mathbb{R}_{>0}$, $b \in \mathbb{R}$ and $c \in \mathbb{R}$, consider the following asymptotically non-negative function of a natural variable:

$$f : \mathbb{N} \rightarrow \mathbb{R}, \quad f : n \mapsto a n^2 + b n + c.$$

Questions

1. Prove, using the definitions, that for all $a \in \mathbb{R}_{>0}$, $b \in \mathbb{R}$ and $c \in \mathbb{R}$:

(a) $a n^2 + b n + c = \Theta(n^2)$;

(b) $a n^2 + b n + c \neq \Theta(n^3)$.

Solution

1. (a) To prove that $a n^2 + b n + c = \Theta(n^2)$, we determine $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^2 \leq a n^2 + b n + c \leq c_2 n^2, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^2 and since $c_1 > 0$, this is equivalent to

$$0 < c_1 \leq a + \frac{b}{n} + \frac{c}{n^2} \leq c_2, \quad \forall n \geq n_1.$$

We bound the two terms $\frac{b}{n}$ and $\frac{c}{n^2}$ separately.

- If $n \geq \frac{2|b|}{a}$, then $\frac{|b|}{n} \leq \frac{a}{2}$, hence $-\frac{a}{2} \leq \frac{b}{n} \leq \frac{a}{2}$.
- If $n \geq 2\sqrt{\frac{|c|}{a}}$, then $n^2 \geq \frac{4|c|}{a}$ and $\frac{|c|}{n^2} \leq \frac{a}{4}$, hence $-\frac{a}{4} \leq \frac{c}{n^2} \leq \frac{a}{4}$.

Therefore, for all $n \geq 2 \max \left\{ \frac{|b|}{a}, \sqrt{\frac{|c|}{a}} \right\}$,

$$0 < \underbrace{a - \frac{1}{2}a - \frac{1}{4}a}_{=\frac{1}{4}a} \leq a + \frac{b}{n} + \frac{c}{n^2} \leq \underbrace{a + \frac{1}{2}a + \frac{1}{4}a}_{=\frac{7}{4}a}.$$

Thus, by choosing for example $c_1 = \frac{1}{4}a > 0$, $c_2 = \frac{7}{4}a > 0$ and

$$n_1 = \max \left\{ 1, \left\lceil 2 \max \left\{ \frac{|b|}{a}, \sqrt{\frac{|c|}{a}} \right\} \right\rceil \right\},$$

we have

$$0 \leq \frac{1}{4}a n^2 \leq a n^2 + b n + c \leq \frac{7}{4}a n^2, \quad \forall n \geq n_1.$$

An alternative way is to compute the limit

$$\lim_{n \rightarrow +\infty} \frac{a n^2 + b n + c}{n^2} = a + \lim_{n \rightarrow +\infty} \underbrace{\frac{b}{n}}_{\rightarrow 0} + \lim_{n \rightarrow +\infty} \underbrace{\frac{c}{n^2}}_{\rightarrow 0} = a \in \mathbb{R}_{>0}.$$

- (b) To prove that $a n^2 + b n + c \neq \Theta(n^3)$, we proceed by contradiction and we suppose that there exist $c_1 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^3 \leq a n^2 + b n + c, \quad \forall n \geq n_1.$$

With $n \geq 1$, dividing by n^3 and since $c_1 > 0$, this is equivalent to

$$0 < c_1 \leq \frac{a}{n} + \frac{b}{n^2} + \frac{c}{n^3}, \quad \forall n \geq n_1.$$

Passing to the limit as $n \rightarrow +\infty$, we obtain $0 < c_1 \leq 0$, which is impossible: the inequality cannot hold for arbitrarily large n , since c_1 is a positive constant.

Exercise 9.13 — Asymptotic notation (E13)

Consider four asymptotically positive functions f , g , p and q of a natural variable and the following implications:

$$f(n) = \Theta(p(n)) \text{ and } g(n) = \Theta(q(n)) \implies f(n) \cdot g(n) = \Theta(p(n) \cdot q(n)), \quad (9.1)$$

$$f(n) = \Theta(p(n)) \text{ and } g(n) = \Theta(q(n)) \implies f(n) + g(n) = \Theta(p(n) + q(n)). \quad (9.2)$$

Questions

1. Prove that the implication (9.1) is true and write its contraposition.
2. Prove that the converse of the implication (9.1) is not true, that is,

$$f(n) \cdot g(n) = \Theta(p(n) \cdot q(n)) \not\Rightarrow f(n) = \Theta(p(n)) \text{ and } g(n) = \Theta(q(n)).$$

3. Prove that the implication (9.2) is true and write its contraposition.
4. Prove that the converse of the implication (9.2) is not true, that is,

$$f(n) + g(n) = \Theta(p(n) + q(n)) \not\Rightarrow f(n) = \Theta(p(n)) \text{ and } g(n) = \Theta(q(n)).$$

Solution

1. If $f(n) = \Theta(p(n))$ and $g(n) = \Theta(q(n))$, there exist $c_1, c_2 \in \mathbb{R}_{>0}$, $n_0 \in \mathbb{N}$ and $c_3, c_4 \in \mathbb{R}_{>0}$, $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 p(n) \leq f(n) \leq c_2 p(n), \quad \forall n \geq n_0 \quad \text{and} \quad 0 \leq c_3 q(n) \leq g(n) \leq c_4 q(n), \quad \forall n \geq n_1.$$

Since all the terms are non-negative, we can multiply the two chains of inequalities term by term:

$$0 \leq \underbrace{c_1 c_3}_{>0} p(n) q(n) \leq f(n) \cdot g(n) \leq \underbrace{c_2 c_4}_{>0} p(n) q(n), \quad \forall n \geq \max\{n_0, n_1\},$$

and, as a consequence, $f(n) \cdot g(n) = \Theta(p(n) \cdot q(n))$.

It follows that “ $f(n) = \Theta(p(n))$ and $g(n) = \Theta(q(n))$ ” is a sufficient condition for “ $f(n) \cdot g(n) = \Theta(p(n) \cdot q(n))$ ”, and the latter is a necessary condition for the former.

By contraposition of (9.1) we have

$$f(n) \cdot g(n) \neq \Theta(p(n) \cdot q(n)) \implies f(n) \neq \Theta(p(n)) \text{ or } g(n) \neq \Theta(q(n)).$$

2. A counterexample is enough. Let

$$f(n) = n^2 + 1, \quad g(n) = n^2 - 1, \quad p(n) = n, \quad q(n) = n^3.$$

The four functions are asymptotically positive ($g(n) > 0$ for $n \geq 2$). We have

$$f(n) \cdot g(n) = (n^2 + 1)(n^2 - 1) = n^4 - 1 \quad \text{and} \quad p(n) \cdot q(n) = n^4,$$

and $n^4 - 1 = \Theta(n^4)$: by choosing $c_1 = \frac{1}{2}$, $c_2 = 1$ and $n_1 = 2$, we have $0 \leq \frac{1}{2}n^4 \leq n^4 - 1 \leq n^4$ for all $n \geq 2$ (the lower bound is equivalent to $n^4 \geq 2$). On the other hand,

$$\lim_{n \rightarrow +\infty} \frac{n^2 + 1}{n} = +\infty \quad \text{and} \quad \lim_{n \rightarrow +\infty} \frac{n^2 - 1}{n^3} = 0^+,$$

so that $n^2 + 1 = \omega(n)$ and $n^2 - 1 = o(n^3)$; hence

$$n^2 + 1 \neq \Theta(n) \quad \text{and} \quad n^2 - 1 \neq \Theta(n^3)$$

(a function that is $\omega(n)$ exceeds $c_2 n$ for every constant c_2 , and a function that is $o(n^3)$ is smaller than $c_1 n^3$ for every constant c_1 , for all sufficiently large n).

3. If $f(n) = \Theta(p(n))$ and $g(n) = \Theta(q(n))$, there exist $c_1, c_2 \in \mathbb{R}_{>0}$, $n_0 \in \mathbb{N}$ and $c_3, c_4 \in \mathbb{R}_{>0}$, $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 p(n) \leq f(n) \leq c_2 p(n), \quad \forall n \geq n_0 \quad \text{and} \quad 0 \leq c_3 q(n) \leq g(n) \leq c_4 q(n), \quad \forall n \geq n_1.$$

Adding the two chains of inequalities, we obtain

$$0 \leq c_1 p(n) + c_3 q(n) \leq f(n) + g(n) \leq c_2 p(n) + c_4 q(n), \quad \forall n \geq \max\{n_0, n_1\}.$$

Since $p(n) \geq 0$ and $q(n) \geq 0$ for these values of n , this implies

$$0 \leq \underbrace{\min\{c_1, c_3\}}_{>0} (p(n) + q(n)) \leq f(n) + g(n) \leq \underbrace{\max\{c_2, c_4\}}_{>0} (p(n) + q(n)),$$

$$\forall n \geq \max\{n_0, n_1\},$$

and, as a consequence, $f(n) + g(n) = \Theta(p(n) + q(n))$.

It follows that “ $f(n) = \Theta(p(n))$ and $g(n) = \Theta(q(n))$ ” is a sufficient condition for “ $f(n) + g(n) = \Theta(p(n) + q(n))$ ”, and the latter is a necessary condition for the former. By contraposition of (9.2) we have

$$f(n) + g(n) \neq \Theta(p(n) + q(n)) \implies f(n) \neq \Theta(p(n)) \quad \text{or} \quad g(n) \neq \Theta(q(n)).$$

4. A counterexample is enough. Let

$$f(n) = 5n^3, \quad g(n) = 7n, \quad p(n) = n^3, \quad q(n) = n^2.$$

We have $5n^3 + 7n = \Theta(n^3 + n^2)$: by choosing $c_1 = \frac{5}{2}$, $c_2 = 7$ and $n_1 = 1$, we have

$$0 \leq \frac{5}{2}(n^3 + n^2) \leq 5n^3 + 7n \leq 7(n^3 + n^2), \quad \forall n \geq 1,$$

since $\frac{5}{2}(n^3 + n^2) \leq \frac{5}{2} \cdot 2n^3 = 5n^3$ and $7n \leq 2n^3 + 7n^2$ for $n \geq 1$. On the other hand, $5n^3 = \Theta(n^3)$ (with $c_1 = c_2 = 5$), but

$$\lim_{n \rightarrow +\infty} \frac{7n}{n^2} = 0^+,$$

so that $7n = o(n^2)$ and therefore $7n \neq \Theta(n^2)$.

Exercise 9.14 — Asymptotic notation (E14)

Consider two asymptotically non-negative functions f and g of a natural variable and the following equivalences:

$$f(n) = \Theta(g(n)) \iff f(n) = \mathcal{O}(g(n)) \text{ and } f(n) = \Omega(g(n)), \quad (9.3)$$

$$f(n) = \mathcal{O}(g(n)) \iff g(n) = \Omega(f(n)), \quad (9.4)$$

$$f(n) = o(g(n)) \iff g(n) = \omega(f(n)), \quad (9.5)$$

$$f(n) = \Theta(g(n)) \iff g(n) = \Theta(f(n)). \quad (9.6)$$

Questions

1. Prove that the equivalence (9.3) is true.
2. Prove that the equivalence (9.4) is true.
3. Prove that the equivalence (9.5) is true.
4. Prove that the equivalence (9.6) is true.

Solution

1. If $f(n) = \Theta(g(n))$, there exist $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n), \quad \forall n \geq n_1,$$

and therefore

$$0 \leq f(n) \leq c_2 g(n), \quad \forall n \geq n_1 \implies f(n) = \mathcal{O}(g(n)),$$

$$0 \leq c_1 g(n) \leq f(n), \quad \forall n \geq n_1 \implies f(n) = \Omega(g(n)).$$

Conversely, if $f(n) = \Omega(g(n))$, there exist $c_1 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that $0 \leq c_1 g(n) \leq f(n)$ for all $n \geq n_1$, and if $f(n) = \mathcal{O}(g(n))$, there exist $c_2 \in \mathbb{R}_{>0}$ and $n_2 \in \mathbb{N}$ such that $0 \leq f(n) \leq c_2 g(n)$ for all $n \geq n_2$. Combining the two inequalities, we have

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n), \quad \forall n \geq \max\{n_1, n_2\} \implies f(n) = \Theta(g(n)).$$

It follows that “ $f(n) = \Theta(g(n))$ ” is a necessary and sufficient condition for “ $f(n) = \mathcal{O}(g(n))$ and $f(n) = \Omega(g(n))$ ”.

2. (a) If $f(n) = \mathcal{O}(g(n))$, there exist $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that $0 \leq f(n) \leq c g(n)$ for all $n \geq n_1$. Dividing by $c > 0$,

$$0 \leq \frac{1}{c} f(n) \leq g(n), \quad \forall n \geq n_1 \implies g(n) = \Omega(f(n)),$$

with the constant $\frac{1}{c} > 0$.

- (b) If $g(n) = \Omega(f(n))$, there exist $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that $0 \leq c f(n) \leq g(n)$ for all $n \geq n_1$. Dividing by $c > 0$,

$$0 \leq f(n) \leq \frac{1}{c} g(n), \quad \forall n \geq n_1 \implies f(n) = \mathcal{O}(g(n)).$$

It follows that “ $f(n) = \mathcal{O}(g(n))$ ” is a necessary and sufficient condition for “ $g(n) = \Omega(f(n))$ ”.

3. (a) Let $f(n) = o(g(n))$. To prove that $g(n) = \omega(f(n))$, we fix $c \in \mathbb{R}_{>0}$. By the definition of $f(n) = o(g(n))$ with the constant $\frac{1}{c} > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$0 \leq f(n) < \frac{1}{c} g(n), \quad \forall n \geq n_1 \implies 0 \leq c f(n) < g(n), \quad \forall n \geq n_1.$$

Since c is arbitrary, $g(n) = \omega(f(n))$.

- (b) Let $g(n) = \omega(f(n))$. To prove that $f(n) = o(g(n))$, we fix $c \in \mathbb{R}_{>0}$. By the definition of $g(n) = \omega(f(n))$ with the constant $\frac{1}{c} > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$0 \leq \frac{1}{c} f(n) < g(n), \quad \forall n \geq n_1 \implies 0 \leq f(n) < c g(n), \quad \forall n \geq n_1.$$

Since c is arbitrary, $f(n) = o(g(n))$.

It follows that “ $f(n) = o(g(n))$ ” is a necessary and sufficient condition for “ $g(n) = \omega(f(n))$ ”.

4. (a) If $f(n) = \Theta(g(n))$, there exist $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n), \quad \forall n \geq n_1.$$

Dividing the left inequality by c_1 and the right inequality by c_2 , we obtain

$$0 \leq \frac{1}{c_2} f(n) \leq g(n) \leq \frac{1}{c_1} f(n), \quad \forall n \geq n_1.$$

Since $\frac{1}{c_2} > 0$ and $\frac{1}{c_1} > 0$, this proves that $g(n) = \Theta(f(n))$.

- (b) If $g(n) = \Theta(f(n))$, the same argument, with the roles of f and g exchanged, gives $f(n) = \Theta(g(n))$.

It follows that “ $f(n) = \Theta(g(n))$ ” is a necessary and sufficient condition for “ $g(n) = \Theta(f(n))$ ”.

Exercise 9.15 — Asymptotic notation (E15)

Consider three asymptotically non-negative functions f , g and h of a natural variable and the following implications (**transitivity**):

$$f(n) = \Theta(g(n)) \text{ and } g(n) = \Theta(h(n)) \implies f(n) = \Theta(h(n)), \quad (9.7)$$

$$f(n) = \mathcal{O}(g(n)) \text{ and } g(n) = \mathcal{O}(h(n)) \implies f(n) = \mathcal{O}(h(n)), \quad (9.8)$$

$$f(n) = \Omega(g(n)) \text{ and } g(n) = \Omega(h(n)) \implies f(n) = \Omega(h(n)), \quad (9.9)$$

$$f(n) = o(g(n)) \text{ and } g(n) = o(h(n)) \implies f(n) = o(h(n)), \quad (9.10)$$

$$f(n) = \omega(g(n)) \text{ and } g(n) = \omega(h(n)) \implies f(n) = \omega(h(n)). \quad (9.11)$$

Questions

1. Prove that the implication (9.7) is true and write its contraposition.
2. Prove that the implication (9.8) is true and write its contraposition.
3. Prove that the implication (9.9) is true and write its contraposition.
4. Prove that the implication (9.10) is true and write its contraposition.
5. Prove that the implication (9.11) is true and write its contraposition.

Solution

1. If $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$, there exist $c_1, c_2 \in \mathbb{R}_{>0}$, $n_0 \in \mathbb{N}$ and $c_3, c_4 \in \mathbb{R}_{>0}$, $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n), \quad \forall n \geq n_0 \quad \text{and} \quad 0 \leq c_3 h(n) \leq g(n) \leq c_4 h(n), \quad \forall n \geq n_1.$$

It follows that, for all $n \geq \max\{n_0, n_1\}$,

$$f(n) \geq c_1 g(n) \geq c_1 c_3 h(n) \quad \text{and} \quad f(n) \leq c_2 g(n) \leq c_2 c_4 h(n),$$

hence

$$0 \leq \underbrace{c_1 c_3}_{>0} h(n) \leq f(n) \leq \underbrace{c_2 c_4}_{>0} h(n), \quad \forall n \geq \max\{n_0, n_1\},$$

which proves $f(n) = \Theta(h(n))$. By contraposition of (9.7):

$$f(n) \neq \Theta(h(n)) \implies f(n) \neq \Theta(g(n)) \text{ or } g(n) \neq \Theta(h(n)).$$

2. If $f(n) = \mathcal{O}(g(n))$ and $g(n) = \mathcal{O}(h(n))$, there exist $c_1 \in \mathbb{R}_{>0}$, $n_0 \in \mathbb{N}$ and $c_2 \in \mathbb{R}_{>0}$, $n_1 \in \mathbb{N}$ such that

$$0 \leq f(n) \leq c_1 g(n), \quad \forall n \geq n_0 \quad \text{and} \quad 0 \leq g(n) \leq c_2 h(n), \quad \forall n \geq n_1.$$

It follows that

$$0 \leq f(n) \leq c_1 g(n) \leq \underbrace{c_1 c_2}_{>0} h(n), \quad \forall n \geq \max\{n_0, n_1\},$$

which proves $f(n) = \mathcal{O}(h(n))$. By contraposition of (9.8):

$$f(n) \neq \mathcal{O}(h(n)) \implies f(n) \neq \mathcal{O}(g(n)) \text{ or } g(n) \neq \mathcal{O}(h(n)).$$

3. If $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$, there exist $c_1 \in \mathbb{R}_{>0}$, $n_0 \in \mathbb{N}$ and $c_2 \in \mathbb{R}_{>0}$, $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 g(n) \leq f(n), \quad \forall n \geq n_0 \quad \text{and} \quad 0 \leq c_2 h(n) \leq g(n), \quad \forall n \geq n_1.$$

It follows that

$$0 \leq \underbrace{c_1 c_2}_{>0} h(n) \leq c_1 g(n) \leq f(n), \quad \forall n \geq \max\{n_0, n_1\},$$

which proves $f(n) = \Omega(h(n))$. By contraposition of (9.9):

$$f(n) \neq \Omega(h(n)) \implies f(n) \neq \Omega(g(n)) \text{ or } g(n) \neq \Omega(h(n)).$$

4. Let $f(n) = o(g(n))$ and $g(n) = o(h(n))$, and let $c \in \mathbb{R}_{>0}$. By the definition of $f(n) = o(g(n))$ with the constant 1, and by the definition of $g(n) = o(h(n))$ with the constant c , there exist $n_0, n_1 \in \mathbb{N}$ such that

$$0 \leq f(n) < g(n), \quad \forall n \geq n_0 \quad \text{and} \quad 0 \leq g(n) < c h(n), \quad \forall n \geq n_1.$$

It follows that

$$0 \leq f(n) < g(n) < c h(n), \quad \forall n \geq \max\{n_0, n_1\},$$

that is, $0 \leq f(n) < c h(n)$. Since $c \in \mathbb{R}_{>0}$ is arbitrary, this proves $f(n) = o(h(n))$. By contraposition of (9.10):

$$f(n) \neq o(h(n)) \implies f(n) \neq o(g(n)) \text{ or } g(n) \neq o(h(n)).$$

5. Let $f(n) = \omega(g(n))$ and $g(n) = \omega(h(n))$, and let $c \in \mathbb{R}_{>0}$. By the definition of $f(n) = \omega(g(n))$ with the constant 1, and by the definition of $g(n) = \omega(h(n))$ with the constant c , there exist $n_0, n_1 \in \mathbb{N}$ such that

$$0 \leq g(n) < f(n), \quad \forall n \geq n_0 \quad \text{and} \quad 0 \leq c h(n) < g(n), \quad \forall n \geq n_1.$$

It follows that

$$0 \leq c h(n) < g(n) < f(n), \quad \forall n \geq \max\{n_0, n_1\},$$

that is, $0 \leq c h(n) < f(n)$. Since $c \in \mathbb{R}_{>0}$ is arbitrary, this proves $f(n) = \omega(h(n))$. By contraposition of (9.11):

$$f(n) \neq \omega(h(n)) \implies f(n) \neq \omega(g(n)) \text{ or } g(n) \neq \omega(h(n)).$$

Exercise 9.16 — Asymptotic notation (E16)

Consider an asymptotically non-negative function f of a natural variable and the following **reflexivity** properties:

$$f(n) = \Theta(f(n)), \quad (9.12)$$

$$f(n) = \mathcal{O}(f(n)), \quad (9.13)$$

$$f(n) = \Omega(f(n)). \quad (9.14)$$

Questions

1. Prove that the property (9.12) is true.
2. Prove that the property (9.13) is true.
3. Prove that the property (9.14) is true.

Solution

Since f is asymptotically non-negative, there exists $\tilde{n} \in \mathbb{N}$ such that $f(n) \geq 0$ for all $n \geq \tilde{n}$. In the three proofs we use $n_1 = \tilde{n}$.

1. To prove that $f(n) = \Theta(f(n))$, we determine $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 f(n) \leq f(n) \leq c_2 f(n), \quad \forall n \geq n_1.$$

For $n \geq \tilde{n}$ the inequality $0 \leq 1 \cdot f(n) \leq f(n) \leq 1 \cdot f(n)$ holds trivially. By choosing $c_1 = 1$, $c_2 = 1$ and $n_1 = \tilde{n}$, we have

$$0 \leq f(n) \leq f(n) \leq f(n), \quad \forall n \geq \tilde{n},$$

which proves $f(n) = \Theta(f(n))$.

2. To prove that $f(n) = \mathcal{O}(f(n))$, we determine $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq f(n) \leq c f(n), \quad \forall n \geq n_1.$$

By choosing $c = 1$ and $n_1 = \tilde{n}$, we have $0 \leq f(n) \leq f(n)$ for all $n \geq \tilde{n}$, which proves $f(n) = \mathcal{O}(f(n))$.

3. To prove that $f(n) = \Omega(f(n))$, we determine $c \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c f(n) \leq f(n), \quad \forall n \geq n_1.$$

By choosing $c = 1$ and $n_1 = \tilde{n}$, we have $0 \leq f(n) \leq f(n)$ for all $n \geq \tilde{n}$, which proves $f(n) = \Omega(f(n))$.

Note that the assumption on f is necessary: if $f(n) < 0$ for infinitely many n , the inequality $0 \leq f(n)$ required by the three definitions fails for arbitrarily large n , and none of the three properties holds.

Asymptotic growth

In the exercises of this chapter the orders of growth are compared with the asymptotic notation of chapter 9 «Asymptotic notation» and with the results of the chapter «Asymptotic growth», summarized below. The limits of functions of a natural variable (sequences) are computed with the results of the course *Mathematical Analysis 1*, at the address <https://fabiofurini.github.io/mathematical-analysis-1/>.

Limit criterion. Given two asymptotically positive functions of a natural variable f and g such that the limit $L = \lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)}$ exists:

- $f(n) = o(g(n))$ if and only if $L = 0^+$: f has a **strictly smaller order of growth** than g (f grows slower than g);
- $f(n) = \mathcal{O}(g(n))$ if and only if $L = 0^+$ or $L \in \mathbb{R}_{>0}$: f has a **smaller or equal order of growth** compared to g ;
- $f(n) = \Theta(g(n))$ if and only if $L \in \mathbb{R}_{>0}$: f has the **same order of growth** as g (f grows as fast as g);
- $f(n) = \Omega(g(n))$ if and only if $L \in \mathbb{R}_{>0}$ or $L = +\infty$: f has a **greater or equal order of growth** compared to g ;
- $f(n) = \omega(g(n))$ if and only if $L = +\infty$: f has a **strictly greater order of growth** than g (f grows faster than g).

- **Definition of Θ .** We have $f(n) = \Theta(g(n))$ if and only if there exist $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n), \quad \forall n \in \mathbb{N}_{\geq n_1}.$$

Accordingly, if such constants do not exist, $f(n) \neq \Theta(g(n))$.

- **Hierarchy.** For every $a, b \in \mathbb{R}_{>1}$ and $\alpha, \beta \in \mathbb{R}_{>0}$:

$$\log_a^\beta n = o(n^\alpha), \quad n^\alpha = o(b^n), \quad b^n = o(n!), \quad n! = o(n^n).$$

Exercise 10.1 — Asymptotic growth (E1)

Given $\lambda \in \mathbb{R}_{>0}$ and $\alpha \in \mathbb{R}_{>0}$, consider the following asymptotically positive function of a natural variable:

$$f : \mathbb{N} \rightarrow \mathbb{R}, \quad f : n \mapsto (n + \lambda)^\alpha$$

Given $b \in \mathbb{R}_{>1}$ and $\lambda \in \mathbb{R}_{>0}$, consider the following asymptotically positive function of a natural variable:

$$f : \mathbb{N} \rightarrow \mathbb{R}, \quad f : n \mapsto b^{\lambda n}$$

Given $a \in \mathbb{R}_{>1}$, consider the following asymptotically positive function of a natural variable:

$$f : \mathbb{N}_{\geq 1} \rightarrow \mathbb{R}, \quad f : n \mapsto n \log_a n$$

Questions

1. Prove that there exist $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^\alpha \leq (n + \lambda)^\alpha \leq c_2 n^\alpha, \quad \forall n \in \mathbb{N}_{\geq n_1},$$

that is, that $(n + \lambda)^\alpha = \Theta(n^\alpha)$, and verify the result with the limit criterion, computing the limit of the ratio.

2. Prove that, for every $\lambda \in \mathbb{R}_{>0}$ with $\lambda \neq 1$, there exist no $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 b^n \leq b^{\lambda n} \leq c_2 b^n, \quad \forall n \in \mathbb{N}_{\geq n_1},$$

and accordingly that $b^{\lambda n} \neq \Theta(b^n)$.

3. Prove that, for every $\alpha \in \mathbb{R}_{>1}$, there exist no $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}_{\geq 1}$ such that

$$0 \leq c_1 n^\alpha \leq n \log_a n \leq c_2 n^\alpha, \quad \forall n \in \mathbb{N}_{\geq n_1},$$

and accordingly that $n \log_a n \neq \Theta(n^\alpha)$.

Solution

1. We look for constants $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 n^\alpha \leq (n + \lambda)^\alpha \leq c_2 n^\alpha, \quad \forall n \in \mathbb{N}_{\geq n_1}.$$

Since $\lambda > 0$, we have

$$0 \leq \frac{n}{2} \leq n + \lambda \quad \forall n \in \mathbb{N} \quad \text{and} \quad n + \lambda \leq 2n \quad \text{if } n \geq \lambda,$$

so that, for every $n \geq 2\lambda$ (which implies $n \geq \lambda$), we have:

$$0 \leq \frac{n}{2} \leq n + \lambda \leq 2n$$

Since $\alpha > 0$, the function $x \mapsto x^\alpha$ is increasing on $\mathbb{R}_{\geq 0}$, and we can raise the three terms to the power α without changing the inequalities:

$$\left(\frac{n}{2}\right)^\alpha \leq (n + \lambda)^\alpha \leq (2n)^\alpha \iff \frac{1}{2^\alpha} n^\alpha \leq (n + \lambda)^\alpha \leq 2^\alpha n^\alpha$$

Thus, choosing for example $c_1 = \frac{1}{2^\alpha} > 0$, $c_2 = 2^\alpha > 0$ and $n_1 = \lceil 2\lambda \rceil \geq 1$, we have:

$$0 \leq \frac{1}{2^\alpha} n^\alpha \leq (n + \lambda)^\alpha \leq 2^\alpha n^\alpha, \quad \forall n \in \mathbb{N}_{\geq \lceil 2\lambda \rceil}$$

By the definition of Θ , these constants prove that $(n + \lambda)^\alpha = \Theta(n^\alpha)$. We verify the result with the limit of the ratio:

$$\lim_{n \rightarrow +\infty} \frac{(n + \lambda)^\alpha}{n^\alpha} = \lim_{n \rightarrow +\infty} \underbrace{\left(1 + \frac{\lambda}{n}\right)^\alpha}_{\rightarrow 1} = 1 \in \mathbb{R}_{>0}$$

and the limit criterion gives again $(n + \lambda)^\alpha = \Theta(n^\alpha)$.

2. We proceed by contradiction, and we suppose that there exist $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}$ such that

$$0 \leq c_1 b^n \leq b^{\lambda n} \leq c_2 b^n, \quad \forall n \in \mathbb{N}_{\geq n_1}.$$

Since $b^n > 0$ for every $n \in \mathbb{N}$ (because $b \in \mathbb{R}_{>1}$), dividing by b^n we obtain the equivalent inequalities

$$0 < c_1 \leq \underbrace{\frac{b^{\lambda n}}{b^n}}_{=\left(\frac{b^\lambda}{b}\right)^n = (b^{\lambda-1})^n} \leq c_2, \quad \forall n \in \mathbb{N}_{\geq n_1},$$

which cannot hold for arbitrarily large n , since c_1 and c_2 are positive constants and:

- if $\lambda > 1$, then $b^{\lambda-1} > 1$ and $\lim_{n \rightarrow +\infty} (b^{\lambda-1})^n = +\infty$, so the inequality on the right is violated for n large enough;
- if $0 < \lambda < 1$, then $0 < b^{\lambda-1} < 1$ and $\lim_{n \rightarrow +\infty} (b^{\lambda-1})^n = 0^+$, so the inequality on the left is violated for n large enough.

It follows that, for every $b \in \mathbb{R}_{>1}$ and every $\lambda \in \mathbb{R}_{>0}$ with $\lambda \neq 1$, we have $b^{\lambda n} \neq \Theta(b^n)$, since the constants required by the definition of Θ do not exist.

3. We proceed by contradiction, and we suppose that there exist $c_1, c_2 \in \mathbb{R}_{>0}$ and $n_1 \in \mathbb{N}_{\geq 1}$ such that

$$0 \leq c_1 n^\alpha \leq n \log_a n \leq c_2 n^\alpha, \quad \forall n \in \mathbb{N}_{\geq n_1}.$$

Since $n^\alpha > 0$ for every $n \in \mathbb{N}_{\geq 1}$, dividing by n^α we obtain the equivalent inequalities

$$0 < c_1 \leq \underbrace{\frac{n \log_a n}{n^\alpha}}_{=\frac{\log_a n}{n^{\alpha-1}}} \leq c_2, \quad \forall n \in \mathbb{N}_{\geq n_1},$$

which cannot hold for arbitrarily large n , since c_1 is a positive constant and, by the hierarchy of the orders of growth (since $a > 1$ and $\alpha - 1 > 0$):

$$\lim_{n \rightarrow +\infty} \frac{\log_a n}{n^{\alpha-1}} = 0^+$$

It follows that, for every $\alpha \in \mathbb{R}_{>1}$ and every $a \in \mathbb{R}_{>1}$, we have $n \log_a n \neq \Theta(n^\alpha)$: the same limit shows, by the limit criterion, that $n \log_a n = o(n^\alpha)$.

Exercise 10.2 — Asymptotic growth (E2)

Consider the following function of a natural variable:

$$f : \mathbb{N}_{\geq 2} \rightarrow \mathbb{R}, \quad f : n \mapsto \ln(\ln n)$$

Questions

1. Prove that the function f is asymptotically positive and asymptotically increasing.
2. Prove that $\ln(\ln n) = o(\ln n)$.

Solution

1. The function $x \mapsto \ln x$ is strictly increasing on $\mathbb{R}_{>0}$, so $n \mapsto \ln(\ln n)$ is strictly increasing on $\mathbb{N}_{\geq 2}$. It is not positive on its whole domain, since $\ln 2 < 1$ and therefore $f(2) = \ln(\ln 2) < 0$; but $\ln 3 > 1$, hence

$$f(n) = \ln(\ln n) \geq \ln(\ln 3) > 0, \quad \forall n \in \mathbb{N}_{\geq 3}.$$

The function f is therefore asymptotically positive (with $n_1 = 3$) and asymptotically increasing.

2. Both functions are positive for every $n \in \mathbb{N}_{\geq 3}$. We compute the limit of the ratio by extending the domain to the real numbers, that is, by computing the limit of the function of a real variable $x \mapsto \frac{\ln(\ln x)}{\ln x}$ as $x \rightarrow +\infty$: if this limit exists, the limit of the function of a natural variable is the same. Since both the numerator and the denominator tend to $+\infty$, we can use L'Hôpital's rule (course *Mathematical Analysis 1*, chapter *L'Hôpital's rule and differentiability*):

$$\lim_{x \rightarrow +\infty} \frac{\ln(\ln x)}{\ln x} = \lim_{x \rightarrow +\infty} \frac{\frac{d}{dx} [\ln(\ln x)]}{\frac{d}{dx} [\ln x]}$$

The derivative of the numerator is

$$\frac{d}{dx} [\ln(\ln x)] = \frac{1}{\ln x} \cdot \frac{1}{x} = \frac{1}{x \ln x},$$

and the derivative of the denominator is

$$\frac{d}{dx} [\ln x] = \frac{1}{x}.$$

Substituting, we have:

$$\lim_{x \rightarrow +\infty} \frac{\ln(\ln x)}{\ln x} = \lim_{x \rightarrow +\infty} \frac{\frac{1}{x \ln x}}{\frac{1}{x}} = \lim_{x \rightarrow +\infty} \frac{1}{\ln x} = 0^+$$

It follows that

$$\lim_{n \rightarrow +\infty} \frac{\ln(\ln n)}{\ln n} = 0^+,$$

that is, by the limit criterion, $\ln(\ln n) = o(\ln n)$.

Exercise 10.3 — Asymptotic growth (E3)

Consider the following twelve functions of a natural variable, defined for $n \in \mathbb{N}_{\geq 1}$:

$$\begin{array}{llll} f_1(n) = \log_2(n^5) & f_2(n) = \ln^3 n & f_3(n) = \sqrt{n} & f_4(n) = n \ln n \\ f_5(n) = (n+7)^2 & f_6(n) = 3n^2 + 100n & f_7(n) = n^{10} & f_8(n) = 2^{n/2} \\ f_9(n) = 2^{n+10} & f_{10}(n) = 3^n & f_{11}(n) = n! & f_{12}(n) = n^n \end{array}$$

Questions

- Sort the twelve functions by increasing order of growth, grouping together the functions f_i and f_j such that $f_i(n) = \Theta(f_j(n))$. Justify each comparison between two consecutive groups with the results of the chapter «Asymptotic growth».
- Classify each function: does it have a sub-linear, linear or super-linear order of growth? Which functions have a polylogarithmically bounded order of growth, and which ones a polynomially bounded order of growth?
- The order of growth describes the behaviour of the functions for large values of n . Verify that $f_2(4) > f_3(4)$ and that $f_7(n) > f_8(n)$ for $n = 100$, and explain why this does not contradict the answer to the first question.

Solution

- The order is the following (functions in the same group are Θ of each other):

$$f_1 \prec f_2 \prec f_3 \prec f_4 \prec \{f_5, f_6\} \prec f_7 \prec f_8 \prec f_9 \prec f_{10} \prec f_{11} \prec f_{12}$$

where here $f \prec g$ is a shorthand for $f(n) = o(g(n))$. The justifications are the following.

- $f_1(n) = \log_2(n^5) = 5 \log_2 n = \Theta(\ln n)$ (change of base). Moreover $\frac{f_1(n)}{f_2(n)} = \frac{5 \log_2 n}{\ln^3 n} = \frac{5}{\ln 2} \cdot \frac{1}{\ln^2 n} \rightarrow 0^+$, so $f_1 \prec f_2$.
- $f_2 \prec f_3$: polylogarithms and powers, with $\beta = 3$ and $\alpha = \frac{1}{2}$.
- $f_3 \prec f_4$: $\frac{\sqrt{n}}{n \ln n} = \frac{1}{\sqrt{n} \ln n} \rightarrow 0^+$.
- $f_4 \prec f_5$: by the quasi-linear and powers property (with $\alpha = 2$) and the shift of the argument of a power,

$$\frac{n \ln n}{(n+7)^2} = \underbrace{\frac{n \ln n}{n^2}}_{\rightarrow 0^+} \cdot \underbrace{\frac{n^2}{(n+7)^2}}_{\rightarrow 1} \rightarrow 0^+.$$

- $f_5(n) = \Theta(f_6(n))$: both are $\Theta(n^2)$ (shift of the argument; polynomial with leading term $3n^2$), and indeed

$$\lim_{n \rightarrow +\infty} \frac{(n+7)^2}{3n^2 + 100n} = \lim_{n \rightarrow +\infty} \frac{1 + \frac{14}{n} + \frac{49}{n^2}}{3 + \frac{100}{n}} = \frac{1}{3} \in \mathbb{R}_{>0}.$$

- $f_6 \prec f_7$: $\frac{3n^2+100n}{n^{10}} = \frac{3}{n^8} + \frac{100}{n^9} \rightarrow 0^+$.
- $f_7 \prec f_8$: powers and exponentials, since $2^{n/2} = (\sqrt{2})^n$ with base $\sqrt{2} > 1$.
- $f_8 \prec f_9$: $f_9(n) = 2^{n+10} = \Theta(2^n)$ (shift of the exponent), and $2^{n/2} = o(2^n)$ (scaling of the exponent with $\lambda = \frac{1}{2}$); directly,

$$\frac{2^{n/2}}{2^{n+10}} = \frac{1}{1024} \left(\frac{1}{\sqrt{2}} \right)^n \rightarrow 0^+.$$

- $f_9 \prec f_{10}$: $\frac{2^{n+10}}{3^n} = 1024 \left(\frac{2}{3} \right)^n \rightarrow 0^+$ (different bases).
- $f_{10} \prec f_{11} \prec f_{12}$: exponentials and factorial, factorial and self-power.

2. Comparing each function with n :

- f_1, f_2 and f_3 are $o(n)$, a sub-linear order of growth ($f_1 \prec f_2 \prec f_3$ and $\frac{\sqrt{n}}{n} \rightarrow 0^+$);
- no function of the list is $\Theta(n)$, a linear order of growth;
- $f_4, f_5, \dots, f_{12}$ are $\omega(n)$, a super-linear order of growth (f_4 is quasi-linear, and all the others are ω of f_4).

The functions f_1 (logarithmically bounded, with $\beta = 1$) and f_2 (with $\beta = 3$) have a polylogarithmically bounded order of growth; f_3 does not, since $f_3(n) = \omega(\ln^\beta n)$ for every $\beta \in \mathbb{R}_{>0}$. The functions $f_1, f_2, \dots, f_7$ have a polynomially bounded order of growth (for example f_4 with $\alpha = 2$, and f_7 with $\alpha = 10$); the functions $f_8, f_9, \dots, f_{12}$ do not, since they are $\omega(n^\alpha)$ for every $\alpha \in \mathbb{R}_{>0}$. Among them, f_8, f_9 and f_{10} have an exponentially bounded order of growth.

3. For $n = 4$ we have

$$f_2(4) = \ln^3 4 \approx 2.66 > 2 = \sqrt{4} = f_3(4),$$

and for $n = 100$ we have

$$f_7(100) = 100^{10} = 10^{20} > 2^{50} \approx 1.1 \cdot 10^{15} = f_8(100).$$

This does not contradict the first answer: the comparison of the orders of growth concerns the limit of the ratio, that is, the behaviour for large n . The inequality $\ln^3 n \geq \sqrt{n}$ holds for every n from 4 to 24,128,091, and $\ln^3 n < \sqrt{n}$ for every $n \geq 24,128,092$; similarly, $n^{10} \geq 2^{n/2}$ holds for every n from 2 to 143, and $n^{10} < 2^{n/2}$ for every $n \geq 144$. These thresholds have been computed numerically.

Exercise 10.4 — Asymptotic growth (E4)

Consider the following pairs of functions of a natural variable (f, g) , defined for $n \in \mathbb{N}_{\geq 1}$:

- (a) $f(n) = 2^{n/2}$, $g(n) = 2^n$; (b) $f(n) = 2^{3n/2}$, $g(n) = 2^n$;
 (c) $f(n) = 4^n$, $g(n) = 2^{2n}$; (d) $f(n) = \log_2 n$, $g(n) = \log_{10} n$;
 (e) $f(n) = \ln(n^2 + 1)$, $g(n) = \ln n$; (f) $f(n) = n \ln n$, $g(n) = n^{3/2}$.

Questions

- For each pair, compute $\lim_{n \rightarrow +\infty} \frac{f(n)}{g(n)}$ (for the pair (d), consider $n \in \mathbb{N}_{\geq 2}$, where $g(n) > 0$) and determine, with the limit criterion, whether $f(n) = o(g(n))$, $f(n) = \Theta(g(n))$ or $f(n) = \omega(g(n))$.
- For each pair, indicate, when possible, which property of the chapter «Asymptotic growth» gives the same answer without computing the limit.

Solution

- The limits are the following.

- (a) $\lim_{n \rightarrow +\infty} \frac{2^{n/2}}{2^n} = \lim_{n \rightarrow +\infty} \left(\frac{1}{\sqrt{2}} \right)^n = 0^+$: $f(n) = o(g(n))$.
 (b) $\lim_{n \rightarrow +\infty} \frac{2^{3n/2}}{2^n} = \lim_{n \rightarrow +\infty} (\sqrt{2})^n = +\infty$: $f(n) = \omega(g(n))$.
 (c) $4^n = (2^2)^n = 2^{2n}$, so $\lim_{n \rightarrow +\infty} \frac{4^n}{2^{2n}} = 1$: $f(n) = \Theta(g(n))$ (they are the same function).
 (d) Since $\log_2 n = \frac{\log_{10} n}{\log_{10} 2}$, we have $\lim_{n \rightarrow +\infty} \frac{\log_2 n}{\log_{10} n} = \frac{1}{\log_{10} 2} = \log_2 10 \approx 3.32$: $f(n) = \Theta(g(n))$.
 (e) Since $\ln(n^2 + 1) = \ln(n^2(1 + \frac{1}{n^2})) = 2 \ln n + \ln(1 + \frac{1}{n^2})$, we have

$$\lim_{n \rightarrow +\infty} \frac{\ln(n^2 + 1)}{\ln n} = \lim_{n \rightarrow +\infty} \left(2 + \overbrace{\frac{\ln(1 + \frac{1}{n^2})}{\ln n}}^{\rightarrow 0} \right) = 2 :$$

$$f(n) = \Theta(g(n)).$$

- (f) $\lim_{n \rightarrow +\infty} \frac{n \ln n}{n^{3/2}} = \lim_{n \rightarrow +\infty} \frac{\ln n}{n^{1/2}} = 0^+$: $f(n) = o(g(n))$.

- The properties are the following.

- (a), (b) Scaling of the exponent of an exponential, with $\lambda = \frac{1}{2} < 1$ and $\lambda = \frac{3}{2} > 1$ (or different bases, $\sqrt{2} < 2$ and $2\sqrt{2} > 2$).

- (c) Scaling of the exponent with $\lambda = 2$ would say that $2^{2n} = \omega(2^n)$; here, instead, the base $4 = 2^2$ already absorbs the factor 2, and the two functions coincide.
- (d) Change of base of a logarithm.
- (e) No property of the chapter applies directly, since the argument $n^2 + 1$ is neither of the form $n + \lambda$ nor of the form λn . The identity $\ln(n^2 + 1) = 2 \ln n + \ln\left(1 + \frac{1}{n^2}\right)$ of the first answer plays the same role as the shift of the argument: the term $\ln\left(1 + \frac{1}{n^2}\right)$ tends to 0, and what remains, $2 \ln n$, is $\Theta(\ln n)$.
- (f) Quasi-linear function and powers, with $\alpha = \frac{3}{2} > 1$.